Sharp all-times heat-content comparison and concentration on compact metric graphs

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Abstract

We prove a sharp concentration comparison for the heat equation on a finite compact connected metric graph with a nonempty Dirichlet vertex set and standard Kirchhoff conditions elsewhere. For every nonnegative L^2 initial datum, the decreasing rearrangement of its heat evolution is dominated in integrated mass by the heat evolution of its rearrangement on the interval of the same total length, with a reflecting endpoint and an absorbing endpoint. For initial datum one this yields the sharp heat-content bound at every positive time. For this datum, equality at even one positive time occurs precisely for the mixed interval, up to non-Dirichlet degree-two subdivisions. An exact positive-kernel representation of the deficit separates the contributions of multiple level crossings and unequal crossing slopes. We also give a separate convolution formula for an asymmetric three-edge pitchfork. The proofs adapt established graph coarea, symmetrization and resolvent-to-semigroup methods.

1 Introduction and statements

The heat content of an absorbing metric graph measures the heat remaining from a constant initial temperature. Equivalently, after division by total length it is the survival probability for diffusion started uniformly on the graph. This quantity involves both eigenvalues and the integrals of eigenfunctions. An extremal eigenvalue comparison alone therefore does not give its ordering at every time.

Let G be a finite compact connected metric graph of total length $L > 0$, and let D be a nonempty set of its vertices. We impose zero values at D, continuity and unweighted Kirchhoff conditions elsewhere. Write $T_G(t) = e^{-tH_G}$ for the corresponding heat semigroup. The interval model $I = [0, L]$ has Neumann condition at 0 and Dirichlet condition at L. For $f \geq 0$, its decreasing rearrangement is denoted by f^*, and $\mathcal{M}f(s) = \int_0^s f^*$. Definitions and normalizations are detailed in Section 2. In particular no connectedness of $G \setminus D$ is assumed.

Theorem 1.1 (Heat concentration)**.** *Let $f \geq 0$ belong to $L^2(G)$. If $g \geq 0$ is nonincreasing in $L^2(0, L)$ and*

$$\mathcal{M}f(s) \leq \int_0^s g(x)\,\mathrm{d}x \qquad (0 \leq s \leq L),$$

then for every $t \geq 0$ and $s \in [0, L]$,

$$\mathcal{M}(T_G(t)f)(s) \leq \int_0^s T_I(t)g(x)\,\mathrm{d}x. \tag{1}$$

In particular one may take $g = f^*$.

This is integrated concentration order.

Theorem 1.2 (Sharp heat content and equality)**.** *For every $t \geq 0$,*

$$\mathcal{Q}_G(t) \leq \mathcal{Q}_I(t).$$

If (G, D) is not isometric to an interval with precisely one Dirichlet endpoint and one Neumann endpoint, after suppressing non-Dirichlet degree-two vertices, then for every $t > 0$ and every $s \in (0, L]$,

$$\mathcal{M}(T_G(t)\mathbf{1})(s) < \int_0^s T_I(t)\mathbf{1}(x)\,\mathrm{d}x.$$

Consequently equality in heat content at any one positive time holds if and only if (G, D) is this mixed interval.

The exact defect formula in Theorem 4.2 explains this strictness. Its source is expressed in terms of the actual level slopes of the heat solution, rather than a distance from a family of graphs. Thus it does not provide a uniform geometric stability estimate.

Relation to prior work

Bifulco and Täufer [4, Theorem 2.6 and Remarks 2.8–2.9] prove the fixed-length mixed-interval heat-content comparison in endpoint time regimes: small times under a commensurability assumption and large times in general. Their random-walk expansion also yields an all-times comparison formula for suitable pairs of equilateral graphs [4, Corollary 4.9]. The contribution of Theorem 1.2 relative to that inspected final article is comparison and rigidity at each positive time without a commensurability condition. Bifulco and Mugnolo [3] develop heat-content formulas and graph surgery principles. Their Remark 2.6 already explains the decomposition at Dirichlet vertices; allowing disconnected $G \setminus D$ here is useful scope clarification, rather than a new decomposition method.

The underlying analytic methods have substantial precedent. Friedlander [5, Lemma 3 and Eq. (2.5)] proves the graph coarea and rearrangement estimate on which our level argument rests. The optimal torsional-rigidity inequality is due to earlier work, including [9]. The graph p-torsion preprint [10, Theorem 4.2] is another recent use of this level-set machinery. In the Euclidean setting, parabolic symmetrization and arbitrary-data concentration are classical [2, 8, 1]. In particular retaining the integrated forcing and passing from elliptic resolvents to heat flow follow that tradition. Abstract resolvent and semigroup domination is also established [7]. We give the graph-specific integrated differential inequality, its approximation argument for arbitrary L^2 data, and the continuous-time defect and topology argument explicitly.

