# Scalar Casimir interactions of holes in a Dirichlet screen: many-hole terms, monotone attraction and near-contact laws

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**Abstract**

Kabat, Karabali and Nair computed the leading Casimir interaction of two small holes in a Dirichlet plate for a massless scalar field, $E = -5Q_1Q_2/(32\pi^3 r^7)$, asked how the interaction behaves as the holes approach, raising the possibility of a minimum at finite separation, and speculated that round holes have the largest charge $Q$ for a given area. We settle both for the scalar screen. For $m$ windows at separation scale $L$ in tangential dimension $d$, the connected $m$-body vacuum energy is of order $L^{-m(d+1)-1}$, with an explicit coefficient summing window torsions over directed Hamiltonian cycles, and we obtain its thermal crossover at inverse temperature $\beta = \lambda L$. For $d = 2$ the hole charge is exactly the torsional rigidity of the half-Laplacian: $Q = 4R^3/3$ for a round hole, replacing the published numerical value $1.28R^3$, and $Q = \pi W^2/8$ for a slit. The rearrangement inequality of Almgren and Lieb then confirms the speculation: among holes of equal area the disk has the largest charge, and among Lipschitz holes only the disk attains it. An exact identity between a regularized boundary determinant and the connected heat trace shows strict attraction along any separation path on which all inter-window distances increase; two round holes attract at every disjoint separation, so there is no minimum. As the gap $g$ between two windows closes, the vacuum energy diverges like $-g^{-(d+1)/2}$ times a curvature integral and a positive planar constant. For apertures moving apart along a common normal, the connected determinant is jointly completely monotone in squared separation and squared mass.

**Keywords.** Scalar Casimir interaction, Dirichlet screen, apertures, relative heat trace, fractional torsion, Saint-Venant inequality, Matsubara sum, near-contact asymptotics.
**MSC 2020.** Primary 81T55; Secondary 35R11, 35K08, 60J65.

## Overview

Sections 1–6 treat separated windows: the connected energy and its thermal crossover (Theorem 2.1), strict attraction (Theorem 2.2) and the exact hole charges in dimension two. Sections 7–11 treat the opposite end of the disjoint family, two windows whose gap closes (Theorem 8.1). Both parts rest on the connected heat trace of the screen, whose positivity, screen limit and many-window diffusion law are proved in [21, 22].

## 1 Separated holes: introduction

Kabat, Karabali, and Nair [1, Sections II and V] use the square-root boundary operator of a scalar field on both sides of a planar Dirichlet screen to obtain the leading two-hole vacuum interaction.

Their large-separation analysis of several holes gives the leading pairwise total energy. We take the alternating sum over all subsets of open windows. It removes every proper-subset interaction, and for a fixed number of windows its first surviving coefficient is a sum of cycle products. This coefficient gives both the vacuum law and its temperature dependence on the scale $\beta \asymp L$.

Kabat, Karabali, and Nair also compare the charges of round, square and long rectangular holes of equal area, and they close that comparison with a speculation: it is "tempting to speculate that round holes have the largest charge for a given area" [1, Section II.B]. Their charge [1, eq. (11)] is exactly the torsional rigidity of the half-Laplacian on the hole (Section 6). Proposition 6.1 combines this identity with the fractional Pólya–Szegő inequality of Almgren and Lieb [10, Theorem 9.2] to confirm the speculation for every bounded open hole; the disk is the only maximizer among holes with Lipschitz boundary, by the equality case of Frank and Seiringer [11, Theorem A.1] as used in the fractional Saint-Venant inequality of Brandolini, de Bonis, Ferone, Piscitelli and Volzone [12, Proposition 3.4]. What is new here is the identification of the charge; the shape inequality then follows from known rearrangement results. This third edition adds Proposition 6.1, states this answer and corrects the citation of the round-hole value $1.28R^3$ to [1, eq. (17)]; the other results and proofs of the second edition are unchanged.

The all-hit bridge mechanism and irreducible scattering expansions for obstacles appear in Schaden [2, eqs. (15)–(18), (22)–(23)]. Emig, Graham, Jaffe, and Kardar [3, Sections II.C, III.C, and IV.D] organize scalar vacuum interactions through multiple-scattering paths and low-frequency shape factors. Hanisch, Strohmaier, and Waters [4, Theorems 1.6–1.7] establish the rigorous relative determinant for compact obstacles with the single-object blocks removed. We apply this normalization to the square-root form of a screen with openings, then use full subset inclusion–exclusion. The connected heat representation is the starting point of [22].

Kabat and Karabali [5, eqs. (10), (26)] use the Matsubara sum and Poisson resummation for scalar diffraction at plate edges and a slit. Rahi et al. [6] give general finite-temperature scattering formulas for electromagnetic bodies. We compute the zero mode and the leading low-temperature correction for the connected window coefficient. For the force, the reflection-based attraction results of Kenneth and Klich [7] and Bachas [8] are antecedents. In the screen model, order of the pointwise kernels along a prescribed separation path gives the strict derivative. The path criterion applies to unequal round windows and to the all-window connected interaction.

## 2 The screen model and its connected energy

Fix $d \geq 1$, $m \geq 2$, distinct centers $p_1, \dots, p_m \in \mathbb{R}^d$, and nonempty bounded open windows $K_1, \dots, K_m \subset \mathbb{R}^d$. The windows at separation $L$ are $Lp_i + K_i$. Take $L$ for which their closures are pairwise disjoint. Each window can carry a bounded nonnegative contact profile $\rho_i$, fixed under translation. For $S \subseteq [m]$, write $K_{L,S} = \bigcup_{i \in S}(Lp_i + K_i)$, with the corresponding translated contact on its components. The open domain for this subset is

$$\mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K_{L,S}) \times \{0\}).$$

Its contact adds the surface integral of $\rho_i |f|^2$ on each open window to the Dirichlet form. Denote the heat gain relative to the two cut Dirichlet half-spaces by $D_{L,S}(t)$, defined by the integrated diagonal difference for generator $\Delta$. Let $D_{L,\varnothing} = 0$ and

$$\mathcal{I}_L(t) = \sum_{S \subseteq [m]} (-1)^{m-|S|} D_{L,S}(t).$$

As in [22], this counts bridges visiting every aperture, with the Feynman–Kac weight of the contacts. It is finite and strictly positive for every $t > 0$. For fixed separated windows it is

bounded at small $t$ by $Ct^{-N}e^{-c/t}$ for some $N, c > 0$. The contact-free single-screen law [22] bounds each $D_{L,S}(t)$ by $O(t^{-(d+1)/2})$ at infinity, since nonnegative contact only reduces the heat gain. We use $\hbar = c = k_{\mathrm{B}} = 1$ and set

$$\mathcal{E}_L = -\frac{1}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \mathcal{I}_L(t)\, dt, \tag{1}$$

$$\mathcal{F}_{L,\beta} = -\frac{1}{2\beta} \int_0^\infty t^{-1} \mathcal{I}_L(t) \sum_{n\in\mathbb{Z}} e^{-t(2\pi n/\beta)^2}\, dt, \qquad \beta > 0. \tag{2}$$

Both converge absolutely by these bounds and are strictly negative. The first is the connected relative energy of the scalar function $\frac{1}{2}\sqrt{a}$. The second corresponds to $\frac{1}{2}\sqrt{a} + \beta^{-1}\log(1 - e^{-\beta\sqrt{a}})$ at spectral energy $a$. In fact the derivative of the Matsubara logarithm representation at $a > 0$ is

$$\frac{1}{2\beta} \sum_{n\in\mathbb{Z}} \frac{1}{a + (2\pi n/\beta)^2} = \frac{1}{4\sqrt{a}} \coth\!\left(\frac{\beta\sqrt{a}}{2}\right),$$

the derivative of that free-energy function. Spectrally constant terms cancel in the connected difference.

