The Riemann Hypothesis through Arithmetic Weil Positivity

Scott Schubr
Luminary Oddities

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Abstract

We develop a quantitative covariance formulation of the Riemann hypothesis through arithmetic Weil positivity. In the original odd weighted realization, RH is equivalent to $\liminf_{M\to\infty}\sqrt{M}(\gamma_M-1)_+=0$, where γ_M is the complete covariance norm at ridge $1/M$ after any sequence of finite genuine-null restrictions. We prove that the complete covariance norm differs from one by $O_{b,\varepsilon}(M^{-1/2+\varepsilon})$ for every $\varepsilon>0$ at each prescribed ridge M^{-b}, $b>0$ fixed, uniformly over all finite genuine-null restrictions. The estimate includes every negative-source frequency and all mixed terms. Its proof combines a complete arithmetic source majorant, theta-tail bounds, the Vinogradov–Korobov prime-number error and a uniform actual-inverse construction. Genuine derivative nulls then provide complete source coordinates and regularized signed certificates. An explicit factorial bound in the original norm controls their growing physical lift cost, while small-set inverse estimates retain the full frequency exterior. The critical covariance excess $o(M^{-1/2})$ yields vanishing original-norm allowances and hence RH, with complete signed certificates formulated in the original source metric.

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1 RH through arithmetic Weil positivity

The Weil criterion expresses the Riemann hypothesis as positivity of a complete quadratic form. We study that criterion in a fixed odd reciprocal-theta realization, retaining its original norm and minimal closed form domain. The purpose is to make the approach to positivity quantitative: a complete signed lower bound with an allowance tending to zero establishes RH, and genuine operator-null projections preserve this implication exactly.

The paper connects this endpoint to the complete arithmetic covariance. Positive averaging of the prime cutoff gives

$$H = P_M - N_M + W_M, \qquad N_M = E_M E_M^*, \qquad \|W_M\| \le d_{M,1},$$

where P_M is the actual positive receiver and E_M contains the entire negative observation. For a finite genuine-null subspace K and ridge $\kappa > 0$, write

$$\gamma_M(K;\kappa) = \|E_{M,K}^*(P_{M,K} + \kappa I)^{-1} E_{M,K}\|.$$

The subscript denotes same-form restriction to $K^\perp$ and exact projection of the source. Section 2 specifies the original realization; Section 4 fixes the source metric.

The RH condition and the quantitative theorem. For any sequence of finite genuine-null spaces K_M, the original positivity criterion takes the exact form

$$\mathrm{RH} \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M(K_M; M^{-1}) - 1)_+ = 0. \tag{1}$$

Corollary 6.2 proves this characterization using the original-domain Weil criterion. Our principal unconditional estimate is

$$\sup_K |\gamma_M(K; M^{-b}) - 1| \le \frac{C_b\sqrt{\log M}\exp(48\sqrt{b\log M})}{\sqrt{M}}, \qquad b > 0 \text{ fixed}, \quad M \ge M_b. \tag{2}$$

The supremum includes every finite genuine-null choice, including moving and growing ones. Theorem 6.1 proves this complete-source bound. Equivalently, for each $\varepsilon > 0$ its right-hand side is $O_{b,\varepsilon}(M^{-1/2+\varepsilon})$. Together, Theorem 6.1 and Corollary 6.2 give a complete covariance estimate and an exact characterization of RH in the original weighted realization.

The proof retains the actual arithmetic positive operator throughout. A complete hard-cutoff source majorizes the possible negative sign of H. The classical Vinogradov–Korobov prime-number error admits a growing comparison band, and an exact polar Green identity supplies a uniform preimage estimate on its entire moving coefficient unit ball. Complete theta tails and the ridge subtraction pay the return to the original form. No finite frequency sampling substitutes for the full covariance.

Complete signed certificates and the RH criterion. Genuine derivative nulls generate a dense system in the complete negative source at each fixed M (Theorem 7.2). Their exact signed identity cancels the artificial ridge charge before estimation. Theorems 7.3 and 8.1 turn this identity into full trace and regularized approximation certificates, including every mixed block. The explicit factorial norm bound in Proposition 8.2 controls the physical lift of growing null families. Uniform small-set inverse estimates then allow collars around poorly conditioned frequencies while retaining the entire exterior. These results give quantitative signed criteria for the covariance condition (1).

