Row-dependent bounds for strong permanent contraction on four-by-eight supports

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Abstract

We prove that every complex positive semidefinite matrix A supported in $K_{4,8}$ satisfies $\mathrm{per}(|A|^2) \le \mathrm{per}(A)\prod_i a_{ii}$. The proof extends an established four-row framework by retaining a necessary lower bound on squared row moments and choosing a convex permanent bound as a polynomial of the row norms. An exact two-point obstruction shows why one fixed mixture in the inherited scalar framework is insufficient. A small-row reduction and a rational tensor Bernstein certificate cover the capped-entry region; a published all-column large-entry theorem covers its complement. The result is computer-assisted and includes arbitrary phases and singular endpoints.

1 Statement and relation to earlier work

For Hermitian A, put $\mathcal{P}(A) = \mathrm{per}(A)$, $\mathcal{Q}(A) = \mathrm{per}(|A|^2)$ and $\mathrm{d}(A) = \prod_i a_{ii}$, with absolute value and square entrywise. A support graph specifies allowed off-diagonal positions, which may have zero entries.

Theorem 1 (Computer-assisted)**.** *For every complex 4×8 matrix X with $\|X\|_{\mathrm{op}} \le 1$, let*

$$A(X) = \begin{pmatrix} I_4 & X \\ X^* & I_8 \end{pmatrix}.$$

Then $\mathcal{Q}(A(X)) \le \mathcal{P}(A(X))$. Equivalently, every Hermitian positive semidefinite matrix B supported in $K_{4,8}$ satisfies

$$\mathcal{Q}(B) \le \mathrm{d}(B)\mathcal{P}(B). \tag{1}$$

Missing edges, zero rows and singular endpoints are included.

This is a structural case of the classical strong self-inequality discussed by Beasley [1, Lemma 2.2]. His order-three and entrywise nonnegative first-factor results do not exhaust complex bipartite supports with cycles: a padded Hadamard contraction has a negative four-cycle product, invariant under diagonal unitary conjugation. Chollet's conjecture concerns $\mathrm{per}(A \circ B) \le \mathrm{per}(A)\,\mathrm{per}(B)$ for positive semidefinite inputs [2]. The strong self-bound implies its self-case but fails for unrestricted positive semidefinite matrices [3].

The present note is a continuation of the author's four-row results through six and seven columns [9, 10]. The overlap bound, trace budget and retained third-order Gram estimate are inherited. The new steps are a row-dependent mixture of two valid upper bounds and use of the necessary moment constraint $R_a \ge r_a^2/8$ in the scalar domain. These repair an exact obstruction to a single fixed mixture pair; the obstruction does not rule out all other approaches.

We use the classical rectangular inequality of Carlen–Lieb–Loss [4]. Structural graph results of Pant and Singh [5] include a bipartite Z-matrix theorem with real sign hypotheses. The available abstract of Sa-nguansin and Rodtes [6] describes annular correlation classes and order-five ridge results; we do not infer exclusion of every possible overlap from an abstract alone. Exact rational Bernstein certificates in Chollet research precede this work: Zeng, Liu and Ratnavelu [7, Section 6, Lemma 4] use them for a structured real order-five self-Chollet ratio. The basis method is established.

The large-entry region uses the published input [9, Lemma 4]: for every finite q, a complex $4\times q$ contraction with squared entry $t\ge 7/10$ satisfies

$$\mathcal{P}(A(X))-\mathcal{Q}(A(X))\ge \frac{7}{2000}(1-t)\mathcal{Q}(A(D))\ge 0, \tag{2}$$

where D is the compression obtained by deleting that entry's row and column. This input uses Lieb's block inequality and the published three-row theorem [8]; it is not an all-column four-row theorem. Our new argument treats all remaining squared entries at most $7/10$.

2 Matrix budgets and necessary row moments

Rows are indexed by $0,1,2,3$. Set $W=|X|^2$, $w_{aj}=|x_{aj}|^2$ and

$$r_a=\sum_j w_{aj},\quad c_j=\sum_a w_{aj},\quad R_a=\sum_j w_{aj}^2,\quad \delta_a=r_a-R_a,\quad G=XX^*,\quad K=X^*X.$$

Contraction implies $r_a,c_j\le 1$ and $0\preceq G\preceq I$. Write e_k for elementary symmetric polynomials in four variables and $r^2=(r_a^2)_a$. Define square-minor sums

$$F_k=\sum_{|I|=|J|=k}|\operatorname{per}X[I,J]|^2,\qquad Q_k=\sum_{|I|=|J|=k}\operatorname{per}W[I,J]^2.$$