Let $F_{K_i}$ be the exterior-Dirichlet half-Laplacian, the compression of the Fourier multiplier $|\xi|$ to the closure of zero extensions of $C_c^\infty(K_i)$ in $H^{1/2}(\mathbb{R}^d)$. Let

$$T_i = \langle \mathbf{1}, (2F_{K_i} + \rho_i)^{-1}\mathbf{1}\rangle, \qquad C_d = \frac{1}{\sqrt{\pi}(4\pi)^{d/2}}, \qquad c_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}.$$

For $\ell > 0$, define the positive functions

$$k_{d,\ell}(t) = C_d t^{-(d+3)/2} e^{-\ell^2/(4t)}, \qquad K_\ell(z) = \int_0^\infty e^{-zt} k_{d,\ell}(t)\, dt \quad (z \geq 0). \tag{3}$$

Write $\ell_{ij} = |p_i - p_j|$. Let $\mathcal{H}_m$ be the directed Hamiltonian cycles on the window labels modulo cyclic rotations. The two directions are distinct when $m \geq 3$, and there is one cycle when $m = 2$. Put

$$P(q) = \sum_{\gamma\in\mathcal{H}_m} \prod_{(i,j)\in\gamma} K_{\ell_{ij}}(q^2), \qquad q \geq 0.$$

**Theorem 2.1.** *For fixed shapes, contacts, and centers as above, as $L \to \infty$,*

$$L^{m(d+1)+1}\mathcal{E}_L \longrightarrow \mathcal{E}_* = -\frac{\prod_i T_i}{2\pi} \int_0^\infty P(q)\, dq, \tag{4}$$

$$L^{m(d+1)+1}\mathcal{F}_{L,\lambda L} \longrightarrow \mathcal{F}_*(\lambda) = -\frac{\prod_i T_i}{2\lambda} \sum_{n\in\mathbb{Z}} P(2\pi|n|/\lambda), \tag{5}$$

$$L^{m(d+1)}\mathcal{F}_{L,\beta} \longrightarrow -\frac{\prod_i T_i}{2\beta} P(0) \qquad (\beta > 0 \text{ fixed}). \tag{6}$$

*The limit in* (5) *is locally uniform for $\lambda > 0$. The zero-mode factors are $K_\ell(0) = 2c_d/\ell^{d+1}$, and*

$$\mathcal{F}_*(\lambda) \sim -\frac{\prod_i T_i}{2\lambda} P(0) \quad (\lambda \downarrow 0), \qquad \mathcal{F}_*(\lambda) \longrightarrow \mathcal{E}_* \quad (\lambda \to \infty).$$

*For each cycle, put $\mu_{ij} = 2c_d/\ell_{ij}^{d+1}$ and let*

$$A = C_d \left(\prod_i T_i\right) \sum_{\gamma\in\mathcal{H}_m} \sum_{e\in\gamma} \prod_{\substack{f\in\gamma \\ f\neq e}} \mu_f,$$

*where $\mu_{(i,j)} = \mu_{ij}$. The first low-temperature correction to the limiting coefficient is*

$$\mathcal{F}_*(\lambda) - \mathcal{E}_* \sim -\frac{A\,4^{(d+2)/2}}{2\sqrt{\pi}}\Gamma((d+2)/2)\zeta(d+2)\lambda^{-(d+2)} \qquad (\lambda \to \infty). \tag{7}$$

**Theorem 2.2.** *Suppose that at a given $L$ the closures are separated and*

$$\partial_L |L(p_i - p_j) + y - y'| > 0 \quad \text{for every } i \neq j,\ y \in \overline{K}_i,\ y' \in \overline{K}_j. \tag{8}$$

*Then $\partial_L \mathcal{E}_L > 0$ and $\partial_L \mathcal{F}_{L,\beta} > 0$ for each fixed $\beta > 0$. The generalized force along increasing $L$ is therefore attractive. For two round windows centered at the two translation points with unit center separation before scaling, the criterion holds for every disjoint distance $L > R_1 + R_2$, where $R_i$ are their radii.*

## 3 The connected determinant on the positive axis

For $z \geq 0$, identify the window spaces by translation with $\mathcal{H}_i = L^2(K_i)$. Let $N_i(z)$ be the form compression of $2\sqrt{|D|^2 + z}$ to $K_i$ plus $\rho_i$, where $|D|$ has symbol $|\xi|$. For a subset $S$, $N_S(z)$ is the same response on the union with its full contact. Its form domain decomposes into the individual aperture form spaces by smooth cutoffs around the separated windows. Normalize by the diagonal square roots of the isolated responses. This gives $I + X_S(z)$, with zero diagonal in $X_S(z)$ and off-diagonal blocks $N_i(z)^{-1/2} N_{ij}(z) N_j(z)^{-1/2}$. Positive definiteness of the full aperture form implies positivity and injectivity of the normalized form. The off-diagonal blocks are Hilbert–Schmidt by the separated kernel formula below. Thus compactness of $X_S(z)$ gives invertibility of $I + X_S(z)$. Set

$$G_L(z) = \sum_{S \subseteq [m]} (-1)^{m-|S|} \log \det{}_2(I + X_S(z)),$$

with zero for the empty subset and the real logarithm. We use the regularized determinant for Hilbert–Schmidt perturbations, for which $\det_2(I+X) = \det(I+X)e^{-\operatorname{tr} X}$ when $X$ is trace class.

For $x \neq 0$ the free two-arm square-root kernel is

$$n_z(x) = -C_d \int_0^\infty t^{-(d+3)/2} e^{-zt - |x|^2/(4t)}\, dt = -K_{|x|}(z). \tag{9}$$

This is the off-diagonal part of the square-root subordination formula, with the identity term absent between distinct windows. In two identical transverse Dirichlet cubes the corresponding off-diagonal kernel is the integral of $-\pi^{-1/2} t^{-3/2} e^{-zt}$ times the cube heat kernel. It is bounded in magnitude by the free kernel integral.

**Proposition 3.1.** *For every separated configuration and $z \geq 0$,*

$$G_L(z) = -\int_0^\infty t^{-1} e^{-zt} \mathcal{I}_L(t)\, dt. \tag{10}$$

*Consequently*

$$\mathcal{E}_L = \frac{1}{2\pi}\int_0^\infty G_L(\kappa^2)\, d\kappa, \qquad \mathcal{F}_{L,\beta} = \frac{1}{2\beta}\sum_{n \in \mathbb{Z}} G_L((2\pi n/\beta)^2). \tag{11}$$

*Proof.* First use identical bounded transverse Dirichlet cubes containing all windows. Denote the connected function, heat gain and measure in the cubes by the same symbols in this part of the proof. Choose finite-dimensional spaces $W_{i,n}$ in the individual aperture form domains with form-dense increasing union. Use their direct sums for the subset responses. In finite dimensions, summing the aperture log determinants over subsets cancels every isolated diagonal log. The block normalization is a congruence whose isolated determinants are exactly the terms removed. The surviving off-diagonal matrices have trace zero, so their ordinary and regularized determinants agree.

For each $z_0 > 0$, this passage to the Galerkin limit is locally holomorphic around $z_0$. In fixed coordinates $N_i(z_0)^{1/2}W_{i,n}$, let $P_{i,n}$ denote the orthogonal projections, which converge strongly to the identity. The full bounded forms $N_i(z_0)^{-1/2}N_i(z)N_i(z_0)^{-1/2}$ are $I + O(|z - z_0|)$. The inverses of their compressions, extended by zero, converge strongly together with their adjoints, locally uniformly on a smaller disk by the uniform Neumann series. The fixed-normalized off-diagonal kernels are holomorphic there in Hilbert–Schmidt norm by the cube heat-kernel formula. Multiplying these compact factors by the bounded inverse compressions preserves locally uniform Hilbert–Schmidt convergence, by finite-rank approximation. The resulting determinant is $\det_2$ of identity plus the diagonal inverse times the off-diagonal matrix. On the real axis this agrees with the symmetrically normalized determinant by cyclic products. In the Galerkin spaces it agrees by similarity. Local convergence of the regularized determinants and nonvanishing at $z_0$ give convergence of the logs with the real value there. Cauchy's formula then gives all fixed derivatives.

Choose an integer $k > (d + 1)/2$. The higher-resolvent reconstruction of [21] gives the $k$-th derivative of the aperture log in the Galerkin limit for each subset:

$$(-1)^{k-1}(k-1)! \int_0^\infty (z+u)^{-k}\, d\nu_S(u).$$

Here $\nu_S$ is the positive relative measure of its product domain. The reconstruction uses the Schatten bounds for the differentiated Poisson and boundary factors and strong convergence of the uniformly bounded Galerkin inverses in form-dense coordinates. These apply to the chosen direct sums because each is form-dense in its subset, with the positive zero-energy normalization of the union. Thus the alternating combination of the displayed derivatives is $G_L^{(k)}(z)$ by the analytic determinant convergence. It is also the $k$-th derivative of the right side of (10), by the heat transform of each $\nu_S$. The two functions can differ only by a polynomial of degree at most $k - 1$. Both vanish at $+\infty$: the determinant by the separated kernel integral and the uniform positive-axis inverse bounds, and the heat integral by the all-hit Gaussian bound. The polynomial vanishes.

