Scalar Casimir interactions of holes in a Dirichlet screen: many-hole terms, monotone attraction and near-contact laws

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November 2026

Abstract

Kabat, Karabali and Nair computed the leading Casimir interaction of two small holes in a Dirichlet plate for a massless scalar field, $E = -5Q_1Q_2/(32\pi^3 r^7)$, found that several distant holes interact pairwise at leading order, and asked how the interaction behaves as the holes approach one another, raising the possibility of a minimum at finite separation. We answer these questions for the scalar screen in every tangential dimension d. First, for m windows at separation scale L, the genuinely m-body (connected) vacuum energy is of order $L^{-m(d+1)-1}$, with an explicit coefficient given by a sum over directed Hamiltonian cycles of products of window torsions and Euclidean kernels; we also obtain its full thermal crossover at inverse temperature $\beta = \lambda L$, the fixed-temperature zero-mode law and the first low-temperature correction. For $d = 2$ and two transparent holes the coefficient is exactly the Kabat–Karabali–Nair result, and the hole charge is an exact fractional torsion: for a round hole $Q = 4R^3/3$, which replaces the published variational estimate $1.28R^3$, and for a slit $Q = \pi W^2/8$. Second, an exact identity between a regularized boundary determinant and the connected heat trace shows that the interaction is strictly attractive along any separation path on which all point-to-point distances between the windows increase. In particular two round holes of any radii attract at every disjoint separation, so the interaction has no minimum. Third, as the gap g between two windows closes, the vacuum energy diverges like $-g^{-(d+1)/2}$ times a curvature integral and a positive planar all-hit constant; finite-type faces change the exponent to $d - \sum_j p_j^{-1}$, and for slit apertures in $d = 1$ the gap-scale thermal law is $-\log(1/g)/(8\lambda g)$.

Keywords. Scalar Casimir interaction, Dirichlet screen, apertures, relative heat trace, fractional torsion, Matsubara sum, near-contact asymptotics.

Overview

Sections 1–6 treat separated windows: the connected energy and its thermal crossover (Theorem 2.1), strict attraction (Theorem 2.2) and the exact hole charges in dimension two. Sections 7–11 treat the opposite end of the disjoint family, two windows whose gap closes (Theorem 8.1). Both parts rest on the connected heat trace of the screen, whose positivity, screen limit and many-window diffusion law are proved in [20, 21].

1 Separated holes: introduction

Kabat, Karabali, and Nair [1, Sections II and V] use the square-root boundary operator of a scalar field on both sides of a planar Dirichlet screen to obtain the leading two-hole vacuum interaction.

Their large-separation analysis of several holes gives the leading pairwise total energy. We take the alternating sum over all subsets of open windows. It removes every proper-subset interaction, and for a fixed number of windows its first surviving coefficient is a sum of cycle products. This coefficient gives both the vacuum law and its temperature dependence on the scale $\beta \asymp L$.

The all-hit bridge mechanism and irreducible scattering expansions for obstacles appear in Schaden [2, eqs. (15)–(18), (22)–(23)]. Emig, Graham, Jaffe, and Kardar [3, Sections II.C, III.C, and IV.D] organize scalar vacuum interactions through multiple-scattering paths and low-frequency shape factors. Hanisch, Strohmaier, and Waters [4, Theorems 1.6–1.7] establish the rigorous relative determinant for compact obstacles with the single-object blocks removed. We apply this normalization to the square-root form of a screen with openings, then use full subset inclusion–exclusion. The connected heat representation is the starting point of [21].

Kabat and Karabali [5, eqs. (10), (26)] use the Matsubara sum and Poisson resummation for scalar diffraction at plate edges and a slit. Rahi et al. [6] give general finite-temperature scattering formulas for electromagnetic bodies. We compute the zero mode and the leading low-temperature correction for the connected window coefficient. For the force, the reflection-based attraction results of Kenneth and Klich [7] and Bachas [8] are antecedents. In the screen model, order of the pointwise kernels along a prescribed separation path gives the strict derivative. The path criterion applies to unequal round windows and to the all-window connected interaction.

2 The screen model and its connected energy

Fix $d \geq 1$, $m \geq 2$, distinct centers $p_1, \dots, p_m \in \mathbb{R}^d$, and nonempty bounded open windows $K_1, \dots, K_m \subset \mathbb{R}^d$. The windows at separation L are $Lp_i + K_i$. Take L for which their closures are pairwise disjoint. Each window can carry a bounded nonnegative contact profile ρ_i, fixed under translation. For $S \subseteq [m]$, write $K_{L,S} = \bigcup_{i \in S} (Lp_i + K_i)$, with the corresponding translated contact on its components. The open domain for this subset is

$$\mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K_{L,S}) \times \{0\}).$$

Its contact adds the surface integral of $\rho_i |f|^2$ on each open window to the Dirichlet form. Denote the heat gain relative to the two cut Dirichlet half-spaces by $D_{L,S}(t)$, defined by the integrated diagonal difference for generator Δ. Let $D_{L,\varnothing} = 0$ and

$$\mathcal{I}_L(t) = \sum_{S \subseteq [m]} (-1)^{m-|S|} D_{L,S}(t).$$

As in [21], this counts bridges visiting every aperture, with the Feynman–Kac weight of the contacts. It is finite and strictly positive for every $t > 0$. For fixed separated windows it is bounded at small t by $Ct^{-N} e^{-c/t}$ for some $N, c > 0$. The contact-free single-screen law [21] bounds each $D_{L,S}(t)$ by $O(t^{-(d+1)/2})$ at infinity, since nonnegative contact only reduces the heat gain. We use $\hbar = c = k_{\mathrm{B}} = 1$ and set

$$\mathcal{E}_L = -\frac{1}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \mathcal{I}_L(t)\, dt, \tag{1}$$

$$\mathcal{F}_{L,\beta} = -\frac{1}{2\beta} \int_0^\infty t^{-1} \mathcal{I}_L(t) \sum_{n \in \mathbb{Z}} e^{-t(2\pi n/\beta)^2}\, dt, \qquad \beta > 0. \tag{2}$$

Both converge absolutely by these bounds and are strictly negative. The first is the connected relative energy of the scalar function $\frac{1}{2}\sqrt{a}$. The second corresponds to $\frac{1}{2}\sqrt{a} + \beta^{-1} \log(1 - e^{-\beta\sqrt{a}})$

at spectral energy a. In fact the derivative of the Matsubara logarithm representation at $a > 0$ is

$$\frac{1}{2\beta}\sum_{n\in\mathbb{Z}}\frac{1}{a+(2\pi n/\beta)^2} = \frac{1}{4\sqrt{a}}\coth\left(\frac{\beta\sqrt{a}}{2}\right),$$

the derivative of that free-energy function. Spectrally constant terms cancel in the connected difference.

