Root-Laguerre iterates of the Riemann kernel: the Csordas–Dimitrov concavity conjecture and depth four

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Abstract

Csordas and Dimitrov (2000) conjectured, as their Problem 3.3, a second-level concavity property of the Jacobi theta kernel Φ of the Riemann Ξ-function: with $s(t) = \Phi(\sqrt{t})$ and $Tf = f'^2 - ff''$, that $T^2 s > 0$ on $(0, \infty)$. The first level, $Ts > 0$, is due to Csordas and Varga (1988); Planat and Solé (2026) announced the first proof of the second. We give an independent rigorous proof and determine the depth of this root-Laguerre hierarchy: $T^r s > 0$ on $[0, \infty)$ for $r = 1, 2, 3, 4$, whereas $T^5 s$ is negative on $[0, 0.00969]$ and positive on $[0.009901, \infty)$. The proof combines an analytic argument for $t \geq 1$, where the first theta term dominates and Cauchy estimates control the logarithmic corrections, with interval-arithmetic Taylor models on $[0, 1]$ whose remainders are bounded by majorant series. We give a complete proof of the Csordas–Dimitrov moment theorem with explicit hypotheses, including a hand proof of the identity in their Lemma 2.3; this makes precise their remark that monotonicity of the kernel is not needed. As a corollary we answer the question closing their list of problems: for no value of the de Bruijn–Newman parameter do the Taylor data of the heat-flow deformations of Ξ fail the double Turán inequalities. The answer follows from any proof of Problem 3.3, because both concavity conditions are invariant under the Gaussian tilts that Csordas and Varga used for the ordinary Turán inequalities.

Keywords. Riemann Ξ-function; Jacobi theta kernel; logarithmic concavity; Laguerre inequalities; double Turán inequalities; de Bruijn–Newman constant; interval arithmetic.
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1 Introduction

1.1 The questions

Let

$$\Phi(u) = \sum_{n=1}^{\infty} a_n(u), \qquad a_n(u) = \pi n^2 \bigl(2\pi n^2 e^{4u} - 3\bigr) \exp\bigl(5u - \pi n^2 e^{4u}\bigr), \tag{1.1}$$

be the Riemann–Jacobi kernel, so that $\frac{1}{8}\Xi(x/2) = \int_0^\infty \Phi(u)\cos(xu)\,du$ [3, (2.1)–(2.2)]. Throughout, for a function F of one real variable we write

$$TF = (F')^2 - FF''. \tag{1.2}$$

For $F > 0$, the inequality $TF > 0$ is equivalent to $(\log F)'' < 0$. Let $s(t) = \Phi(\sqrt{t})$ for $t \geq 0$. Csordas and Varga proved that $\log s$ is strictly concave on $(0, \infty)$ [5, Theorem 2.1], that is, $Ts > 0$ there, and Csordas and Dimitrov posed the next level.

Problem 1.1 (Csordas–Dimitrov [3, Problem 3.3, (3.4)]). *Let $s(t) = \Phi(\sqrt{t})$ and $f(t) = s'(t)^2 - s(t)s''(t)$. Show that $(\log f(t))'' < 0$ for $t > 0$.*

Csordas restated it as [2, Open Problem 4.14]. In the notation (1.2), Problem 1.1 asks for $T^2 s > 0$ on $(0,\infty)$, and the natural continuation is the sign of the *root-Laguerre iterates* $T^r s$ for $r \geq 3$.

The motivation of Problem 1.1 is a moment inequality. For a real parameter λ put

$$H_\lambda(x) = \int_0^\infty e^{\lambda u^2} \Phi(u) \cos(xu)\, du, \qquad b_k(\lambda) = \int_0^\infty e^{\lambda u^2} u^{2k} \Phi(u)\, du, \qquad \gamma_k(\lambda) = \frac{k!\, b_k(\lambda)}{(2k)!}. \tag{1.3}$$

These are the functions and Taylor data of Csordas and Dimitrov [3, (3.8), (3.11)], who write t for our λ. De Bruijn [6] proved that H_λ has only real zeros for $\lambda \geq \frac{1}{2}$, and Newman [11] showed that there is a constant $\Lambda \leq \frac{1}{2}$ such that H_λ has only real zeros if and only if $\lambda \geq \Lambda$; Rodgers and Tao [13] proved $\Lambda \geq 0$. With $T_k = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1}$ ($k \geq 1$), the *double Turán inequalities* for a positive sequence (γ_k) are

$$E_k = T_k^2 - T_{k-1} T_{k+1} \geq 0 \qquad (k \geq 2) \tag{1.4}$$

[3, (1.12)]. By a theorem of Craven and Csordas [1] (see [3, Corollary 1.3]) they hold for the Maclaurin coefficients, in the normalization $\sum \gamma_k x^k/k!$, of every function of type I in the Laguerre–Pólya class; so for $\lambda \geq \Lambda$ they are necessary consequences of the reality of the zeros of H_λ. Csordas and Dimitrov conjectured them for $\lambda = 0$ [3, Problem 3.2, (3.3)], and proved a moment theorem [3, Theorem 2.4] that yields them from the two concavity conditions $Ts > 0$ and $T^2 s > 0$. They close their list of problems with the following question.

Problem 1.2 (Csordas–Dimitrov [3, §3, after Problem 3.6, with (3.11)]). *Determine the values of λ, if any, for which the sequence $\{\gamma_k(\lambda)\}_{k \geq 0}$ fails to satisfy the double Turán inequalities.*

1.2 Main results

Theorem 1.3 (Depth of the root-Laguerre hierarchy). *Let $s(t) = \Phi(\sqrt{t})$ for $t \geq 0$.*

(i) $T^r s(t) > 0$ for every $t \geq 0$ and $r = 1, 2, 3, 4$. Equivalently, $\log T^r s$ is strictly concave on $[0,\infty)$ for $r = 0, 1, 2, 3$.

(ii) $T^5 s(t) < 0$ for $0 \leq t \leq 0.00969$ and $T^5 s(t) > 0$ for $t \geq 0.009901$. In particular $T^5 s$ changes sign, and all its zeros in $[0,\infty)$ lie in $(0.00969, 0.009901)$.

Part (i) with $r = 1$ is the theorem of Csordas and Varga [5, Theorem 2.1]. Part (i) with $r = 2$ answers Problem 1.1 affirmatively; the first proof was announced by Planat and Solé [12, Theorem 1.1], and our proof is independent of theirs (see §1.3). The cases $r = 3, 4$ and part (ii) are new. Part (ii) shows that the hierarchy has depth exactly four; numerically, the sign change of $T^5 s$ is at $t \approx 0.00983$ (Remark 7.4).

Corollary 1.4 (Answer to Problem 1.2). *For every real λ and every $k \geq 2$, $E_k(\lambda) = T_k(\lambda)^2 - T_{k-1}(\lambda) T_{k+1}(\lambda) > 0$. Hence there is no real λ for which $\{\gamma_k(\lambda)\}$ fails the double Turán inequalities.*

The ordinary Turán inequalities $T_k(\lambda) > 0$, $k \geq 1$, for every real λ are due to Csordas and Varga [5, Theorem 2.4] (take $f \equiv 1$ there). Corollary 1.4 follows from the case $r = 2$ of Theorem 1.3 by the tilt identity (Lemma 2.2) and the moment theorem (Theorem 3.1). In particular any proof of Problem 1.1, such as that of Planat and Solé, answers Problem 1.2; we record the answer explicitly.