Now exhaust the plane by cubes. For $z > 0$, the cube heat kernel tends to the free heat kernel and is dominated by it. Off-diagonal kernels thus converge in Hilbert–Schmidt norm by the separated integral. For each diagonal response the minimum of $\int(|\nabla f|^2 + z|f|^2)$ over extensions with fixed trace converges to the half-space minimum: cut off the half-space minimizer on large cubes to get the reverse comparison to the free lower bound. The common aperture form domain and monotone form convergence give strong convergence of the bounded inverses. Their inverse square roots converge strongly as well, with uniform bounds, so the normalized off-diagonal operators converge in Hilbert–Schmidt norm. The regularized determinants then converge. The all-hit bridge representation on cubes converges to the full-screen one with the same contact weights, and is dominated by the free all-hit bound. This gives convergence of the heat integral with $z > 0$ and proves (10) there. At zero, the isolated and subset fractional forms have a positive gap on their bounded apertures. Their bounded $O(\sqrt{z})$ form perturbations and

the separated kernel formula give determinant continuity. The heat side extends by monotone convergence and its large-time bound. This proves (10) for $z = 0$.

Finally $-G_L$ has a positive heat representation. Tonelli in (1) and (2), using $\int_0^\infty e^{-t\kappa^2}\, d\kappa = \sqrt{\pi}/(2\sqrt{t})$, proves (11). □

## 4 Large separation and the thermal crossover

Take $z = q^2/L^2$. By (9) and heat scaling, the free cross kernel at $y \in K_i,\ y' \in K_j$ is

$$-L^{-(d+1)} K_{|p_i - p_j + (y-y')/L|}(q^2).$$

The lengths stay in a compact interval away from zero for large $L$. The Bessel formula for (3), with $\alpha = (d+1)/2$, is

$$K_\ell(q^2) = 2C_d (4q^2/\ell^2)^{\alpha/2} \mathsf{K}_\alpha(\ell q),$$

where $\mathsf{K}_\alpha$ is the modified Bessel function of the second kind. Its small-argument bound and large-argument exponential decay give a uniform bound on the cross blocks in Hilbert–Schmidt norm by $CL^{-(d+1)}(1+q)^M e^{-cq}$, with $c > 0$. The real-axis diagonal forms satisfy $N_i(z) \geq N_i(0) > 0$, so the normalization preserves this bound. In particular $\|X_S(z)\| < 1$ uniformly for all $q \geq 0$ and large $L$. Therefore

$$\log \det{}_2(I + X_S) = \sum_{k \geq 2} \frac{(-1)^{k+1}}{k} \operatorname{tr} X_S^k. \tag{12}$$

Each trace is a sum over rooted closed walks in the window labels. Inclusion–exclusion retains exactly the walks visiting all labels. Degrees below $m$ vanish. The $m$-th degree consists of Hamiltonian cycles, each with $m$ possible roots canceling the denominator $m$.

For fixed $q$, divide each cross block by $L^{-(d+1)}$. Its free kernel converges uniformly on the aperture pairs to the constant $-K_{\ell_{ij}}(q^2)$, giving a Hilbert–Schmidt rank-one limit between the individual constant vectors. Each isolated inverse tends in norm to $N_i(0)^{-1}$, uniformly on bounded $q$-intervals. In the leading closed traces the normalized factors combine to one isolated inverse between each pair of kernels. The contractions give $\prod_i T_i$. The $m$ negative kernels and the log-series sign give a single minus, so

$$L^{m(d+1)} G_L(q^2/L^2) \longrightarrow -(\prod_i T_i) P(q) \tag{13}$$

locally uniformly for $q \geq 0$. For $k \geq 2$, $|\operatorname{tr} X_S^k| \leq \|X_S\|_2^2 \|X_S\|^{k-2}$. In particular the convergent series starting in degree $m$ gives a uniform bound on the scaled absolute value in (13) by $C(1+q)^{M'} e^{-c'q}$, with $c' > 0$. Substitution $q = L\kappa$ in (11) and dominated convergence establish (4). At $\beta = \lambda L$, the scaled Matsubara variables are $2\pi|n|/\lambda$. The same bound gives uniform exponential summability when $\lambda$ ranges over a compact positive interval, proving (5). At fixed $\beta$, the nonzero scaled modes have $q = 2\pi|n|L/\beta$ and are exponentially small as $L \to \infty$. This proves (6) directly.

The mass of $k_{d,\ell}$ is $2c_d/\ell^{d+1}$, by $v = \ell^2/(4t)$. Thus the $n = 0$ term gives the small-$\lambda$ asymptotic in Theorem 2.1. The other terms are exponentially small up to a polynomial in $1/\lambda$. The Riemann sum as $\lambda \to \infty$ recovers the vacuum coefficient, with the full sum over positive and negative frequencies.

To obtain the first correction there, use the connected heat profile of [22],

$$H(\theta) = \left(\prod_i T_i\right) \theta \sum_{\gamma \in \mathcal{H}_m} \left(*_{(i,j)\in\gamma} k_{d,\ell_{ij}}\right)(\theta).$$

It is exponentially small up to a power as $\theta \downarrow 0$. Its late-time law is $H(\theta) \sim A\theta^{-(d+1)/2}$, with $A$ as in Theorem 2.1. The coefficient $\mathcal{F}_*(\lambda)$ is the positive-time representation (2) with $H$ in place of $\mathcal{I}_L$ and period $\lambda$, by the convolution formula for $P$. Apply Poisson summation to the Matsubara Gaussian. The zero winding term is $\mathcal{E}_*$, and the two signs of nonzero winding give

$$\mathcal{F}_*(\lambda) - \mathcal{E}_* = -\frac{1}{2\sqrt{\pi}} \sum_{k \geq 1} \int_0^\infty \theta^{-3/2} H(\theta) e^{-\lambda^2 k^2/(4\theta)} \, d\theta.$$

The two threshold-time regimes give a global majorant $C\theta^{-(d+1)/2}$ for $H$. The change $v = \lambda^2 k^2/(4\theta)$ and dominated convergence yield the Gamma integral with the summable factor $k^{-(d+2)}$, giving exactly (7).

## 5 Strict force under proportional separation

We prove Theorem 2.2 at each admissible $L$. For real $z \geq 0$, put $Y_S = -X_S$. The free positive cross kernel $-n_z(x)$ is strictly decreasing in $|x| > 0$ by (9). Each diagonal normalizer $N_i(z)^{-1/2}$ preserves positivity. Indeed the compressed square-root form is a Dirichlet form with killing by heat subordination and zero extension. Adding the nonnegative contact preserves this property, and the inverse square root is the Laplace integral of the corresponding positivity-preserving semigroup. Thus the normalized cross blocks of $Y_S$ preserve positivity. Their Hilbert–Schmidt kernels are nonnegative almost everywhere, as follows by testing measurable rectangles.

The operator $Y_S$ is compact and self-adjoint, and $I - Y_S > 0$ is invertible. Also

$$|\langle f, Y_S f \rangle| \leq \langle |f|, Y_S |f| \rangle.$$

Its operator norm therefore equals its top spectral endpoint, which is strictly less than one. For any separated $L$, the regularized log equals $-\sum_{k \geq 2} \operatorname{tr}(Y_S^k)/k$. Inclusion–exclusion again retains just the closed walks visiting every label. Their traces are nonnegative, since the kernels are nonnegative. For instance the cyclic trace can be paired in Hilbert–Schmidt norm between two factors or products, with the kernel formula justified by Tonelli.

Under (8), the cross kernels strictly decrease pointwise before normalization, and the diagonal normalizers are independent of $L$. Each normalized cross derivative is nonpositive in kernel order. Each surviving walk therefore contributes a nonnegative derivative to the negative log series. For strictness choose a Hamiltonian cycle and differentiate just one edge. Before normalization both the negative kernel derivative on that edge and the undifferentiated positive kernels on the other edges have strictly positive lower bounds on their compact aperture pairs, at fixed $z$. After normalization each dominates a positive constant times the rectangular rank-one kernel formed from $N_i(z)^{-1/2}\mathbf{1}$ and $N_j(z)^{-1/2}\mathbf{1}$. These vectors are nonzero and nonnegative. The product of those rank-one kernels around the cycle has positive trace, giving a strictly positive contribution to $\partial_L G_L(z)$.