These comparisons refer to the particular sources and versions listed below.

2 Graphs, concentration and the interval model

A finite compact metric graph G is obtained from finitely many intervals $[0, \ell_e]$, with $\ell_e > 0$, by identifying their endpoints. We assume that G is connected, has at least one edge, and has total length $L = \sum_e \ell_e > 0$. Integration is with respect to arclength on the edges; vertices have measure zero. Loops and multiple edges are allowed. Each loop has two incident germs at its vertex.

Let D be a nonempty subset of the vertex set. The closed form

$$a_G(\phi, \psi) = \sum_e \int_0^{\ell_e} \phi_e' \overline{\psi_e'}\,\mathrm{d}x, \qquad \operatorname{dom} a_G = \{\phi \in H^1(G) : \phi|_D = 0\}$$

defines a nonnegative self-adjoint operator H_G. Here $H^1(G)$ consists of continuous, edgewise H^1 functions. The operator acts as $-\phi''$ on each edge; outside D its domain satisfies continuity

and the unweighted Kirchhoff condition. At a vertex in D there is no additional Kirchhoff condition. A degree-one vertex outside D is consequently Neumann. Write $T_G(t) = e^{-tH_G}$. The operator and its semigroup are unchanged by inserting or suppressing non-Dirichlet vertices of degree two.

We do not assume that $G \setminus D$ is connected. Cutting all Dirichlet incidences decomposes the form into finitely many connected positive-length components, each with a Dirichlet endpoint. Indeed a path in the original connected graph from any point to D meets such an endpoint. On each component the form has trivial kernel. This decomposition is already recorded in [3, Remark 2.6].

The form contraction property gives a positive sub-Markovian semigroup and positive resolvents. On each cut component the resolvent is positivity improving. To recall the elementary maximum-principle argument, if $u = (1 + hH_G)^{-1} f \geq 0$, $f \geq 0$, has an interior zero, then $u' = 0$ there and $u'' \leq u/h$. The integral form of this inequality, with zero initial value and slope, forces u to vanish on a short interval and then on the edge. At a zero non-Dirichlet vertex the outgoing derivatives are nonnegative and have zero sum, so all vanish and the same propagation applies on each germ. Connectedness forces $u = 0$, and hence $f = 0$, on that component. Therefore nonzero nonnegative forcing gives strict positivity off its Dirichlet endpoints. The standard equivalence between resolvent and semigroup positivity improvement [6, Theorem A.2] gives positivity improvement of the heat semigroup on each component. Here improvement means strict positivity almost everywhere; this is sufficient below.

The comparison operator H_I acts on $I = [0, L]$, with

$$H_I = -\frac{d^2}{dx^2}, \qquad \phi'(0) = 0, \quad \phi(L) = 0.$$

Thus decreasing data are placed with their largest values at the reflecting endpoint. Our diffusion generator is d^2/dx^2, without a factor $1/2$. The heat content means

$$\mathcal{Q}_G(t) = \langle \mathbf{1}, T_G(t)\mathbf{1} \rangle_{L^2(G)};$$

it is not the heat trace. In particular

$$\mathcal{Q}_I(t) = \frac{8L}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp\bigl(-(2n+1)^2 \pi^2 t/(4L^2)\bigr)}{(2n+1)^2}. \tag{2}$$

Definition 2.1. For a nonnegative function f on a nonatomic space of measure L, let f^* be its nonincreasing rearrangement on $(0, L)$, and set

$$\mathcal{M}f(s) = \int_0^s f^*(r)\,\mathrm{d}r = \sup_{|E|=s} \int_E f, \qquad 0 \leq s \leq L.$$

The order $f \preceq g$ means $\mathcal{M}f(s) \leq \mathcal{M}g(s)$ for all s.

The supremum permits measurable subsets of a level plateau. Two continuity facts will be useful:

$$\|\mathcal{M}f - \mathcal{M}k\|_\infty \leq \|f - k\|_1 \leq \sqrt{L}\,\|f - k\|_2, \tag{3}$$

$$\|f^* - k^*\|_2 \leq \|f - k\|_2. \tag{4}$$

The first follows directly from the supremum formula. For the second, equimeasurability preserves the squared norms and Hardy–Littlewood gives $\int fk \leq \int f^* k^*$. Expanding the squared distances proves the claim.