The positive-majorant and raw-exterior obstructions in Section 10 explain why the signed structure matters. These are obstructions to particular sufficient estimates for the signed criteria. The finite-coordinate implementations and calibrated examples are retained in the appendices so that the principal argument remains the passage from arithmetic covariance to Weil positivity.

Relation to earlier work. The original weighted realization, null compression and certified lower bounds are developed in [1, 2, 3]; the complete hard-cutoff source estimates also enter [24]. The present whole-source rate strengthens the origin estimate and complementary criterion by placing an unconditional bound on the full covariance. Mellin smoothing, inverse preimages, Schur elimination and regularization are classical tools [20, 18, 12, 19]. Related exact-form constructions use truncated preimages and theta-tail payments [22, Section 7], and collective exact-Gram covariance with pole compensation [23, Sections 6–7]. Here their quantitative use retains one original domain, true source metrics and simultaneous control of finite genuine-null restrictions. The classical explicit formula and Weil criterion remain the number-theoretic foundations [5, 16, 6].

Organization. Sections 2–3 establish the original form and arithmetic realization. Sections 4–6 prove the actual inverse estimates and complete covariance theorem. Sections 7–9 construct the signed null-source certificates and exterior controls. Section 10 treats their feasibility. The appendices give the odd Weil equivalence, reference calibrations, complete hard-cutoff and finite-source reductions, conditioning and reproducible numerical applications.

2 The positivity endpoint and the finite reduction

The Riemann hypothesis has a positivity endpoint in a fixed odd Weil realization. We retain this realization throughout the complete covariance estimates and the constructive signed certificates.

We first specify the original norm and domain; the later reductions return to precisely these objects.

Use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi}\,\mathrm{d}u$ and Plancherel measure $\mathrm{d}\xi/(2\pi)$. For $u \geq 0$ define

$$\psi(u) = e^{u/2}\sum_{n\geq 1}(4r_n^2 - 6r_n)e^{-r_n}, \quad r_n = \pi n^2 e^{2u}, \qquad s^2 = \psi/\cosh(u/2), \tag{3}$$

with the even continuation given by theta inversion. Let $\mathcal{K}$ be the odd subspace of $L^2(\mathbb{R})$ satisfying

$$\langle h, s\sinh(u/2)\rangle = 0.$$

Write $\Psi = \Gamma'/\Gamma$ and

$$a(\xi) = \Re\Psi(1/4 + i\xi/2) - \Psi(1/4) \geq 0, \qquad m_0 = \Psi(1/4) - \log\pi.$$

The domain V is the completion of the smooth compact odd polar-free core for

$$\|h\|_V^2 = \|h\|_2^2 + \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}. \tag{4}$$

The complete form on this domain is the closure of

$$\begin{aligned} q_H[h] = &\int_{\mathbb{R}}(m_0 + a)|\widehat{sh}|^2\frac{\mathrm{d}\xi}{2\pi} \\ &- \sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\langle sh, (S_{\log n} + S_{-\log n})sh\rangle, \qquad S_b f(u) = f(u+b). \end{aligned} \tag{5}$$

Inner products are conjugate-linear first. For $f = sh$ the original physical norm is

$$\|h\|_2^2 = \int_{\mathbb{R}}|f(u)|^2\frac{\cosh(u/2)}{\psi(u)}\,\mathrm{d}u.$$

These are the conventions of the weighted-Weil and complementary-receiver predecessors [1, 2, 3].

