Roughness and boundary-density thresholds for small balls of the Kloosterman process

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Luminary Oddities

Abstract

The Kloosterman limit process K is a random Fourier series with independent bounded Sato–Tate coefficients. Around admissible half-Hölder centers its uniform small-ball probabilities have negative logarithm of order ϵ^{-2}. We determine how the roughness of the center and the flatness of the coefficient density at its boundary change this order. For admissible γ-Hölder classes the worst-case negative logarithm has order $\epsilon^{-\max(2,1/\gamma)}$; for $\gamma < 1/2$ a Rudin–Shapiro signed train of narrow spikes supplies the matching upper estimate. For bounded symmetric unimodal coefficient laws a weighted boundary integral gives the ϵ^{-2} order at every half-Hölder center. For the flat-edge family $f \propto \exp[-A(2-|x|)^{-a}]$ we obtain the complete power diagram: for $0 < \gamma < 1$ the worst-case order is $\epsilon^{-\max(2,1/\gamma,a)}$ away from the equality line $a = 1/\gamma$, $\gamma \le 1/2$, and on that line a single fixed admissible center carries an additional logarithmic factor. At the critical point $\gamma = 1/2$, $a = 2$, the negative logarithm at the fixed witness lies between $\epsilon^{-2}\log^2(1/\epsilon)$ and $\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)$, and the log-cubed payment is optimal for positive Fejér mixtures with termwise Hölder error control. Further results give a necessary 1/4-Hölder condition for the ϵ^{-2} order at a fixed center and an increment-energy upper bound.

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1 Introduction

The Kloosterman process is the limit in law, in $C([0,1])$, of the polygonal paths of normalized Kloosterman sums [1]. It is the random Fourier series (2.2) with independent Sato–Tate coefficients, recalled in Section 2. Around every admissible half-Hölder center g, including boundary coefficients and endpoints, the companion paper [8, Theorem 1.1] proves

$$-\log \mathbb{P}(\|K-g\|_\infty < \epsilon) \asymp_L \epsilon^{-2}.$$

That paper uses this theorem to prove a Wasserstein rate for Kloosterman paths. Here we determine how the roughness of the center and the flatness of the coefficient density at its boundary change the small-ball order.

The first main result determines the worst-case scale for every Hölder exponent. Let $\mathcal{H}_{\gamma,L}$ retain the same endpoint and Fourier restrictions, with Hölder exponent γ in place of $1/2$. Theorem 3.1 proves that

$$-\log \inf_{g\in\mathcal{H}_{\gamma,L}} \mathbb{P}(\|K-g\|_\infty < \epsilon) \asymp_{\gamma,L} \epsilon^{-\max(2,1/\gamma)}.$$

For $\gamma < 1/2$, a Rudin–Shapiro signed train of narrow triangular spikes supplies the matching upper estimate. This is a worst-case statement: the extremal center may vary with ϵ. The signs and their

use in Kloosterman support are established tools, including in [3]; the new ingredient here is their constrained approximation-probability consequence.

Section 4 gives a sufficient boundary condition for other bounded symmetric unimodal coefficient laws. For the family $f(x) \propto \exp[-A(2-|x|)^{-a}]$, Sections 5 and 6 determine the lower-order property for $0 < \gamma \le 1/2$: it holds precisely when $a < 1/\gamma$, and at equality it fails at a single fixed admissible boundary center. At $\gamma = 1/2$, $a = 2$, Theorem 5.2 supplies a uniform critical lower estimate, so that at the fixed witness the negative logarithm lies between $\epsilon^{-2}\log^2(1/\epsilon)$ and $\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)$; Proposition 5.3 shows that the log-cubed payment is optimal for positive Fejér mixtures with termwise Hölder error control. Section 7 completes the power orders for $0 < \gamma < 1$: the worst-case negative logarithm has order $\epsilon^{-\max(2,1/\gamma,a)}$ away from the equality line $a = 1/\gamma$, $\gamma \le 1/2$. In particular the smoother classes $1/2 < \gamma < 1$ have order $\epsilon^{-\max(2,a)}$ for every $a > 0$, with no additional logarithm at $a = 2$.

Supremum-norm small-ball estimates for independent symmetric unimodal wavelet series were established by Reiss and Schmidt-Hieber [4, Lemma 4.1]. The harmonic Fourier basis used here calls for the full-boundary argument below, since a change of basis need not preserve coefficient independence. Regularity-dependent shifted concentration for Gaussian processes is established through approximation in their reproducing-kernel Hilbert spaces [5, 6]; the present theorems give the corresponding worst-case theory for the bounded Sato–Tate harmonic class and its unimodal relatives.

Section 2 recalls the process and the results of [8] used below. Sections 3–7 prove the threshold theorems, and the appendices contain the continuity and concentration obstructions.

2 The Kloosterman process and inputs

Let $(X_h)_{h\in\mathbb{Z}}$ be independent real random variables of density

$$f(x) = \frac{\sqrt{4-x^2}}{2\pi}\,\mathbf{1}_{[-2,2]}(x). \tag{2.1}$$

Set $v_0(t) = t$, and, for $h \neq 0$, set

$$v_h(t) = \frac{\mathrm{e}^{2\pi i h t} - 1}{2\pi i h}, \qquad 0 \le t \le 1.$$

The Kloosterman process is

$$K(t) = \sum_{h\in\mathbb{Z}} X_h v_h(t), \tag{2.2}$$

where the nonzero frequencies are summed symmetrically. Almost sure uniform convergence is known from the work of Kowalski–Sawin [1]; a proof in the form needed here is [8, Lemma 2.1]. We use $\|w\|_\infty = \sup_{0\le t\le 1}|w(t)|$ for complex-valued paths, and $\widehat{u}(h) = \int_0^1 u(t)\mathrm{e}^{-2\pi i h t}\,dt$ for Fourier coefficients.

For $L \ge 1$, let $\mathcal{H}_L$ be the class of continuous $g : [0,1] \to \mathbb{C}$ with the following properties:

1. $g(0) = 0$ and $a = g(1)$ belongs to the real interval $[-2, 2]$;
2. $|g(t) - g(s)| \le L\sqrt{|t-s|}$ for all $s, t \in [0,1]$;
3. with $u(t) = g(t) - at$, the numbers

$$\beta_h = 2\pi i h \widehat{u}(h), \qquad h \in \mathbb{Z}\setminus\{0\}, \tag{2.3}$$

are real and belong to $[-2, 2]$.

The endpoint and Fourier conditions are the established support characterization of Kowalski–Sawin [2, Theorem 1.1]; we add a uniform Hölder bound.

Theorem 2.1 ([8, Theorem 1.1]). *For every $L \geq 1$, every $g \in \mathcal{H}_L$, and $0 < \epsilon \leq 1/4$,*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \geq \exp\{-26000(L+2)^2\epsilon^{-2}\}. \tag{2.4}$$

Let

$$\varpi = 1 - \pi^{-1}, \qquad \delta = \frac{\varpi}{128\pi^2}, \qquad \epsilon_0 = \sqrt{\delta/2}, \qquad c = \frac{\varpi\delta}{16}. \tag{2.5}$$

Then, for every deterministic $g \in C([0,1], \mathbb{C})$,

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \leq \exp\{-c\epsilon^{-2}\}, \qquad 0 < \epsilon \leq \epsilon_0. \tag{2.6}$$

The remaining inputs from [8] are the following.

2.1 Convergence and centered comparison

A centered variable W with values in $[-2,2]$ satisfies the bounded-variable exponential estimate $\mathbb{E}e^{\lambda W} \leq \cosh(2\lambda) \leq e^{2\lambda^2}$, with equality in the last bound for $N(0,4)$.

Lemma 2.2 ([8, Lemma 2.1]). *If the independent coefficients W_h are either all distributed as (2.1) or all distributed as $N(0,4)$, the symmetric partial sums $\sum_{|h|\leq H} W_h v_h$ converge almost surely uniformly on $[0,1]$, and in expected uniform norm. The limit is continuous. A finite set of coefficients is independent of the continuous series formed from its complement.*

The proof in [8] uses the coefficient law only through the exponential estimate above. Hence Lemma 2.2 holds for any independent centered coefficients in $[-2,2]$.

Lemma 2.3 ([8, Lemma 2.3]). *Let $Z_1, \dots, Z_m$ be independent variables whose densities are even, bounded, supported on $[-2,2]$, and nonincreasing on $[0,2)$. Let Y_j be independent $N(0,4)$ variables. For every real linear map T and convex symmetric Borel set A,*

$$\mathbb{P}(TZ \in A) \geq \mathbb{P}(TY \in A).$$

The conclusion also applies to the inverse image of a closed uniform-norm ball under a finite continuous-function map.

Let Y_h be independent $N(0,4)$ variables, $G = \sum_h Y_h v_h$, and $R_M = \sum_{|h|>M} X_h v_h$. By [8, Lemma 3.1],

$$\mathbb{P}(\|G\|_\infty \leq r) \geq \exp(-64/r^2), \qquad 0 < r \leq 1, \tag{2.7}$$

and, for every $M \geq 0$,

$$\mathbb{P}(\|R_M\|_\infty \leq r) \geq \exp(-256/r^2), \qquad 0 < r \leq 1. \tag{2.8}$$

The second bound follows from the first by the centered comparison and Gaussian translation comparison. We also use the classical identity

$$\sum_{h\geq 1} \frac{\cos(2\pi hx)}{h^2} = \pi^2(x^2 - x + 1/6), \qquad 0 \leq x \leq 1. \tag{2.9}$$

2.2 Fejér approximation with graded margins

Fix g with the endpoint and Fourier properties of $\mathcal{H}_L$, put $a = g(1)$ and $u = g - at$. If g has Hölder constant L at exponent $1/2$, then u has Hölder constant at most $L + 2$ and, as a periodic function,

$$|u(t) - u(s)| \le \sqrt{2}(L+2)\sqrt{d(t-s,\mathbb{Z})}, \tag{2.10}$$

by passing through the common zero endpoint for a wrapped pair. The degree-M Fejér kernel F_M has integral one, is nonnegative, and obeys

$$F_M(t) \le M + 1, \qquad F_M(t) \le \frac{1}{4(M+1)d(t,\mathbb{Z})^2}.$$

For $\sigma_M u = F_M * u$ these bounds and (2.10) give $\|\sigma_M u - \sigma_M u(0) - u\|_\infty \le 14\sqrt{2}(L+2)/(3\sqrt{M+1})$ [8, Section 4]. For $0 < \epsilon \le 1/4$ and an integer $M \ge 1$ set

$$b_0 = (1 - \epsilon/8)a, \qquad x_h = \frac{|h|}{M+1}, \qquad b_h = (1 - x_h)\beta_h \quad (0 < |h| \le M), \tag{2.11}$$

and $q_M = \sum_{|h|\le M} b_h v_h$, so that $q_M = b_0 t + \sigma_M u - \sigma_M u(0)$.