Let F_{K_i} be the exterior-Dirichlet half-Laplacian, the compression of the Fourier multiplier $|\xi|$ to the closure of zero extensions of $C_c^\infty(K_i)$ in $H^{1/2}(\mathbb{R}^d)$. Let

$$T_i = \langle \mathbf{1}, (2F_{K_i} + \rho_i)^{-1}\mathbf{1}\rangle, \qquad C_d = \frac{1}{\sqrt{\pi}(4\pi)^{d/2}}, \qquad c_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}.$$

For $\ell > 0$, define the positive functions

$$k_{d,\ell}(t) = C_d t^{-(d+3)/2} e^{-\ell^2/(4t)}, \qquad K_\ell(z) = \int_0^\infty e^{-zt} k_{d,\ell}(t)\,dt \quad (z \geq 0). \tag{3}$$

Write $\ell_{ij} = |p_i - p_j|$. Let $\mathcal{H}_m$ be the directed Hamiltonian cycles on the window labels modulo cyclic rotations. The two directions are distinct when $m \geq 3$, and there is one cycle when $m = 2$. Put

$$P(q) = \sum_{\gamma\in\mathcal{H}_m}\prod_{(i,j)\in\gamma} K_{\ell_{ij}}(q^2), \qquad q \geq 0.$$

Theorem 2.1. *For fixed shapes, contacts, and centers as above, as $L \to \infty$,*

$$L^{m(d+1)+1}\mathcal{E}_L \longrightarrow \mathcal{E}_* = -\frac{\prod_i T_i}{2\pi}\int_0^\infty P(q)\,dq, \tag{4}$$

$$L^{m(d+1)+1}\mathcal{F}_{L,\lambda L} \longrightarrow \mathcal{F}_*(\lambda) = -\frac{\prod_i T_i}{2\lambda}\sum_{n\in\mathbb{Z}} P(2\pi|n|/\lambda), \tag{5}$$

$$L^{m(d+1)}\mathcal{F}_{L,\beta} \longrightarrow -\frac{\prod_i T_i}{2\beta}P(0) \qquad (\beta > 0 \text{ fixed}). \tag{6}$$

The limit in (5) *is locally uniform for $\lambda > 0$. The zero-mode factors are $K_\ell(0) = 2c_d/\ell^{d+1}$, and*

$$\mathcal{F}_*(\lambda) \sim -\frac{\prod_i T_i}{2\lambda}P(0) \quad (\lambda \downarrow 0), \qquad \mathcal{F}_*(\lambda) \longrightarrow \mathcal{E}_* \quad (\lambda \to \infty).$$

For each cycle, put $\mu_{ij} = 2c_d/\ell_{ij}^{d+1}$ and let

$$A = C_d\left(\prod_i T_i\right)\sum_{\gamma\in\mathcal{H}_m}\sum_{e\in\gamma}\prod_{\substack{f\in\gamma\\ f\neq e}}\mu_f,$$

where $\mu_{(i,j)} = \mu_{ij}$. The first low-temperature correction to the limiting coefficient is

$$\mathcal{F}_*(\lambda) - \mathcal{E}_* \sim -\frac{A\,4^{(d+2)/2}}{2\sqrt{\pi}}\Gamma((d+2)/2)\zeta(d+2)\lambda^{-(d+2)} \qquad (\lambda \to \infty). \tag{7}$$

Theorem 2.2. *Suppose that at a given L the closures are separated and*

$$\partial_L|L(p_i - p_j) + y - y'| > 0 \quad \text{for every } i \neq j,\ y \in \overline{K}_i,\ y' \in \overline{K}_j. \tag{8}$$

Then $\partial_L\mathcal{E}_L > 0$ and $\partial_L\mathcal{F}_{L,\beta} > 0$ for each fixed $\beta > 0$. The generalized force along increasing L is therefore attractive. For two round windows centered at the two translation points with unit center separation before scaling, the criterion holds for every disjoint distance $L > R_1 + R_2$, where R_i are their radii.

3 The connected determinant on the positive axis

For $z \geq 0$, identify the window spaces by translation with $\mathcal{H}_i = L^2(K_i)$. Let $N_i(z)$ be the form compression of $2\sqrt{|D|^2 + z}$ to K_i plus ρ_i, where $|D|$ has symbol $|\xi|$. For a subset S, $N_S(z)$ is the same response on the union with its full contact. Its form domain decomposes into the individual aperture form spaces by smooth cutoffs around the separated windows. Normalize by the diagonal square roots of the isolated responses. This gives $I + X_S(z)$, with zero diagonal in $X_S(z)$ and off-diagonal blocks $N_i(z)^{-1/2} N_{ij}(z) N_j(z)^{-1/2}$. Positive definiteness of the full aperture form implies positivity and injectivity of the normalized form. The off-diagonal blocks are Hilbert–Schmidt by the separated kernel formula below. Thus compactness of $X_S(z)$ gives invertibility of $I + X_S(z)$. Set

$$G_L(z) = \sum_{S \subseteq [m]} (-1)^{m-|S|} \log \det{}_2(I + X_S(z)),$$

with zero for the empty subset and the real logarithm. We use the regularized determinant for Hilbert–Schmidt perturbations, for which $\det_2(I+X) = \det(I+X)e^{-\operatorname{tr} X}$ when X is trace class.

For $x \neq 0$ the free two-arm square-root kernel is

$$n_z(x) = -C_d \int_0^\infty t^{-(d+3)/2} e^{-zt - |x|^2/(4t)}\, dt = -K_{|x|}(z). \tag{9}$$

This is the off-diagonal part of the square-root subordination formula, with the identity term absent between distinct windows. In two identical transverse Dirichlet cubes the corresponding off-diagonal kernel is the integral of $-\pi^{-1/2} t^{-3/2} e^{-zt}$ times the cube heat kernel. It is bounded in magnitude by the free kernel integral.