There is a second, reversed mixed operator in the proof. On the *mass coordinate* $s \in [0, L]$, let

$$\mathsf{A} = -\frac{d^2}{ds^2}, \qquad W(0) = 0, \quad W'(L) = 0, \qquad S_h = (1 + h\mathsf{A})^{-1}.$$

The mixed maximum principle shows that S_h is positive and satisfies

$$\|S_h F - S_h K\|_\infty \le \|F - K\|_\infty \quad (F, K \in C[0, L]). \tag{5}$$

For example, subtract the constant $\|F - K\|_\infty$ from the difference and use its nonpositive value at 0 and zero derivative at L. The distinction between the physical conditions N at 0, D at L, and the integrated conditions D at 0, N at L, is essential.

Lemma 2.2 (The decreasing interval cone)**.** *If $g \ge 0$ is nonincreasing and belongs to $L^2(0, L)$, then $w = (1 + hH_I)^{-1} g$ is nonnegative and nonincreasing. Moreover*

$$\int_0^s w(x)\,\mathrm{d}x = S_h\left(\int_0^{\cdot} g(x)\,\mathrm{d}x\right)(s). \tag{6}$$

Proof. For smooth decreasing g, positivity gives $w \ge 0$. Its derivative $p = w'$ satisfies $p - hp'' = g' \le 0$, $p(0) = 0$, and $p(L) \le 0$, the latter because $w(L) = 0$ and $w \ge 0$. The scalar maximum principle gives $p \le 0$. For general g, truncate, extend constantly past both endpoints, and mollify with a nonnegative kernel. This gives nonnegative smooth decreasing approximants in L^2. Resolvent continuity and closedness of the decreasing cone prove the assertion. Integrating $w - hw'' = g$, using $w'(0) = 0$ and $w(L) = 0$, proves (6), also for L^2 data. □

3 Resolvent concentration and the heat comparison

The graph coarea estimate in the next argument is the one used by Friedlander [5, Lemma 3, pp. 203–205]. We retain the integrated forcing term, as in classical Talenti and parabolic concentration arguments [8, 1]. An approximation step allows arbitrary data without assuming concavity or finite critical-level structure for their resolvents.

Lemma 3.1 (A dense analytic forcing class)**.** *The nonnegative cone of $L^2(G)$ has a dense subset consisting of functions that are strictly positive nonconstant real polynomials on every closed edge. If f belongs to this subset and $u = (1+hH_G)^{-1} f$, then u is positive off D, analytic and nonconstant on each closed edge, and has only finitely many vertex and critical values.*

Proof. On an edge of length ℓ, choose real polynomials a_j converging to $\sqrt{f}$ in L^4, and choose $\varepsilon_j \downarrow 0$. Put

$$f_j(x) = a_j(x)^2 + \varepsilon_j(1 + x/\ell).$$

Then $f_j \ge \varepsilon_j$, and

$$\|a_j^2 - f\|_2 \le \|a_j - \sqrt{f}\|_4(\|a_j\|_4 + \|\sqrt{f}\|_4) \longrightarrow 0.$$

Each f_j is nonconstant: a nonconstant a_j^2 has leading even degree at least two, while a constant a_j leaves the added linear term. Apply the construction independently on the finitely many edges. Forcings need no vertex compatibility.

The weak maximum principle gives $u \ge 0$. An interior zero would contradict $u - hu'' = f > 0$ at a nonnegative minimum. At a non-Dirichlet zero vertex, all outgoing derivatives are nonnegative; Kirchhoff makes all zero, and the edge equation gives $u'' = -f/h < 0$ there, again contradicting nonnegativity immediately inside an edge. Thus $u > 0$ off D. On each edge the solution is a polynomial particular solution plus two exponentials. It extends analytically beyond the closed edge and cannot be constant, since its forcing is nonconstant. Its derivative therefore has finitely many zeros on that closed edge. The asserted finiteness follows. □

Lemma 3.2 (Inverse-distribution regularity)**.** *Suppose u has the regularity and positivity properties of Lemma 3.1, with $u|_D = 0$. Then its rearrangement $q = u^*$, extended by $q(0) = \max u$ and $q(L) = 0$, is Lipschitz. Consequently $V = \mathcal{M}u \in W^{2,\infty}(0, L)$, with $V(0) = 0$ and $V'(L) = 0$.*

Proof. Every level set has zero length, and $\mu(z) = |\{u > z\}|$ is continuous. Let $M = \max_e \|u_e'\|_\infty > 0$. Choose a simple arclength path from a point of D to a maximizer of u. For $0 \le \alpha < \beta \le \max u$, choose the *first* β crossing on this path, and the *last* preceding α crossing. On the open segment between them, $\alpha < u < \beta$. The segment is counted only once in edge measure and has length at least $(\beta - \alpha)/M$. Hence

$$\mu(\alpha) - \mu(\beta) \ge \frac{\beta - \alpha}{M}. \tag{7}$$

In particular μ strictly decreases from L to 0, and its inverse q is M-Lipschitz, including at its endpoints. Integrating q proves the claim. Paths may pass through other Dirichlet vertices; connectedness of $G \setminus D$ has not been used. □