Locally around a separated $L$, $Y_S$ and its $L$ derivative vary continuously in Hilbert–Schmidt norm, uniformly on compact $z$-intervals, including zero. On such intervals $\|Y_S\|$ is uniformly below one, and

$$|\partial_L \operatorname{tr}(Y_S^k)| \leq k \|\partial_L Y_S\|_2 \|Y_S\|_2 \|Y_S\|^{k-2}.$$

This justifies termwise differentiation. At large $\kappa = \sqrt{z}$, the Bessel formula and its spatial derivative give polynomial times exponential decay for the cross kernels and their $L$ derivatives, locally uniformly in separated $L$. The diagonal inverse factors are uniformly bounded on the real

positive axis. These estimates justify differentiation under the integral and sum in (11). Both derivatives are strictly positive because $\partial_L G_L(z) > 0$ for every $z \geq 0$.

For two centered round windows, write $p_2 - p_1 = e$, $|e| = 1$. The $L$-derivative of one half the squared point-to-point separation is $L + e \cdot (y' - y) \geq L - R_1 - R_2 > 0$. Thus the strict sign holds at every disjoint separation, also for unequal radii. This excludes an interior minimum of the relative pair potential on that disjoint range, the finite-separation issue raised by Kabat, Karabali, and Nair [1, Section V].

## 6 Dimension two and explicit torsions

In tangential dimension two the half-integer Bessel formula gives

$$K_\ell(q^2) = \frac{(1 + \ell q)e^{-\ell q}}{\pi \ell^3}.$$

For a cycle with edge lengths $\ell_1, \dots, \ell_m$ and perimeter $P_\gamma = \sum_j \ell_j$, its contribution to $\int_0^\infty P(q)\, dq$ is therefore

$$\frac{1}{\pi^m \prod_j \ell_j^3} \sum_{r=0}^{m} \frac{r!\, e_r(\ell_1, \dots, \ell_m)}{P_\gamma^{r+1}},$$

where $e_r$ is the elementary symmetric polynomial and $e_0 = 1$. This follows by multiplying the $m$ polynomials $1 + \ell_j q$ and integrating against $e^{-P_\gamma q}$.

For a pair at unit scaled center separation this yields

$$\mathcal{E}_* = -\frac{5 T_1 T_2}{8\pi^3} \qquad (d = 2).$$

For transparent windows, Kabat, Karabali, and Nair [1, eqs. (10)–(11)] use $Q_i = \langle \mathbf{1}, F_{K_i}^{-1} \mathbf{1} \rangle = 2T_i$, since their charge uses the one-side root. Their two-hole interaction is $-5Q_1Q_2/(32\pi^3 L^7)$ to leading order, exactly the normalization above.

For a ball of radius $R$, set the contact to zero. Dyda's power formula [9, Theorem 1] gives $F_{B_R}[(R^2 - |y|^2)_+^{1/2}] = \sqrt{\pi}\, \Gamma((d+1)/2)/\Gamma(d/2)$ inside the ball. It follows by integration that the transparent charge $Q = 2T$ is

$$Q(B_R) = \frac{\pi^{d/2} \Gamma(d/2)}{2\Gamma((d+3)/2)\Gamma((d+1)/2)} R^{d+1}.$$

In particular the circular charge is $4R^3/3$, and the charge of an interval of width $W$ is $\pi W^2/8$. These are exact evaluations of the charge defined in [1], eq. (11): the slit value agrees with the numerical $0.393W^2$ there, while the closed form $4R^3/3 \approx 1.333R^3$ for a round hole replaces the numerical lattice value $1.28R^3$ of [1, eq. (17)]. Indeed the torsion function of the disk for $(-\Delta)^{1/2}$ is $u(x) = (2/\pi)(R^2 - |x|^2)^{1/2}$, by the classical identity $(-\Delta)^{1/2}(1 - |x|^2)_+^{1/2} = \pi/2$ in the unit disk, and $\int u = 4R^3/3$. For transparent apertures with fixed individual measures and fixed centers, the fractional Saint-Venant bound [12, Proposition 3.4] maximizes the leading energy magnitude at balls in its bounded Lipschitz class in $d \geq 2$. The comparison also holds for arbitrary bounded open apertures in these dimensions: each compactly supported test in the torsion variational quotient lies in a smooth open subdomain compactly contained in the aperture. Apply the bound there, compare the balls by their volumes, and pass to the form closure. Every cycle coefficient contains the same product of the positive torsions, which gives the shape conclusion.

For holes in the plate of [1] this settles the speculation recorded there. Write $Q(\Omega)$ for the charge of a bounded open hole $\Omega \subset \mathbb{R}^2$ defined in [1, eq. (11)], that is, the matrix element of the inverse of the compression of $\sqrt{-\Delta}$ to $\Omega$ between functions equal to one on $\Omega$. We use the form domain of Section 2, the closure $\mathcal{D}(\Omega)$ of $C_c^\infty(\Omega)$ in $H^{1/2}(\mathbb{R}^2)$, and write $F_\Omega$ for the compression of $|D|$ to it; by Cauchy–Schwarz in the form inner product,

$$Q(\Omega) = \langle \mathbf{1}, F_\Omega^{-1}\mathbf{1}\rangle = \sup_{0\neq\eta\in\mathcal{D}(\Omega)} \frac{(\int \eta)^2}{\langle \eta, |D|\eta\rangle}.$$

Since $Q(\Omega) = \langle \mathbf{1}, F_\Omega^{-1}\mathbf{1}\rangle$ and $u_\Omega = F_\Omega^{-1}\mathbf{1}$, also $Q(\Omega) = \int_\Omega u_\Omega$; for Lipschitz holes this is the fractional torsional rigidity of order $s = 1/2$ in the normalization of [12, eqs. (1.7) and (3.7)], whose form domain coincides with $\mathcal{D}(\Omega)$ for such holes. The charge also has a classical meaning: if the plate $\Omega$ moves normal to itself with unit speed in an ideal fluid of density $\rho$, the odd velocity potential vanishes on the plane off $\Omega$ and its normal derivative on $\Omega$ is $|D|$ applied to its trace, so the normal added mass equals $2\rho\, Q(\Omega)$. For the disk this reproduces Lamb's value $\frac{8}{3}\rho R^3$, an independent check of $4R^3/3$.

**Proposition 6.1** (Round holes have the largest charge)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open hole and let $\Omega^*$ be a disk with the same area. Then*

$$Q(\Omega) \le Q(\Omega^*) = \frac{4}{3}\left(\frac{|\Omega|}{\pi}\right)^{3/2}.$$

*The same bound holds if $\mathcal{D}(\Omega)$ is replaced by the larger space of $\eta \in H^{1/2}(\mathbb{R}^2)$ vanishing almost everywhere outside $\Omega$. If $\Omega$ has Lipschitz boundary, equality holds only when $\Omega$ is a disk. Consequently, at fixed hole areas and fixed large separation, the leading two-hole energy $-5Q_1Q_2/(32\pi^3 r^7)$ of [1, eq. (10)] has the largest magnitude when both holes are round, and among Lipschitz holes only then.*

*Proof.* Let $\eta \neq 0$ lie in either space. Then $|\eta|$ lies in the same space, $\int|\eta| \ge |\int\eta|$, and $\langle|\eta|, |D||\eta|\rangle \le \langle\eta, |D|\eta\rangle$, since the form is a multiple of the Gagliardo seminorm of order $1/2$. Let $\eta^*$ be the symmetric decreasing rearrangement of $|\eta|$. It has the same integral, vanishes almost everywhere outside the closed disk $\overline{\Omega^*}$, and $\langle\eta^*, |D|\eta^*\rangle \le \langle|\eta|, |D||\eta|\rangle$ by the fractional Pólya–Szegő inequality [10, Theorem 9.2]. For the disk the two spaces coincide, so $\eta^* \in \mathcal{D}(\Omega^*)$ and the quotient of $\eta$ is at most $Q(\Omega^*)$. Taking the supremum gives $Q(\Omega) \le Q(\Omega^*)$. For a Lipschitz hole, $Q(\Omega)$ is the fractional torsional rigidity of [12], and the equality case of the fractional Saint-Venant inequality [12, Proposition 3.4], which rests on [11, Theorem A.1], shows that equality holds only for a disk. The value of $Q(\Omega^*)$ is the circular charge $4R^3/3$ with $\pi R^2 = |\Omega|$. The final assertion follows because the leading energy is a fixed negative multiple of $Q_1Q_2$ and both charges are positive. □

The numerical charges of [1, eqs. (17) and (22)], namely $0.230\,|\Omega|^{3/2}$ for the round hole and $0.228\,|\Omega|^{3/2}$ for the square hole, are consistent with Proposition 6.1; the exact round value is $Q(\Omega^*)/|\Omega|^{3/2} = 4/(3\pi^{3/2}) \approx 0.2394$. Some regularity is needed for the equality statement: removing a segment from a disk leaves the charge unchanged, because a segment is polar for three-dimensional Brownian motion and so does not change $\mathcal{D}(\Omega)$.