Proposition 3.1. *For every separated configuration and $z \geq 0$,*

$$G_L(z) = -\int_0^\infty t^{-1} e^{-zt} \mathcal{I}_L(t)\, dt. \tag{10}$$

Consequently

$$\mathcal{E}_L = \frac{1}{2\pi} \int_0^\infty G_L(\kappa^2)\, d\kappa, \qquad \mathcal{F}_{L,\beta} = \frac{1}{2\beta} \sum_{n \in \mathbb{Z}} G_L((2\pi n/\beta)^2). \tag{11}$$

Proof. First use identical bounded transverse Dirichlet cubes containing all windows. Denote the connected function, heat gain and measure in the cubes by the same symbols in this part of the proof. Choose finite-dimensional spaces $W_{i,n}$ in the individual aperture form domains with form-dense increasing union. Use their direct sums for the subset responses. In finite dimensions, summing the aperture log determinants over subsets cancels every isolated diagonal log. The block normalization is a congruence whose isolated determinants are exactly the terms removed. The surviving off-diagonal matrices have trace zero, so their ordinary and regularized determinants agree.

For each $z_0 > 0$, this passage to the Galerkin limit is locally holomorphic around z_0. In fixed coordinates $N_i(z_0)^{1/2} W_{i,n}$, let $P_{i,n}$ denote the orthogonal projections, which converge strongly to the identity. The full bounded forms $N_i(z_0)^{-1/2} N_i(z) N_i(z_0)^{-1/2}$ are $I + O(|z - z_0|)$. The inverses of their compressions, extended by zero, converge strongly together with their adjoints, locally uniformly on a smaller disk by the uniform Neumann series. The fixed-normalized off-diagonal kernels are holomorphic there in Hilbert–Schmidt norm by the cube heat-kernel formula. Multiplying these compact factors by the bounded inverse compressions preserves locally uniform

Hilbert–Schmidt convergence, by finite-rank approximation. The resulting determinant is $\det_2$ of identity plus the diagonal inverse times the off-diagonal matrix. On the real axis this agrees with the symmetrically normalized determinant by cyclic products. In the Galerkin spaces it agrees by similarity. Local convergence of the regularized determinants and nonvanishing at z_0 give convergence of the logs with the real value there. Cauchy's formula then gives all fixed derivatives.

Choose an integer $k > (d+1)/2$. The higher-resolvent reconstruction of [20] gives the k-th derivative of the aperture log in the Galerkin limit for each subset:

$$(-1)^{k-1}(k-1)! \int_0^\infty (z+u)^{-k}\, d\nu_S(u).$$

Here ν_S is the positive relative measure of its product domain. The reconstruction uses the Schatten bounds for the differentiated Poisson and boundary factors and strong convergence of the uniformly bounded Galerkin inverses in form-dense coordinates. These apply to the chosen direct sums because each is form-dense in its subset, with the positive zero-energy normalization of the union. Thus the alternating combination of the displayed derivatives is $G_L^{(k)}(z)$ by the analytic determinant convergence. It is also the k-th derivative of the right side of (10), by the heat transform of each ν_S. The two functions can differ only by a polynomial of degree at most $k-1$. Both vanish at $+\infty$: the determinant by the separated kernel integral and the uniform positive-axis inverse bounds, and the heat integral by the all-hit Gaussian bound. The polynomial vanishes.

Now exhaust the plane by cubes. For $z > 0$, the cube heat kernel tends to the free heat kernel and is dominated by it. Off-diagonal kernels thus converge in Hilbert–Schmidt norm by the separated integral. For each diagonal response the minimum of $\int(|\nabla f|^2 + z|f|^2)$ over extensions with fixed trace converges to the half-space minimum: cut off the half-space minimizer on large cubes to get the reverse comparison to the free lower bound. The common aperture form domain and monotone form convergence give strong convergence of the bounded inverses. Their inverse square roots converge strongly as well, with uniform bounds, so the normalized off-diagonal operators converge in Hilbert–Schmidt norm. The regularized determinants then converge. The all-hit bridge representation on cubes converges to the full-screen one with the same contact weights, and is dominated by the free all-hit bound. This gives convergence of the heat integral with $z > 0$ and proves (10) there. At zero, the isolated and subset fractional forms have a positive gap on their bounded apertures. Their bounded $O(\sqrt{z})$ form perturbations and the separated kernel formula give determinant continuity. The heat side extends by monotone convergence and its large-time bound. This proves (10) for $z = 0$.

Finally $-G_L$ has a positive heat representation. Tonelli in (1) and (2), using $\int_0^\infty e^{-t\kappa^2}\, d\kappa = \sqrt{\pi}/(2\sqrt{t})$, proves (11). □

4 Large separation and the thermal crossover

Take $z = q^2/L^2$. By (9) and heat scaling, the free cross kernel at $y \in K_i,\ y' \in K_j$ is

$$-L^{-(d+1)} K_{|p_i - p_j + (y-y')/L|}(q^2).$$

The lengths stay in a compact interval away from zero for large L. The Bessel formula for (3), with $\alpha = (d+1)/2$, is

$$K_\ell(q^2) = 2C_d (4q^2/\ell^2)^{\alpha/2} \mathsf{K}_\alpha(\ell q),$$

where K_α is the modified Bessel function of the second kind. Its small-argument bound and large-argument exponential decay give a uniform bound on the cross blocks in Hilbert–Schmidt norm by $CL^{-(d+1)}(1+q)^M e^{-cq}$, with $c>0$. The real-axis diagonal forms satisfy $N_i(z) \geq N_i(0) > 0$, so the normalization preserves this bound. In particular $\|X_S(z)\| < 1$ uniformly for all $q \geq 0$ and large L. Therefore

$$\log \det{}_2(I + X_S) = \sum_{k\geq 2} \frac{(-1)^{k+1}}{k} \operatorname{tr} X_S^k. \tag{12}$$

Each trace is a sum over rooted closed walks in the window labels. Inclusion–exclusion retains exactly the walks visiting all labels. Degrees below m vanish. The m-th degree consists of Hamiltonian cycles, each with m possible roots canceling the denominator m.