Lemma 2.4 ([8, Lemma 4.1])**.** *For $T_M z = \sum_{|h|\le M} z_h v_h$ and the coefficients in (2.11),*

$$\mathbb{P}(\|T_M(X - b)\|_\infty \le r) \ge c_e \epsilon^{3/2} \exp\{-6M - 64/r^2\}, \qquad 0 < r \le 1,$$

where $c_e = \sqrt{3/8}/(8\pi) > \mathrm{e}^{-5}$.

Its proof uses only the endpoint and Fourier bounds on a and β_h, the finite density comparison on the box $\prod_{|h|\le M}[-r_h, r_h]$ with $r_h = \min(1/2, x_h)$, and the factorial estimate

$$\log \frac{M!}{(M+1)^M} \ge -M, \qquad c_*^{2M} \left(\frac{M!}{(M+1)^M}\right)^3 \ge \mathrm{e}^{-6M}, \quad c_* = \frac{\sqrt{3}}{2\pi}. \tag{2.12}$$

Lemma 2.5 (Assembly, [8, Section 5])**.** *Let g satisfy the endpoint and Fourier properties of $\mathcal{H}_L$, let $0 < \epsilon \le 1/4$ and $M \ge 1$, and suppose $\|q_M - g\|_\infty \le 3\epsilon/8$. Then*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge c_e \epsilon^{3/2} \exp[-6M - 5120\epsilon^{-2}].$$

Proof. Lemma 2.4 with $r = \epsilon/4$ and the independent event $\|R_M\|_\infty \le \epsilon/4$ of (2.8) have probabilities at least $c_e\epsilon^{3/2}\exp\{-6M - 1024\epsilon^{-2}\}$ and $\exp(-4096\epsilon^{-2})$. On their intersection $\|K - g\|_\infty \le \epsilon/4 + \epsilon/4 + 3\epsilon/8 < \epsilon$. □

2.3 The upper-bound mechanism

For an arbitrary $g \in C([0,1], \mathbb{C})$ put $a = g(1)$, $u = g - at$ and $\beta_h = 2\pi i h \widehat{u}(h)$, now possibly complex. On the event $\|K - g\|_\infty < \epsilon$, the endpoint identity $K(1) = X_0$ and Parseval give, for every $N \ge 1$ [8, Section 5],

$$\sum_{h=1}^{N} |X_h - \beta_h|^2 \le N^2 \sum_{h=1}^{N} \frac{|X_h - \beta_h|^2}{h^2} \le 16\pi^2 N^2 \epsilon^2. \tag{2.13}$$

If independent indicators $I_1, \dots, I_N$ have success probabilities at least $q \in (0,1]$, exponential Markov at $\lambda = \log 2$ gives $\mathbb{P}(\sum_h I_h \le qN/2) \le \mathrm{e}^{-qN/8}$.

2.4 Arithmetic paths

For an odd prime q and $(a,b) \in (\mathbb{F}_q^\times)^2$, let $K_q(\cdot;a,b)$ be the polygonal Kloosterman path of [8, (6.1)], and let μ_q be its law for independent uniform nonzero parameters. Write $B(g,\epsilon)$ for the open uniform ball. The law μ_q converges weakly to the law of K [1, Theorem 1.5].

Corollary 2.6 ([8, Corollary 6.8])**.** *Fix $L \geq 1$ and $0 < \epsilon \leq 1/4$. Then*

$$\liminf_{q\to\infty} \inf_{g\in\mathcal{H}_L} \mu_q(B(g,\epsilon)) \geq \exp\{-104000(L+2)^2\epsilon^{-2}\}.$$

Its proof uses the compactness of $\mathcal{H}_L$ in the uniform norm, a finite $\epsilon/4$-net, the portmanteau theorem and Theorem 2.1.

3 The exact worst-case scale for Hölder centers

For $0 < \gamma \leq 1$ and $L \geq 1$, let $\mathcal{H}_{\gamma,L}$ have the same endpoint and normalized Fourier restrictions as $\mathcal{H}_L$, with its regularity condition replaced by $|g(t)-g(s)| \leq L|t-s|^\gamma$. Thus $\mathcal{H}_{1/2,L} = \mathcal{H}_L$. Define

$$F_{\gamma,L}(\epsilon) = \inf_{g\in\mathcal{H}_{\gamma,L}} \mathbb{P}(\|K-g\|_\infty < \epsilon).$$

Theorem 3.1. *For every fixed $0 < \gamma \leq 1$ and $L \geq 1$, there are constants $0 < c_\gamma, C_{\gamma,L} < \infty$ and $\epsilon_\gamma > 0$ such that*

$$\exp[-C_{\gamma,L}\epsilon^{-\max(2,1/\gamma)}] \leq F_{\gamma,L}(\epsilon) \leq \exp[-c_\gamma\epsilon^{-\max(2,1/\gamma)}], \qquad 0 < \epsilon \leq \epsilon_\gamma. \tag{3.1}$$

For $0 < \gamma < 1/2$, one may take $c_\gamma = 64^{-1/\gamma}/2048$.

The rough upper bound is a worst-case statement: the center used to bound the infimum can change with ϵ. Every center constructed below is piecewise linear, hence individually 1/2-Hölder with a nonuniform constant. The uniform bound on the weaker γ-Hölder norm is what makes them one admissible family. Theorem 2.1 retains its stronger each-center conclusion at exponent 1/2.

Proof. First suppose $0 < \gamma < 1/2$. Put

$$E_\gamma = \frac{2}{\gamma+1} + \frac{1}{2(1-\gamma)}, \qquad D_\gamma = 2^{2-\gamma}E_\gamma.$$

For $u = g - at$, the circular Hölder constant is at most $2^{1-\gamma}(L+2)$, by passing through the zero endpoint as in (2.10). The Fejér bounds in Section 2.2, with $n = M+1$, give

$$\int_0^1 F_M(t)d(t,\mathbb{Z})^\gamma\,dt \leq E_\gamma n^{-\gamma}.$$

Indeed the two inner intervals contribute at most $2n\int_0^{1/n} t^\gamma\,dt = 2n^{-\gamma}/(\gamma+1)$, and the outer intervals contribute at most $(2n)^{-1}\int_{1/n}^{1/2} t^{\gamma-2}\,dt \leq n^{-\gamma}/(2(1-\gamma))$. Pinning at zero therefore gives error at most $D_\gamma(L+2)(M+1)^{-\gamma}$. Choose

$$M = \left\lceil [8D_\gamma(L+2)]^{1/\gamma}\epsilon^{-1/\gamma} \right\rceil.$$

This error is at most $\epsilon/8$. Lemma 2.5 applies to this cutoff and gives

$$\mathbb{P}(\|K-g\|_\infty < \epsilon) \geq c_e\epsilon^{3/2}\exp[-6M - 5120\epsilon^{-2}].$$

Since $1/\gamma > 2$, the required lower bound follows for $0 < \epsilon \le 1/4$, for example with

$$C_{\gamma,L} = 6\{[8D_\gamma(L+2)]^{1/\gamma} + 1\} + 5127. \tag{3.2}$$

We now construct centers giving the opposite estimate. For every dyadic $J = 2^m$, there are classical Rudin–Shapiro signs $\sigma_1, \dots, \sigma_J$ with

$$\sup_{|z|=1} \left| \sum_{j=1}^{J} \sigma_j z^j \right| \le \sqrt{2J}. \tag{3.3}$$

The elementary construction, also used in [3], starts with $P_0 = Q_0 = 1$ and takes

$$P_{m+1} = P_m + z^{2^m} Q_m, \qquad Q_{m+1} = P_m - z^{2^m} Q_m.$$

The coefficients are signs on disjoint concatenated supports, and on the unit circle the parallelogram identity gives $|P_m|^2 + |Q_m|^2 = 2^{m+1} = 2J$. Multiplying P_m by z proves (3.3). These signs and their cancellation are classical.

Set

$$\lambda = \frac{1}{8\sqrt{2}}, \quad N = \lceil J^{1/(2\gamma)} \rceil, \quad A = \lambda J^{-1/2}, \quad t_j = \frac{j}{4J}, \quad \phi(x) = (1 - |x|)_+,$$

and choose J large enough that $N \ge 16J$. Define

$$g_J(t) = A \sum_{j=1}^{J} \sigma_j \{\phi(N(t - t_j)) - \phi(N(t - (1 - t_j)))\}. \tag{3.4}$$

All supports are disjoint and interior to $[0,1]$: the left centers lie in $[1/(4J), 1/4]$, their spacing is $1/(4J)$ and their radii are at most $1/(16J)$; the remaining supports are their reflections. Thus g_J is continuous, real, zero at both endpoints, and odd under $t \mapsto 1 - t$. It has norm at most A and global Lipschitz constant at most AN, because its piecewise linear slopes have that bound. It follows that

$$|g_J(t) - g_J(s)| \le \min(AN|t - s|, 2A) \le 2^{1-\gamma} AN^\gamma |t - s|^\gamma \le 2\lambda |t - s|^\gamma.$$

The last step uses $N \le 2J^{1/(2\gamma)}$.