Proposition 3.3 (Shifted resolvent comparison)**.** *For every $h > 0$ and $f \ge 0$ in $L^2(G)$,*

$$\mathcal{M}\big((1 + hH_G)^{-1} f\big) \le S_h \mathcal{M} f = \mathcal{M}\big((1 + hH_I)^{-1} f^*\big) \quad \text{on } [0, L]. \tag{8}$$

Proof. First take f from Lemma 3.1, set $u = (1 + hH_G)^{-1} f$ and $V = \mathcal{M}u$, and let $z \in (0, \max u)$ avoid all vertex and critical values. Write $s = \mu(z)$ and, over *all* crossings of the level z, define

$$A(z) = \sum_{u=z} |u'|, \qquad B(z) = \sum_{u=z} \frac{1}{|u'|}, \qquad N(z) = \#\{u = z\}.$$

There is at least one crossing by the path in Lemma 3.2. Coarea on the finitely many monotone edge pieces and inverse differentiation give

$$-\mu'(z) = B(z), \qquad V''(s) = -\frac{1}{B(z)}, \qquad A(z)B(z) \ge N(z)^2 \ge 1. \tag{9}$$

The last inequality is Cauchy–Schwarz, as in [5, Eq. (2.5)].

Integrate $u - hu'' = f$ over the superlevel set $E_z = \{u > z\}$. Every hot non-Dirichlet vertex includes all its germs and contributes zero by Kirchhoff. Dirichlet vertices do not occur in a positive hot set. Every boundary crossing has outward flux $-|u'|$. Thus

$$V(s) + hA(z) = \int_{E_z} f \le \mathcal{M} f(s). \tag{10}$$

Disconnected hot sets cause no change to this identity. Combining (9)–(10) yields

$$V - hV'' \le \mathcal{M} f \quad \text{almost everywhere}, \qquad V(0) = 0, \quad V'(L) = 0.$$

Lemma 3.2 justifies the weak equation across all exceptional mass coordinates. If $Z = V - S_h \mathcal{M} f$, testing $Z - hZ'' \le 0$ with Z_+ gives

$$\int_0^L Z_+^2 + h \int_0^L |(Z_+)'|^2 \le 0.$$

The boundary terms vanish, so $V \le S_h \mathcal{M} f$.

For arbitrary $f \ge 0$, approximate by Lemma 3.1. The graph resolvent is an L^2 contraction; (3) gives uniform convergence of both mass profiles, and (5) permits the limit on the right. The proof uses the stated approximation of the resolvent. Finally Lemma 2.2 identifies $S_h \mathcal{M} f$ with the last expression in (8). □

Proof of Theorem 1.1. Fix $h > 0$. Set $u_m = (1 + hH_G)^{-m} f$ and $v_m = (1 + hH_I)^{-m} g$, where g is the decreasing interval datum in the theorem. Let $V_m = \mathcal{M} u_m$ and $F_m(s) = \int_0^s v_m$. Lemma 2.2 makes every v_m nonnegative and decreasing, and Proposition 3.3 gives

$$V_m \le S_h V_{m-1}, \qquad F_m = S_h F_{m-1}.$$

Since $V_0 \le F_0$ and S_h is positive, induction yields $V_m \le F_m$ for every m.

For fixed $t > 0$, set $h = t/n$ and $m = n$. Spectral calculus gives, strongly in the respective L^2 spaces,

$$(1 + tH_G/n)^{-n} f \longrightarrow T_G(t) f, \qquad (1 + tH_I/n)^{-n} g \longrightarrow T_I(t) g.$$

Indeed the scalar spectral multipliers converge and are bounded by one. Equation (3) passes the comparison uniformly in s. The decreasing cone is closed in L^2, so the interval heat solution is still decreasing. At $t = 0$ the claimed inequality is the hypothesis; for $g = f^*$ it is equality. □

Remark 3.4. The argument proves concentration at every repeated-resolvent step before taking the heat limit. It does not infer timewise order from a scalar Laplace-transform inequality or from ordering two quadratic forms. General resolvent–semigroup domination frameworks, including decreasing rearrangements, are prior work [7]; the explicit graph comparison above verifies the structure needed here.