Corollary 1.5 (Problem 3.2 of [3]). *The Taylor data $\gamma_k = \gamma_k(0)$ of the Riemann Ξ-function satisfy $E_k > 0$ for all $k \geq 2$.*

Corollary 1.5 was first derived by Planat and Solé [12, §7] from their proof of Problem 1.1. The next consequence shows what Corollary 1.4 says about the de Bruijn–Newman constant, which Csordas and Dimitrov hoped Problem 1.2 might illuminate [3, §3].

Corollary 1.6. *For every $\lambda < 0$ the function H_λ has nonreal zeros, while $T_k(\lambda) > 0$ for $k \geq 1$ and $E_k(\lambda) > 0$ for $k \geq 2$. In particular the double Turán inequalities for $\{\gamma_k(\lambda)\}$ hold on both sides of Λ.*

Proof. For $\lambda < 0 \leq \Lambda$ the function H_λ has nonreal zeros by the theorems of Newman [11] and Rodgers–Tao [13]. The inequalities are [5, Theorem 2.4] and Corollary 1.4. □

1.3 Relation to earlier work and route of the proof

The moment theorem. Corollary 1.4 rests on [3, Theorem 2.4]. Csordas and Dimitrov remark that an examination of its proof shows that it remains valid under less restrictive assumptions on the kernel [3, Remark 2.5], and its part (a) is [5, Proposition 2.2] (as they note), which Csordas and Varga state for every real λ. Section 3 makes this precise. Theorem 3.1 lists the hypotheses the proof uses (positivity, evenness, smoothness and decay of the kernel, with no monotonicity) and gives a complete proof, including a hand proof (Lemma 3.3) of the identity [3, Lemma 2.3, (2.11)], whose verification the authors omitted and suggested checking with a symbol-manipulating program. The monotonicity condition [3, Definition 2.2(iv)] matters here because the tilted kernels $e^{\lambda u^2}\Phi(u)$ are monotone on $(0,\infty)$ exactly for $\lambda \leq \lambda_* = -\Phi''(0)/(2\Phi(0)) = 37.4538\ldots$ (Proposition 7.1). The invariance of the first concavity condition under these tilts is the observation of Csordas and Varga [5, (2.23)–(2.25)] behind their Theorem 2.4. Lemma 2.2 iterates it, $T^r(e^{\beta t}F) = e^{2^r\beta t}T^rF$, so that the second condition is invariant too; the author recorded this iterated form for general kernels in [14, Lemma 2.1]. Remark 7.2 gives a second route to Corollary 1.4, with non-strict inequalities for $\lambda > \lambda_*$, through de Bruijn's theorem and the original statement of [3, Theorem 2.4].

Related inequalities. The Turán inequalities at $\lambda = 0$ were first proved by Csordas, Norfolk and Varga [4]. Katkova [7] recorded the double Turán inequalities for Ξ as an open question raised by Csordas, and proved multiple positivity of order 44 for the Maclaurin coefficients of a function attached to Ξ [7, Theorem 1]. Liu and Mao [8] proved that the ordinary Taylor coefficients of $F(x) = \sum \gamma_k x^k/k!$ at $\lambda = 0$, which are $\gamma_k/k!$, are strictly infinitely log-concave. The double Turán inequalities iterate the Turán differences of the γ_k themselves, and the factor $1/k!$ does not commute with this iteration, so the two families of inequalities are different. A different positivity hierarchy for the same kernel, the Pólya frequency property of the translation kernel $\Phi(|u|)$, fails at order five [9].

The hierarchy. We write $T^r s$ through curvatures: if $\kappa_0 = -(\log s)''$ and $\kappa_{j+1} = 2\kappa_j - (\log\kappa_j)''$, then $T^r s = s^{2^r}\prod_{j<r}\kappa_j^{2^{r-1-j}}$ (Lemma 2.3). The same doubling-with-correction structure organizes the author's finite-depth constructions for general kernels [16]; nothing from there is used. For $t \geq 1$ the first theta term dominates, $\kappa_j \approx 2^j\pi e^{4\sqrt{t}}(4\sqrt{t}-1)t^{-3/2}$, and Cauchy estimates on nested complex disks bound the logarithmic corrections (Proposition 4.2). On $[0,1]$ we use Taylor models on intervals with dyadic endpoints: at each centre we compute the Taylor coefficients of s in interval arithmetic, tilt them by $e^{\beta h}$ to remove the first-order variation, compute the coefficients of $T^r s$ exactly by power-series arithmetic, and bound the remainder by an explicit majorant series that comes from a single bound for $|s|$ on a circle (Lemma 2.4, Propositions 5.2 and 6.1). The point $t = 0$ is treated like any other centre, because s is analytic at 0.

Comparison with Planat and Solé. They prove the case $r = 2$ by two arguments, a direct theta-series estimate with exact Bernstein bounds and an argument through Jacobi's differential equation for θ_3, which share a degree-90 Taylor certificate on $0 < t \le 0.022$ [12]. Our computation is independent of theirs. Near $t = 0$ both arguments use a Taylor certificate with interval evaluation and a Cauchy-type remainder. Ours uses tilted Taylor models with majorant remainders across all of $[0, 1]$, and an analytic argument for $t \ge 1$ replaces their theta-series and Jacobi-equation continuations; it also yields the levels $r = 3, 4$ and the sign of $T^5 s$.

Software and files. The computations use the interval arithmetic of `mpmath` [10] with outward rounding. The scripts and their outputs accompany the paper; each proposition names the script that carries it.

2 Preliminaries

2.1 The kernel and its root profile

We use the following classical facts [3, Theorem 2.1], [2, Theorem 4.1]: Φ is even, $\Phi(u) > 0$ for real u, and the series (1.1) converges locally uniformly on the strip $S = \{\sigma \in \mathbb{C} : |\operatorname{Im}\sigma| < \pi/8\}$, where Φ is analytic. For $\sigma = a + ib \in S$ each term satisfies

$$|a_n(\sigma)| \le \pi n^2 \big(2\pi n^2 e^{4a} + 3\big) e^{5a} \exp\big(-\pi n^2 e^{4a} \cos 4b\big), \tag{2.1}$$

because $|\exp(-\pi n^2 e^{4\sigma})| = \exp(-\pi n^2 e^{4a} \cos 4b)$ and $\cos 4b > 0$ on S.

Lemma 2.1 (Analyticity of the root profile)**.** *Let $\Omega = \{\sigma^2 : \sigma \in S\}$. The function $s(t) = \Phi(\sqrt{t})$, $t \ge 0$, extends analytically to Ω by $s(\tau) = \Phi(\sigma)$ for any $\sigma \in S$ with $\sigma^2 = \tau$; in particular $s(\tau) = \Phi(\sqrt{\tau})$ with the principal root. For every $t_c \ge 0$ the open disk $\{|\tau - t_c| < \pi^2/64\}$ lies in Ω.*

Proof. If $\sigma \in S$ then $-\sigma \in S$, and $\Phi(-\sigma) = \Phi(\sigma)$, so the definition does not depend on the root. Near a point $\tau_0 \ne 0$ of Ω a branch of $\sqrt{\tau}$ with values in S is analytic, hence so is s. Near $\tau_0 = 0$ the even Taylor series $\Phi(\sigma) = \sum_j \phi_{2j} \sigma^{2j}$ converges for $|\sigma| < \pi/8$, so $s(\tau) = \sum_j \phi_{2j} \tau^j$ is analytic for $|\tau| < \pi^2/64$. Writing $\sigma = a + ib$ gives $\sigma^2 = a^2 - b^2 + 2iab$, and eliminating a shows that $\Omega = \{x + iy : x + r > y^2/(4r)\}$ with $r = \pi^2/64$. If $|\tau - t_c| < r$ with $t_c \ge 0$, put $\tau = x + iy$ and $\xi = x - t_c \in (-r, r)$. Then $y^2/(4r) < (r - \xi)(r + \xi)/(4r) < (r + \xi)/2 < x + r$. □