The full-line Fourier transform of ϕ is $\widehat{\phi}(\xi) = (\sin(\pi\xi)/(\pi\xi))^2$. Since the bumps are interior, their periodic Fourier coefficients give

$$\widehat{g}_J(h) = -\frac{2iA}{N} \widehat{\phi}(h/N) \sum_{j=1}^{J} \sigma_j \sin(2\pi h t_j), \qquad h \ne 0.$$

Consequently their normalized coefficients are real and equal to

$$\beta_h = 4\pi A \frac{h}{N} \widehat{\phi}(h/N) \sum_{j=1}^{J} \sigma_j \sin(2\pi h t_j). \tag{3.5}$$

The inequality $\sin^2 x \le \min(x^2, 1) \le |x|$ gives $|\xi \widehat{\phi}(\xi)| \le 1/\pi$. The sine sum is an imaginary part of the polynomial in (3.3), so $|\beta_h| \le 4A\sqrt{2J} = 1/2$. Thus every g_J lies in the single class $\mathcal{H}_{\gamma,1}$, and hence in $\mathcal{H}_{\gamma,L}$ for $L \ge 1$.

The upper probability estimate uses orthogonal increments in coefficient space, not independent process increments. Write, for $0 \le t \le 1/2$,

$$\operatorname{Re} K(t) = \sum_h c_h(t) X_h, \qquad c_0(t) = t, \quad c_h(t) = \frac{\sin(2\pi h t)}{2\pi h} \ (h \neq 0).$$

For $0 \le s \le t \le 1/2$, identity (2.9) and product-to-sum show

$$\sum_h c_h(s) c_h(t) = st + \frac{1}{2\pi^2} \sum_{h \ge 1} \frac{\sin(2\pi h s)\sin(2\pi h t)}{h^2} = \frac{s}{2}. \tag{3.6}$$

For clarity, the sine sum is half the difference of the cosine series at $t - s$ and $t + s$, and equals $\pi^2(s - 2st)$. Let $s_j = t_j - 1/N$. The intervals $[s_j, t_j]$ are disjoint and inside $(0, 1/2)$. For

$$S = \sum_{j=1}^{J} \sigma_j \{\operatorname{Re} K(t_j) - \operatorname{Re} K(s_j)\} = \sum_h d_h X_h,$$

equation (3.6) gives $\sum_h d_h^2 = J/(2N)$: each diagonal contributes half the interval length, and each disjoint off-diagonal pair contributes zero. The independent bounded scalar coefficients and their exponential estimate, followed by Fatou through the convergent symmetric sums, yield $\mathbb{E}e^{uS} \le \exp(u^2 J/N)$.

If $\|K - g_J\|_\infty < \epsilon$ and $\epsilon \le A/4$, each signed target increment is A and its two endpoint errors total less than 2ϵ. Hence $S \ge JA/2$. Exponential Markov inequality at $u = AN/4$ proves

$$\mathbb{P}(\|K - g_J\|_\infty < \epsilon) \le \exp(-JA^2 N/16) = \exp(-\lambda^2 N/16). \tag{3.7}$$

To cover every sufficiently small radius, let J_γ be a dyadic integer at least $16^{1/(1/(2\gamma)-1)}$. For $0 < \epsilon \le \lambda/(4\sqrt{2J_\gamma})$, choose the largest dyadic $J \le (\lambda/(4\epsilon))^2$. Then $J \ge J_\gamma$, $4\epsilon \le A < 4\sqrt{2}\epsilon$, and

$$N \ge J^{1/(2\gamma)} > \left(\frac{\lambda}{4\sqrt{2}}\right)^{1/\gamma} \epsilon^{-1/\gamma}.$$

Taking the infimum in (3.7) proves the rough upper bound with $c_\gamma = (\lambda^2/16)(\lambda/(4\sqrt{2}))^{1/\gamma} = 64^{-1/\gamma}/2048$.

Finally, if $1/2 \le \gamma \le 1$, then $\mathcal{H}_{\gamma,L} \subseteq \mathcal{H}_L$, since $|t - s|^\gamma \le |t - s|^{1/2}$. Theorem 2.1 supplies the uniform lower bound. The zero center belongs to every class, and its upper bound in that theorem bounds the infimum above by $\exp(-c\epsilon^{-2})$. This completes (3.1). □

For fixed γ, L, the class $\mathcal{H}_{\gamma,L}$ is compact by the same equicontinuity and closed-coefficient argument used in Corollary 2.6. Hence its lower estimate also gives

$$\liminf_{q\to\infty} \inf_{g \in \mathcal{H}_{\gamma,L}} \mu_q(B(g,\epsilon)) \ge \exp[-2^s C_{\gamma,L} \epsilon^{-s}], \qquad s = \max(2, 1/\gamma),$$

for each fixed sufficiently small ϵ: repeat that corollary's finite $\epsilon/4$-net proof with radius $\epsilon/2$. This fixed-radius display follows from the qualitative finite-net argument. The effective and shrinking-radius statements for the half-Hölder class are proved in [8, Section 6] for the same two-parameter prime ensemble.

4 A weighted boundary condition for other coefficient laws

The same matching mechanism extends beyond Sato–Tate. Fractional Fejér tapers improve the sufficient condition on how rapidly a bounded symmetric unimodal density can vanish at its boundary.

Proposition 4.1. *Let f be an even probability density supported on $[-2,2]$, nonincreasing on $[0,2)$, positive in its interior, and bounded above by $B<\infty$. Use its right-continuous monotone representative on $[0,2)$ and even extension, and put*

$$\Psi(s)=\max\{0,-\log f(2-s)\},\qquad 0<s\le 1.$$

Suppose that for some fixed $1/2<r<1$,

$$I_r=\int_0^1 \Psi(t^r)\,dt=\frac{1}{r}\int_0^1 \Psi(s)s^{1/r-1}\,ds<\infty. \tag{4.1}$$

Let the independent X_h have density f, and define $K_f=\sum_h X_h v_h$ with the same basis and symmetric summation as before. For every $L\ge 1$, $g\in\mathcal{H}_L$, and $0<\epsilon\le 1/4$,

$$\mathbb{P}(\|K_f-g\|_\infty<\epsilon)\ge\exp\{-C_{f,L,r}\epsilon^{-2}\}, \tag{4.2}$$

where one possible constant is

$$T_r=3+\frac{2r(1-r)}{r-1/2},\qquad D_r=\frac{14\sqrt{2}}{3}T_r,\qquad Q_r=64D_r^2+1,$$
$$C_{f,L,r}=(2r+2I_r)Q_r(L+2)^2+5120+\log 4+1+8^{1/r}I_r.$$

For an upper estimate, put

$$\eta=\frac{1}{4B},\qquad p_0=\frac{1}{2},\qquad \delta_0=\frac{p_0\eta^2}{32\pi^2},\qquad c_0=\frac{p_0\delta_0}{16},\qquad \epsilon_f=\sqrt{\delta_0/2}.$$

Then, for every deterministic $g\in C([0,1],\mathbb{C})$,

$$\mathbb{P}(\|K_f-g\|_\infty<\epsilon)\le\exp(-c_0\epsilon^{-2}),\qquad 0<\epsilon\le\epsilon_f. \tag{4.3}$$

The class $\mathcal{H}_L$ retains its exact definition from Section 1; no separate support characterization for f is needed.

Proof. We first record the approximation kernel and its full estimate. For an integer $N\ge 2$ let $w_h=(1-(h/N)^r)_+$ for integers $h\ge 0$, and put

$$\alpha_m=(m+1)(w_m-2w_{m+1}+w_{m+2}),\qquad 0\le m\le N-1.$$

The decreasing sequence w_h is convex: its second derivative on $(0,N)$ is $r(1-r)N^{-r}h^{r-2}>0$, and its slope jumps upward to zero at N. Thus $\alpha_m\ge 0$. With $d_m=w_m-w_{m+1}$, telescoping gives

$$\sum_{m=0}^{N-1}\alpha_m=\sum_{m=0}^{N-1}(m+1)(d_m-d_{m+1})=1,$$
$$\sum_{m=h}^{N-1}\alpha_m\left(1-\frac{h}{m+1}\right)=\sum_{m=h}^{N-1}(m+1-h)(d_m-d_{m+1})=w_h.$$

Consequently $F_{N,r} = \sum_{m=0}^{N-1} \alpha_m F_m$ is a nonnegative integral-one kernel with Fourier coefficients $w_{|h|}$.

For $0 < \gamma < r$, these mixture coefficients satisfy

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-\gamma} \le \left(3 + \frac{2r(1-r)}{r-\gamma}\right) N^{-\gamma}. \tag{4.4}$$

Indeed $\alpha_0 = (2-2^r)N^{-r} \le 2N^{-\gamma}$, while the last coefficient contributes at most $N^{-\gamma}$ since $\alpha_{N-1} \le 1$. For $1 \le m \le N-2$, integrating the second derivative of x^r gives

$$2(m+1)^r - m^r - (m+2)^r = r(1-r)\int_0^1 \int_0^1 (m+s+t)^{r-2}\, ds\, dt \le r(1-r)m^{r-2}.$$

It follows that $\alpha_m (m+1)^{-\gamma} \le 2r(1-r)N^{-r}m^{r-\gamma-1}$. Since $r-\gamma-1 \in (-1,0)$, $\sum_{m=1}^{N-2} m^{r-\gamma-1} \le N^{r-\gamma}/(r-\gamma)$, which proves (4.4).