# 7 Closing gaps: introduction

Kabat, Karabali, and Nair [1, Sections II and V] compute the distant two-hole scalar interaction in one Dirichlet screen and ask how the potential behaves as the holes approach one another.

The connected scalar energy of separated round holes has a strict attractive force for all disjoint center distances (Theorem 2.2). We address the other endpoint of that disjoint family. As the gap closes, the bridge paths contributing at leading order concentrate near the two facing edges. The transverse directions along the edges scale more slowly than the plane-normal and gap directions. In the two directions that remain diffusive, the limiting barrier is a retained segment with two open exterior rays.

In the coplanar whisker geometry of Kabat, Karabali, and Nair [13, Section 3 and Figure 2], the two exterior rays are instead the retained parts of a screen with an open slit. At the path level their two-obstacle interaction counts paths hitting both exterior rays. Our planar integral imposes the additional condition that those paths avoid the retained central segment. This changes the connected heat function even though the two endpoint positions coincide. For complementary-boundary scattering relations see Maghrebi, Abravanel, and Jaffe [14]. Worldline edge calculations and the all-hit cancellation have antecedents in Gies and Klingmüller [15] and Schaden [2, eqs. (15)–(18)]. The localization here uses the connected screen heat gain of [22].

A local geometry factor also enters derivative expansions for facing mirrors, including the scalar vacuum and thermal calculations of Fosco, Lombardo, and Mazzitelli [16, 17]. Here the gap is lateral, and the local heat coefficient is a planar all-hit integral for a segment. The temperature analysis uses the Matsubara and Poisson representations, as in Kabat and Karabali [5] for plate edges and a slit. We give the gap-scale profile for two windows and the borderline logarithm for intervals.

## 8 Heat observables and local geometry

In a flat screen in $\mathbb{R}^{d+1}$, write $(s, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ for the coordinates in the plane, with $s$ along the common supporting normal within the plane. Let $x \in \mathbb{R}$ be the coordinate normal to the screen. Two bounded transparent apertures have closures meeting only at $(s, y) = (0, 0)$ before translation, on opposite sides of $s = 0$. Translate the left one by $-g/2$ and the right one by $g/2$ along $s$, with $g > 0$. The boundaries are Dirichlet away from the open apertures. There are no surface contacts.

For a bounded open aperture $K \subset \mathbb{R}^d$, use $D_K(t)$ for the integrated diagonal difference of the heat kernel on $\mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K) \times \{0\})$ and that of the cut half-spaces. The generator is $\Delta$, with free diagonal $(4\pi t)^{-(d+1)/2}$. With $K_{1,g}, K_{2,g}$ the translated windows, put

$$\mathcal{I}_g(t) = D_{K_{1,g} \cup K_{2,g}}(t) - D_{K_{1,g}}(t) - D_{K_{2,g}}(t).$$

This is the integral of the free diagonal times the probability that a bridge avoids the closed screen and visits both apertures on the plane [22]. In units $\hbar = c = k_{\mathrm{B}} = 1$, the connected scalar observables of Section 1 and the following section are

$$\mathcal{E}_g = -\frac{1}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \mathcal{I}_g(t)\, dt, \tag{14}$$

$$\mathcal{F}_{g,\beta} = -\frac{1}{2\beta} \int_0^\infty t^{-1} \mathcal{I}_g(t) \sum_{n \in \mathbb{Z}} e^{-t(2\pi n/\beta)^2}\, dt, \qquad \beta > 0. \tag{15}$$

For $h > 0$, let $J_h(\tau)$ be the planar all-hit heat integral with a retained Dirichlet segment of length $h$ on an axis. A planar bridge based at $(S, X)$ contributes when it hits that axis in both open exterior rays and avoids the closed intervening segment. Integrate the probability against $(4\pi\tau)^{-1} dS\, dX$. This definition uses a single integrable probability. By heat scaling, $J_h(\tau) = J_1(\tau/h^2)$.

We use either of the following local geometries for $d \geq 2$. In both, the windows are convex, the closure contact before translation is unique, and near the origin the facing graphs before translation are $s = -a_1(y)$ and $s = a_2(y)$ with the windows on the outer sides.

- For $C^2$ faces, let $B_i$ be the Hessian of $a_i$ at zero and assume $B_1 + B_2$ is positive definite. Put
$$\alpha = \frac{d-1}{2}, \qquad H(w) = 1 + \frac{1}{2} w^{\mathsf{T}} (B_1 + B_2) w.$$
Round windows of radii $R_1, R_2$ have $B_i = R_i^{-1} I$.

- For separable finite-type faces, assume in orthonormal tangential coordinates that
$$a_i(y) = \sum_{j=1}^{d-1} c_{ij} |y_j|^{p_j} + o\left( \sum_{j=1}^{d-1} |y_j|^{p_j} \right), \quad p_j > 1, \quad c_{ij} \geq 0, \quad c_j = c_{1j} + c_{2j} > 0.$$
Put $\alpha = \sum_j 1/p_j$ and $H(w) = 1 + \sum_j c_j |w_j|^{p_j}$.

For $d = 1$, take intervals of lengths $2R_1, 2R_2$. The tangent space is zero dimensional, with $\alpha = 0$, $H = 1$ and its spatial integral interpreted as evaluation at its unique point.

**Theorem 8.1.** *For the geometries above, as $g \downarrow 0$,*
$$\mathcal{E}_g \sim -g^{-(d-\alpha)} D_d \int_{\mathbb{R}^{d-1}} H(w)^{-d} \, dw, \qquad D_d = \frac{(4\pi)^{-(d-1)/2}}{4\sqrt{\pi}} \int_0^\infty \tau^{-(d+2)/2} J_1(\tau) \, d\tau \in (0, \infty). \tag{16}$$
*At fixed $\beta > 0$ the difference $\mathcal{F}_{g,\beta} - \mathcal{E}_g$ stays bounded, so the free energy has the same leading term.*

*For $d \geq 2$, as $g \downarrow 0$ locally uniformly for $\lambda > 0$,*
$$g^{d-\alpha} \mathcal{F}_{g,\lambda g} \longrightarrow -B_H(\lambda), \tag{17}$$
*where*
$$B_H(\lambda) = \frac{(4\pi)^{-(d-1)/2}}{2\lambda} \int_{\mathbb{R}^{d-1}} \int_0^\infty \tau^{-(d+1)/2} J_{H(w)}(\tau) \sum_{n \in \mathbb{Z}} e^{-\tau (2\pi n/\lambda)^2} \, d\tau \, dw. \tag{18}$$
*As $\lambda \to \infty$, $B_H(\lambda) \to D_d \int H^{-d}$. As $\lambda \downarrow 0$,*
$$\lambda B_H(\lambda) \longrightarrow \frac{(4\pi)^{-(d-1)/2}}{2} \left( \int_0^\infty \tau^{-(d+1)/2} J_1(\tau) \, d\tau \right) \int_{\mathbb{R}^{d-1}} H(w)^{-(d-1)} \, dw. \tag{19}$$
*For $d = 1$, the gap-scale law is instead*
$$\mathcal{F}_{g,\lambda g} \sim -\frac{\log(1/g)}{8\lambda g} \tag{20}$$
*locally uniformly for $\lambda > 0$.*

For $C^2$ faces the vacuum geometry factor is
$$\int_{\mathbb{R}^{d-1}} H^{-d} = \det\left((B_1 + B_2)/2\right)^{-1/2} \pi^{(d-1)/2} \frac{\Gamma((d+1)/2)}{\Gamma(d)}.$$
For round windows this gives the factor $\left((R_1^{-1} + R_2^{-1})/2\right)^{-(d-1)/2}$ times the same Gamma quotient and power of $\pi$. For finite-type faces,
$$\int_{\mathbb{R}^{d-1}} H^{-d} = 2^{d-1} \frac{\Gamma(d-\alpha)}{\Gamma(d)} \prod_{j=1}^{d-1} c_j^{-1/p_j} \Gamma(1 + 1/p_j).$$

## 9 Gaussian localization and the vacuum law

Choose an enclosing ball $Q$ in the screen containing the two shifted windows for all small $g$. Because a bridge visiting both windows also contributes to $D_Q$, we have $0 \le \mathcal{I}_g(t) \le D_Q(t)$. By the area and torsion estimates of [22], this enclosure gives $O(t^{-d/2})$ at small time and $O(t^{-(d+1)/2})$ at large time, uniformly in $g$.