2.2 The operator T, tilts and curvatures

Lemma 2.2 (Tilt identity)**.** *For every real β, every $r \ge 1$ and every C^{2r} function F, $T^r(e^{\beta t} F) = e^{2^r \beta t} T^r F$.*

Proof. With $G = e^{\beta t} F$ we have $G' = e^{\beta t}(F' + \beta F)$ and $G'' = e^{\beta t}(F'' + 2\beta F' + \beta^2 F)$, so $G'^2 - GG'' = e^{2\beta t}(F'^2 - FF'')$: the terms $2\beta FF'$ and $\beta^2 F^2$ cancel. Induction on r, with 2β in place of β, gives the claim. □

Lemma 2.3 (Curvature recursion)**.** *For a positive C^2 function F put $\kappa(F) = -(\log F)''$. Then $TF = F^2 \kappa(F)$. If F is C^4 and $\kappa(F) > 0$, then $TF > 0$ and*

$$\kappa(TF) = 2\kappa(F) - \big(\log \kappa(F)\big)''. \tag{2.2}$$

Consequently, if $s > 0$ is C^{2r} on an interval and $\kappa_0 = \kappa(s)$, $\kappa_{j+1} = 2\kappa_j - (\log \kappa_j)''$ are positive for $j < r$, then $T^j s > 0$, $\kappa(T^j s) = \kappa_j$ for $j \le r$, and

$$T^r s = s^{2^r} \prod_{j=0}^{r-1} \kappa_j^{\,2^{r-1-j}}. \tag{2.3}$$

Conversely, if $T^j s > 0$ for $j < r$, then the sign of $T^r s$ is the sign of κ_{r-1}.

Proof. $(\log F)'' = (FF'' - F'^2)/F^2$ gives $TF = F^2\kappa(F)$. Taking logarithms, $\log TF = 2\log F + \log\kappa(F)$, and differentiating twice gives (2.2). The remaining statements follow by induction on r. □

In particular, Problem 1.1 is the statement $\kappa_1 > 0$ on $(0,\infty)$, and Theorem 1.3(i) is the statement $\kappa_0, \kappa_1, \kappa_2, \kappa_3 > 0$ on $[0,\infty)$.

2.3 Majorant series

For power series $F = \sum f_j h^j$ and $G = \sum g_j h^j$ with $g_j \geq 0$ we write $F \ll G$ if $|f_j| \leq g_j$ for all j. If $F \ll G$ then $F' \ll G'$, and products and sums of majorized series are majorized by the products and sums of the majorants.

Lemma 2.4 (Majorants of the iterates)**.** *Let F be analytic on a neighbourhood of the closed disk $|h| \leq R$ and $|F| \leq M$ on $|h| = R$. Define $C_0 = M$, $p_0 = 1$ and*

$$C_{r+1} = C_r^2 R^{-2} p_r(2p_r + 1), \qquad p_{r+1} = 2p_r + 2.$$

Then $T^r F \ll C_r(1 - h/R)^{-p_r}$ for every $r \geq 0$. Consequently, for real h with $|h| \leq \zeta R$, $0 < \zeta < 1$, and every $J \geq 0$,

$$\Bigl| T^r F(h) - \sum_{j=0}^{J} [h^j]\, T^r F \cdot h^j \Bigr| \leq C_r \sum_{j>J} \binom{j + p_r - 1}{j} \zeta^j. \tag{2.4}$$

Proof. By Cauchy's inequality $|[h^j]F| \leq MR^{-j}$, that is, $F \ll M(1 - h/R)^{-1}$. If $F \ll G$ then $TF = F'^2 - FF'' \ll G'^2 + GG''$. For $G = C(1 - h/R)^{-p}$ one computes $G'^2 + GG'' = C^2R^{-2}(p^2 + p(p+1))(1-h/R)^{-2p-2}$, which gives the recursion. The coefficient of h^j in $(1-h/R)^{-p}$ is $\binom{j+p-1}{j}R^{-j}$, which gives (2.4). □

For $r = 0, 1, \ldots, 5$ the exponents are $p_r = 1, 4, 10, 22, 46, 94$, and $C_r = A_r M^{2^r} R^{-w_r}$ with $w_r = 0, 2, 6, 14, 30, 62$ and

$$(A_0, A_1, A_2, A_3) = (1,\ 3,\ 324,\ 22\,044\,960), \qquad A_{r+1} = A_r^2\, p_r(2p_r + 1).$$

3 The moment theorem of Csordas and Dimitrov

We restate [3, Theorem 2.4] with the hypotheses its proof uses. For a function K on $\mathbb{R}$ we write $b_k = \int_0^\infty x^{2k} K(x)\,dx$, $\gamma_k = k!\, b_k/(2k)!$, $s(t) = K(\sqrt{t})$ $(t > 0)$ and $f = Ts = s'^2 - ss''$.

Theorem 3.1 (Csordas–Dimitrov [3, Theorem 2.4 and Remark 2.5]; part (a): Csordas–Varga [5, Proposition 2.2])**.** *Let $K : \mathbb{R} \to (0,\infty)$ be even and C^∞, and suppose that $x^m K^{(i)}(x)$ is integrable on $[0,\infty)$ and tends to 0 as $x \to \infty$ for all $m \geq 0$ and $0 \leq i \leq 2$.*

(a) If $(\log s)'' < 0$ on $(0,\infty)$, then $T_k = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} > 0$ for all $k \geq 1$.

(b) If moreover $(\log f)'' < 0$ on $(0,\infty)$, then $E_k = T_k^2 - T_{k-1}T_{k+1} > 0$ for all $k \geq 2$.

Csordas and Dimitrov state the theorem for admissible kernels [3, Definition 2.2], and remark that an examination of the proof shows that it remains valid under less restrictive assumptions on the kernel [3, Remark 2.5]; as they note, part (a) goes back to Csordas and Varga [5, Proposition 2.2], who work with the kernels $e^{\lambda u^2} f(iu)\Phi(u)$ for every real λ and deduce the Turán inequalities for all of them [5, Theorem 2.4]. Theorem 3.1 makes the hypotheses explicit. Every admissible kernel satisfies them, and the monotonicity condition $K' < 0$ of [3, Definition 2.2(iv)] is not among them. The proof below follows the structure of theirs, with the identity of [3, Lemma 2.3] proved by hand and the sign of the kernel determinant obtained from a divided difference.

3.1 Moment determinants

Put $p_m = (2m)!/m!$, so that $\gamma_m = b_m/p_m$ and $p_{m+1} = 2(2m+1)p_m$.