Take $\gamma = 1/2$ and write $A_{N,r}u = F_{N,r} * u$, where $u = g - at$. The ordinary pinned Fejér error proved in Section 2.2 is at most $14\sqrt{2}(L+2)/(3\sqrt{m+1})$ for $m \ge 1$. For $m = 0$ its pinned mean is zero and $\|u\|_\infty \le L+2$, so that bound is still valid. Mixing the errors by (4.4) yields

$$\|A_{N,r}u - A_{N,r}u(0) - u\|_\infty \le D_r(L+2)N^{-1/2}.$$

Choose $N = \lceil 64 D_r^2 (L+2)^2 \epsilon^{-2} \rceil$, so this error is at most $\epsilon/8$ and $N \le Q_r(L+2)^2\epsilon^{-2}$. For $0 < |h| \le N-1$ put

$$x_h = (|h|/N)^r, \qquad b_h = (1-x_h)\beta_h, \qquad r_h = \min(1/2, x_h).$$

Keep $b_0 = (1-\epsilon/8)a$ and $r_0 = \epsilon/8$. The pinned polynomial $q = b_0 t + A_{N,r}u - A_{N,r}u(0) = \sum_{|h|\le N-1} b_h v_h$ has $\|q-g\|_\infty \le 3\epsilon/8$.

Radial monotonicity makes Ψ nonincreasing; it is nonnegative by definition. On the hth coefficient box, $|b_h + z| \le 2 - x_h$; hence $f(b_h+z) \ge \exp[-\Psi(x_h)]$. The local factor against the centered uniform density is at least $x_h \exp[-\Psi(x_h)]$, since $2r_h \ge x_h$. The decreasing function $t \mapsto \Psi(t^r)$ obeys

$$\sum_{h=1}^{N-1} \Psi((h/N)^r) \le N \int_0^{(N-1)/N} \Psi(t^r)\, dt \le N I_r.$$

Together with the factorial estimate used in (2.12), the two-sign product is at least

$$\left(\frac{(N-1)!}{N^{N-1}}\right)^{2r} \exp\left[-2\sum_{h=1}^{N-1} \Psi((h/N)^r)\right] \ge \exp[-(2r+2I_r)N].$$

The endpoint factor is at least $(\epsilon/4)\exp[-\Psi(\epsilon/8)]$. Moreover (4.1) implies

$$I_r \ge \int_0^{s^{1/r}} \Psi(t^r)\, dt \ge s^{1/r}\Psi(s), \qquad \Psi(\epsilon/8) \le 8^{1/r} I_r \epsilon^{-1/r}.$$

The bounded symmetric coefficients satisfy the same endpoint-chord exponential bound as in Lemma 2.2, so K_f converges uniformly by that proof. Lemma 2.3 gives the centered finite-sum comparison. The Gaussian reference and the limiting tail argument therefore retain the bounds (2.7) and (2.8). The auxiliary uniform variables in finite density matching are themselves log-concave, so

that step is unchanged. Finite density matching, followed by the centered comparison, now bounds the low-frequency event of radius $\epsilon/4$ below by

$$\frac{\epsilon}{4}\exp[-\Psi(\epsilon/8)-(2r+2I_r)N-1024\epsilon^{-2}].$$

The independent tail event of radius $\epsilon/4$ contributes $\exp(-4096\epsilon^{-2})$. Their intersection has norm at most $7\epsilon/8<\epsilon$. Since $1/r<2$ and $\epsilon\le 1$, $\epsilon^{-1/r}\le\epsilon^{-2}$. Inserting the bound on N and $\log(4/\epsilon)\le(\log 4+1)\epsilon^{-2}$ proves (4.2) with the stated constant.

For the upper bound and an arbitrary continuous complex center, define a,u,β_h as in the proof of Theorem 2.1. The endpoint/Parseval inequality (2.13) is unchanged. Every interval of length 2η has mass at most $2B\eta=1/2$. Thus the indicators $\mathbf{1}_{\{|X_h-\operatorname{Re}\beta_h|\ge\eta\}}$ have independent success probabilities at least p_0. Their sum on the small-ball event is at most $16\pi^2\eta^{-2}N^2\epsilon^2$. With $N=\lfloor\delta_0\epsilon^{-2}\rfloor$, this is at most $p_0N/2$. The exponential-moment bound of Section 2 bounds its probability by $\mathrm{e}^{-p_0N/8}$. For $\epsilon^2\le\delta_0/2$ this is at most $\exp[-(p_0\delta_0/16)\epsilon^{-2}]$, as asserted. □

An unweighted integrable boundary logarithm is sufficient for (4.1): $t^r\ge t$ and monotonicity give $\Psi(t^r)\le\Psi(t)$. In particular every polynomial bound $f(x)\ge d(2-|x|)^\kappa$, $d>0$, $\kappa\ge 0$, works, since $\Psi(s)\le\max(0,-\log d)+\kappa\log(1/s)$. This includes the symmetric Jacobi densities

$$f_\kappa(x)=\frac{\Gamma(\kappa+3/2)}{2^{2\kappa+1}\sqrt{\pi}\,\Gamma(\kappa+1)}(4-x^2)^\kappa\mathbf{1}_{[-2,2]}(x),\qquad \kappa\ge 0.$$

Their normalization follows from the beta integral, and concavity of $\log(4-x^2)$ proves log-concavity. The cases $\kappa=0$ and $\kappa=1/2$ are uniform and Sato–Tate, respectively.

The density hypotheses also include laws that are not log-concave. For example,

$$f(x)=\begin{cases}1/3, & |x|<1,\\ 1/6, & 1\le|x|<2,\\ 0, & |x|\ge 2,\end{cases}$$

is the mixture of a uniform law on $[-1,1]$ of weight $1/3$ and a uniform law on $[-2,2]$ of weight $2/3$. It has an interior jump and therefore is not log-concave, but its boundary function satisfies $\Psi(s)=\log 6$ for $0<s\le 1$, so the weighted condition holds for every permitted r.

There is also an exact threshold for the uniform lower order in a faster-vanishing family. For $A>0$ and $a>0$, put

$$f_{A,a}(x)=Z_{A,a}^{-1}\exp[-A(2-|x|)^{-a}]\mathbf{1}_{(-2,2)}(x). \tag{4.5}$$

This density is even, positive in the interior, bounded and log-concave: $2-|x|$ is concave, and $y\mapsto -Ay^{-a}$ is increasing and concave on positive y. Its boundary function is at most $|\log Z_{A,a}|+As^{-a}$. If $a<2$, choose $1/2<r<\min(1,1/a)$; then $ar<1$ and (4.1) holds. Proposition 4.1 proves the uniform ϵ^{-2} logarithmic order throughout this range.

For $a\ge 2$, Proposition 5.1 supplies a fixed-center obstruction in every prescribed $\mathcal{H}_L$, including the critical case $a=2$. Thus $a<2$ is the exact threshold for the existence of a uniform lower bound of order $\exp(-C\epsilon^{-2})$ in this family. Theorem 5.2 complements this obstruction with a uniform critical lower bound.

Corollary 4.2. *Fix $0<\gamma<1/2$ and $L\ge 1$. Let f satisfy the density hypotheses of Proposition 4.1, but suppose more generally that (4.1) holds for some $\gamma<r<1$. Then*

$$\inf_{g\in\mathcal{H}_{\gamma,L}}\mathbb{P}(\|K_f-g\|_\infty<\epsilon)$$

has negative logarithm of order $\epsilon^{-1/\gamma}$, for all sufficiently small ϵ, with constants depending on the fixed parameters and law. For the family (4.5), this conclusion holds whenever $a < 1/\gamma$. Theorem 6.1 proves failure at and beyond equality in every prescribed $\mathcal{H}_{\gamma,L}$.

Proof. The mixture and its moment estimate (4.4) hold for every $0 < \gamma < r < 1$, including $r \le 1/2$. With D_γ from Section 3 and $T_{\gamma,r} = 3 + 2r(1-r)/(r-\gamma)$, the mixed pinned error is at most $D_\gamma T_{\gamma,r}(L+2)N^{-\gamma}$. For the degree-zero component the same estimate follows from $\|u\|_\infty \le L+2$, so no mixture term is omitted. Choose

$$N = \left\lceil [8D_\gamma T_{\gamma,r}(L+2)]^{1/\gamma}\epsilon^{-1/\gamma} \right\rceil .$$

The density matching proof gives the same nonzero-coefficient factor $\exp[-(2r+2I_r)N]$ and endpoint factor $(\epsilon/4)\exp[-\Psi(\epsilon/8)]$. Now $\Psi(\epsilon/8) \le 8^{1/r} I_r \epsilon^{-1/r}$ is absorbed by $\epsilon^{-1/\gamma}$ since $r > \gamma$; the Gaussian and tail cost $5120\epsilon^{-2}$ is absorbed since $\gamma < 1/2$. For example a lower-bound constant is

$$(2r+2I_r)\{[8D_\gamma T_{\gamma,r}(L+2)]^{1/\gamma}+1\} + 5120 + \log 4 + 1 + 8^{1/r} I_r.$$

The signed-spike upper proof in Section 3 uses only independent centered coefficients in $[-2,2]$. It therefore applies to f without change, with the same c_γ and small-radius threshold. This proves matching worst-case order.

For (4.5) and $a < 1/\gamma$, choose $\gamma < r < \min(1, 1/a)$; then $ar < 1$ makes the weighted integral finite. The separate negative assertion is Theorem 6.1. □

5 Flat-edge threshold and critical-rate bounds

The sufficient condition in Section 4 excludes the critical case $a = 2$ of (4.5). In that family the exclusion reflects an actual obstruction. Infinitely many boundary coefficients can force an additional harmonic-sum cost while classical Rudin–Shapiro cancellation keeps their center 1/2-Hölder. Rapidly vanishing origin distributions and weighted-sum small deviations belong to classical Laplace-transform theory; see, for example, Rozovsky [7]. The elementary estimate constructs a fixed admissible harmonic boundary center.