4 The exact defect and equality at a single time

In this section the initial datum is $\mathbf{1}$. Put

$$u(t) = T_G(t)\mathbf{1}, \quad v(t) = T_I(t)\mathbf{1}, \quad U(t,s) = \mathcal{M}u(t)(s), \quad F(t,s) = \int_0^s v(t,x)\,\mathrm{d}x, \quad W = F - U.$$

Theorem 1.1 gives $W \ge 0$. We will identify its source of strict positivity directly in continuous time.

Lemma 4.1 (Concavity of the survival profile)**.** *For $t > 0$, $u(t) > 0$ off D, and $u_{xx}(t,x) = u_t(t,x) < 0$ almost everywhere on every open edge. In particular each edge profile is strictly concave. The derivatives used here are bounded on each closed edge, locally uniformly for positive times.*

Proof. Sub-Markovianity and positivity imply $u(t + h) = T_G(t)T_G(h)\mathbf{1} \le T_G(t)\mathbf{1} = u(t)$. For positive time, spectral calculus puts $u(t)$ in every power domain of H_G; edgewise regularity gives the asserted smoothing and local bounds. In particular $w(t) = -u_t(t) = H_G u(t) \ge 0$.

Work on any component obtained by cutting at D. Its Dirichlet operator has trivial kernel, while $u(r)$ is nonzero there by injectivity of e^{-rH_G}. Thus $w(r)$ is nonzero on that component. For $0 < r < t$, $w(t) = T_G(t-r)w(r)$, so componentwise positivity improvement gives $w(t) > 0$ almost everywhere on each component. The edge heat equation $u_t = u_{xx} = -w$ implies strict concavity on every edge. Since $u \ge 0$, strict concavity gives positivity in every open edge. A zero value at a non-Dirichlet vertex would give a positive derivative into each incident edge, contrary to Kirchhoff. Thus $u(t) > 0$ at those vertices as well. □

Each positive-time edge profile is strictly concave, so there are finitely many critical and vertex levels. Lemma 3.2, with the same first-upper-crossing selection, applies at each time: its proof uses finite critical levels, positivity and a slope bound, not the resolvent equation. Thus $U(t,\cdot) \in W^{2,\infty}$ locally uniformly for $t > 0$, with $U(t,0) = 0$ and $U_s(t,L) = 0$.

At a regular level $z = u(t)^*(s)$, write $\alpha_1, \dots, \alpha_N > 0$ for the absolute crossing slopes and set

$$A = \sum_i \alpha_i, \qquad B = \sum_i \alpha_i^{-1}.$$

Coarea gives $U_{ss} = -1/B$. The unique top-s maximizing set is the hot set $E(t,s) = \{u(t) > u(t)^*(s)\}$, up to null sets. Its envelope derivative is

$$U_t(t,s) = \int_{E(t,s)} u_t(t,x)\,\mathrm{d}x = -A. \tag{11}$$

For clarity, the first equality does not omit a moving-boundary term. Bracket the time difference by evaluating at the old and new maximizing sets. Uniform convergence of u and continuity of its inverse distribution place their symmetric difference in a shrinking band around a null level set. Both difference quotients have the same limit. The second equality is the integrated heat equation, with the same Kirchhoff cancellation as in (10). Also $|U(t,s) - U(r,s)| \le \|u(t) - u(r)\|_1$ gives local Lipschitz regularity in time.

Theorem 4.2 (Exact nonnegative defect)**.** *There is a nonnegative measurable function $d(t,s)$, bounded on each compact positive-time cylinder, such that*

$$d(t,s) = A - \frac{1}{B} = \frac{N^2-1}{B} + \frac{1}{B}\sum_{i<j} \frac{(\alpha_i - \alpha_j)^2}{\alpha_i \alpha_j} \tag{12}$$

at almost every mass coordinate of every positive-time profile, and

$$W_t - W_{ss} = d, \qquad W(t,0) = 0, \quad W_s(t,L) = 0.$$

Let K be the heat kernel for A, with Dirichlet condition at the mass endpoint 0 and Neumann condition at L. Then

$$W(t,s) = \int_0^t \int_0^L K(t-r;s,y) d(r,y)\,\mathrm{d}y\,\mathrm{d}r, \qquad t > 0, \quad 0 \le s \le L. \tag{13}$$

The integral is the finite limit of nonnegative integrals starting at $\varepsilon > 0$; no separate integrability assumption at time zero is needed.