The empty permanent is one. Crossing permutations of $A(X)$ choose equal-size subsets from the two parts and two independent bijections. Therefore

$$\mathcal{P}(A(X))=\sum_{k=0}^{4}F_k,\qquad \mathcal{Q}(A(X))=\sum_{k=0}^{4}Q_k. \tag{3}$$

For $D_{ab}=\sum_j w_{aj}w_{bj}$ and $E_{ab}=\sum_j w_{aj}^2w_{bj}^2$, direct expansion gives

$$F_2=e_2(r)+\sum_{a<b}|G_{ab}|^2-2\sum_{a<b}D_{ab}, \tag{4}$$

$$Q_2=e_2(R)+\sum_{a<b}D_{ab}^2-2\sum_{a<b}E_{ab}. \tag{5}$$

Put $C=\sum_j c_j(1-c_j)$ and $O=\sum_{a<b}|G_{ab}|^2$. The trace identity $\operatorname{Tr}G^2=\operatorname{Tr}K^2$ gives

$$C+O=\frac12\sum_a r_a(1-r_a)+\frac12\sum_j c_j(1-c_j)+\sum_{j<l}|K_{jl}|^2\ge\frac12\sum_a r_a(1-r_a). \tag{6}$$

Since $F_1-Q_1=C+2\sum D_{ab}$, dropping $F_3,F_4\ge 0$ yields

$$\mathcal{P}-\mathcal{Q}\ge\frac12\sum_a r_a(1-r_a)+e_2(r)-Q_2-Q_3-Q_4. \tag{7}$$

We recall the elementary four-vertex overlap estimate. If six nonnegative pair weights z_{ab} have degrees $h_a \le \delta_a$, then

$$\sum_{a<b} z_{ab}^2 \le M(\delta) := \max_{\pi} \sum_{\{a,b\}\in\pi} \delta_a \delta_b, \tag{8}$$

where π ranges over the three perfect pairings. For completeness, choose a largest edge $a = z_{12}$, its opposite $f = z_{34}$ and the others $b = z_{13}, c = z_{14}, d = z_{23}, e = z_{24}$. Expansion gives

$$h_1h_2 + h_3h_4 = a^2 + f^2 + (a+f)(b+c+d+e) + (b+c)(d+e) + (b+d)(c+e).$$

The product of a with the four other edges dominates their squared sum. Apply the degree bounds and the maximum over pairings to obtain (8).

Column sums give $\sum_{b\neq a} D_{ab} = \sum_j w_{aj}(c_j - w_{aj}) \le \delta_a$. Eight-column Cauchy–Schwarz gives $E_{ab} \ge D_{ab}^2/8$. Consequently

$$Q_2 \le e_2(R) + \frac{3}{4} M(\delta). \tag{9}$$

The rectangular CLL inequality [4, Theorem 3.1] gives

$$Q_3 \le \frac{63}{16} e_3(R), \qquad Q_4 \le \frac{315}{32} e_4(R), \tag{10}$$

with squared constants $\binom{8}{k}(k!)^2/8^k$. Independently,

$$Q_k \le e_k(r^2). \tag{11}$$

For fixed rows I, the sum $\sum_J \operatorname{per} W[I,J]$ allows only distinct column choices and is at most $\prod_{a\in I} r_a$. Its sum of nonnegative squared summands is at most the square of its sum. Sum over I to prove (11).

Assume now $w_{aj} \le 7/10$. Convex concentration at fixed row sum, together with eight-column Cauchy–Schwarz, gives

$$\frac{r_a^2}{8} \le R_a \le f(r_a), \qquad f(t) = \begin{cases} t^2, & 0 \le t \le 7/10, \\ 49/100 + (t - 7/10)^2, & 7/10 \le t \le 1. \end{cases} \tag{12}$$

For the upper bound there is at most one filled cap and one remainder, since $r_a \le 1 < 2(7/10)$. The small-row proof may enlarge the lower endpoint to zero; the balanced-row proof will retain the necessary lower endpoint in (12).

3 The small-row region

Lemma 2. *If all squared entries are at most* $7/10$ *and some* $r_i \le 5/8$*, then* $\mathcal{Q}(A(X)) \le \mathcal{P}(A(X))$.