Proposition 5.1. *For every $L \ge 1$, there is a fixed $g_L \in \mathcal{H}_L$ such that, for the coefficient law (4.5) with $A > 0$ and $a \ge 2$, some $c_{A,a,L} > 0$ and $\epsilon_{A,a,L} > 0$ satisfy*

$$\mathbb{P}(\|K_f - g_L\|_\infty < \epsilon) \le \exp[-c_{A,a,L}\epsilon^{-2}\log^2(1/\epsilon)], \qquad 0 < \epsilon \le \epsilon_{A,a,L}. \tag{5.1}$$

In particular the order ϵ^{-2} fails already at this one center. Consequently, within (4.5), a uniform lower bound of that order on $\mathcal{H}_L$ holds if and only if $a < 2$, for every fixed $L \ge 1$.

Proof. Use the polynomials P_m, Q_m in the proof of (3.3). The first 2^m coefficients of P_{m+1} equal those of P_m, so they define infinite signs $s_0, s_1, \ldots$; put $r_h = s_{h-1}$. A prefix of P_m or Q_m decomposes recursively into signed, shifted complete P_j or Q_j blocks, with at most one block of each dyadic size. An entire block is kept intact. Otherwise a prefix in the first half is a prefix of P_{m-1}; a longer one consists of that whole first block and a signed shifted prefix of Q_{m-1}. Iteration follows its binary length expansion. Each complete block of length 2^j has norm at most $\sqrt{2\cdot 2^j}$ by the complementary identity. Thus, uniformly on $|z| = 1$,

$$\left|\sum_{h=1}^{n} r_h z^h\right| \le C_{\mathrm{RS}}\sqrt{n}, \qquad C_{\mathrm{RS}} = \frac{\sqrt{2}}{1-2^{-1/2}} < 5. \tag{5.2}$$

The shift from index zero multiplies by z and preserves the norm.

Consider the deterministic center

$$g(t) = \frac{2}{\pi} \sum_{h \geq 1} \frac{r_h \sin(2\pi h t)}{h}.$$

Its uniform convergence and regularity can be checked directly. For $S_n(t) = \sum_{h=1}^{n} r_h \mathrm{e}^{2\pi i h t}$, summation by parts gives

$$\sum_{h=M+1}^{H} \frac{r_h \mathrm{e}^{2\pi i h t}}{h} = \frac{S_H(t)}{H} - \frac{S_M(t)}{M} + \sum_{h=M}^{H-1} S_h(t) \left(\frac{1}{h} - \frac{1}{h+1} \right).$$

The first term tends uniformly to zero as $H \to \infty$, and $\sum_{h=M}^{\infty} \sqrt{h}/(h(h+1)) \leq 3M^{-1/2}$. The complex-series tail is therefore at most $4C_{\mathrm{RS}} M^{-1/2}$, and the sine-series tail at most $8C_{\mathrm{RS}}/(\pi\sqrt{M})$. The center is consequently continuous, real, zero at both endpoints, and has normalized coefficients $\beta_h = \beta_{-h} = 2r_h$ for $h \geq 1$.

Its degree-M partial sum has derivative $4\sum_{h=1}^{M} r_h \cos(2\pi h t)$, of norm at most $4C_{\mathrm{RS}}\sqrt{M}$. For $\delta = |t - s| \in (0, 1]$, choose $M = \lfloor \delta^{-1} \rfloor$. The partial sum and its two tails give

$$|g(t) - g(s)| \leq 4C_{\mathrm{RS}}\sqrt{M}\,\delta + \frac{16C_{\mathrm{RS}}}{\pi\sqrt{M}} \leq \left(4 + \frac{16\sqrt{2}}{\pi}\right) C_{\mathrm{RS}}\sqrt{\delta} < 80\sqrt{\delta}.$$

Thus we may fix $L_0 = 80$.

Extend g periodically. Its zero endpoints give circular 1/2-Hölder constant at most $\sqrt{2}L_0$, by the same wrapped-pair argument as in (2.10). For a positive integer q, put $g_q(t) = q^{-1}g(qt)$. This has Hölder constant at most $\sqrt{2}L_0/\sqrt{q}$, zero endpoints, and normalized coefficients

$$\beta_{qk} = \beta_{-qk} = 2r_k \quad (k \geq 1), \qquad \beta_h = 0 \quad (q \nmid h).$$

The formula follows directly from its sine series, or by partitioning the Fourier integral into q periods. Choose the fixed integer $q = \max\{1, \lceil 2L_0^2/L^2 \rceil\}$, and let $g_L = g_q$. It belongs to $\mathcal{H}_L$ and is independent of the radius.

On $\|K_f - g_q\|_\infty < \epsilon$, the endpoint identity gives $|X_0| < \epsilon$. Thus $D = K_f - tX_0 - g_q$ has norm less than 2ϵ. Parseval, retaining positive frequencies qk for $1 \leq k \leq M$, gives

$$\sum_{k=1}^{M} \frac{(X_{qk} - 2r_k)^2}{(qk)^2} \leq 16\pi^2\epsilon^2.$$

Write $d_k = 2 - r_k X_{qk} = |X_{qk} - 2r_k| \in (0, 4)$ almost surely, and $H_M = \sum_{k=1}^{M} 1/k$. Cauchy's inequality implies

$$\left(\frac{H_M}{q}\right)^2 \leq \left(\sum_{k=1}^{M} \frac{d_k^2}{(qk)^2}\right) \left(\sum_{k=1}^{M} d_k^{-2}\right), \qquad \sum_{k=1}^{M} d_k^{-2} \geq \frac{H_M^2}{16\pi^2 q^2 \epsilon^2}.$$

For $a \geq 2$ and $\delta_x = 2 - |x| \in (0, 2]$, $\delta_x^{-a} \geq 2^{2-a}\delta_x^{-2}$. Also $\delta_{X_{qk}} \leq d_k$ for either sign r_k. With $A_* = A2^{2-a}$ and $Z = Z_{A,a}$, the marginal density is at most $Z^{-1}\exp(-A_*/d_k^2)$. Independence of these M coefficients, integration over their finite necessary set, and the full box volume 4^M therefore give

$$\mathbb{P}(\|K_f - g_q\|_\infty < \epsilon) \leq (4/Z)^M \exp\left[-\frac{A_* H_M^2}{16\pi^2 q^2 \epsilon^2}\right]. \tag{5.3}$$

All unselected coefficients were discarded through a necessary-event upper bound; no infinite density product is used.

Take $M = \lfloor \epsilon^{-2} \rfloor$ for $\epsilon \le 1/4$. Then $H_M \ge \log(M+1) \ge 2\log(1/\epsilon)$ and $M \le \epsilon^{-2}$. Put $b = \log(4/Z) > 0$. The right side of (5.3) is at most

$$\exp\left\{\epsilon^{-2}\left[b - \frac{A_*}{4\pi^2 q^2}\log^2(1/\epsilon)\right]\right\}.$$

For $\epsilon \le \min\{1/4, \exp[-\sqrt{8\pi^2 q^2 b/A_*}]\}$, this proves (5.1) with $c_{A,a,L} = A_*/(8\pi^2 q^2)$. The extra unbounded factor $\log^2(1/\epsilon)$ excludes every finite-C lower bound of order $\exp(-C\epsilon^{-2})$ at g_L. For $a < 2$, Proposition 4.1 already proves the positive assertion, completing the stated threshold. □

5.1 A lower bound at the critical law

For $a = 2$, a borderline positive taper supplies a complementary uniform lower bound. Write K_A for the process with density $f_{A,2}$, and $Z_A = Z_{A,2}$.

Theorem 5.2. *For every fixed $A > 0$ and $L \ge 1$, there is an explicitly finite $C_{A,L}$ such that*

$$\mathbb{P}(\|K_A - g\|_\infty < \epsilon) \ge \exp[-C_{A,L}\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)], \qquad g \in \mathcal{H}_L, \quad 0 < \epsilon \le 1/4. \tag{5.4}$$

For the single fixed center g_L of Proposition 5.1, therefore,

$$c_{A,L}\epsilon^{-2}\log^2(1/\epsilon) \le -\log \mathbb{P}(\|K_A - g_L\|_\infty < \epsilon) \le C_{A,L}\epsilon^{-2}\log^3(\mathrm{e}/\epsilon) \tag{5.5}$$

at every sufficiently small radius.

Proof. Use the mixture from Section 4 with $w_h = (1 - \sqrt{h/N})_+$, $N \ge 2$, and $\alpha_m = (m+1)(w_m - 2w_{m+1} + w_{m+2})$. Its positivity, unit sum and exact Fejér representation hold also at this borderline exponent. For $1 \le m \le N-2$, the second-difference integral gives

$$\alpha_m(m+1)^{-1/2} \le \tfrac{1}{4}N^{-1/2}(m+1)^{1/2}m^{-3/2} \le \tfrac{1}{2}N^{-1/2}/m.$$

The $m = 0$ and $m = N-1$ contributions are at most $2N^{-1/2}$ and $N^{-1/2}$, respectively. Hence, with $H_0 = 0$,

$$\sum_{m=0}^{N-1} \alpha_m(m+1)^{-1/2} \le N^{-1/2}[3 + \tfrac{1}{2}H_{N-2}] \le N^{-1/2}[4 + \tfrac{1}{2}\log N]. \tag{5.6}$$

Put

$$D = \frac{14\sqrt{2}}{3}, \quad S = 8D(L+2), \quad B = 256S^2, \quad \ell = \log(\mathrm{e}/\epsilon), \quad N = \lceil B^2\epsilon^{-2}\ell^2 \rceil.$$

These constants satisfy

$$\sqrt{N} \ge B\epsilon^{-1}\ell, \quad N \le 2B^2\epsilon^{-2}\ell^2, \quad \log N \le \log 2 + 2\log B + 4\ell.$$

In particular $4 + \frac{1}{2}\log N \le (7 + \log B)\ell$ and $B \ge S(7 + \log B)$: use $S \ge 1$, $7 + \log 256 + 2\log S \le 13S$ and $B/S = 256S$. For $u = g - g(1)t$, the ordinary pinned Fejér error is at most $D(L+2)/\sqrt{m+1}$, including $m = 0$. Its mixture and (5.6) consequently give error at most $\epsilon/8$.