Proof. Equations (9) and (11) give $U_t - U_{ss} = -(A - 1/B)$. Moreover

$$AB - N^2 = \sum_{i<j} \left(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} - 2 \right),$$

which proves (12). In particular $d \ge 0$ and $d = 0$ precisely when $N = 1$ at a regular level. If $M(t) = \max_e \|u_x(t)\|_\infty$, then $d \le A \le 2|\mathsf{E}(G)|M(t)$. The weak derivatives define a jointly measurable version of d. For each time only finitely many mass values are exceptional, so the identities hold almost everywhere on positive-time cylinders. The established Sobolev and Lipschitz bounds exclude a singular residual.

The interval profile satisfies $F_t = F_{ss}$ with the displayed mixed mass conditions. On $[\varepsilon, t]$, variation of constants for the weak mixed heat problem gives

$$\begin{aligned} W(t,s) = &\int_0^L K(t-\varepsilon;s,y) W(\varepsilon,y)\,\mathrm{d}y \\ &+ \int_\varepsilon^t \int_0^L K(t-r;s,y) d(r,y)\,\mathrm{d}y\,\mathrm{d}r. \end{aligned} \tag{14}$$

This identity first holds in L^2; smoothing and a final-time-strip bound by the local L^∞ norm of d give continuous spatial representatives and equality at each s, including L. Both $U(\varepsilon,\cdot)$ and $F(\varepsilon,\cdot)$ tend uniformly to s by strong L^2 continuity and (3). The positive mixed semigroup is an L^∞ contraction. Hence the first term of (14) is nonnegative and tends uniformly to zero. The second increases as $\varepsilon \downarrow 0$. Monotone convergence proves (13) and its finiteness. □

Lemma 4.3 (A multiple-crossing band)**.** *Unless (G, D) is the DN interval up to non-Dirichlet degree-two subdivisions, every positive-time profile has an open interval of positive levels on which $N(z) \geq 2$ at every regular level.*

Proof. Let $m_D = \sum_{v \in D} \deg(v)$, counting germs. If $m_D \geq 2$, strict concavity and positivity give a positive derivative away from every Dirichlet incidence: it exceeds a positive secant slope. Choose disjoint short germ neighborhoods. They each cross every sufficiently small positive level, so $N(z) \geq m_D \geq 2$ there.

If $m_D = 1$, D is one leaf. A finite connected graph with a leaf and all degrees at most two is a path, giving precisely the DN interval. Otherwise there is a non-Dirichlet vertex v of degree $k \geq 3$. Put $c = u(t, v) > 0$ and let p_i be its outgoing derivatives. Kirchhoff gives $\sum_i p_i = 0$. A positive p_i gives a short increasing germ above c; a negative p_i gives a decreasing germ below c. A zero p_i also gives a strictly decreasing germ below c, by strict concavity. If all slopes vanish, all k germs cross a small band below c. Otherwise the numbers of upward and downward germs sum to k, and at least one of these two numbers is at least two. Choose the corresponding band above or below c, with a below band contained in $(0, c)$. Germ neighborhoods can be chosen disjoint even for loops. Omitting the finitely many exceptional levels does not remove the positive-width band. □

Proof of Theorem 1.2. For every positive time, Lemma 4.3 and (12) make $d(t, \cdot) > 0$ on a set of positive measure, unless (G, D) is the DN interval. Level bands correspond to mass bands of positive width by the strictly decreasing distribution. On every time interval $0 < a < b < t$, measurability and Fubini therefore give $\int_a^b \int_0^L d(r, y)\, \mathrm{d}y\, \mathrm{d}r > 0$.

The mixed kernel satisfies $K(\tau; s, y) > 0$ for $\tau > 0$ and $s, y \in (0, L]$. One way to see this is reflection about L:

$$K(\tau; s, y) = K_{DD,2L}(\tau; s, y) + K_{DD,2L}(\tau; s, 2L - y),$$

where $K_{DD,2L}$ is the positive Dirichlet kernel on $(0, 2L)$. Thus (13) gives $W(t, s) > 0$ for every $s > 0$. Conversely the stated interval and its non-Dirichlet subdivisions have the same operator as H_I, and $u(t)$ is already decreasing from its Neumann endpoint. Hence $W = 0$ there. At $s = L$ this proves the heat-content equality assertion. □

Corollary 4.4 (A source-dependent lower bound)**.** *Fix $t > 0$, $s > 0$, $0 < a < b < t$ and $0 < \delta < L$. Then*

$$W(t, s) \geq \kappa \int_a^b \int_\delta^L d(r, y)\, \mathrm{d}y\, \mathrm{d}r, \qquad \kappa = \min_{\substack{r \in [a,b] \\ y \in [\delta, L]}} K(t - r; s, y) > 0.$$

For a noninterval graph, some δ makes the integral positive. This is a graph-dependent defect estimate, not a uniform geometric stability theorem.