Lemma 3.2. *Under the hypotheses of Theorem 3.1, let $\psi_1(x) = (K'(x)/x)'$, $\psi_2(x) = -K'(x)$ and $\psi_3(x) = xK(x)$ for $x > 0$. Then, for $m \geq 1$ and $m \geq 2$ respectively,*

$$\int_0^\infty x^{2m-1}\psi_2(x)\,dx = (2m-1)b_{m-1}, \qquad \int_0^\infty x^{2m-1}\psi_1(x)\,dx = (2m-1)(2m-3)b_{m-2}, \tag{3.1}$$

and $\int_0^\infty x^{2m-1}\psi_3\,dx = b_m$. Moreover, with the Hankel determinant $\Delta_{k-2} = \det(\gamma_{k-2+i+j})_{i,j=0}^{2}$, for $k \geq 1$ and $k \geq 2$ respectively,

$$-T_k = \frac{2}{p_k p_{k+1}} \det\begin{pmatrix} (2k-1)b_{k-1} & b_k \\ (2k+1)b_k & b_{k+1} \end{pmatrix}, \tag{3.2}$$

$$\Delta_{k-2} = \frac{8}{p_k p_{k+1} p_{k+2}} \det\begin{pmatrix} (2k-1)(2k-3)b_{k-2} & (2k-1)b_{k-1} & b_k \\ (2k+1)(2k-1)b_{k-1} & (2k+1)b_k & b_{k+1} \\ (2k+3)(2k+1)b_k & (2k+3)b_{k+1} & b_{k+2} \end{pmatrix}, \tag{3.3}$$

and $E_k = -\gamma_k \Delta_{k-2}$ for $k \geq 2$.

Proof. Since K is even and smooth, $K'(x)/x = \int_0^1 K''(\theta x)\,d\theta$ is smooth on $[0,\infty)$, so all integrands are integrable near 0. Integration by parts gives (3.1); the boundary terms are multiples of $x^{2m-1}K(x)$, $x^{2m-2}K'(x)$ and $x^{2m-3}K(x)$, which vanish at 0 for the stated m and at ∞ by hypothesis. For (3.2) and (3.3), the relation $p_m = 2(2m-1)p_{m-1} = 4(2m-1)(2m-3)p_{m-2}$ gives $(2m-1)b_{m-1} = \frac{1}{2}p_m\gamma_{m-1}$ and $(2m-1)(2m-3)b_{m-2} = \frac{1}{4}p_m\gamma_{m-2}$. Hence row i of the matrix in (3.3) is $p_{k+i}(\frac{1}{4}\gamma_{k+i-2}, \frac{1}{2}\gamma_{k+i-1}, \gamma_{k+i})$, and its determinant is $\frac{1}{8}p_k p_{k+1} p_{k+2}\Delta_{k-2}$; (3.2) is the same computation for the 2×2 matrix. Finally, Sylvester's identity for the 3×3 Hankel matrix $(\gamma_{k-2+i+j})$ and its central entry γ_k gives $\gamma_k\Delta_{k-2} = (-T_{k-1})(-T_{k+1}) - (-T_k)^2 = -E_k$. □

The script `01_cd_identities.py` also checks all identities of Lemma 3.2 symbolically, with k a free symbol. We use the continuous Cauchy–Binet formula (Andréief's identity): if $f_i\varphi_j$ is integrable on $(0,\infty)$ for $1 \leq i,j \leq n$, then

$$\det\Big(\int_0^\infty f_i\varphi_j\Big)_{i,j=1}^{n} = \frac{1}{n!}\int_{(0,\infty)^n} \det\big(f_i(x_l)\big)_{i,l}\,\det\big(\varphi_j(x_l)\big)_{j,l}\,dx. \tag{3.4}$$

Indeed, the Leibniz formula and Fubini's theorem write the left side as the integral of $\prod_i f_i(x_i)\cdot \det(\varphi_j(x_i))$ over $(0,\infty)^n$, and averaging this integral over the $n!$ permutations of the variables gives the right side.

3.2 The identity of Lemma 2.3 of Csordas–Dimitrov

For $x > 0$ put

$$h_2(x) = -\frac{K'(x)}{xK(x)}, \qquad h_3(x) = -\frac{(K'(x)/x)'}{xK(x)}, \qquad w = h_2' h_3'' - h_3' h_2''. \tag{3.5}$$

The function $w(t)$ of [3, Lemma 2.3] is the same determinant written for $-h_2$ and $-h_3$; changing the sign of both rows does not change its value, so it equals our w.

Lemma 3.3 ([3, Lemma 2.3, (2.11)])**.** *Let $\ell = \log s$, $f = Ts$ and $g = (\log f)''$, where $s(t) = K(\sqrt{t})$ and $K > 0$ is C^6. Then for $x > 0$*

$$h_2(x) = -2\ell'(x^2), \qquad h_3(x) = -4\big(\ell''(x^2) + \ell'(x^2)^2\big), \qquad w(x) = \frac{64\, x^3\, g(x^2)\, f(x^2)^2}{K(x)^4}.$$

Proof. From $K(x) = s(x^2)$ we get $K'(x)/x = 2s'(x^2)$ and $(K'(x)/x)' = 4xs''(x^2)$; dividing by $xK(x) = xs(x^2)$ gives h_2 and h_3, since $s''/s = \ell'' + \ell'^2$. For functions $A(x) = \tilde{A}(x^2)$, $B(x) = \tilde{B}(x^2)$ one has $A' = 2x\tilde{A}'$ and $A'' = 4x^2\tilde{A}'' + 2\tilde{A}'$, hence

$$A'B'' - B'A'' = 8x^3(\tilde{A}'\tilde{B}'' - \tilde{B}'\tilde{A}'')(x^2).$$

With $\tilde{A} = -2\ell'$ and $\tilde{B} = -4(\ell'' + \ell'^2)$ this gives $w(x) = 64x^3\, W(x^2)$, where

$$W = \ell''(\ell'''' + 2\ell''^2 + 2\ell'\ell''') - \ell'''(\ell''' + 2\ell'\ell'') = \ell''\ell'''' - \ell'''^2 + 2\ell''^3.$$

On the other hand $f = s'^2 - ss'' = -s^2\ell''$, so $\log f = 2\ell + \log(-\ell'')$ where $f > 0$, and

$$g = 2\ell'' + \frac{\ell''''\ell'' - \ell'''^2}{\ell''^2}, \qquad \text{that is,} \qquad g\,\ell''^2 = W.$$

Since $\ell''^2 = f^2/s^4$, we obtain $W = gf^2/s^4$, and $s(x^2) = K(x)$ gives the formula for w. (Where $f = 0$, both sides of $g\ell''^2 = W$ are read as the polynomial expression W, and the formula for w holds by continuity whenever g is defined.) □

Lemma 3.4 (Sign of the kernel determinant)**.** *Assume $(\log s)'' < 0$ and $(\log f)'' < 0$ on $(0, \infty)$. Then for $0 < x_1 < x_2 < x_3$,*

$$V(x_1, x_2, x_3) := \det\big(\psi_i(x_l)\big)_{i,l=1}^{3} < 0.$$

Proof. Dividing column l by $x_l K(x_l) > 0$ and exchanging the first and third rows gives $V = \prod_l x_l K(x_l) \cdot \det(1;\, h_2(x_l);\, h_3(x_l))$, because $\psi_1/(xK) = -h_3$, $\psi_2/(xK) = h_2$ and $\psi_3/(xK) = 1$, and the exchange and the sign of the row $-h_3$ cancel. By Lemma 3.3, $h_2'(x) = -4x\ell''(x^2) > 0$, so h_2 is a C^2 diffeomorphism of $(0, \infty)$ onto an interval and $y_l = h_2(x_l)$ satisfy $y_1 < y_2 < y_3$. Let $H = h_3 \circ h_2^{-1}$. Then

$$\det \begin{pmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ H(y_1) & H(y_2) & H(y_3) \end{pmatrix} = v\, H[y_1, y_2, y_3] = \tfrac{1}{2}\, v\, H''(\eta), \qquad v = (y_2 - y_1)(y_3 - y_1)(y_3 - y_2) > 0,$$

for some $\eta \in (y_1, y_3)$, by the mean value theorem for second divided differences. The chain rule gives $H'' = w/h_2'^3$ at the point $h_2^{-1}(\eta)$, and $w < 0$ by Lemma 3.3 because $g < 0$ and $x^3 f^2/K^4 > 0$. Hence the determinant is negative, and so is V. □