For $0 < |h| \le N-1$, put

$$x_h = \sqrt{|h|/N}, \quad b_h = (1 - x_h)\beta_h, \quad r_h = \min(1/2, x_h),$$

and put $b_0 = (1-\epsilon/8)g(1)$, $r_0 = \epsilon/8$. The pinned polynomial $q = \sum_{|h|\le N-1} b_h v_h$ has $\|q-g\|_\infty \le 3\epsilon/8$. For $|z| \le r_h$, $h \ne 0$, $|b_h + z| \le 2 - x_h$, so comparison with the centered uniform law on $[-r_h, r_h]$ pays at least

$$x_h \exp[-AN/|h| - z_A], \qquad z_A = \max(0, \log Z_A).$$

The factorial estimate, over both frequency signs, gives the nonzero-frequency product at least

$$\exp[-(1+2z_A)N - 2ANH_{N-1}].$$

For $h = 0$ the factor is at least $(\epsilon/4)\exp[-64A\epsilon^{-2} - z_A]$.

The centered uniform acceptance event of radius $\epsilon/4$ has probability at least $\exp(-1024\epsilon^{-2})$ by the peakedness and Gaussian lemmas. The independent tail of K_A has acceptance probability at least $\exp(-4096\epsilon^{-2})$. Indeed the even log-concave density $f_{A,2}$ dominates $N(0,4)$ in central peakedness, exactly as in Section 4. The deterministic and two random errors total $7\epsilon/8$. Thus the strict-ball probability is at least

$$\frac{\epsilon}{4}\exp[-z_A - (1+2z_A)N - 2ANH_{N-1} - (5120+64A)\epsilon^{-2}].$$

Since $H_{N-1} \le (6 + 2\log B)\ell$, one admissible constant is

$$C_{A,L} = 2B^2[1 + 2z_A + 2A(6 + 2\log B)] + 5120 + 64A + \log 4 + 1 + z_A.$$

This proves (5.4); combining it with (5.1) gives (5.5). □

The extra harmonic sum in (5.6) explains the loss in this lower-bound proof. The next proposition identifies the log-cubed payment as optimal for this method.

5.2 A limitation of the positive-mixture error payment

Positive Fejér mixtures with termwise Hölder error control pay at least the log-cubed cost. The following statement concerns this approximation certificate.

Proposition 5.3. *Let $\lambda_0, \dots, \lambda_N \ge 0$ sum to one, and define*

$$E = \sum_{m=0}^{N} \frac{\lambda_m}{\sqrt{m+1}}, \qquad x_h = \sum_{m=0}^{N} \lambda_m \min\{1, h/(m+1)\} \quad (h \ge 1).$$

For $0 < \tau \le 1/4$, suppose $E \le \tau$, and put

$$J = \lfloor \log_2(\tau^{-2}) \rfloor - 1, \qquad C_* = \frac{2}{1 - 2^{-1/2}}.$$

Then $2^J \le N$ and

$$\sum_{h=1}^{2^J} x_h^{-2} \ge \frac{J^3}{2C_*^2 E^2} \ge \frac{J^3}{2C_*^2\tau^2}. \tag{5.7}$$

Proof. Since $E \ge (N+1)^{-1/2}$, we have $N+1 \ge \tau^{-2}$ and $2^J \le \tau^{-2}/2 \le N$. Set $y_i = x_{2^i}/\sqrt{2^i}$. With $a = m+1$ and $e_m = \lambda_m/\sqrt{a}$,

$$y_i = \sum_m e_m \min\{\sqrt{2^i/a}, \sqrt{a/2^i}\}.$$

Choose an integer k with $2^k \le a < 2^{k+1}$. The terms for $i \le k$ are bounded by $2^{(i-k)/2}$, and those for $i \ge k+1$ by $2^{(k+1-i)/2}$. Their sum over all integer i is at most C_*. Nonnegativity therefore gives $\sum_{i=1}^{J} y_i \le C_* E$. On $2^{i-1} < h \le 2^i$, monotonicity gives $x_h \le x_{2^i}$, so

$$\sum_{h=1}^{2^J} x_h^{-2} \ge \tfrac{1}{2} \sum_{i=1}^{J} y_i^{-2} \ge \frac{J^3}{2(\sum_{i=1}^{J} y_i)^2}.$$

The last step is Jensen for the convex function t^{-2}. This proves the claim. □

For this mixture the inward multiplier margin is exactly x_h, and the pinned Hölder approximation estimate used above pays error at most $D(L+2)E$, where $D = 14\sqrt{2}/3$. To certify error $\epsilon/8$ by that estimate requires $E \le \tau = \epsilon/[8D(L+2)]$. Equation (5.7) then forces an inverse-square margin payment of order at least $\epsilon^{-2}\log^3(1/\epsilon)$, as in the shifted finite-box density comparison.

6 The exact joint roughness–density threshold

Corollary 4.2 gives a sufficient range for rough centers. The equality case in the flat-edge family can also be decided. The following construction uses classical Rudin–Shapiro cancellation and positive Fejér tests to force a separate density penalty at every geometric frequency scale.

Theorem 6.1. *Fix $0 < \gamma < 1/2$, $L \ge 1$, $A > 0$, and put $a = 1/\gamma$. There is one fixed real zero-endpoint $g_{\gamma,L} \in \mathcal{H}_{\gamma,L}$ such that, for the law $f_{A,a}$ in (4.5),*

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le \exp[-c_{A,\gamma,L}\epsilon^{-1/\gamma}\log(1/\epsilon)] \tag{6.1}$$

for every sufficiently small ϵ, with $c_{A,\gamma,L} > 0$. The same center excludes a bound of the form

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-1/\gamma})$$

with finite C for all sufficiently small radii, for every law $f_{A,a'}$ with $a' \ge 1/\gamma$. Consequently, within this family a uniform bound

$$\inf_{g \in \mathcal{H}_{\gamma,L}} \mathbb{P}(\|K_f - g\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-1/\gamma})$$

with a finite constant and a sufficiently small radius exists exactly when $a' < 1/\gamma$.

Proof. For $j \ge 1$, put

$$N_j = 4^j, \quad J_j = 2^{\lfloor 2\gamma \log_2 N_j \rfloor}, \quad m_j = N_j/J_j, \quad B_j = 4N_j.$$

Here J_j, m_j are positive integers, $\frac{1}{2}N_j^{2\gamma} < J_j \le N_j^{2\gamma}$, and $J_j m_j = N_j$. Choose independently for each band a deterministic Rudin–Shapiro polynomial $R_{J_j}(z) = \sum_{k=0}^{J_j-1} \sigma_k z^k$ of norm at most $\sqrt{2J_j}$, using the complete recursion in (3.3). No consistency between different finite sign blocks is needed.

In band j, assign $r_h = \sigma_k$ at

$$h = B_j + 2m_j k + \ell, \qquad 0 \le k < J_j, \quad |\ell| \le m_j - 1.$$

Subblocks leave a one-frequency gap; all band frequencies lie in $[3N_j, 6N_j]$, while the next band lies in $[12N_j, 24N_j]$. Thus every assignment is unambiguous. With $D_{m-1}(t) = \sum_{\ell=-m+1}^{m-1} \mathrm{e}^{2\pi i \ell t}$,

$$T_j(t) := \sum_{h \text{ in band } j} r_h \mathrm{e}^{2\pi i h t} = \mathrm{e}^{2\pi i B_j t} D_{m_j-1}(t) R_{J_j}(\mathrm{e}^{2\pi i 2m_j t}), \qquad \|T_j\|_\infty \le 2\sqrt{2} m_j \sqrt{J_j}.$$

Define

$$g_j(t) = \frac{2}{\pi} \sum_{h \text{ in band } j} \frac{r_h \sin(2\pi h t)}{h}, \qquad g(t) = \sum_{j \geq 1} g_j(t).$$

If $b_j \geq 3N_j$ is the smallest band frequency, write $T_j(z) = z^{b_j} Q_j(z)$. Maximum modulus gives $|Q_j(z)| \leq \|T_j\|_\infty$ for $|z| \leq 1$, and

$$\sum_{h \text{ in band } j} \frac{r_h z^h}{h} = z^{b_j} \int_0^1 t^{b_j - 1} Q_j(tz)\, dt.$$

Therefore

$$\|g_j\|_\infty \leq \frac{4\sqrt{2}}{3\pi\sqrt{J_j}} \leq \frac{8}{3\pi} N_j^{-\gamma} < N_j^{-\gamma}, \qquad \left\|g_j'\right\|_\infty \leq 16 N_j^{1-\gamma}.$$

This proves uniform convergence, zero endpoints, reality, and exact normalized coefficients $\beta_h = \beta_{-h} = 2r_h$ on the bands, zero elsewhere. No prefix estimate for a truncated band is assumed.

Splitting bands at $N_k \leq |t-s|^{-1} < N_{k+1}$ proves a Hölder-γ bound with constant

$$L_\gamma^* = \frac{16}{1 - 4^{-(1-\gamma)}} + \frac{2}{1 - 4^{-\gamma}}.$$

Indeed low-band increments are bounded by $16|t-s| \sum_{j \leq k} N_j^{1-\gamma}$, and the two tails by $2 \sum_{j>k} N_j^{-\gamma}$; their geometric sums give the two displayed terms. For $|t-s| \geq 1/4$, the full tail bound gives the second term directly. Extend g periodically. Its circular constant is at most $2^{1-\gamma} L_\gamma^*$ by splitting wrapped pairs at the common zero endpoint. Fix an integer

$$q \geq \max\left\{1, (2^{1-\gamma} L_\gamma^* / L)^{1/(1-\gamma)}\right\}$$

and set $g_{\gamma,L}(t) = q^{-1} g(qt)$. Its Hölder constant is at most $2^{1-\gamma} L_\gamma^* q^{\gamma - 1} \leq L$, and its normalized boundary coefficients at qh are still $2r_h$. Thus the center is fixed in the required class.