Corollary 4.5 (Integrated defect)**.** *The exact identity*

$$\int_0^\infty W(t, s)\, \mathrm{d}t = \int_0^\infty \int_0^L \min(s, y) d(r, y)\, \mathrm{d}y\, \mathrm{d}r$$

holds. In particular

$$\frac{L^3}{3} - \langle \mathbf{1}, H_G^{-1} \mathbf{1} \rangle = \int_0^\infty \int_0^L y\, d(r, y)\, \mathrm{d}y\, \mathrm{d}r.$$

Proof. Use Tonelli in (13) and $\int_0^\infty K(\tau; s, y)\, \mathrm{d}\tau = \min(s, y)$. The left side is finite since $0 \leq W \leq F$ and H_I has a positive spectral gap. At $s = L$, integrate the heat contents and use the interval torsion value $L^3/3$. □

The optimal torsion inequality is prior work [9]; Corollary 4.5 records its relation to this heat-level defect and is not presented as a new torsion maximization theorem.

5 An independent formula for the asymmetric pitchfork

Consider a stem of length $a > 0$ joining a Dirichlet leaf to a Kirchhoff junction, with two further arms of lengths $b, c > 0$ ending at Neumann leaves. Write $P = P(a; b, c)$ and $L = a+b+c$. This example admits a separate positive convolution proof. It is a supporting exact calculation; general pendant-edge surgery principles are treated in [3, Section 6].

For $s > 0$ put $k = \sqrt{s}$, $A = \tanh(ka)$, $B = \tanh(kb)$ and $C = \tanh(kc)$. Let $Z_P(s)$ be the Laplace transform of the absorption time for diffusion on P started at the junction, and let $Z_I(s)$ be the corresponding transform on the mixed interval started at distance a from its absorbing endpoint. This point has coordinate $L - a$ in our Neumann-at-zero convention. Solving $Z'' = sZ$, with $Z = 1$ at the absorbing endpoint and the usual conditions elsewhere, gives

$$Z_P(s) = \frac{1}{\cosh(ka) + (B + C)\sinh(ka)}, \qquad Z_I(s) = \frac{\cosh(k(b + c))}{\cosh(kL)}. \tag{15}$$

The same calculation gives the heat-content transforms

$$\widehat{\mathcal{Q}}_P(s) = \frac{L}{s} - \frac{A + B + C}{sk(1 + A(B + C))}, \qquad \widehat{\mathcal{Q}}_I(s) = \frac{L}{s} - \frac{\tanh(kL)}{sk}. \tag{16}$$

For completeness, $\widehat{u} = (1 - Z)/s$ solves the heat resolvent with initial datum one. Integrating $Z'' = sZ$ over the graph cancels all Kirchhoff and Neumann fluxes. On the stem, with distance x from the absorbing endpoint, $Z(x) = \cosh(kx) + d\sinh(kx)$; the junction condition gives $d = -(A+B+C)/(1+A(B+C))$. Hence $\int_P Z = -Z'(0)/s = (A+B+C)/(k(1+A(B+C)))$, proving (16).

Proposition 5.1 (Pitchfork convolution identity)**.** *For $\ell > 0$ define the positive locally integrable function*

$$h_\ell(t) = \frac{2}{\ell}\sum_{n=0}^{\infty} \exp\left(-\frac{(n + 1/2)^2\pi^2 t}{\ell^2}\right), \qquad t > 0.$$

Let μ_P, μ_I be the absorption-time probability measures whose Laplace transforms are (15)*. Then*

$$\mathcal{Q}_I(t) - \mathcal{Q}_P(t) = \big(h_b * h_c * h_{b+c} * \mu_P * \mu_I\big)(t), \qquad t > 0. \tag{17}$$

In particular the deficit is nonnegative, and

$$\int_0^\infty (\mathcal{Q}_I(t) - \mathcal{Q}_P(t))\,\mathrm{d}t = bc(b + c).$$

Proof. The interval mixed eigenfunction expansion, or the partial fraction expansion of tanh, gives

$$\int_0^\infty e^{-st} h_\ell(t)\,\mathrm{d}t = \frac{\tanh(\ell\sqrt{s})}{\sqrt{s}}, \qquad \int_0^\infty h_\ell(t)\,\mathrm{d}t = \ell.$$

The transforms Z_P, Z_I are those of probability measures: absorption is almost sure on these finite anchored graphs. This also follows by letting $s \downarrow 0$ in (15). Subtract (16), use $\tanh(k(b+c)) = (B + C)/(1 + BC)$, and simplify to obtain

$$\widehat{\mathcal{Q}}_I(s) - \widehat{\mathcal{Q}}_P(s) = \frac{\tanh(kb)}{k}\frac{\tanh(kc)}{k}\frac{\tanh(k(b + c))}{k} Z_P(s)Z_I(s).$$