For fixed q, divide each cross block by $L^{-(d+1)}$. Its free kernel converges uniformly on the aperture pairs to the constant $-K_{\ell_{ij}}(q^2)$, giving a Hilbert–Schmidt rank-one limit between the individual constant vectors. Each isolated inverse tends in norm to $N_i(0)^{-1}$, uniformly on bounded q-intervals. In the leading closed traces the normalized factors combine to one isolated inverse between each pair of kernels. The contractions give $\prod_i T_i$. The m negative kernels and the log-series sign give a single minus, so

$$L^{m(d+1)} G_L(q^2/L^2) \longrightarrow -(\prod_i T_i) P(q) \tag{13}$$

locally uniformly for $q \geq 0$. For $k \geq 2$, $|\operatorname{tr} X_S^k| \leq \|X_S\|_2^2 \|X_S\|^{k-2}$. In particular the convergent series starting in degree m gives a uniform bound on the scaled absolute value in (13) by $C(1+q)^{M'} e^{-c'q}$, with $c' > 0$. Substitution $q = L\kappa$ in (11) and dominated convergence establish (4). At $\beta = \lambda L$, the scaled Matsubara variables are $2\pi|n|/\lambda$. The same bound gives uniform exponential summability when λ ranges over a compact positive interval, proving (5). At fixed β, the nonzero scaled modes have $q = 2\pi|n|L/\beta$ and are exponentially small as $L \to \infty$. This proves (6) directly.

The mass of $k_{d,\ell}$ is $2c_d/\ell^{d+1}$, by $v = \ell^2/(4t)$. Thus the $n = 0$ term gives the small-λ asymptotic in Theorem 2.1. The other terms are exponentially small up to a polynomial in $1/\lambda$. The Riemann sum as $\lambda \to \infty$ recovers the vacuum coefficient, with the full sum over positive and negative frequencies.

To obtain the first correction there, use the connected heat profile of [21],

$$H(\theta) = \left(\prod_i T_i\right) \theta \sum_{\gamma \in \mathcal{H}_m} \left(*_{(i,j)\in\gamma} k_{d,\ell_{ij}}\right)(\theta).$$

It is exponentially small up to a power as $\theta \downarrow 0$. Its late-time law is $H(\theta) \sim A\theta^{-(d+1)/2}$, with A as in Theorem 2.1. The coefficient $\mathcal{F}_*(\lambda)$ is the positive-time representation (2) with H in place of $\mathcal{I}_L$ and period λ, by the convolution formula for P. Apply Poisson summation to the Matsubara Gaussian. The zero winding term is $\mathcal{E}_*$, and the two signs of nonzero winding give

$$\mathcal{F}_*(\lambda) - \mathcal{E}_* = -\frac{1}{2\sqrt{\pi}} \sum_{k\geq 1} \int_0^\infty \theta^{-3/2} H(\theta) e^{-\lambda^2 k^2/(4\theta)}\, d\theta.$$

The two threshold-time regimes give a global majorant $C\theta^{-(d+1)/2}$ for H. The change $v = \lambda^2 k^2/(4\theta)$ and dominated convergence yield the Gamma integral with the summable factor $k^{-(d+2)}$, giving exactly (7).

5 Strict force under proportional separation

We prove Theorem 2.2 at each admissible L. For real $z \geq 0$, put $Y_S = -X_S$. The free positive cross kernel $-n_z(x)$ is strictly decreasing in $|x| > 0$ by (9). Each diagonal normalizer $N_i(z)^{-1/2}$ preserves positivity. Indeed the compressed square-root form is a Dirichlet form with killing by heat subordination and zero extension. Adding the nonnegative contact preserves this property, and the inverse square root is the Laplace integral of the corresponding positivity-preserving semigroup. Thus the normalized cross blocks of Y_S preserve positivity. Their Hilbert–Schmidt kernels are nonnegative almost everywhere, as follows by testing measurable rectangles.

The operator Y_S is compact and self-adjoint, and $I - Y_S > 0$ is invertible. Also

$$|\langle f, Y_S f\rangle| \leq \langle |f|, Y_S |f|\rangle.$$

Its operator norm therefore equals its top spectral endpoint, which is strictly less than one. For any separated L, the regularized log equals $-\sum_{k\geq 2} \operatorname{tr}(Y_S^k)/k$. Inclusion–exclusion again retains just the closed walks visiting every label. Their traces are nonnegative, since the kernels are nonnegative. For instance the cyclic trace can be paired in Hilbert–Schmidt norm between two factors or products, with the kernel formula justified by Tonelli.

Under (8), the cross kernels strictly decrease pointwise before normalization, and the diagonal normalizers are independent of L. Each normalized cross derivative is nonpositive in kernel order. Each surviving walk therefore contributes a nonnegative derivative to the negative log series. For strictness choose a Hamiltonian cycle and differentiate just one edge. Before normalization both the negative kernel derivative on that edge and the undifferentiated positive kernels on the other edges have strictly positive lower bounds on their compact aperture pairs, at fixed z. After normalization each dominates a positive constant times the rectangular rank-one kernel formed from $N_i(z)^{-1/2}\mathbf{1}$ and $N_j(z)^{-1/2}\mathbf{1}$. These vectors are nonzero and nonnegative. The product of those rank-one kernels around the cycle has positive trace, giving a strictly positive contribution to $\partial_L G_L(z)$.

Locally around a separated L, Y_S and its L derivative vary continuously in Hilbert–Schmidt norm, uniformly on compact z-intervals, including zero. On such intervals $\|Y_S\|$ is uniformly below one, and

$$|\partial_L \operatorname{tr}(Y_S^k)| \leq k\|\partial_L Y_S\|_2 \|Y_S\|_2 \|Y_S\|^{k-2}.$$

This justifies termwise differentiation. At large $\kappa = \sqrt{z}$, the Bessel formula and its spatial derivative give polynomial times exponential decay for the cross kernels and their L derivatives, locally uniformly in separated L. The diagonal inverse factors are uniformly bounded on the real positive axis. These estimates justify differentiation under the integral and sum in (11). Both derivatives are strictly positive because $\partial_L G_L(z) > 0$ for every $z \geq 0$.