For a band, suppress indices and put $N = Jm$, $B = 4N$. The analytic test

$$P_j(t) = \mathrm{e}^{2\pi i B t} F_{m-1}(t) R_J(\mathrm{e}^{2\pi i 2mt})$$

has coefficient $r_h w_h$ on the band, with $w_h = 1 - |\ell|/m \in (0, 1]$, and

$$W_j := \sum_h w_h = N, \qquad \|P_j\|_1 \leq \sqrt{2J}.$$

The derivative satisfies

$$\|P_j'\|_1 \leq DN\sqrt{J}, \qquad D = 4\pi(3 + \mathrm{e})\sqrt{2}. \tag{6.2}$$

For completeness, $F_{m-1} = |D_m^+|^2/m$ with $D_m^+ = \sum_{k=0}^{m-1} \mathrm{e}^{2\pi i k t}$; Cauchy–Schwarz gives $\|F_{m-1}'\|_1 \leq 4\pi m$. An analytic polynomial R of degree $d \geq 1$ obeys $\|R'\|_{|z|=1} \leq \mathrm{e} d \|R\|_{|z|=1}$: reverse the polynomial to bound its modulus outside the unit disk by $|z|^d \|R\|$, and use Cauchy's formula on radius $1/d$. For degree zero the derivative vanishes. Differentiating the three factors now bounds $\|P_j'\|_1$ by

$$2\pi B\sqrt{2J} + 4\pi m\sqrt{2J} + 4\pi \mathrm{e} m(J-1)\sqrt{2J},$$

which proves (6.2), including $m = 1$ or $J = 1$.

On the small-ball event, $|X_0| < \epsilon$ and $U = K_f - tX_0 - g_{\gamma,L}$ has norm less than 2ϵ. For a selected positive frequency define $d_h = 2 - r_h X_{qh} > 0$. Fourier multiplication gives the exact nonnegative identity

$$S_j := \sum_{h \text{ in band } j} w_h d_h = \int_0^1 U(t) \overline{\frac{d}{dt} P_j(qt)}\, dt \le 2qD\epsilon N_j \sqrt{J_j}.$$

In fact $\widehat{U}(qh) = (X_{qh} - 2r_h)/(2\pi i q h)$, and the conjugate derivative coefficient is $-2\pi i q h\, r_h w_h$, giving precisely $w_h d_h$. The L^1 derivative norm after integer dilation is $q\|P_j'\|_1$, so no further factor appears.

For $a = 1/\gamma$, weighted Jensen gives

$$\sum_{h \text{ in band } j} d_h^{-a} \ge \sum_h w_h d_h^{-a} \ge \frac{W_j^{a+1}}{S_j^a} \ge \frac{N_j}{(2qD\epsilon\sqrt{J_j})^a} \ge c_* \epsilon^{-a}, \qquad c_* = (2qD)^{-a}.$$

The last inequality uses $J_j^{a/2} \le N_j$. The actual distance to the coefficient boundary is at most d_h, so its density is at most $Z_{A,a}^{-1} \exp(-A d_h^{-a})$.

Select the first K bands, containing

$$V_K = \sum_{j=1}^K J_j(2m_j - 1) \le 2 \sum_{j=1}^K N_j \le \tfrac{8}{3} N_K$$

independent positive-frequency coefficients. Integrate just these finite marginals over their necessary set; the enclosing box has volume at most 4^{V_K}. It follows that

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le (4/Z_{A,a})^{V_K} \exp[-A c_* K \epsilon^{-a}].$$

This uses independence of scalar coefficients, not of process increments.

Put

$$b = \log(4/Z_{A,a}) > 0, \qquad K = \lfloor \log_4 \epsilon^{-a} \rfloor, \qquad K_0 = \max(2, \lceil 16b/(3Ac_*) \rceil).$$

For $0 < \epsilon \le \epsilon_0$, where $\epsilon_0 = \min(1/4, 4^{-\gamma(K_0+1)})$, the conditions are

$$N_K \le \epsilon^{-a}, \qquad K \ge K_0, \qquad K \ge \frac{a \log(1/\epsilon)}{2 \log 4}.$$

Absorbing the finite volume term therefore gives

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le \exp\left[-\frac{Aac_*}{4 \log 4} \epsilon^{-a} \log(1/\epsilon) \right].$$

For $a' \ge a$, the inequality $\delta^{-a'} \ge 2^{a-a'} \delta^{-a}$ on $(0, 2]$ applies the same proof with A replaced by $A 2^{a-a'}$ and with the actual normalization constant. The center depends only on γ, L, not on the law parameters or the radius. The positive assertion for $a' < 1/\gamma$ is Corollary 4.2; the extra logarithm excludes every finite-constant lower bound at and beyond equality. □

Together with Proposition 5.1, this determines the flat-edge lower-order property for all $0 < \gamma \le 1/2$: its threshold is $a < 1/\gamma$. Below $\gamma = 1/2$ the theorem gives the critical obstruction $\epsilon^{-1/\gamma} \log(1/\epsilon)$ at one fixed center.

7 Supercritical laws and smoother centers

The same positive taper also covers the range beyond the joint threshold. Its approximation order becomes slower when its exponent is below the Hölder exponent, but the resulting density cost has exactly the required power. This also distinguishes the critical half-Hölder case from smoother centers.

Theorem 7.1. *Fix $0 < \gamma < 1$, $L \geq 1$, $A > 0$, and $a > 1/\gamma$. For the law $f_{A,a}$ of (4.5), there are explicitly finite constants C_a, C_2 such that*

$$\mathbb{P}(\|K_f - g\|_\infty < \epsilon) \geq \exp[-C_a\epsilon^{-a} - C_2\epsilon^{-2}], \qquad g \in \mathcal{H}_{\gamma,L}, \quad 0 < \epsilon \leq 1/4. \tag{7.1}$$

If $a > \max(2, 1/\gamma)$, a single fixed smooth center in $\mathcal{H}_{\gamma,L}$ gives a matching upper bound $\exp(-c\epsilon^{-a})$ at all sufficiently small radii. Consequently the worst-case negative logarithm has exact order ϵ^{-a} in this range. For $1/2 < \gamma < 1$ and $a = 2$, it instead has exact order ϵ^{-2}.

Proof. Choose $1/a < r < \gamma$. Use w_h and α_m from Section 4. The positive-kernel representation is unchanged, but the mixture moment is now

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-\gamma} \leq TN^{-r}, \qquad T = 3 + 2r(1-r)\left(1 + \frac{1}{\gamma - r}\right). \tag{7.2}$$

Indeed the initial term is at most $2N^{-r}$, and the last term is at most $N^{-\gamma} \leq N^{-r}$. For $1 \leq m \leq N-2$, the same second-difference integral gives

$$\alpha_m (m+1)^{-\gamma} \leq 2r(1-r)N^{-r}m^{r-\gamma-1}.$$

The sum of $m^{-1-(\gamma-r)}$ is at most $1 + 1/(\gamma - r)$, which proves (7.2).

Put

$$E_\gamma = \frac{2}{\gamma+1} + \frac{1}{2(1-\gamma)}, \qquad D_\gamma = 2^{2-\gamma}E_\gamma, \qquad U = 8D_\gamma T(L+2), \qquad N = \left\lceil (U/\epsilon)^{1/r} \right\rceil.$$

The elementary Fejér moment calculation in Section 3 is valid for every $0 < \gamma < 1$. Thus the ordinary pinned approximation to $u = g - g(1)t$ has error at most $D_\gamma(L+2)(m+1)^{-\gamma}$, including $m = 0$. Mixing by (7.2) gives error at most $\epsilon/8$.

As before set, for $0 < |h| < N$,

$$x_h = (|h|/N)^r, \qquad b_h = (1 - x_h)\beta_h, \qquad \rho_h = \min(1/2, x_h),$$

and set $b_0 = (1 - \epsilon/8)g(1)$, $\rho_0 = \epsilon/8$. The pinned polynomial $q = \sum_{|h|<N} b_h v_h$ has $\|q - g\|_\infty \leq 3\epsilon/8$. Writing $z = \max(0, \log Z_{A,a})$, the local density factor against the centered uniform comparison is at least

$$x_h \exp[-AN^{ar}|h|^{-ar} - z] \quad (h \neq 0).$$

Now $ar > 1$, so

$$\sum_{h=1}^{N-1} h^{-ar} \leq S := 1 + \frac{1}{ar - 1}.$$

The factorial estimate and both frequency signs give a joint factor at least

$$\exp[-2rN - 2zN - 2ASN^{ar}] \geq \exp[-(2r + 2z + 2AS)N^{ar}].$$

The endpoint factor is at least $(\epsilon/4)\exp[-A8^a\epsilon^{-a}-z]$. The unchanged centered acceptance and independent tail cost at most $5120\epsilon^{-2}$ in the negative logarithm. Their two errors are $\epsilon/4$ each, so the total error is $7\epsilon/8<\epsilon$. Since $N\le 2(U/\epsilon)^{1/r}$, one possible choice in (7.1) is

$$C_a=(2r+2z+2AS)2^{ar}U^a+A8^a,\qquad C_2=5120+z+\log 4+1.$$

For the opposite estimate, choose a fixed positive integer q and put

$$g_q(t)=\frac{2}{\pi q}\sin(2\pi qt).$$

This center is real and has zero endpoints, with normalized coefficients $\beta_q=\beta_{-q}=2$ and all others zero. Its increments satisfy

$$|g_q(t)-g_q(s)|\le\min(4|t-s|,4/(\pi q))\le 4\pi^{\gamma-1}q^{\gamma-1}|t-s|^\gamma.$$

Taking $q\ge\max\{1,(4\pi^{\gamma-1}/L)^{1/(1-\gamma)}\}$ places it in $\mathcal{H}_{\gamma,L}$. On the small-ball event, $|X_0|<\epsilon$, and the qth Fourier coefficient of $K_f-tX_0-g_q$ gives $|X_q-2|\le 4\pi q\epsilon=\delta$. For $0<\delta\le 1$, this necessary event has probability at most

$$\frac{\delta}{Z_{A,a}}\exp(-A\delta^{-a})\le\exp\left[\max(0,-\log Z_{A,a})-A(4\pi q)^{-a}\epsilon^{-a}\right].$$

At sufficiently small radii this is bounded by $\exp[-\frac{A}{2}(4\pi q)^{-a}\epsilon^{-a}]$. Combining with (7.1) proves the supercritical assertion. For $\gamma>1/2, a=2$, both powers in (7.1) equal two, and the same witness supplies the reverse bound. □

For clarity, set

$$F^{A,a}_{\gamma,L}(\epsilon)=\inf_{g\in\mathcal{H}_{\gamma,L}}\mathbb{P}(\|K_f-g\|_\infty<\epsilon).$$

Theorem 7.1, the earlier subcritical results, and the inclusion $\mathcal{H}_{\gamma,L}\subseteq\mathcal{H}_{1/2,L}$ for $\gamma>1/2$ give the following complete noncritical power orders. All parameters in the table are fixed, with $A>0, L\ge 1$.