Tonelli and uniqueness of Laplace transforms prove the convolution identity almost everywhere. It holds everywhere by continuity: $h_\ell(t) = O(t^{-1/2})$ near zero and decays exponentially at infinity, so the triple convolution extends to a continuous bounded function on the real line when set to zero for $t \le 0$. Convolving with the finite measures preserves continuity. Finally integrate by Tonelli and multiply the five total masses. Strict positivity for each $t > 0$ also follows from Theorem 1.2. □

6 Scope and remaining literature questions

The comparison concerns finite compact connected unweighted graphs with positive edge lengths, a nonempty Dirichlet vertex set, and standard Kirchhoff conditions at all remaining vertices. It includes loops, multiple edges and disconnected Dirichlet-cut interiors. Equality is classified only for the constant initial datum. Arbitrary-data equality can have additional degeneracies and is not addressed.

The source-dependent lower bound in Corollary 4.4 depends on the heat solution. Turning it into a uniform estimate in terms of edge lengths, topology or distance from an interval would require additional work. The survival-probability interpretation is exact for the stated boundary conditions and scalar diffusion operator.

Our literature comparison was checked through 20 October 2026. The final journal versions of [4, 3, 5, 9] and the original texts of [8, 1] were inspected. The abstract framework of [7] was checked in arXiv:1711.07225v1 (20 November 2017); the positivity criterion in [6] was checked in arXiv:1101.0373v2 (18 December 2012). The full text of the earlier symmetrization article [2] was not available for a direct theorem-by-theorem comparison; we credit its methodological precedent.

A Potsdam announcement for Täufer's seminar of 8 May 2024 [11] uses the title "Anti-Faber–Krahn property for the heat content on metric graphs". A full proof or slides resolving its relationship to the present theorem were not located. Related later talk abstracts are insufficient to resolve the issue. This is a remaining priority and interpretation uncertainty. Any earlier equivalent theorem or proof should receive full credit, with the contribution here narrowed to its actual additional content. The complete proofs above make the validity of the stated results independently assessable despite that unresolved literature question.

References

[1] A. Alvino, G. Trombetti and P.-L. Lions, Comparison results for elliptic and parabolic equations via Schwarz symmetrization, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **7** (1990), no. 2, 37–65. doi:10.1016/S0294-1449(16)30303-1.

[2] C. Bandle, On symmetrizations in parabolic equations, *J. Anal. Math.* **30** (1976), 98–112. doi:10.1007/BF02786706.

[3] P. Bifulco and D. Mugnolo, On the Heat Content of Compact Quantum Graphs, *Ann. Henri Poincaré*, published online 9 September 2026. doi:10.1007/s00023-026-01755-3.

[4] P. Bifulco and M. Täufer, Faber–Krahn inequality for the heat content on quantum graphs via random walk expansion, *Electron. J. Probab.* **30** (2025), paper 158, 1–19. doi:10.1214/25-EJP1414.

[5] L. Friedlander, Extremal properties of eigenvalues for a metric graph, *Ann. Inst. Fourier* **55** (2005), no. 1, 199–211. doi:10.5802/aif.2095.

[6] M. Keller, D. Lenz, H. Vogt and R. K. Wojciechowski, Note on basic features of large time behaviour of heat kernels, *J. Reine Angew. Math.* **708** (2015), 73–95. doi:10.1515/crelle-2013-0070; version checked: arXiv:1101.0373v2, 18 December 2012.

[7] D. Lenz, M. Schmidt and M. Wirth, Domination of quadratic forms, *Math. Z.* **296** (2020), 761–786. doi:10.1007/s00209-019-02440-4; version checked: arXiv:1711.07225v1, 20 November 2017.

[8] J. Mossino and J.-M. Rakotoson, Isoperimetric inequalities in parabolic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **13** (1986), no. 1, 51–73. NUMDAM original text.

[9] D. Mugnolo and M. Plümer, On torsional rigidity and ground-state energy of compact quantum graphs, *Calc. Var. Partial Differential Equations* **62** (2023), article 27; published online 9 November 2022. doi:10.1007/s00526-022-02363-9.

[10] S. Özcan, On the p-torsional rigidity of compact metric graphs: a sharp Kohler–Jobin inequality, arXiv:2607.12333v2, 15 July 2026 (preprint).

[11] M. Täufer, Anti-Faber–Krahn property for the heat content on metric graphs (or "Building better batteries"), seminar announcement, University of Potsdam, 8 May 2024. Potsdam announcement, accessed 20 October 2026.