3.3 Proof of Theorem 3.1

(a) By (3.1), (3.2) and (3.4) with $f_1 = x^{2k}$, $f_2 = x^{2k+2}$, $\varphi_1 = -K'(x)/x$ and $\varphi_2 = K$,

$$-T_k = \frac{1}{p_k p_{k+1}} \int_0^\infty \int_0^\infty x_1^{2k} x_2^{2k} (x_2^2 - x_1^2) K(x_1) K(x_2) \big(h_2(x_1) - h_2(x_2)\big)\, dx_1 dx_2.$$

The integrand is symmetric in (x_1, x_2) and negative for $x_1 < x_2$ because h_2 is strictly increasing. Hence $T_k > 0$.

(b) By (3.1), (3.3) and (3.4) with $f_i = x^{2k-1+2(i-1)}$ and $\varphi_i = \psi_i$,

$$\Delta_{k-2} = \frac{8}{3!\, p_k p_{k+1} p_{k+2}} \int_{(0,\infty)^3} (x_1 x_2 x_3)^{2k-1}\, M(x_1, x_2, x_3)\, V(x_1, x_2, x_3)\, dx,$$

where $M = \det(x_l^{2(i-1)}) = (x_2^2 - x_1^2)(x_3^2 - x_1^2)(x_3^2 - x_2^2)$. Both M and V change sign under a transposition of the variables, so the integrand is symmetric and Δ_{k-2} equals $\frac{8}{p_k p_{k+1} p_{k+2}}$ times the integral over $0 < x_1 < x_2 < x_3$. There $M > 0$ and $V < 0$ by Lemma 3.4, so $\Delta_{k-2} < 0$ and $E_k = -\gamma_k \Delta_{k-2} > 0$. □

4 The range $t \geq 1$

For complex τ near the positive axis we write $\sigma = \sqrt{\tau}$ (principal root), $x = \pi e^{4\sigma}$ and

$$m(\tau) = \pi e^{4\sigma} (4\sigma - 1) \tau^{-3/2}.$$

For real $t > 1/16$, $m(t) = \pi \frac{d^2}{dt^2} e^{4\sqrt{t}} > 0$ is the curvature of the leading part $-\pi e^{4\sqrt{t}}$ of $-\log s$. For $t \geq 1$ and $0 \leq \rho \leq \frac{1}{2}$ let $D(t, \rho) = \{|\tau - t| \leq \rho\}$ and

$$\mu(t, \rho) = \frac{\pi e^{4\sqrt{t-\rho}} (4\sqrt{t-\rho} - 1)}{(t+\rho)^{3/2}}, \qquad \Lambda(t, \rho) = \frac{1}{(t-\rho)^{3/2}} + \frac{3}{2(t-\rho)^2} + \frac{8 + (t-\rho)^{-1/2}}{\big(4(t-\rho) - (t+\rho)^{1/2}\big)^2}.$$

Lemma 4.1. *Let $t \geq 1$, $0 \leq \rho \leq \frac{1}{2}$ and $\tau \in D(t, \rho)$. Then $\operatorname{Re}\sigma \geq \sqrt{t-\rho}$, $|\operatorname{Im}\sigma| \leq \rho/(2\sqrt{t-\rho})$, $|m(\tau)| \geq \mu(t, \rho)$ and $|(\log m)''(\tau)| \leq \Lambda(t, \rho)$.*

Proof. $\operatorname{Re}\tau \geq t - \rho > 0$ and $(\operatorname{Re}\sigma)^2 = (|\tau| + \operatorname{Re}\tau)/2 \geq \operatorname{Re}\tau$; also $|\operatorname{Im}\sigma| = |\operatorname{Im}\tau|/(2\operatorname{Re}\sigma)$. Then $|e^{4\sigma}| = e^{4\operatorname{Re}\sigma}$, $|4\sigma - 1| \geq 4\operatorname{Re}\sigma - 1 > 0$ and $|\tau| \leq t + \rho$ give the bound for $|m|$. Since $\log m = \log\pi + 4\sigma + \log(4\sigma - 1) - \frac{3}{2}\log\tau$ is analytic on $D(t, \rho)$,

$$(\log m)'' = -\tau^{-3/2} + \tfrac{3}{2}\tau^{-2} - \frac{2(4 - \frac{1}{2}\tau^{-1/2})}{(4\tau - \tau^{1/2})^2},$$

and $|\tau| \geq t - \rho$, $|4\tau - \tau^{1/2}| \geq 4|\tau| - |\tau|^{1/2} \geq 4(t-\rho) - (t+\rho)^{1/2} > 0$ give the bound Λ. □

Proposition 4.2. *Let κ_j be the curvatures of Lemma 2.3 for $s = \Phi(\sqrt{t})$. For every $t \geq 1$ and $j = 0, 1, 2, 3$,*

$$\Big|\frac{\kappa_j(t)}{2^j m(t)} - 1\Big| \leq \eta_j, \qquad (\eta_0, \eta_1, \eta_2, \eta_3) = (0.0831,\ 0.1337,\ 0.1527,\ 0.1587).$$

In particular $\kappa_0, \dots, \kappa_3 > 0$ and $T^r s > 0$ on $[1, \infty)$ for $r = 1, 2, 3, 4$. With five disks instead of four the same argument gives $|\kappa_4/(16m) - 1| \leq 0.349$, hence also $T^5 s > 0$ on $[1, \infty)$.

Proof. Fix $t \ge 1$ and the radii $\rho_j = (4-j)/8$, $j = 0, \dots, 4$, so that consecutive disks $D_j = D(t, \rho_j)$ differ by $\frac{1}{8}$ and $D_4 = \{t\}$.

Step 1: the theta tail. On D_0, Lemma 4.1 gives $\operatorname{Re}\sigma \ge a_0 = \sqrt{t - 1/2} \ge 2^{-1/2}$ and $|\operatorname{Im}\sigma| \le b_0 = 1/(4a_0) \le 2^{-3/2}$, so $4|\operatorname{Im}\sigma| \le \sqrt{2} < \pi/2$ and $D_0 \subset \Omega$. Write $\Phi = a_1(1+q)$ with $q = \sum_{n\ge 2} a_n/a_1$ and $a_1(\sigma) = 2\pi^2 e^{9\sigma - x}(1 - \frac{3}{2x})$. With $z_0 = 3/(2\pi e^{4a_0})$ and $X_0 = \pi e^{4a_0}\cos 4b_0$,

$$\Big|\frac{3}{2x}\Big| \le z_0, \qquad |q| \le \frac{1+z_0}{1-z_0}\sum_{n\ge 2} n^4 e^{-(n^2-1)X_0} =: q_0,$$

because $|a_n/a_1| = n^2|2n^2x - 3|\,|2x-3|^{-1}|e^{-(n^2-1)x}|$, $|2n^2x - 3| \le n^2(2|x| + 3)$, $|2x - 3| \ge 2|x| - 3$, $|x| \ge \pi e^{4a_0}$ and $\operatorname{Re} x \ge X_0$. Both z_0 and q_0 are below 1, so $s = \Phi(\sigma)$ has no zero on D_0 and

$$\log s(\tau) = \log(2\pi^2) + 9\sqrt{\tau} - \pi e^{4\sqrt{\tau}} + \varepsilon(\tau), \qquad \varepsilon = \log\Big(1 - \frac{3}{2x}\Big) + \log(1+q), \tag{4.1}$$

is an analytic branch with $|\varepsilon| \le \varepsilon_0 := z_0/(1-z_0) + q_0/(1-q_0)$ on D_0.