For two centered round windows, write $p_2 - p_1 = e$, $|e| = 1$. The L-derivative of one half the squared point-to-point separation is $L + e \cdot (y' - y) \geq L - R_1 - R_2 > 0$. Thus the strict sign holds at every disjoint separation, also for unequal radii. This excludes an interior minimum of the relative pair potential on that disjoint range, the finite-separation issue raised by Kabat, Karabali, and Nair [1, Section V].

6 Dimension two and explicit torsions

In tangential dimension two the half-integer Bessel formula gives

$$K_\ell(q^2) = \frac{(1 + \ell q)e^{-\ell q}}{\pi \ell^3}.$$

For a cycle with edge lengths $\ell_1, \dots, \ell_m$ and perimeter $P_\gamma = \sum_j \ell_j$, its contribution to $\int_0^\infty P(q)\,dq$ is therefore

$$\frac{1}{\pi^m \prod_j \ell_j^3} \sum_{r=0}^{m} \frac{r!\, e_r(\ell_1, \dots, \ell_m)}{P_\gamma^{r+1}},$$

where e_r is the elementary symmetric polynomial and $e_0 = 1$. This follows by multiplying the m polynomials $1 + \ell_j q$ and integrating against $e^{-P_\gamma q}$.

For a pair at unit scaled center separation this yields

$$\mathcal{E}_* = -\frac{5T_1T_2}{8\pi^3} \qquad (d = 2).$$

For transparent windows, Kabat, Karabali, and Nair [1, eqs. (10)–(11)] use $Q_i = \langle \mathbf{1}, F_{K_i}^{-1}\mathbf{1}\rangle = 2T_i$, since their charge uses the one-side root. Their two-hole interaction is $-5Q_1Q_2/(32\pi^3 L^7)$ to leading order, exactly the normalization above.

For a ball of radius R, set the contact to zero. Dyda's power formula [9, Theorem 1] gives $F_{B_R}[(R^2 - |y|^2)_+^{1/2}] = \sqrt{\pi}\,\Gamma((d+1)/2)/\Gamma(d/2)$ inside the ball. It follows by integration that the transparent charge $Q = 2T$ is

$$Q(B_R) = \frac{\pi^{d/2}\Gamma(d/2)}{2\Gamma((d+3)/2)\Gamma((d+1)/2)} R^{d+1}.$$

In particular the circular charge is $4R^3/3$, and the charge of an interval of width W is $\pi W^2/8$. These are exact evaluations of the charge defined in [1], eq. (11): the slit value agrees with the numerical $0.393W^2$ there, while the closed form $4R^3/3 \approx 1.333R^3$ for a round hole replaces the variational estimate $1.28R^3$ of [1, eq. (25)]. Indeed the torsion function of the disk for $(-\Delta)^{1/2}$ is $u(x) = (2/\pi)(R^2 - |x|^2)^{1/2}$, by the classical identity $(-\Delta)^{1/2}(1 - |x|^2)_+^{1/2} = \pi/2$ in the unit disk, and $\int u = 4R^3/3$. For transparent apertures with fixed individual measures and fixed centers, the fractional Saint-Venant bound [10, Proposition 3.4] maximizes the leading energy magnitude at balls in its bounded Lipschitz class in $d \geq 2$. The comparison also holds for arbitrary bounded open apertures in these dimensions: each compactly supported test in the torsion variational quotient lies in a smooth open subdomain compactly contained in the aperture. Apply the bound there, compare the balls by their volumes, and pass to the form closure. Every cycle coefficient contains the same product of the positive torsions, which gives the shape conclusion.

7 Closing gaps: introduction

Kabat, Karabali, and Nair [1, Sections II and V] compute the distant two-hole scalar interaction in one Dirichlet screen and ask how the potential behaves as the holes approach one another. The connected scalar energy of separated round holes has a strict attractive force for all disjoint center distances (Theorem 2.2). We address the other endpoint of that disjoint family. As the gap closes, the bridge paths contributing at leading order concentrate near the two facing edges. The transverse directions along the edges scale more slowly than the plane-normal and gap directions. In the two directions that remain diffusive, the limiting barrier is a retained segment with two open exterior rays.

In the coplanar whisker geometry of Kabat, Karabali, and Nair [11, Section 3 and Figure 2], the two exterior rays are instead the retained parts of a screen with an open slit. At the path level their two-obstacle interaction counts paths hitting both exterior rays. Our planar integral imposes the additional condition that those paths avoid the retained central segment.

This changes the connected heat function even though the two endpoint positions coincide. For complementary-boundary scattering relations see Maghrebi, Abravanel, and Jaffe [12]. Worldline edge calculations and the all-hit cancellation have antecedents in Gies and Klingmüller [13] and Schaden [2, eqs. (15)–(18)]. The localization here uses the connected screen heat gain of [21].

A local geometry factor also enters derivative expansions for facing mirrors, including the scalar vacuum and thermal calculations of Fosco, Lombardo, and Mazzitelli [14, 15]. Here the gap is lateral, and the local heat coefficient is a planar all-hit integral for a segment. The temperature analysis uses the Matsubara and Poisson representations, as in Kabat and Karabali [16] for plate edges and a slit. We give the gap-scale profile for two windows and the borderline logarithm for intervals.

8 Heat observables and local geometry

In a flat screen in $\mathbb{R}^{d+1}$, write $(s, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ for the coordinates in the plane, with s along the common supporting normal within the plane. Let $x \in \mathbb{R}$ be the coordinate normal to the screen. Two bounded transparent apertures have closures meeting only at $(s, y) = (0, 0)$ before translation, on opposite sides of $s = 0$. Translate the left one by $-g/2$ and the right one by $g/2$ along s, with $g > 0$. The boundaries are Dirichlet away from the open apertures. There are no surface contacts.