An all-hit bridge based at $(s, y, x)$ must move normally by at least $|x|$. It must also move in the $s$ coordinate by at least $|s| + g/2$, since it reaches both supported sides. These two requirements give Gaussian bounds in $x$, $s$, and $g$. For $d \ge 2$, let

$$r(y) = \begin{cases} |y|^2, & C^2 \text{ faces}, \\ \sum_j |y_j|^{p_j}, & \text{finite-type faces}. \end{cases}$$

For a base $y$ in a fixed small tangent neighborhood, we claim that an all-hit path must move by at least $c\, r(y)$ in the tangential directions or in the $s$ coordinate.

For the $C^2$ case, if either tangent hit differs from $y$ by more than $\delta_1|y|$ with small fixed $\delta_1 > 0$, that alone controls $|y|^2$. Otherwise the sum of the two graph heights at the two hits equals $y^{\mathsf{T}}(B_1 + B_2)y/2 + O(\delta_1|y|^2) + o(|y|^2)$. Its lower bound is $c|y|^2$ by positive definiteness of the sum and a sufficiently small neighborhood. The graph heights also bound deeper aperture points in their respective fibers. Thus the two hit positions in the $s$ coordinate are at least $c|y|^2$ apart, forcing at least half that excursion from the base. For round faces, the underlying graph is $R_i - \sqrt{R_i^2 - |y|^2}$, with the same quadratic term and a quartic error.

In the finite-type case, a tangent displacement comparable to $r(y)$ suffices. Otherwise both hit coordinates differ from $y_j$ by $O(r(y)) = o(r(y)^{1/p_j})$ for each $j$. Since $|y_j| \le r(y)^{1/p_j}$, normalizing by $r(y)^{1/p_j}$ and using uniform continuity of the power function on bounded intervals shows that the resulting changes in each $|y_j|^{p_j}$ are $o(r(y))$. The two small remainders at the hit coordinates are also $o(r(y))$. Consequently the sum of the two graph heights is at least $cr(y)$, by $c_j > 0$. This proves the claim in either geometry.

Combining the Gaussian excursion bounds for the three requirements by a geometric mean yields the following majorant for the all-hit probability in the local tangent neighborhood:

$$C \exp[-c(x^2 + s^2 + g^2 + r(y)^2)/t]. \tag{21}$$

For $d = 1$ omit $r(y)$, and the estimate holds for every base in the plane $(s, x)$. The underlying one-coordinate bridge deviation bound is $2e^{-b^2/t}$ for displacement at least $b$, with the generator-$\Delta$ convention.

For the complement of the local tangent neighborhood, either a hit requires a fixed tangent excursion, or both hits have tangent positions outside a smaller neighborhood of the contact point. The latter pieces are separated by a positive distance at $g = 0$, by compactness of the closures and uniqueness of the contact. This separation persists for small $g$, forcing a fixed excursion in the screen coordinates. The aperture hits also remain in a bounded region, giving another Gaussian bound at large $|y|$ with distance $(|y| - M)_+$ for a fixed $M$. Combine these estimates with the normal and $s$ bounds and integrate the free diagonal. On any fixed small time interval the far part has a majorant $Ct^{-N}e^{-c_0/t}$ for some $N, c_0 > 0$ independent of $g$. Its vacuum contribution is uniformly bounded. At times bounded away from zero the entire contribution is uniformly bounded by the enclosure. These estimates also prove the absolute convergence of (14) and (15) at each fixed $g > 0$.

For the local part put

$$s = gS, \quad x = gX, \quad t = g^2\tau, \qquad y_j = \begin{cases} g^{1/2} w_j, & C^2 \text{ faces}, \\ g^{1/p_j} w_j, & \text{finite-type faces}. \end{cases}$$

For fixed $(S, X, w)$ and $\tau$, the scaled $s, x$ bridges are duration-$\tau$ bridges based at $S, X$. Tangent increments in the $w_j$ variables have size $g^{1/2}$ or $g^{1-1/p_j}$, respectively. On a common realization by fixed-time bridges, they tend uniformly to zero almost surely. The limiting graph thresholds on the normal zero times are

$$S_{\text{bridge}} < -1/2 - A_1(w), \qquad S_{\text{bridge}} > 1/2 + A_2(w),$$

where $A_i(w) = w^{\mathsf{T}} B_i w/2$ in the quadratic case and $A_i(w) = \sum_j c_{ij} |w_j|^{p_j}$ in the finite-type case. Their gap is $H(w)$. There is no tangent variable in $d = 1$, and the two thresholds there are $-1/2, 1/2$.

For $X \neq 0$ the normal zero times form a compact subset of $(0, \tau)$. The planar bridge in $(S, X)$ almost surely misses both limiting endpoints, by planar point polarity and absolute continuity with respect to Brownian motion on strict initial time intervals. Therefore its $S$ values at the normal zeros stay separated from both thresholds if there are any such zeros. The graph expansions hold uniformly on the bounded path ranges after scaling, so the indicators of allowed hits and of visiting both rays converge. The set $X = 0$ is null in the diagonal integral.

The volume element contributes $g^{2+\alpha}$, the free diagonal contributes $g^{-(d+1)}$, and $t^{-3/2} dt$ contributes $g^{-1}$. After multiplying the vacuum integral by $g^{d-\alpha}$, its local scalar integrand is dominated by

$$C\tau^{-(d+4)/2} e^{-c(1+S^2+X^2+r(w)^2)/\tau}.$$

In $d = 1$ drop $r(w)$. The $S, X$ integral contributes $O(\tau)$. The tangent integral is $O(\tau^{\alpha/2})$, by scaling each $w_j$ by $\tau^{1/(2p_j)}$ in the finite-type case and by $\tau^{1/4}$ in the quadratic case. The time bound is therefore $C\tau^{-(d+2-\alpha)/2} e^{-c/\tau}$, integrable since $\alpha < d$.

Dominated convergence reduces the diagonal integral in $S, X$ at fixed $w, \tau$ to the planar all-hit integral $J_{H(w)}(\tau)$, after translating the center of its segment. The free tangential factor is $(4\pi\tau)^{-(d-1)/2}$. In the remaining time integral use $\tau = H(w)^2 v$, giving exactly $H(w)^{-d}$ and the universal factor $D_d$ in (16). For unit gap the $S, X$ Gaussian bounds give $J_1(\tau) \leq Ce^{-c/\tau}$, since integrating in $S, X$ cancels the planar diagonal factor. A loop that crosses each ray through an interior interval has a small tube of allowed bridges with positive probability. Thus $J_1(\tau) > 0$ for $\tau > 0$ and the moment defining $D_d$ is finite and positive.

For the $C^2$ shape integral, use the square root of the positive definite matrix $(B_1 + B_2)/2$ to reduce the integral to a radial beta integral. For finite-type faces and any $b > \alpha$, apply the Gamma representation of $H^{-b}$ and Tonelli. Each coordinate integral of $e^{-vc_j |w_j|^{p_j}}$ gives $2(vc_j)^{-1/p_j} \Gamma(1 + 1/p_j)$. The remaining Gamma integral gives the stated shape factor at $b = d$. The same formulas at $b = d - 1$ give the shape factor in the thermal zero mode when $d \geq 2$.

## 10 Temperature at and above the gap scale

Poisson summation in (15) for a fixed $\beta > 0$ gives

$$\mathcal{F}_{g,\beta} - \mathcal{E}_g = -\frac{1}{2\sqrt{\pi}} \sum_{k \geq 1} \int_0^\infty t^{-3/2} \mathcal{I}_g(t) e^{-\beta^2 k^2/(4t)} \, dt. \tag{22}$$

The sum of the image factors is exponentially small at $t = 0$ and $O_\beta(\sqrt{t})$ as $t \to \infty$. The enclosure bound from [22] then gives an integrable majorant independent of $g$, proving the fixed-temperature assertion. It also works uniformly for $\beta$ bounded below by any positive number.

Now let $\beta = \lambda g$ and $d \geq 2$. The same local spatial change of variables gives the power $g^{\alpha-d}$, since $t^{-1}dt/(2\beta)$ contributes $g^{-1}$. The Matsubara factor becomes $\sum_{n\in\mathbb{Z}} e^{-\tau(2\pi n/\lambda)^2}$, bounded by $C(1 + \lambda/\sqrt{\tau})$. The integrated local time majorant is therefore a constant times

$$\tau^{-(d+1-\alpha)/2} e^{-c/\tau}(1 + \lambda/\sqrt{\tau}),$$

with the positive prefactor $1/(2\lambda)$ outside. It is integrable at infinity because $\alpha < d - 1$, and uniformly integrable for $\lambda$ on compact positive intervals. The far spatial region at small $t$ costs $O(g^{-1})$ on those intervals by its fixed Gaussian and the bound $C(1 + \lambda g/\sqrt{t})$ on the unscaled sum. Times bounded away from zero also cost $O(g^{-1})$ by the enclosure. Since $d - \alpha > 1$, both are lower order. Dominated convergence now gives (17) and (18), with local uniformity in $\lambda$ because the bridge integrand convergence is independent of $\lambda$.