Proof. Use convex weight $17/20$ on (10) and $3/20$ on (11). It suffices by (7) and (9) to prove each fixed-pairing budget

$$\begin{aligned} B_\pi(r,R) = \frac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - e_2(R) - \frac{3}{4}\sum_{\{a,b\}\in\pi} \delta_a\delta_b \\ - \frac{1071}{320} e_3(R) - \frac{1071}{128} e_4(R) - \frac{3}{20}\big[e_3(r^2) + e_4(r^2)\big] \ge 0. \end{aligned} \tag{13}$$

For fixed r, separate affinity in R reduces to $R_i = 0$ or $f(r_i)$. At an inactive row $R_i = 0$, replace $r_i = t$ by zero. With partner p, $S = \sum_{a\neq i} r_a$ and $H = e_2(r^2_{\neq i}) + e_3(r^2_{\neq i})$, the exact budget difference is

$$\frac{1}{2}t(1-t) + tS - \frac{3}{4}t\delta_p - \frac{3}{20}t^2 H.$$

For three variables in $[0,1]$, $e_2(u^2) \le \sum u$ and $e_3(u^2) \le (\sum u)/3$: use $u^2v^2 \le uv \le (u+v)/2$ and $u^2v^2z^2 \le uvz \le (u+v+z)/3$. Thus $H \le 4S/3$, while $\delta_p \le S$. The difference is at least $t(1-t)/2 + tS/20 \ge 0$. Delete every inactive row. Existence of a small row is preserved, and $f(0) = 0$.

On substituting $R = f(r)$, the coefficient of $f(t)$ in the coordinate dependence is

$$L = -\sum_{a\neq i} f(r_a) + \frac{3}{4}\delta_p - \frac{1071}{320}e_2(f(r_{\neq i})) - \frac{1071}{128}e_3(f(r_{\neq i})) \le \frac{3}{4}r_p - \frac{7}{4}f(r_p) \le \frac{9}{112}.$$

The lower-interval maximum occurs at $3/14$; the upper quadratic has maximum $-353/1400$ at $32/35$. Apart from $Lf(t)$, coordinate dependence is linear minus $t^2(1/2 + 3H/20)$. The second derivative on either interval is therefore at most $-1 + 2(9/112) = -47/56$.

Choose a distinguished row in $[0, 5/8]$ and split other row intervals at $7/10$. Separate concavity reduces to the distinguished endpoints $0, 5/8$ and other endpoints $0, 7/10, 1$. A sufficient larger collection comprises the 30 count patterns in $\{0, 5/8, 7/10, 1\}^4$ containing a zero or $5/8$. Evaluate (13) at $R = f(r)$ and take the minimum over pairings. Every value is nonnegative; the smallest positive value is $2598855957/40960000000$. The all-zero and single-unit-row patterns have value zero. The supplied `verify_k48_certificate.py` reconstructs these rational evaluations, including all 720 labelled corner/pairing cases. The affinity, deletion and concavity reductions now prove (13) throughout the region. □

4 The inherited third-order estimate

The following column-independent estimate is [10, Lemma 4]:

$$F_2 + F_3 \ge e_2(r) + e_3(r) - 2\sum_{a<b}(1 + r_c + r_d)D_{ab} + \frac{1}{3}O, \tag{14}$$

where c, d are the other two rows. We briefly recall its mechanism so that the new scalar reduction can be checked. Let $T_I = \sum_j \prod_{a\in I} w_{aj}$ for triples I, $z_{ab} = r_c + r_d$ and $H_{ab} = \sum_j (w_{cj} + w_{dj})x_{aj}\overline{x_{bj}}$. Grouping Gram column assignments by multiplicities $1+1+1$, $2+1$ and 3 gives

$$F_3 = \sum_{|I|=3} \operatorname{per} G[I] - 2\sum_{a<b} z_{ab}D_{ab} - 4\Re\sum_{a<b}\overline{G_{ab}}H_{ab} + 12\sum_{|I|=3} T_I.$$

For a triple, double columns contribute twice the squared sum of three products. Extending their sum to equal columns adds $18T_I$, while the original triple-column collision contributes $6T_I$, giving the rearranged coefficient 12. Expansion gives the displayed coefficient -4 and conjugation.

Since $0 \preceq G \preceq I$, off-diagonal Gram entries have modulus at most $1/2$. Each triple's cubic Gram term is at least minus one third of its off-diagonal squared sum, by $u^2 + v^2 + z^2 \ge 6uvz$ for $u, v, z \le 1/2$. Each pair occurs in two triples. Adding (4) leaves coefficient $1/3 + z_{ab}$ on each $|G_{ab}|^2$. Weighted Cauchy–Schwarz gives $|H_{ab}|^2 \le z_{ab}(T_{abc} + T_{abd})$. Reserve $1/3$ of each square and complete the remaining z_{ab} part, costing at most $4(T_{abc} + T_{abd})$. Each triple is charged three times,

exactly consuming $12T_I$. If $z_{ab} = 0$, its corresponding terms vanish directly. This proves (14), with actual common phases retained.