For a bounded open aperture $K \subset \mathbb{R}^d$, use $D_K(t)$ for the integrated diagonal difference of the heat kernel on $\mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K) \times \{0\})$ and that of the cut half-spaces. The generator is Δ, with free diagonal $(4\pi t)^{-(d+1)/2}$. With $K_{1,g}, K_{2,g}$ the translated windows, put

$$\mathcal{I}_g(t) = D_{K_{1,g} \cup K_{2,g}}(t) - D_{K_{1,g}}(t) - D_{K_{2,g}}(t).$$

This is the integral of the free diagonal times the probability that a bridge avoids the closed screen and visits both apertures on the plane [21]. In units $\hbar = c = k_{\mathrm{B}} = 1$, the connected scalar observables of Section 1 and the following section are

$$\mathcal{E}_g = -\frac{1}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \mathcal{I}_g(t)\, dt, \tag{14}$$

$$\mathcal{F}_{g,\beta} = -\frac{1}{2\beta} \int_0^\infty t^{-1} \mathcal{I}_g(t) \sum_{n \in \mathbb{Z}} e^{-t(2\pi n/\beta)^2}\, dt, \qquad \beta > 0. \tag{15}$$

For $h > 0$, let $J_h(\tau)$ be the planar all-hit heat integral with a retained Dirichlet segment of length h on an axis. A planar bridge based at (S, X) contributes when it hits that axis in both open exterior rays and avoids the closed intervening segment. Integrate the probability against $(4\pi\tau)^{-1} dS\, dX$. This definition uses a single integrable probability. By heat scaling, $J_h(\tau) = J_1(\tau/h^2)$.

We use either of the following local geometries for $d \geq 2$. In both, the windows are convex, the closure contact before translation is unique, and near the origin the facing graphs before translation are $s = -a_1(y)$ and $s = a_2(y)$ with the windows on the outer sides.

- For C^2 faces, let B_i be the Hessian of a_i at zero and assume $B_1 + B_2$ is positive definite. Put
$$\alpha = \frac{d-1}{2}, \qquad H(w) = 1 + \frac{1}{2} w^{\mathsf{T}} (B_1 + B_2) w.$$
Round windows of radii R_1, R_2 have $B_i = R_i^{-1} I$.

- For separable finite-type faces, assume in orthonormal tangential coordinates that

$$a_i(y) = \sum_{j=1}^{d-1} c_{ij}|y_j|^{p_j} + o\left(\sum_{j=1}^{d-1}|y_j|^{p_j}\right), \quad p_j > 1, \quad c_{ij} \geq 0, \quad c_j = c_{1j} + c_{2j} > 0.$$

  Put $\alpha = \sum_j 1/p_j$ and $H(w) = 1 + \sum_j c_j|w_j|^{p_j}$.

For $d = 1$, take intervals of lengths $2R_1, 2R_2$. The tangent space is zero dimensional, with $\alpha = 0$, $H = 1$ and its spatial integral interpreted as evaluation at its unique point.

Theorem 8.1. *For the geometries above, as $g \downarrow 0$,*

$$\mathcal{E}_g \sim -g^{-(d-\alpha)} D_d \int_{\mathbb{R}^{d-1}} H(w)^{-d}\, dw, \qquad D_d = \frac{(4\pi)^{-(d-1)/2}}{4\sqrt{\pi}} \int_0^\infty \tau^{-(d+2)/2} J_1(\tau)\, d\tau \in (0, \infty). \tag{16}$$

At fixed $\beta > 0$ the difference $\mathcal{F}_{g,\beta} - \mathcal{E}_g$ stays bounded, so the free energy has the same leading term.

For $d \geq 2$, as $g \downarrow 0$ locally uniformly for $\lambda > 0$,

$$g^{d-\alpha}\mathcal{F}_{g,\lambda g} \longrightarrow -B_H(\lambda), \tag{17}$$

where

$$B_H(\lambda) = \frac{(4\pi)^{-(d-1)/2}}{2\lambda} \int_{\mathbb{R}^{d-1}} \int_0^\infty \tau^{-(d+1)/2} J_{H(w)}(\tau) \sum_{n\in\mathbb{Z}} e^{-\tau(2\pi n/\lambda)^2}\, d\tau\, dw. \tag{18}$$

As $\lambda \to \infty$, $B_H(\lambda) \to D_d \int H^{-d}$. As $\lambda \downarrow 0$,

$$\lambda B_H(\lambda) \longrightarrow \frac{(4\pi)^{-(d-1)/2}}{2} \left(\int_0^\infty \tau^{-(d+1)/2} J_1(\tau)\, d\tau\right) \int_{\mathbb{R}^{d-1}} H(w)^{-(d-1)}\, dw. \tag{19}$$

For $d = 1$, the gap-scale law is instead

$$\mathcal{F}_{g,\lambda g} \sim -\frac{\log(1/g)}{8\lambda g} \tag{20}$$

locally uniformly for $\lambda > 0$.

For C^2 faces the vacuum geometry factor is

$$\int_{\mathbb{R}^{d-1}} H^{-d} = \det\left((B_1 + B_2)/2\right)^{-1/2} \pi^{(d-1)/2} \frac{\Gamma((d+1)/2)}{\Gamma(d)}.$$

For round windows this gives the factor $\left((R_1^{-1} + R_2^{-1})/2\right)^{-(d-1)/2}$ times the same Gamma quotient and power of π. For finite-type faces,

$$\int_{\mathbb{R}^{d-1}} H^{-d} = 2^{d-1} \frac{\Gamma(d-\alpha)}{\Gamma(d)} \prod_{j=1}^{d-1} c_j^{-1/p_j} \Gamma(1 + 1/p_j).$$

9 Gaussian localization and the vacuum law

Choose an enclosing ball Q in the screen containing the two shifted windows for all small g. Because a bridge visiting both windows also contributes to D_Q, we have $0 \le \mathcal{I}_g(t) \le D_Q(t)$. By the area and torsion estimates of [21], this enclosure gives $O(t^{-d/2})$ at small time and $O(t^{-(d+1)/2})$ at large time, uniformly in g.

An all-hit bridge based at (s, y, x) must move normally by at least $|x|$. It must also move in the s coordinate by at least $|s| + g/2$, since it reaches both supported sides. These two requirements give Gaussian bounds in x, s, and g. For $d \ge 2$, let

$$r(y) = \begin{cases} |y|^2, & C^2 \text{ faces}, \\ \sum_j |y_j|^{p_j}, & \text{finite-type faces}. \end{cases}$$

For a base y in a fixed small tangent neighborhood, we claim that an all-hit path must move by at least $c\, r(y)$ in the tangential directions or in the s coordinate.