Step 2: the first curvature. Differentiating (4.1) twice gives, on D_0,

$$\kappa_0 = -(\log s)'' = m + \tfrac{9}{4}\tau^{-3/2} - \varepsilon''.$$

For $\tau \in D_1$ the disk $D(\tau, \frac{1}{8})$ lies in D_0, so Cauchy's estimate gives $|\varepsilon''(\tau)| \le 128\varepsilon_0$. Hence $\kappa_0 = m(1+\delta_0)$ on D_1 with

$$|\delta_0| \le \eta_0 := \frac{\frac{9}{4}(t-\rho_1)^{-3/2} + 128\,\varepsilon_0}{\mu(t,\rho_1)}.$$

Step 3: induction. Suppose that $\kappa_j = 2^j m(1+\delta_j)$ on D_{j+1} with $|\delta_j| \le \eta_j < 1$. Then $\log\kappa_j = j\log 2 + \log m + \log(1+\delta_j)$ with the principal logarithm, $|\log(1+\delta_j)| \le L(\eta_j) := -\log(1-\eta_j)$, and by (2.2)

$$\kappa_{j+1} = 2^{j+1}m(1+\delta_j) - (\log m)'' - \big(\log(1+\delta_j)\big)''.$$

For $\tau \in D_{j+2}$, Cauchy's estimate on $D(\tau, \frac{1}{8}) \subset D_{j+1}$ and Lemma 4.1 give $\kappa_{j+1} = 2^{j+1}m(1+\delta_{j+1})$ with

$$|\delta_{j+1}| \le \eta_{j+1} := \eta_j + \frac{\Lambda(t,\rho_{j+2}) + 128\,L(\eta_j)}{2^{j+1}\mu(t,\rho_{j+2})}.$$

Step 4: uniformity in t. As t increases, a_0 increases and b_0 decreases, so z_0, q_0 and ε_0 decrease; $(t-\rho)^{-3/2}$ and $\Lambda(t,\rho)$ decrease (the denominator of the last term of Λ has derivative $4 - \frac{1}{2}(t+\rho)^{-1/2} > 0$); and $\mu(t,\rho)$ increases, since its logarithmic derivative is at least $2(t-\rho)^{-1/2} - \frac{3}{2}(t+\rho)^{-1} > 0$. Since η_{j+1} increases with η_j and Λ and decreases with μ, every η_j is a nonincreasing function of t on $[1,\infty)$, and it suffices to evaluate the bounds at $t = 1$. There, interval arithmetic (`03_large_t.py`) gives

$$z_0 < 0.02823, \quad X_0 > 8.288, \quad q_0 < 2.7\cdot 10^{-10}, \quad \varepsilon_0 < 0.02905,$$

	$\rho_1 = \frac{3}{8}$	$\rho_2 = \frac{1}{4}$	$\rho_3 = \frac{1}{8}$	$\rho_4 = 0$
$\mu(1,\rho)$	99.53	176.96	304.37	514.57
$\Lambda(1,\rho)$		6.792	4.706	3.5

and $\eta_0 < 0.08310$, $\eta_1 < 0.13366$, $\eta_2 < 0.15261$, $\eta_3 < 0.15861$. Since $m(t) > 0$ for real $t \ge 1$, $\kappa_j(t) = 2^j m(t)(1+\delta_j(t)) > 0$, and Lemma 2.3 gives $T^r s > 0$. The statement for κ_4 follows in the same way with $\rho_j = (5-j)/10$; the script prints $\eta_4 < 0.3490$. □

5 The range $0 \le t \le 1$

Lemma 5.1 (Tilted Taylor model)**.** *Let $t_c \ge 0$, $0 < R < \pi^2/64$ and $\beta \in \mathbb{R}$. Put $\tilde{s}(h) = e^{\beta h} s(t_c + h)$, let M be an upper bound of $|\tilde{s}|$ on the circle $|h| = R$, and let C_r, p_r be as in Lemma 2.4. If $J \ge 0$, $0 < \zeta < 1$ and*

$$\min_{|h| \le \zeta R,\ t_c + h \ge 0} \sum_{j=0}^{J} [h^j]\, T^r \tilde{s} \cdot h^j \ > \ C_r \sum_{j > J} \binom{j + p_r - 1}{j} \zeta^j,$$

then $T^r s > 0$ on $[t_c - \zeta R, t_c + \zeta R] \cap [0, \infty)$.

Proof. By Lemma 2.1, $\tilde{s}$ is analytic on a neighbourhood of $|h| \le R$. Lemma 2.4 bounds the difference between $T^r \tilde{s}(h)$ and its Taylor polynomial of degree J by the right side, so $T^r \tilde{s} > 0$ on the interval, and $T^r s(t_c + h) = e^{-2^r \beta h} T^r \tilde{s}(h)$ by Lemma 2.2. □

The inputs of Lemma 5.1 are computed as follows. All arithmetic is interval arithmetic with outward rounding at 110 significant digits (at 120 in Section 6); binomial coefficients enter as exact integers, and all interval endpoints are dyadic rationals, hence exact binary numbers.

Taylor coefficients of Φ. Let $\sigma_c = \sqrt{t_c}$, $X_n = \pi n^2 e^{4\sigma_c}$. For $n \le n_{\max}$,

$$a_n(\sigma_c + y) = \pi n^2 \big(2X_n e^{4y} - 3\big) e^{5\sigma_c - X_n} \exp\big(5y - X_n(e^{4y} - 1)\big)$$

is expanded by exact power-series recursions for exp and products. The tail $\sum_{n > n_{\max}} a_n(\sigma_c + y)$ is analytic for $|y| \le 0.3$ (since $4 \cdot 0.3 < \pi/2$), and (2.1) bounds it there by an explicit sum B, evaluated term by term up to $n_{\max} + 40$ and by a geometric series beyond. Hence its k-th Taylor coefficient lies in $[-B\, 0.3^{-k}, B\, 0.3^{-k}]$, and this interval is added to the k-th coefficient ϕ_k of Φ at σ_c.

Taylor coefficients of s. For $t_c = 0$ the coefficients of s are $c_j = \phi_{2j}$ (Lemma 2.1). For $t_c > 0$ they are obtained by composing $\sum_k \phi_k y^k$ with $y(h) = \sqrt{t_c + h} - \sqrt{t_c} = \sqrt{t_c} \sum_{i \ge 1} \binom{1/2}{i} (h/t_c)^i$; since $y(0) = 0$, the coefficients $c_0, \dots, c_N$ involve only $\phi_0, \dots, \phi_N$.

Tilt and iterates. We take β to be a double-precision approximation of $-c_1/c_0 = -s'(t_c)/s(t_c)$; any real β is admissible in Lemma 5.1, and this choice removes the first-order variation of $\tilde{s}$ and with it most of the size of M. The coefficients of $\tilde{s}$ are those of $e^{\beta h}$ times those of s, and $T^r \tilde{s}$ is computed by power-series arithmetic; if $\tilde{s}$ is known through degree N, then $T^r \tilde{s}$ is known through degree $J = N - 2r$.