It follows from (3) and (14) that

$$\mathcal{P} - \mathcal{Q} \geq C + O/3 + e_2(r) + e_3(r) - Q_2 - Q_3 - Q_4 - 2\sum_{a<b} D_{ab}(r_c + r_d). \tag{15}$$

Here $C + O/3 \geq (C + O)/3 \geq \sum r_a(1 - r_a)/6$. On the balanced domain $r_a \geq 5/8$, write $S = \sum r_a$ and $y_a = S/2 - r_a \geq 0$, since the other three rows sum to at least $15/8 > 1 \geq r_a$. Then

$$\sum_{a<b} D_{ab}(r_c + r_d) = \sum_a y_a \sum_{b\neq a} D_{ab} \leq \sum_a \delta_a y_a. \tag{16}$$

The sign of y is indispensable; we use this allowance only in the balanced region.

5 Why one fixed mixture is insufficient

Mix (15) with (7), giving weight $\theta \in [0, 1]$ to the new bound, and use weight $\lambda \in [0, 1]$ on CLL rather than (11). Equations (9) and (16) give the fixed-pairing sufficient budget

$$\begin{aligned} J_{\pi,\theta,\lambda} = {} & \left(\frac{1}{2} - \frac{\theta}{3}\right)\sum_a r_a(1 - r_a) + e_2(r) + \theta e_3(r) - e_2(R) - \frac{3}{4}\sum_{\{a,b\}\in\pi} \delta_a\delta_b \\ & - \lambda\left[\frac{63}{16}e_3(R) + \frac{315}{32}e_4(R)\right] - (1 - \lambda)\big[e_3(r^2) + e_4(r^2)\big] - 2\theta\sum_a \delta_a(S/2 - r_a). \end{aligned} \tag{17}$$

No single constant pair (θ, λ) makes this budget nonnegative on the whole balanced scalar domain. At $r = (7/10)^4$, $R = (49/100)^4$, its value is

$$\frac{132500599}{100000000} - \frac{6055041083}{3200000000}\lambda - \frac{21}{250}\theta.$$

If $\lambda \geq 3/4$, this is at most $-1205046577/12800000000$. At $r = (1, 1, 1, 1)$, $R = (29/50)^4$, its value is

$$-\frac{1283}{1000} + \frac{32520737}{40000000}\lambda + \frac{16}{25}\theta.$$

If $\lambda \leq 3/4$ and $\theta \leq 1$, it is at most $-5317789/160000000$. Both evaluations are independent of the pairing. This is a limitation of this sufficient scalar template with one fixed pair, not a counterexample to the matrix inequality or an impossibility theorem for other methods.

6 A row-dependent bound and its exact certificate

Fix $\theta = 9/10$ and choose the polynomial weight

$$\lambda(r) = \frac{1}{2} + \frac{2}{5}e_4(r). \tag{18}$$

Since $0 \leq r_a \leq 1$, one has $1/2 \leq \lambda(r) \leq 9/10$. At every actual row vector this is a legitimate convex mixture of the same two upper bounds. Its dependence on r does not introduce any dependence on R.

Substitution in (17) gives

$$\mathcal{R}_\pi(r,R) = \frac{1}{5}\sum_a r_a(1-r_a) + e_2(r) + \frac{9}{10}e_3(r) - e_2(R) - \frac{3}{4}\sum_{\{a,b\}\in\pi}\delta_a\delta_b \\ - \lambda(r)\left[\frac{63}{16}e_3(R) + \frac{315}{32}e_4(R)\right] - (1-\lambda(r))\big[e_3(r^2) + e_4(r^2)\big] - \frac{9}{5}\sum_a \delta_a(S/2 - r_a). \tag{19}$$

It suffices to prove every $\mathcal{R}_\pi \geq 0$ on $r \in [5/8,1]^4$ and $r_a^2/8 \leq R_a \leq f(r_a)$. Separate affinity in R is preserved, so endpoints are

$$R_a = \epsilon_a f(r_a) + (1-\epsilon_a)r_a^2/8, \qquad \epsilon \in \{0,1\}^4. \tag{20}$$

The lower endpoint here is the necessary eight-column moment bound.