For the C^2 case, if either tangent hit differs from y by more than $\delta_1|y|$ with small fixed $\delta_1 > 0$, that alone controls $|y|^2$. Otherwise the sum of the two graph heights at the two hits equals $y^{\mathsf{T}}(B_1 + B_2)y/2 + O(\delta_1|y|^2) + o(|y|^2)$. Its lower bound is $c|y|^2$ by positive definiteness of the sum and a sufficiently small neighborhood. The graph heights also bound deeper aperture points in their respective fibers. Thus the two hit positions in the s coordinate are at least $c|y|^2$ apart, forcing at least half that excursion from the base. For round faces, the underlying graph is $R_i - \sqrt{R_i^2 - |y|^2}$, with the same quadratic term and a quartic error.

In the finite-type case, a tangent displacement comparable to $r(y)$ suffices. Otherwise both hit coordinates differ from y_j by $O(r(y)) = o(r(y)^{1/p_j})$ for each j. Since $|y_j| \le r(y)^{1/p_j}$, normalizing by $r(y)^{1/p_j}$ and using uniform continuity of the power function on bounded intervals shows that the resulting changes in each $|y_j|^{p_j}$ are $o(r(y))$. The two small remainders at the hit coordinates are also $o(r(y))$. Consequently the sum of the two graph heights is at least $cr(y)$, by $c_j > 0$. This proves the claim in either geometry.

Combining the Gaussian excursion bounds for the three requirements by a geometric mean yields the following majorant for the all-hit probability in the local tangent neighborhood:

$$C \exp[-c(x^2 + s^2 + g^2 + r(y)^2)/t]. \tag{21}$$

For $d = 1$ omit $r(y)$, and the estimate holds for every base in the plane (s, x). The underlying one-coordinate bridge deviation bound is $2e^{-b^2/t}$ for displacement at least b, with the generator-Δ convention.

For the complement of the local tangent neighborhood, either a hit requires a fixed tangent excursion, or both hits have tangent positions outside a smaller neighborhood of the contact point. The latter pieces are separated by a positive distance at $g = 0$, by compactness of the closures and uniqueness of the contact. This separation persists for small g, forcing a fixed excursion in the screen coordinates. The aperture hits also remain in a bounded region, giving another Gaussian bound at large $|y|$ with distance $(|y| - M)_+$ for a fixed M. Combine these estimates with the normal and s bounds and integrate the free diagonal. On any fixed small time interval the far part has a majorant $Ct^{-N}e^{-c_0/t}$ for some $N, c_0 > 0$ independent of g. Its vacuum contribution is uniformly bounded. At times bounded away from zero the entire contribution is uniformly bounded by the enclosure. These estimates also prove the absolute convergence of (14) and (15) at each fixed $g > 0$.

For the local part put

$$s = gS, \quad x = gX, \quad t = g^2\tau, \qquad y_j = \begin{cases} g^{1/2} w_j, & C^2 \text{ faces}, \\ g^{1/p_j} w_j, & \text{finite-type faces}. \end{cases}$$

For fixed (S, X, w) and τ, the scaled s, x bridges are duration-τ bridges based at S, X. Tangent increments in the w_j variables have size $g^{1/2}$ or g^{1-1/p_j}, respectively. On a common realization by fixed-time bridges, they tend uniformly to zero almost surely. The limiting graph thresholds on the normal zero times are

$$S_{\text{bridge}} < -1/2 - A_1(w), \qquad S_{\text{bridge}} > 1/2 + A_2(w),$$

where $A_i(w) = w^{\mathsf{T}} B_i w/2$ in the quadratic case and $A_i(w) = \sum_j c_{ij} |w_j|^{p_j}$ in the finite-type case. Their gap is $H(w)$. There is no tangent variable in $d = 1$, and the two thresholds there are $-1/2, 1/2$.

For $X \neq 0$ the normal zero times form a compact subset of $(0, \tau)$. The planar bridge in (S, X) almost surely misses both limiting endpoints, by planar point polarity and absolute continuity with respect to Brownian motion on strict initial time intervals. Therefore its S values at the normal zeros stay separated from both thresholds if there are any such zeros. The graph expansions hold uniformly on the bounded path ranges after scaling, so the indicators of allowed hits and of visiting both rays converge. The set $X = 0$ is null in the diagonal integral.

The volume element contributes $g^{2+\alpha}$, the free diagonal contributes $g^{-(d+1)}$, and $t^{-3/2} dt$ contributes g^{-1}. After multiplying the vacuum integral by $g^{d-\alpha}$, its local scalar integrand is dominated by

$$C\tau^{-(d+4)/2} e^{-c(1+S^2+X^2+r(w)^2)/\tau}.$$

In $d = 1$ drop $r(w)$. The S, X integral contributes $O(\tau)$. The tangent integral is $O(\tau^{\alpha/2})$, by scaling each w_j by $\tau^{1/(2p_j)}$ in the finite-type case and by $\tau^{1/4}$ in the quadratic case. The time bound is therefore $C\tau^{-(d+2-\alpha)/2} e^{-c/\tau}$, integrable since $\alpha < d$.

Dominated convergence reduces the diagonal integral in S, X at fixed w, τ to the planar all-hit integral $J_{H(w)}(\tau)$, after translating the center of its segment. The free tangential factor is $(4\pi\tau)^{-(d-1)/2}$. In the remaining time integral use $\tau = H(w)^2 v$, giving exactly $H(w)^{-d}$ and the universal factor D_d in (16). For unit gap the S, X Gaussian bounds give $J_1(\tau) \leq Ce^{-c/\tau}$, since integrating in S, X cancels the planar diagonal factor. A loop that crosses each ray through an interior interval has a small tube of allowed bridges with positive probability. Thus $J_1(\tau) > 0$ for $\tau > 0$ and the moment defining D_d is finite and positive.

For the C^2 shape integral, use the square root of the positive definite matrix $(B_1 + B_2)/2$ to reduce the integral to a radial beta integral. For finite-type faces and any $b > \alpha$, apply the Gamma representation of H^{-b} and Tonelli. Each coordinate integral of $e^{-vc_j |w_j|^{p_j}}$ gives $2(vc_j)^{-1/p_j} \Gamma(1 + 1/p_j)$. The remaining Gamma integral gives the stated shape factor at $b = d$. The same formulas at $b = d - 1$ give the shape factor in the thermal zero mode when $d \geq 2$.