As $\lambda \to \infty$, the Gaussian sum divided by $\lambda$ tends to $(2\sqrt{\pi\tau})^{-1}$ by Poisson summation. For $\lambda \geq 1$ it is bounded by $C(1 + \tau^{-1/2})$, so the same integrated majorant applies to the limiting profile. This yields the vacuum coefficient. As $\lambda \downarrow 0$, the Gaussian sum tends to one and is bounded by $C(1 + \tau^{-1/2})$ for $\lambda \leq 1$. After canceling the prefactor $1/\lambda$ and rescaling the time by $H(w)^2$, the zero mode gives (19).

## 11 The logarithm for interval apertures

Take $d = 1$. Write $J_0$ for the zero-gap planar all-hit integral for the two complementary rays, omitting the common point by planar point polarity. This value is independent of time by scaling. To evaluate it, denote by $p_+$ and $p_-$ the plane heat diagonals with just the right or left ray open, and by $p_{\mathrm{cut}}$ and $p_{\mathrm{free}}$ the cut and free plane diagonals. The connected diagonal is $p_{\mathrm{free}} - p_+ - p_- + p_{\mathrm{cut}}$. At time one the relative half-plane area baselines for the two rays are

$$A_\pm(S, X) = (4\pi)^{-1} \mathbf{1}_{\{\pm S>0\}}\, e^{-X^2}.$$

They sum to $p_{\mathrm{free}} - p_{\mathrm{cut}}$ by reflection. Each integrated slit defect $p_\pm - p_{\mathrm{cut}} - A_\pm$ is $-1/16$, with absolute convergence, by the planar slit computation in [22]. Decompose the slit plane by reflection. The odd Dirichlet sector cancels from the relative trace and the even sector gives the classical regular Dirichlet–Neumann junction constant of Avramidi [18, eq. (169)] and Nursultanov, Rowlett, and Sher [19, Appendix B, eq. (B.8)]. It follows that the connected all-hit integral is $J_0 = 1/8$.

The zero Matsubara term is proportional to $\int_0^\infty \mathcal{I}_g(t)\, dt/t$. By (21) in dimension one, $\mathcal{I}_g(t) \leq Ce^{-cg^2/t}$, so the portion $t < Ag^2$ is $O_A(1)$. The portion $t > b > 0$ is $O_b(1)$ by the fixed enclosure. For $Ag^2 \leq t \leq b$, scale the planar bridge and its spatial integral to time one. The actual scaled left aperture is

$$\left(-\frac{g}{2\sqrt{t}} - \frac{2R_1}{\sqrt{t}}, -\frac{g}{2\sqrt{t}}\right).$$

It contains the fixed interval $(-2R_1/\sqrt{b}, -1/(2\sqrt{A}))$ when $A$ is large and $b$ small, and lies in the negative half-line. The reflected analogous inclusions hold for the right aperture. The all-hit integrals are monotone in each ray opening because enlarging that opening preserves the allowed hits and all required visits. Thus throughout the middle time range $\mathcal{I}_g(t)$ is squeezed between $J_0$ and the all-hit integral of the fixed truncated pair. The lower bound tends to $J_0$ as $A \to \infty$ and

$b \downarrow 0$, by monotone convergence and point polarity. The logarithmic measure of the middle range is $2\log(1/g) + O_{A,b}(1)$. First letting $g$ tend to zero and then improving the fixed truncation gives

$$\int_0^\infty \mathcal{I}_g(t)\,\frac{dt}{t} \sim 2J_0 \log(1/g).$$

The remaining Matsubara terms are $O_\lambda(g^{-1})$ locally uniformly in $\lambda > 0$. Indeed $\mathcal{I}_g(g^2\tau) \le Ce^{-c/\tau}$, and after removing $n = 0$ the Gaussian sum is $O(\tau^{-1/2})$ for small $\tau$ and exponentially small for large $\tau$, uniformly on compact $\lambda$-intervals. Multiplying the zero-mode integral by $-1/(2\lambda g)$ and inserting $J_0 = 1/8$ proves (20). Any fixed length used to make the logarithm dimensionless changes only the lower-order scale $g^{-1}$. The same static logarithmic coefficient occurs for the critical interaction of two retained Dirichlet needles with an open gap [20, eq. (13)]. In the aperture problem it arises from the connected heat subtraction with a retained central segment and the gap-scale Matsubara limit.

## 12 Joint complete monotonicity of the connected energy

Consider the fixed physical aperture model of [1, 23] and Sections 1–6. Keep all supported aperture regions and nonnegative contact profiles fixed in their own coordinates. Translate two facing regions apart by a gap $g > 0$ along a common supporting normal. Their cross distances have the form

$$r_g(X,Y)^2 = (g + u_X + v_Y)^2 + |X_\parallel - Y_\parallel|^2, \qquad u_X, v_Y \ge 0.$$

At mass $\kappa \ge 0$, let $R_\pm$ be the supported inverses of $2\sqrt{-\Delta + \kappa^2} + V_\pm$, with the nonnegative contacts $V_\pm$ on their corresponding apertures. The positive cross kernel is $B_\kappa(g) = 2j_\kappa(r_g)$. The connected observable is

$$q_\kappa(g) = -\log\det(I - X^*X), \qquad X = R_+^{1/2} B_\kappa(g) R_-^{1/2}.$$

The trace in supported energy coordinates defines this expression also when an isolated inverse is unbounded as an $L^2$ operator. Let $q_{\kappa,\Omega}(g)$ denote the corresponding quantity when either or both supported aperture regions are restricted to fixed windows $\Omega$. The basic statement concerns bounded open apertures with bounded nonnegative contacts and the strict-contraction hypothesis at every $g > 0$. For unbounded apertures the conclusion applies when the convergent energy trace admits the same positive cyclic kernel expansion, for example by a separately justified exhaustion.

**Theorem 12.1** (Gap and material mass)**.** *Then $q_\kappa(\sqrt{s})$, $q_{\kappa,\Omega}(\sqrt{s})$, and the truncation loss $q_\kappa(\sqrt{s}) - q_{\kappa,\Omega}(\sqrt{s})$ are completely monotone in $s > 0$. For a nonempty pair of aperture regions, the full observable obeys strict alternating derivative inequalities of every order. The same holds for the loss when a genuine truncation removes an open subset on at least one side. At fixed physical temperature, a finite nonnegative Matsubara sum of losses, or a convergent sum over all modes, retains the property. The same is true of the convergent vacuum frequency integral.*

## 13 Proof

The off-diagonal square-root jump density is a positive Gaussian mixture. Up to its positive dimensional constant,

$$j_\kappa(r) = \int_0^\infty t^{-(d+3)/2} e^{-\kappa^2 t - r^2/(4t)}\,dt.$$

At $g = \sqrt{s}$, the $s$-dependent factor in each integrand is

$$\exp\left(-\frac{s}{4t} - \frac{u_X + v_Y}{2t}\sqrt{s}\right).$$

The first factor is the Laplace transform of an atom at $1/(4t)$. The second is the Laplace transform of a positive 1/2-stable law. For parameter $\gamma > 0$ its density at $\lambda > 0$ is

$$\frac{\gamma}{2\sqrt{\pi}}\lambda^{-3/2}\exp\left(-\frac{\gamma^2}{4\lambda}\right).$$

For $\gamma = 0$ the measure is a unit atom at zero. Each cross-kernel element therefore has a positive Laplace measure carried by strictly positive times.

The supported full and restricted resolvent kernels $R_\pm$ and $R_{\pm,\Omega}$, each extended by zero to the original coordinate set, obey the pointwise inequalities $0 \le R_{\pm,\Omega}(X,Y) \le R_\pm(X,Y)$ almost everywhere. This follows from the killed semigroup comparison detailed below. Contacts are kept identical on common support. The determinant expansion gives

$$q_\kappa(g) = \sum_{m\ge 1}\frac{1}{m}\operatorname{Tr}\left[(R_+ B_\kappa(g) R_- B_\kappa(g)^*)^m\right],$$

with the corresponding restricted expression. Restricting a region is implemented by replacing its zero-extended resolvent kernel. For bounded openings the fractional Poincaré inequality bounds each supported inverse even at zero mass. The separated cross kernel is bounded and Hilbert–Schmidt. Approximate it by bounded nonnegative finite-rank rectangular conditional expectations. The cyclic traces converge by Hilbert–Schmidt and trace-norm continuity. The kernel integrals agree at finite rank and converge by dominated convergence along an almost-everywhere subsequence, since every resolvent kernel is integrable over its bounded pair of domains by $\int R = \langle 1, R1\rangle < \infty$. This proves the cyclic formula at zero mass as well.