Split each row interval into $[5/8, 7/10]$ and $[7/10, 1]$, indexed by $u \in \{0,1\}^4$. Set $(l_i, h_i) = (5/8, 7/10)$ for $u_i = 0$, and $(7/10, 1)$ otherwise. Use $r_i = l_i + (h_i - l_i)x_i$ in (20) and (19), with $f_{\text{low}}(r) = r^2$ and $f_{\text{high}}(r) = r^2 - (7/5)r + 49/50$. Each resulting polynomial $p(x) = \sum_a c_a x^a$ has degree at most three in every coordinate: λ has coordinate degree one, R_i and r_i^2 have degree two, and the dual term is at most cubic.

Fix pairing 01/23; coordinate permutations cover the other pairings on the identical domain. There are 16 endpoint choices and 16 interval boxes, hence 256 polynomials. Their tensor Bernstein coefficients of degree three in every coordinate are

$$b_k = \sum_{a\leq k} c_a \prod_{i=0}^{3} \frac{\binom{k_i}{a_i}}{\binom{3}{a_i}}, \qquad p(x) = \sum_{0\leq k_i\leq 3} b_k \prod_{i=0}^{3} \binom{3}{k_i} x_i^{k_i}(1-x_i)^{3-k_i}. \tag{21}$$

The basis functions are nonnegative and sum to one on $[0,1]^4$. Thus nonnegative coefficients establish the continuum inequality on each closed box.

Exact rational computation gives the per-endpoint-choice minima in Table 1. All $256\cdot 4^4 = 65536$ coefficients are positive, with global minimum $23953423/4000000000$. The supplied standalone `verify_k48_certificate.py` reconstructs the coefficients from direct evaluations of (19) and exact tensor interpolation, checks positivity, and checks reconstruction away from the interpolation grid. It also reconstructs the small-row evaluations of (13). No numerical optimization or tolerance enters these calculations. Together with (21), this finite exact certificate proves (19); affinity then extends the endpoint conclusion to every R in (12).

7 Exhaustion, normalization and limits

Proof of Theorem 1. If any squared entry is at least $7/10$, apply (2). Otherwise every entry is capped. If some $r_i \leq 5/8$, Lemma 2 applies; otherwise all rows belong to the balanced region proved above. Each bound includes its boundary, so the union is exhaustive.

For positive-diagonal B supported in $K_{4,8}$, normalize as $B = EA(X)E$, with $E_{ii} = \sqrt{b_{ii}}$. The diagonal blocks become identities and the Schur complement gives $\|X\|_{\text{op}} \leq 1$. Scaling gives $\mathcal{P}(B) = \mathrm{d}(B)\mathcal{P}(A(X))$ and $\mathcal{Q}(B) = \mathrm{d}(B)^2\mathcal{Q}(A(X))$, proving (1). If a diagonal vanishes, positivity forces its row and column to vanish, so both permanents are zero. This includes singular and missing-edge cases. □

Table 1: Minimum Bernstein coefficient over all interval boxes for each endpoint choice ϵ, with pairing 01/23.

$\epsilon_0\epsilon_1\epsilon_2\epsilon_3$	Minimum coefficient
0000	851985385888273/4503599627370496
0001	374762722318481/562949953421312
0010	374762722318481/562949953421312
0011	64110219984657/70368744177664
0100	374762722318481/562949953421312
0101	70275846864657/70368744177664
0110	70275846864657/70368744177664
0111	8809158275729/8796093022208
1000	374762722318481/562949953421312
1001	70275846864657/70368744177664
1010	70275846864657/70368744177664
1011	8809158275729/8796093022208
1100	64110219984657/70368744177664
1101	8809158275729/8796093022208
1110	8809158275729/8796093022208
1111	23953423/4000000000

Zero-row or zero-column padding adds isolated unit diagonal vertices, so smaller four-row rectangles through eight columns are included.

A paired Chollet consequence holds when *both* factors belong to proved strong classes. Permutation Cauchy–Schwarz gives $|\operatorname{per}(A \circ B)|^2 \le \mathcal{Q}(A)\mathcal{Q}(B)$. In correlation normalization, strong contraction yields $\mathcal{P}(A) \ge \mathcal{Q}(A) \ge 1$, and likewise for B, whence $\operatorname{per}(A \circ B) \le \sqrt{\mathcal{P}(A)\mathcal{P}(B)} \le \mathcal{P}(A)\mathcal{P}(B)$. Scaling and zero-row cases extend this assertion.

Reproducibility and acknowledgment

The theorem uses analytic continuum reductions and an exact finite rational certificate. The verifier proves the finite corner and coefficient calculations; the matrix reductions, domain predicates and positive Bernstein basis remain mathematical parts of the proof. AI tools assisted mathematical research, drafting and verification. The author is responsible for the statements, proofs, attribution and final text.

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