Lemma 2.1 (Complete elementary envelopes)**.** *For $r = \pi e^{2|u|}$,*

$$2r^2e^{-r} \leq s(u)^2 \leq 169e^{4|u|-\pi e^{2|u|}}. \tag{6}$$

The arithmetic sum in (5) *converges in original operator norm. Moreover $0 \leq a(\xi) \leq 17\xi^2$.*

Proof. Each theta atom is positive. The first atom, $4r^2 - 6r \geq 2r^2$, and $e^{|u|/2}/\cosh(u/2) \geq 1$ give the lower bound. For the upper, $e^{|u|/2}/\cosh(u/2) \leq 2$ and $\sum_{n\geq 1} n^4e^{-(n^2-1)r} < 2$: the consecutive ratios are bounded by $16e^{-3r} < 1/2$. Hence $s^2 < 16r^2e^{-r} < 169e^{4|u|-r}$. For $b = \log n$, put $x = |u| + |u+b| \geq b$. The upper envelope gives

$$s(u)s(u+b) \leq 169e^{2x-\pi e^x} \leq 169n^2e^{-\pi n}.$$

Thus the norm of the full omitted arithmetic sum is bounded by

$$\delta_M = 338\sum_{n>M} n^{3/2}\log n\, e^{-\pi n} \longrightarrow 0. \tag{7}$$

This sums all integers and therefore every omitted prime power. In particular, for every integer $M \geq 2$,

$$\delta_M < 451(M+1)^2e^{-\pi(M+1)}. \tag{8}$$

Indeed $\log n \le \sqrt{n}$, and the consecutive ratios of $n^2 e^{-\pi n}$ are at most $4e^{-\pi} < 1/4$. Summing the resulting geometric majorant gives the coefficient $1352/3 < 451$. Boundedness of s and this convergence make (4) the original semibounded form topology. Finally the digamma series is

$$a(\xi) = \sum_{n\ge 0} \frac{(\xi/2)^2}{(n+1/4)((n+1/4)^2 + (\xi/2)^2)}.$$

The $n = 0$ coefficient is at most $16\xi^2$, and the others total at most $\xi^2 \sum_{n\ge 1} n^{-3}/4 \le 3\xi^2/8$. □

Theorem 2.2 (Cofinal complete certificates)**.** *For each j let Π_j be a finite orthogonal projection onto genuine null vectors of the complete associated operator H, and put $P_j = I - \Pi_j$. Suppose a complete signed certificate on the original domain gives*

$$q_H[P_j h] \ge -\epsilon_j \|P_j h\|_2^2 \quad (h \in V), \qquad 0 \le \epsilon_j \longrightarrow 0 \tag{9}$$

along an unbounded sequence of indices. Then $q_H \ge 0$ on V, and the Riemann hypothesis holds. Each Π_j may use its own finite genuine-null space.

Proof. For a genuine associated-operator null vector $z \in \operatorname{Dom} H$, $q_H[z, h] = 0$ for every $h \in V$. Consequently finite null projections preserve V, $q_H[P_j h] = q_H[h]$ and $\|P_j h\|_2 \le \|h\|_2$. For each fixed h, (9) gives $q_H[h] \ge -\epsilon_j \|h\|_2^2$; taking the limit proves positivity. Lemma A.1 supplies the endpoint implication. □

The following arithmetic realization and covariance estimates return to precisely this norm and domain. The finite-coordinate constructions in the appendices provide alternative implementations of the same endpoint. Related compact-window reductions retain a Fourier envelope and leading–tail coupling [7], or source-exact positive directions in a block–Schur certificate [8]. Our main estimates act on the complete odd weighted realization; the compact-window and whole-line recipients are distinct.

3 Positive arithmetic smoothing and a cofinal schedule

Write M_s for multiplication by the complete theta weight s in (3). All identities in this section begin on the original polar compact core and extend through its minimal closure V. Put

$$m_R(\xi) = m_0 + a(\xi) - 2\sum_{n\le R} \Lambda(n) n^{-1/2} \cos(\xi \log n), \quad p_R = (m_R)_+, \quad n_R = (-m_R)_+.$$

The cutoff is real, with $W_R = W_{\lfloor R \rfloor}$. For integers $M \ge 2$, $k \ge 1$, average $R = Me^v$ with density $v^{k-1}e^{-v}/(k-1)!$. The survival weights are

$$w_{M,k}(n) = \begin{cases} 1, & n \le M, \\ \dfrac{M}{n} \displaystyle\sum_{j=0}^{k-1} \frac{\log(n/M)^j}{j!}, & n > M. \end{cases} \tag{10}$$

Let $m_{M,k} = m_0 + a - 2\Re S_{M,k}$, where $S_{M,k}(\xi) = \sum_n \Lambda(n) n^{-1/2+i\xi} w_{M,k}(n)$, and take $p_{M,k} = (m_{M,k})_+$, $n_{M,k} = (-m_{M,k})_+$ *after* averaging. The series is absolutely convergent at fixed M, k.