Hölder exponent	Density exponent	Order of $-\log F^{A,a}_{\gamma,L}$
$0<\gamma\le 1/2$	$a<1/\gamma$	$\epsilon^{-1/\gamma}$
$0<\gamma\le 1/2$	$a>1/\gamma$	ϵ^{-a}
$1/2<\gamma<1$	every $a>0$	$\epsilon^{-\max(2,a)}$

On the equality line $a=1/\gamma$, $0<\gamma\le 1/2$, the logarithmic obstruction of Theorem 6.1 holds. In particular, the fixed half-Hölder RS center still has the bracket (5.5) at $a=2$. For that same fixed center, however, the new theorem and any one of its saturated coefficients give exact order ϵ^{-a} for every $a>2$.

A A continuous center without the lower order

Proposition A.1. *There is a continuous real g with $g(0)=g(1)=0$ and every normalized Fourier coefficient real in $[-1,1]$ for which no finite C satisfies $\mathbb{P}(\|K-g\|_\infty<\epsilon)\ge\exp(-C\epsilon^{-2})$ for all sufficiently small ϵ. The center is not $1/2$-Hölder.*

Proof. Put $a_j = 16^{-j}$, $M_j = a_j^{-8} = 16^{8j}$, and

$$g_j(t) = a_j \sum_{h=M_j}^{2M_j} \frac{\sin(2\pi h t)}{\pi h}, \qquad g(t) = \sum_{j\geq 1} g_j(t).$$

Because $\|g_j\|_\infty \leq 2a_j/\pi$, this is a uniformly convergent continuous real series, zero at both endpoints. The frequency blocks are disjoint. Its normalized Fourier coefficients at both signs of a frequency in the jth block are a_j, and all others are zero, hence are strictly inside $[-1, 1]$.

At $t_j = 1/(8M_j)$, all sines of the current block are at least $\sqrt{2}/2$, and those of earlier blocks are nonnegative. The later blocks have total norm at most $2a_j/(15\pi)$. Therefore

$$g(t_j) \geq \left(\frac{\sqrt{2}}{4\pi} - \frac{2}{15\pi}\right) a_j \geq a_j/16. \tag{A.1}$$

For the real part of $K(t)$, the real scalar coefficients are $c_0 = t$ and $c_h = \sin(2\pi h t)/(2\pi h)$ for $h \neq 0$. If $0 < t \leq 1/2$, identity (2.9) gives

$$\sum_h c_h^2 = t^2 + \frac{1}{2\pi^2} \sum_{h\geq 1} \frac{\sin^2(2\pi h t)}{h^2} = t/2.$$

Hoeffding's estimate, independence, and Fatou through the convergent symmetric sums yield $\mathbb{E}\exp(\lambda \operatorname{Re} K(t)) \leq \exp(\lambda^2 t)$. Optimizing exponential Markov inequality gives

$$\mathbb{P}(\operatorname{Re} K(t) \geq A) \leq \exp(-A^2/(4t)), \quad A > 0.$$

For $\epsilon_j = a_j/64$, equations (A.1) and the small-ball event imply $\operatorname{Re} K(t_j) > 3a_j/64$. Consequently

$$\mathbb{P}(\|K - g\|_\infty < \epsilon_j) \leq \exp\{-\tfrac{9}{2048} a_j^2 M_j\} = \exp\{-\tfrac{9}{2048} a_j^{-6}\}.$$

Since $\epsilon_j^{-2} = 4096 a_j^{-2}$, this contradicts the proposed lower bound for every fixed C when j is sufficiently large. Finally, $g(t_j)/\sqrt{t_j} \geq (\sqrt{8}/16) a_j^{-3} \to \infty$, so this center is not 1/2-Hölder. □

The proposition shows that the regularity hypothesis cannot be replaced by continuity alone, even strictly inside the permitted Fourier interval. Theorem 3.1 determines the corresponding uniform worst-case scale within the Hölder hierarchy.

B Center-dependent concentration obstructions

The next consequences expose the bounded-coefficient concentration mechanism behind the spike and continuity examples. They use classical exponential estimates.

Proposition B.1 (An increment-energy obstruction)**.** *Let $g \in C([0,1], \mathbb{C})$, let $\epsilon > 0$, and let $[s_j, t_j] \subset [0, 1/2]$ be finitely many intervals with disjoint interiors and positive lengths $\ell_j = t_j - s_j$. Then*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \leq \exp\left\{-\frac{1}{4} \sum_j \frac{(|g(t_j) - g(s_j)| - 2\epsilon)_+^2}{\ell_j}\right\}.$$

In particular, the same bound holds with the supremum of the displayed sum over all such finite collections; an infinite supremum gives probability zero.

Proof. Write $v_h = c_h + id_h$, so

$$c_0(t) = t, \quad d_0(t) = 0, \qquad c_h(t) = \frac{\sin(2\pi ht)}{2\pi h}, \quad d_h(t) = \frac{1-\cos(2\pi ht)}{2\pi h} \quad (h \neq 0).$$

For $0 \le s \le t \le 1/2$, the cosine-series identity (2.9) gives

$$\sum_h c_h(s)c_h(t) = \sum_h d_h(s)d_h(t) = s/2, \qquad \sum_h c_h(s)d_h(t) = 0.$$

The first identity for c_h was used in (3.6). For d_h, write the product of $1-\cos(2\pi hs)$ and $1-\cos(2\pi ht)$ using the cosine series at $0, s, t, t-s, t+s$; its value is $s/2$. The cross sum vanishes by pairing h and $-h$. Consequently, for arbitrary complex weights z_j, the real scalar variable

$$S = \sum_j \mathrm{Re}\{\overline{z_j}(K(t_j) - K(s_j))\}$$

has a coefficient vector of squared norm $\frac{1}{2}\sum_j |z_j|^2\ell_j$. Independence and the bounded-variable exponential estimate from Section 2.1 therefore give

$$\mathbb{E}\mathrm{e}^{uS} \le \exp\{u^2 \sum_j |z_j|^2\ell_j\}.$$

For the infinite series this follows by Fatou from the symmetric partial sums, which converge almost surely at all the finitely many endpoints.

Put $\Delta_j = g(t_j) - g(s_j)$ and $\eta_j = (|\Delta_j| - 2\epsilon)_+$. Take $z_j = 0$ if $\eta_j = 0$, and otherwise take $z_j = \eta_j\Delta_j/(\ell_j|\Delta_j|)$. If $E = \sum_j \eta_j^2/\ell_j > 0$, the strict small-ball event implies $S > E$, whereas $\sum_j |z_j|^2\ell_j = E$. Exponential Markov inequality at $u = 1/2$ yields $\mathbb{P}(S > E) \le \mathrm{e}^{-E/4}$. If $E = 0$ the conclusion is trivial. Taking the infimum over these bounds proves the final assertion. □

Corollary B.2 (A necessary regularity condition at a fixed center)**.** *Suppose $g \in C([0,1], \mathbb{C})$ and there exist $C, \epsilon_* > 0$ such that*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-2}), \qquad 0 < \epsilon \le \epsilon_*.$$

Then g is $1/4$-Hölder on $[0,1]$. Combined with Theorem 2.1, the half-Hölder and quarter-Hölder conditions bracket the centers with the displayed lower order.

Proof. For $s < t$ in $[0, 1/2]$, put $d = |g(t) - g(s)|$. When $0 < d \le 4\epsilon_*$, apply the preceding proposition to the single interval and take $\epsilon = d/4$. The assumed lower bound and the resulting upper bound imply

$$\exp(-16C/d^2) \le \mathbb{P}(\|K - g\|_\infty < d/4) \le \exp\{-d^2/[16(t-s)]\}, \qquad d^4 \le 256C(t-s).$$

By uniform continuity, the restriction $d \le 4\epsilon_*$ holds for all sufficiently short intervals. The longer intervals are controlled using the boundedness of g. Thus g is $1/4$-Hölder on the first half interval.

Every sample path satisfies $K(t) + \overline{K(1-t)} = K(1)$: the identity holds termwise for the symmetric partial sums and passes to their uniform limit. The assumed positive small-ball probabilities for arbitrarily small radii force the same identity for g. Indeed on such an event the uniform defect in that identity is less than 3ϵ. Reflection therefore gives the same Hölder bound on $[1/2, 1]$. Splitting an interval at $1/2$ and using $x^{1/4} + y^{1/4} \le 2^{3/4}(x+y)^{1/4}$ completes the proof. □

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