For each cyclic integral, the product of cross factors is a Laplace transform of a positive convolution measure in $s$. The resolvent factors are independent of $s$. In the difference of full and restricted terms, telescoping the product of the $2m$ resolvent kernel factors leaves a sum of nonnegative products with one factor $R_\pm - R_{\pm,\Omega}$. The cross factors remain the same in every product. Tonelli gives positive representing measures for the full, restricted and difference observables. Finiteness at every $s > 0$ justifies all derivatives by exponential domination on compact $s$-intervals.

Strictness follows already from the $m = 1$ term. Test with nonzero nonnegative functions $h, k \le 1$ compactly supported in the two openings. For a truncation loss choose one of them in a removed open subset, where the corresponding resolvent difference has the full kernel. The cross density has a positive lower bound between the compact test supports. The two integrals $\langle h, R_+ h\rangle$ and $\langle k, R_- k\rangle$ are strictly positive by positive definiteness. Hence the relevant trace term, and its representing measure, is nonzero. Each derivative has a strictly positive alternating sign because that measure charges strictly positive times.

Fixed physical frequency integration and summation preserve this representation by Tonelli. The conclusion applies to fixed physical openings and contacts. In a family that rescales the contact profile or the inverse temperature with the gap, those factors depend on $s$ and require a separate analysis.

## 14 Joint gap and mass monotonicity

**Theorem 14.1** (Joint derivatives)**.** *For the supported square-root model with a fixed nonnegative multiplication contact, the same observables are jointly completely monotone in the two variables $s = g^2$ and $\mu = \kappa^2$. The representing measures are positive on $(0, \infty)^2$. Consequently every mixed alternating derivative is strictly positive under the same nontriviality conditions, at $s, \mu > 0$. Values at $\mu = 0$, when finite, are obtained by monotone limits.*

To prove the joint statement, represent the operator $2\sqrt{-\Delta + \mu}$ by Brownian motion $B$ with generator $\Delta$, run on an independent stable clock $S_t$ with Laplace exponent $2\sqrt{z}$. Then $Y_t = B_{S_t}$ is the massless jump process, and tilting paths by $e^{-\mu S_t}$ yields the massive semigroup. This follows from the Brownian Fourier multiplier and the killed subordinate process construction [24, 25]. Multiplicativity over independent clock increments gives the same construction up to exit of the jump process from a supported domain. Killing is after subordination. A nonnegative contact $V$ adds the Feynman–Kac factor $e^{-\int_0^t V(Y_r)\,dr}$ independent of $\mu$. Thus the full or restricted resolvent kernel is an integral over $t > 0$ of endpoint densities with the extra Laplace factor $e^{-\mu S_t}$.

The full minus restricted semigroup in common coordinates uses the difference of their survival indicators on the same path, weighted by the same contact up to time $t$ on full survival. This difference is nonnegative. Initial points outside the restricted region have zero restricted indicator. Integrating over $t$ gives a positive representing kernel measure for $R_D - R_\Omega$ in $\mu$, as well as for the individual resolvents. These measures can be formed jointly in the endpoint and clock time. Absolute continuity in the endpoint is inherited from the subordinate Brownian transition law, so disintegration or Tonelli on the full cyclic integral produces the kernel form used above.

Each cross factor is already a simultaneous Laplace transform by the Gaussian subordination formula, because its integrand includes $e^{-\mu t}$ with $t > 0$. A product of kernel factors in the trace gives convolution of their measures in the two variables. The same telescoping loss expansion therefore has a positive joint measure; the cross factors force both Laplace coordinates to be strictly positive. This proves the mixed derivative statement by exponential domination. Replacing $\mu$ by $\mu + \omega^2$ multiplies the measure by a nonnegative frequency factor, so convergent fixed-temperature and vacuum observables inherit the joint property with respect to the common squared material mass.

## 15 Exhaustion in the supported energy spaces

For unbounded physical openings with locally bounded nonnegative contacts, the same proof applies whenever the cross map in supported energy coordinates is Hilbert–Schmidt and a strict contraction for each positive gap. This includes the finite-to-full window comparison in the Poisson aperture model in its trace regime [23, Sections 65–69].

Indeed, exhaust both openings by their intersections with increasing bounded balls. Their supported subspaces are generated by compactly supported smooth functions, so the increasing energy projections tend strongly to the identity. For a fixed reduced window the analogous intersections tend strongly to its energy projection. Left and right compression of a fixed Hilbert–Schmidt cross map by those projections converges in Hilbert–Schmidt norm. The products entering the determinant therefore converge in trace norm, and strict contraction of the full map bounds the compressed norms away from one at the fixed parameters. Thus the bounded-exhaustion determinants tend to the claimed exact observables.

At the kernel level the unbounded limit preserves the positive measures for differences. If $D_R$ exhausts $D$ and the reduced domain is $D_R \cap \Omega$, the path indicator for the resolvent difference is

full survival in $D_R$ through time $t$, together with the condition that the path fails to survive in $\Omega$ through $t$. This event increases with $R$. The analogous full and reduced indicators increase as well. Expand the cyclic words into sums of products of restricted kernels and their positive differences. All path measures increase along the exhaustion in each such product, so Tonelli gives the positive measure of the exact limit, even at zero mass.

## 16 Two-window overlap and fixed contact parameters

Write $q_{ab}$ for the determinant observable with window indicator $a$ on the first opening and $b$ on the second, where 0 means full and 1 means restricted. The overlap

$$q_{00} - q_{10} - q_{01} + q_{11}$$

also has a positive joint Laplace representation. Indeed, in every word expand each full resolvent factor as its restricted kernel plus its positive difference. Inclusion–exclusion keeps exactly the products using at least one difference from each side. For genuine truncations on both sides the trace term with one crossing in each direction makes the overlap nonzero. Thus the combined truncation loss is bounded above by the sum of the two one-sided losses, and the gap between them is jointly completely monotone.

An affine contact family $V_0 + \sum_j \theta_j W_j$, with all $W_j \geq 0$ and $\theta_j \geq 0$, adds path variables $\int_0^t W_j(Y_r)\, dr$. Their exponential factors are Laplace transforms in the additional parameters $\theta_j$. The same proof gives joint complete monotonicity in these contact strengths as well as squared gap and squared mass. This assertion requires the same contact family on overlapping supports. Strictness in an individual contact direction requires positive occupation on a set of contributing paths.

## 17 Several compact apertures moving apart

For $N \geq 2$ bounded openings, let $\mathcal{X}_A$ be the block operator for a subset $A$ of openings, with zero diagonal blocks and off-diagonal blocks $R_i^{1/2} B_{ij} R_j^{1/2}$. Write $Q_A = -\log \det_2(I - \mathcal{X}_A)$ and set $Q_\varnothing = 0$. The connected observable is $\sum_{A \subseteq \{1,\dots,N\}} (-1)^{N-|A|} Q_A$. Use the connected trace expansion in Section 3. Let opening $i$ move with velocity $v_i$, starting at a configuration in which

$$(v_i - v_j) \cdot (X - Y) \geq 0$$

for every $X, Y$ in distinct supported openings $i, j$. For an increment $g > 0$ the pairwise squared distance is

$$|X - Y|^2 + 2g(v_i - v_j) \cdot (X - Y) + g^2 |v_i - v_j|^2.$$

Its Gaussian factor is a positive Laplace transform in $s = g^2$, by the same calculation. In the connected inclusion–exclusion of the negative log determinant, each closed word visiting all apertures survives with coefficient $1/m$ at word length $m$. Every other word cancels by the subset identity. Under the strict-contraction and trace hypotheses this is a sum of positive cyclic kernel integrals. The resulting connected observable therefore has the same joint complete monotonicity in squared separation increment, squared mass and fixed nonnegative affine contact parameters. A closed word visiting all openings gives strictness in $s, \mu$ if all openings are nonempty and the velocities are pairwise distinct. For bounded apertures undergoing proportional separation, the pairwise condition holds once the starting separation is large enough.

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