Theorem 3.1 (Exact smoothed realization and complete remainder)**.** *On the same original domain, and after any finite genuine-null projection,*

$$H = P_{M,k} - N_{M,k} + W_{M,k}, \qquad \|W_{M,k}\| \le d_{M,k} := \frac{451(M+1)e^{-\pi(M+1)}}{M^{k-1}(1-e^{-\pi})^k}. \tag{11}$$

The bound holds for every integer $M \ge 2$, $k \ge 1$. Moreover

$$\|N_{M,k}\| \le B_{M,k} := 9[6 + 8(\log 2)\sqrt{M}\,2^k]. \tag{12}$$

The separately averaged positive and negative receivers have an exact common positive term $J_{M,k}$ such that

$$P_{M,k} = \mathbb{E}P_R - J_{M,k}, \qquad N_{M,k} = \mathbb{E}N_R - J_{M,k}.$$

Proof. Integration of the cutoff indicator gives (10). The elementary estimate $\sum_{n\le R}\Lambda(n)/\sqrt{n} \le 4(\log 2)\sqrt{R}$ is proved in Appendix L. Since $\mathbb{E}e^{v/2} = 2^k$, absolute Fubini applies and gives (12), using $|m_0| < 6$, $a \ge 0$, $\|s\|_\infty < 3$. Every arithmetic perturbation is bounded at fixed M, k; its positive symbol differs from a by a bounded multiplier. The original minimal form domain is therefore unchanged.

Set $q = e^{-\pi}$ and $j = M + l$. The probability of the floor-cutoff cell is

$$\mu_j \le \frac{[(l+1)/M]^{k-1}}{(k-1)!}\frac{M}{j(j+1)}.$$

Indeed $v \le \log((j+1)/M) \le (l+1)/M$ in that cell. Multiply by the complete bound $451(j+1)^2q^{j+1}$ from (8). As $M(j+1)/j \le M+1$ and $(l+1)^{k-1} \le (k-1)!\binom{l+k-1}{k-1}$, summing the binomial generating series proves (11). The remainder is an average of bounded original-Hilbert operators. This extends the core identity through the paired original closure.

Pointwise put $A = \mathbb{E}p_R$, $B = \mathbb{E}n_R$. Then $A - B = m_{M,k}$ and $j = \min(A, B) = A - (m_{M,k})_+ = B - (-m_{M,k})_+ \ge 0$. The bounded operator $J_{M,k} = QM_sj(D)M_sQ$ gives both identities. The positive average is interpreted on V; its difference from the bounded negative average is the original signed expression. □

Corollary 3.2 (Fixed-seed certificate criterion)**.** *Fix an integer $M \ge 2$, $\kappa_k > 0$ and $\eta_k, \tau_k \ge 0$. On the original finite genuine-null complement suppose*

$$0 \preceq N_{M,k} - N_{M,k}^{\mathrm{fin}} \preceq \tau_k I, \qquad N_{M,k}^{\mathrm{fin}} \preceq (1+\eta_k)(P_{M,k} + \kappa_k I).$$

Then on all original V,

$$q_H[h] \ge -\left[d_{M,k} + \kappa_k + \frac{\tau_k + \eta_k B_{M,k}}{1+\eta_k}\right]\|h\|^2. \tag{13}$$

In particular the complete inequalities with $\kappa_k = \tau_k = 2^{-k}$ and $\eta_k = 4^{-k}$ imply RH.