The bound M. The upper half of the circle $|h| = R$ is divided into 256 arcs. On an arc, interval bounds for $\operatorname{Re}\tau$ and $\operatorname{Im}\tau$, $\tau = t_c + h$, give bounds for $a = \operatorname{Re}\sqrt{\tau} = \sqrt{(|\tau| + \operatorname{Re}\tau)/2}$ and $|b| = \sqrt{(|\tau| - \operatorname{Re}\tau)/2}$; the script checks $4|b| < \pi/2$ and adds the bounds (2.1) for all n, times $e^{\beta \operatorname{Re} h}$. Complex conjugation gives the lower half.

The minimum. The minimum in Lemma 5.1 is bounded below by interval Horner evaluation on 16 equal subintervals with dyadic endpoints. For an upper bound (used in Section 6) the maximum is bounded above in the same way.

Proposition 5.2. *$T^r s > 0$ on $[0, 1]$ for $r = 1, 2, 3, 4$.*

Proof. The script `02_compact_certificate.py` applies Lemma 5.1 on 60 closed intervals with endpoints in $2^{-20}\mathbb{Z}$; neighbouring intervals share their endpoints, and the union is exactly $[0,\ 262977 \cdot 2^{-18}] \supset [0, 1]$, which the script checks in exact rational arithmetic. The first interval is $[0, 6291 \cdot 2^{-20}]$ with centre $t_c = 0$; the others have half-width about $0.18R$ (one of them halved once), with $R = 0.06$ on $[0, 0.3]$, $R = 0.05$ on $[0.3, 0.6]$ and $R = 0.04$ beyond. It uses $N = 84$ (and $N = 92$ on the two halved intervals), $4 \le n_{\max} \le 31$, and obtains $M/\tilde{s}(0) \le 2.54$ on every interval. The inequality of

Lemma 5.1 holds on every interval for $r = 1, 2, 3, 4$. Over all intervals the certified lower bounds of $T^r\tilde{s}/s(t_c)^{2^r}$ are at least

$$170.8, \qquad 7.456 \cdot 10^6, \qquad 6.013 \cdot 10^{16}, \qquad 5.165 \cdot 10^{35} \qquad (r = 1, 2, 3, 4),$$

and the majorant remainder is at most $4.6 \cdot 10^{-56}$, $6.6 \cdot 10^{-45}$, $5.3 \cdot 10^{-28}$ and 0.44 times the central value $T^r\tilde{s}(0)$, respectively. □

6 The fifth iterate

Proposition 6.1. *Let* $A = 10171 \cdot 2^{-20} = 0.0096998\ldots$ *and* $B = 10381 \cdot 2^{-20} = 0.0099000\ldots$. *Then* $T^5 s < 0$ *on* $[0, A]$ *and* $T^5 s > 0$ *on* $[B, 1]$. *At* $t = 0$ *the curvatures of Lemma 2.3 are*

$$\kappa_0(0) = 190.6866\ldots, \qquad \kappa_1(0) = 205.0616\ldots, \qquad \kappa_2(0) = 2390.551\ldots,$$
$$\kappa_3(0) = 146.8692\ldots, \qquad \kappa_4(0) = -11322.45\ldots,$$

and $T^5 s(0) = -4.81288\ldots \cdot 10^{65}$.

Proof. The script `05_depth_five.py` applies Lemma 5.1 with $r = 5$, in the form with the sign reversed on $[0, A]$: there it bounds the Taylor polynomial of $T^5\tilde{s}$ from above and adds the majorant remainder. It uses 114 closed intervals with endpoints in $2^{-20}\mathbb{Z}$, 120 significant digits, $R = 0.06$ on $[0, 0.3]$, $R = 0.05$ on $[0.3, 0.6]$ and $R = 0.04$ beyond, $N = 112$ on $[0, A]$ and $N = 120$ on $[B, 1]$, and $M/\tilde{s}(0) \le 2.54$. The 7 intervals of the first kind cover $[0, A]$, the 107 intervals of the second kind cover $[B, 1 + 29 \cdot 2^{-20}]$, and both coverings are checked in exact rational arithmetic. On $[0, A]$ the certified upper bounds of $T^5\tilde{s}/s(t_c)^{32}$ are at most $-7.8 \cdot 10^{74}$, with remainders at most $2.1 \cdot 10^{-28}$ times $|T^5\tilde{s}(0)|$. On $[B, 1]$ the certified lower bounds are at least $4.46 \cdot 10^{74}$, with remainders at most 0.0032 times $T^5\tilde{s}(0)$. Lemma 2.2 transfers both signs from $T^5\tilde{s}$ to $T^5 s$. The values at $t = 0$ are enclosures of polynomials in the coefficients $c_0, \ldots, c_{10}$ of s at 0, computed with 140 digits; in particular $T^5 s(0) < 0$. □

7 Proofs of the main results

Proof of Theorem 1.3. Part (i) on $[0, 1]$ is Proposition 5.2 and on $[1, \infty)$ is Proposition 4.2. The equivalence with strict concavity of $\log T^r s$, $r \le 3$, is Lemma 2.3. Part (ii) is Proposition 6.1 together with the last statement of Proposition 4.2, since $A > 0.00969$ and $B < 0.009901$. □

Proof of Corollary 1.4. Fix $\lambda \in \mathbb{R}$ and let $K_\lambda(u) = e^{\lambda u^2}\Phi(u)$. It is positive, even and C^∞. Each derivative $\Phi^{(i)}$ is a finite combination of terms $e^{5u}e^{-\pi n^2 e^{4u}}P(\pi n^2 e^{4u})$ with polynomials P, so $|\Phi^{(i)}(u)| \le C_i e^{(9+4i)u - \pi e^{4u}}$ for $u \ge 0$, and $x^m K_\lambda^{(i)}(x)$ is integrable and tends to 0 for all m and i. The moments and Taylor data of K_λ in Theorem 3.1 are $b_k(\lambda)$ and $\gamma_k(\lambda)$. Its root profile is $s_\lambda(t) = K_\lambda(\sqrt{t}) = e^{\lambda t}s(t)$, so $(\log s_\lambda)'' = (\log s)'' < 0$ by Theorem 1.3(i) with $r = 1$. By Lemma 2.2, $f_\lambda = Ts_\lambda = e^{2\lambda t}Ts$, so $(\log f_\lambda)'' = (\log Ts)'' < 0$ by Theorem 1.3(i) with $r = 2$. Theorem 3.1 gives $E_k(\lambda) > 0$ (and $T_k(\lambda) > 0$, which is [5, Theorem 2.4]). □

Corollary 1.5 is the case $\lambda = 0$. The next proposition records exactly where the original hypotheses of [3, Theorem 2.4] hold in the family.

Proposition 7.1. *Let* $\lambda_* = -\Phi''(0)/(2\Phi(0))$. *Then* $37.45380985900 < \lambda_* < 37.45380985901$, *and* $K_\lambda = e^{\lambda u^2}\Phi$ *is an admissible kernel in the sense of [3, Definition 2.2] if and only if* $\lambda \le \lambda_*$.