10 Temperature at and above the gap scale

Poisson summation in (15) for a fixed $\beta > 0$ gives

$$\mathcal{F}_{g,\beta} - \mathcal{E}_g = -\frac{1}{2\sqrt{\pi}} \sum_{k \geq 1} \int_0^\infty t^{-3/2} \mathcal{I}_g(t) e^{-\beta^2 k^2/(4t)} \, dt. \tag{22}$$

The sum of the image factors is exponentially small at $t = 0$ and $O_\beta(\sqrt{t})$ as $t \to \infty$. The enclosure bound from [21] then gives an integrable majorant independent of g, proving the fixed-temperature assertion. It also works uniformly for β bounded below by any positive number.

Now let $\beta = \lambda g$ and $d \geq 2$. The same local spatial change of variables gives the power $g^{\alpha-d}$, since $t^{-1}dt/(2\beta)$ contributes g^{-1}. The Matsubara factor becomes $\sum_{n\in\mathbb{Z}} e^{-\tau(2\pi n/\lambda)^2}$, bounded by $C(1 + \lambda/\sqrt{\tau})$. The integrated local time majorant is therefore a constant times

$$\tau^{-(d+1-\alpha)/2} e^{-c/\tau}(1 + \lambda/\sqrt{\tau}),$$

with the positive prefactor $1/(2\lambda)$ outside. It is integrable at infinity because $\alpha < d - 1$, and uniformly integrable for λ on compact positive intervals. The far spatial region at small t costs $O(g^{-1})$ on those intervals by its fixed Gaussian and the bound $C(1 + \lambda g/\sqrt{t})$ on the unscaled sum. Times bounded away from zero also cost $O(g^{-1})$ by the enclosure. Since $d - \alpha > 1$, both are lower order. Dominated convergence now gives (17) and (18), with local uniformity in λ because the bridge integrand convergence is independent of λ.

As $\lambda \to \infty$, the Gaussian sum divided by λ tends to $(2\sqrt{\pi\tau})^{-1}$ by Poisson summation. For $\lambda \geq 1$ it is bounded by $C(1 + \tau^{-1/2})$, so the same integrated majorant applies to the limiting profile. This yields the vacuum coefficient. As $\lambda \downarrow 0$, the Gaussian sum tends to one and is bounded by $C(1 + \tau^{-1/2})$ for $\lambda \leq 1$. After canceling the prefactor $1/\lambda$ and rescaling the time by $H(w)^2$, the zero mode gives (19).

11 The logarithm for interval apertures

Take $d = 1$. Write J_0 for the zero-gap planar all-hit integral for the two complementary rays, omitting the common point by planar point polarity. This value is independent of time by scaling. To evaluate it, denote by p_+ and p_- the plane heat diagonals with just the right or left ray open, and by p_{cut} and p_{free} the cut and free plane diagonals. The connected diagonal is $p_{\mathrm{free}} - p_+ - p_- + p_{\mathrm{cut}}$. At time one the relative half-plane area baselines for the two rays are

$$A_\pm(S, X) = (4\pi)^{-1} \mathbf{1}_{\{\pm S>0\}}\, e^{-X^2}.$$

They sum to $p_{\mathrm{free}} - p_{\mathrm{cut}}$ by reflection. Each integrated slit defect $p_\pm - p_{\mathrm{cut}} - A_\pm$ is $-1/16$, with absolute convergence, by the planar slit computation in [21]. Decompose the slit plane by reflection. The odd Dirichlet sector cancels from the relative trace and the even sector gives the classical regular Dirichlet–Neumann junction constant of Avramidi [17, eq. (169)] and Nursultanov, Rowlett, and Sher [18, Appendix B, eq. (B.8)]. It follows that the connected all-hit integral is $J_0 = 1/8$.

The zero Matsubara term is proportional to $\int_0^\infty \mathcal{I}_g(t)\, dt/t$. By (21) in dimension one, $\mathcal{I}_g(t) \leq Ce^{-cg^2/t}$, so the portion $t < Ag^2$ is $O_A(1)$. The portion $t > b > 0$ is $O_b(1)$ by the fixed enclosure. For $Ag^2 \leq t \leq b$, scale the planar bridge and its spatial integral to time one. The actual scaled left aperture is

$$\left(-\frac{g}{2\sqrt{t}} - \frac{2R_1}{\sqrt{t}}, -\frac{g}{2\sqrt{t}}\right).$$

It contains the fixed interval $(-2R_1/\sqrt{b}, -1/(2\sqrt{A}))$ when A is large and b small, and lies in the negative half-line. The reflected analogous inclusions hold for the right aperture. The all-hit integrals are monotone in each ray opening because enlarging that opening preserves the allowed hits and all required visits. Thus throughout the middle time range $\mathcal{I}_g(t)$ is squeezed between J_0 and the all-hit integral of the fixed truncated pair. The lower bound tends to J_0 as $A \to \infty$ and

$b \downarrow 0$, by monotone convergence and point polarity. The logarithmic measure of the middle range is $2\log(1/g) + O_{A,b}(1)$. First letting g tend to zero and then improving the fixed truncation gives

$$\int_0^\infty \mathcal{I}_g(t)\,\frac{dt}{t} \sim 2J_0 \log(1/g).$$

The remaining Matsubara terms are $O_\lambda(g^{-1})$ locally uniformly in $\lambda > 0$. Indeed $\mathcal{I}_g(g^2\tau) \le Ce^{-c/\tau}$, and after removing $n = 0$ the Gaussian sum is $O(\tau^{-1/2})$ for small τ and exponentially small for large τ, uniformly on compact λ-intervals. Multiplying the zero-mode integral by $-1/(2\lambda g)$ and inserting $J_0 = 1/8$ proves (20). Any fixed length used to make the logarithm dimensionless changes only the lower-order scale g^{-1}. The same static logarithmic coefficient occurs for the critical interaction of two retained Dirichlet needles with an open gap [19, eq. (13)]. In the aperture problem it arises from the connected heat subtraction with a retained central segment and the gap-scale Matsubara limit.

Acknowledgement. OpenAI Codex and Anthropic Claude assisted with mathematical exploration, source comparison, proof checking and manuscript preparation. The author is responsible for the work.

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