Proof. Conditions (i), (ii), (iii) and (v) of [3, Definition 2.2] hold for every λ, by the facts recalled in Section 2 and the decay bound in the proof of Corollary 1.4. For (iv), write $K_\lambda'(u) = uK_\lambda(u)(\Phi'(u)/(u\Phi(u))+2\lambda)$ and $\Phi'(u)/(u\Phi(u)) = 2(\log s)'(u^2)$. This function is strictly decreasing in $u > 0$, because $(\log s)'' < 0$, and tends to $2s'(0)/s(0) = \Phi''(0)/\Phi(0) = -2\lambda_*$ as $u \to 0^+$. Hence $K_\lambda' < 0$ on $(0,\infty)$ if $\lambda \le \lambda_*$, while for $\lambda > \lambda_*$ we have $K_\lambda' > 0$ near 0. The enclosure of λ_* is computed by `04_admissibility_threshold.py` from the enclosures $\Phi(0) = 0.446696900467123\ldots$ and $\Phi''(0) = -33.4610015494065\ldots$, both certified to 60 digits. □

Remark 7.2 (A second route to Corollary 1.4, with $\ge$ in place of $>$ for $\lambda > \lambda_*$). For $\lambda \le \lambda_*$ the original [3, Theorem 2.4(b)] applies to K_λ by Proposition 7.1, with the hypotheses verified as in the proof above. For $\lambda \ge \frac{1}{2}$, de Bruijn's theorem [6] shows that H_λ has only real zeros; then $F_\lambda(x) = \sum_k \gamma_k(\lambda)x^k/k!$, which satisfies $F_\lambda(-z^2) = H_\lambda(z)$, has order $\frac{1}{2}$ and only real nonpositive zeros, so it is of type I in the Laguerre–Pólya class, and the theorem of Craven and Csordas [1] (see [3, Corollary 1.3]) gives $E_k(\lambda) \ge 0$. Since $\lambda_* > \frac{1}{2}$, the two ranges cover $\mathbb{R}$. Both routes rest on the case $r = 2$ of Theorem 1.3.

Remark 7.3 (Depth five along the family). By Lemma 2.2, $T^r(e^{\lambda t}s) = e^{2^r\lambda t}T^r s$ for every λ. Hence every kernel K_λ has positive root iterates of depths 1 to 4 on $[0,\infty)$ and a fifth root iterate that is negative on $[0, 0.00969]$. For $\frac{1}{2} \le \lambda \le \lambda_*$ the kernel K_λ is admissible and H_λ has only real zeros. So, within the de Bruijn–Newman family, positivity of the fifth root iterate is not implied by the reality of the zeros, while the first four iterates are positive for every λ, on both sides of Λ. For general admissible kernels, the author showed that positivity of all root iterates does not force real zeros [14], and constructed admissible kernels that satisfy both concavity hypotheses of Theorem 3.1 while their Fourier transforms have nonreal zeros [15]; for those kernels Theorem 3.1 gives strict Turán and double Turán inequalities, as Corollary 1.6 does for $\lambda < 0$. Theorem 1.3 concerns the Riemann kernel itself.

Remark 7.4 (Numerical cross-checks). The script `06_independent_checks.py` recomputes, in floating point and with code independent of the certificate, the values $\kappa_j(0)$ from the integer-polynomial form $\Phi^{(k)}(u) = \sum_n \pi n^2 e^{5u}e^{-x_n}P_k(x_n)$, $P_0 = 2x - 3$, $P_{k+1} = 4xP_k' + (5 - 4x)P_k$; scans $\kappa_0, \ldots, \kappa_4$ on the grid $t = 0.01, 0.02, \ldots, 3$, where all five are positive; and computes $T_k(\lambda)$ and $E_k(\lambda)$, $k \le 20$, by quadrature for $\lambda \in \{-3, -1, -0.1, 0, 0.5, 10, 40, 100\}$, including the non-admissible values 40 and 100. The script `07_sign_change_scan.py` evaluates κ_4 on $[0.0095, 0.0101]$ in steps of $5\cdot10^{-5}$ and locates its sign change, hence that of $T^5 s$, at $t \approx 0.00983$. All values agree with the theorems.

Statements and Declarations

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Data and code availability. The verification scripts cited in the text are included with this article, each with its output:

- `dbn_lib.py` (shared routines);
- `01_cd_identities.py`, `02_compact_certificate.py`, `03_large_t.py`;
- `04_admissibility_threshold.py`, `05_depth_five.py`;

- `06_independent_checks.py`, `07_sign_change_scan.py`.

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References

[1] T. Craven and G. Csordas. Jensen polynomials and the Turán and Laguerre inequalities. *Pacific J. Math.* 136 (1989), 241–260. https://doi.org/10.2140/pjm.1989.136.241

[2] G. Csordas. Fourier transforms of positive definite kernels and the Riemann ξ-function. *Comput. Methods Funct. Theory* 15 (2015), 373–391. https://doi.org/10.1007/s40315-014-0105-8; arXiv:1309.0055v2.

[3] G. Csordas and D. K. Dimitrov. Conjectures and theorems in the theory of entire functions. *Numer. Algorithms* 25 (2000), 109–122. https://doi.org/10.1023/A:1016604906346

[4] G. Csordas, T. S. Norfolk and R. S. Varga. The Riemann hypothesis and the Turán inequalities. *Trans. Amer. Math. Soc.* 296 (1986), 521–541. https://doi.org/10.1090/S0002-9947-1986-0846596-4

[5] G. Csordas and R. S. Varga. Moment inequalities and the Riemann hypothesis. *Constr. Approx.* 4 (1988), 175–198. https://doi.org/10.1007/BF02075457

[6] N. G. de Bruijn. The roots of trigonometric integrals. *Duke Math. J.* 17 (1950), 197–226. https://doi.org/10.1215/S0012-7094-50-01720-0

[7] O. M. Katkova. Multiple positivity and the Riemann zeta-function. *Comput. Methods Funct. Theory* 7 (2007), 13–31. https://doi.org/10.1007/BF03321628; arXiv:math/0505174.

[8] Y. Liu and J. Mao. Infinite log-concavity of the Taylor coefficients of the Riemann xi-function. Preprint, arXiv:2610.04972v1, 2026. https://arxiv.org/abs/2610.04972

[9] W. Michalowski. On the Pólya frequency order of the de Bruijn–Newman kernel: certified failure at order five. Preprint, arXiv:2602.20313v2, 2026. https://arxiv.org/abs/2602.20313

[10] The mpmath development team. *mpmath: a Python library for arbitrary-precision floating-point arithmetic*, version 1.3.0, 2023. https://mpmath.org

[11] C. M. Newman. Fourier transforms with only real zeros. *Proc. Amer. Math. Soc.* 61 (1976), 245–251. https://doi.org/10.1090/S0002-9939-1976-0434982-5

[12] M. Planat and P. Solé. Second-level concavity of the Riemann Ξ kernel. Preprint, arXiv:2608.19160v1, 2026. https://arxiv.org/abs/2608.19160

[13] B. Rodgers and T. Tao. The de Bruijn–Newman constant is non-negative. *Forum Math. Pi* 8 (2020), e6. https://doi.org/10.1017/fmp.2020.6

[14] S. Schubr. *All-depth root-Laguerre positivity with nonreal Fourier zeros*, edition 3. Luminary Research Archive, 2026. https://doi.org/10.5281/zenodo.24473019

[15] S. Schubr. *Admissible kernel lifts with positive endpoint forms and nonreal Fourier zeros*, edition 2. Luminary Research Archive, 2026. https://doi.org/10.5281/zenodo.24472933

[16] S. Schubr. *Global finite-depth stability and nonreal zeros of admissible Fourier transforms*, edition 2. Luminary Research Archive, 2026. https://doi.org/10.5281/zenodo.24472935