Proof. Rearrange $N \preceq (1+\eta)(P+\kappa I)+\tau I$ using the bounded N: $P - N \succeq -[\kappa + (\tau + \eta\|N\|)/(1+\eta)]I$. Add the complete remainder and use genuine-null transport as in Theorem 2.2. The displayed schedule tends to zero by (11)–(12). □

3.1 An exact full-symbol evaluator

For $F(s) = -\zeta'(s)/\zeta(s)$ and $F_{>M}(s) = F(s) - \sum_{n\le M} \Lambda(n)n^{-s}$, absolute differentiation of the Euler series gives

$$S_{M,k}(\xi) = \sum_{n\le M} \Lambda(n)n^{-1/2+i\xi} + M\sum_{j=0}^{k-1} \frac{(-\partial_s - \log M)^j F_{>M}(s)}{j!}\Big|_{s=3/2-i\xi}. \tag{14}$$

Thus the full arithmetic symbol is evaluable from a finite head and k derivatives in the absolutely convergent half-plane. A Taylor expansion $F_{>M}(s+x) = \sum_l a_l x^l$ makes its jth summand $\sum_{l=0}^{j}(-1)^l a_l(-\log M)^{j-l}/(j-l)!$. The paired operator remainder and the scalar evaluation error are separate quantities; both have explicit roles in (13). For example $d_{16,1} < 5.122\cdot 10^{-20}$, $d_{16,2} < 3.346\cdot 10^{-21}$ and $d_{16,16} < 8.618\cdot 10^{-38}$.

For fixed k the prime continuum has amplitude

$$\frac{e^{i\xi\log M}}{(1/2+i\xi)(1/2-i\xi)^k},$$

whose negative phase mean is $2^{k+2}\pi^{-1}(1+4\xi^2)^{-(k+1)/2}$. This fixed-$k$ continuum motivates the local reference for $k=1$ below. The all-k remainder (11) and the fixed-k mean have their stated, distinct parameter regimes.

4 Complete sources and the actual inverse construction

The arithmetic realization of Section 3 is now fixed at averaging order one. We retain the complete negative source and construct preimages for its actual positive receiver. Throughout the main covariance arguments the original odd polar space is denoted by $\mathcal{X} = \mathcal{K}$, with minimal form domain V.

$$H = P_M - N_M + W_M, \qquad N_M = E_M E_M^*, \qquad \|W_M\| \le \delta_M := d_{M,1},$$
$$R_M(\kappa) = (P_M + \kappa I)^{-1}, \qquad G_M(\kappa) = E_M^* R_M(\kappa) E_M.$$

Here $P_M = P_{M,1}$ and $N_M = N_{M,1}$ in the main argument; the appendices explicitly identify their separate hard-cutoff notation. The source space is $\mathcal{Y}_M = L^2(S_M, d\xi/\pi)$, $S_M = \{\xi > 0 : n_{M,1}(\xi) > 0\}$, with

$$E_M^* h(\xi) = \sqrt{n_{M,1}(\xi)}\int_{\mathbb{R}} s(u)h(u)\sin(\xi u)\,du.$$

For a finite genuine-null subspace K, $P_{M,K}$ is the same-form restriction to $K^\perp$ and $E_{M,K}$ its exactly projected source. All such inverses use the original norm. We write $\gamma_M(K;\kappa) = \|E_{M,K}^*(P_{M,K} + \kappa I)^{-1}E_{M,K}\|$.

An equivalent weighted-source convention is convenient for profiles: for $I \subset [0,\infty)$ and $\phi \in L^2(I, n_{M,1}\,d\xi/\pi)$,

$$g_\phi(u) = \int_I n_{M,1}(\xi)\phi(\xi)\sin(\xi u)\frac{d\xi}{\pi}, \quad E_I\phi = Q(sg_\phi), \quad D_\phi = \int_I n_{M,1}(\xi)|\phi(\xi)|^2\frac{d\xi}{\pi}.$$

Multiplication by $\sqrt{n_{M,1}}$ identifies this convention isometrically with the unweighted source on its positive support. Thus neither notation changes a covariance cap or omits mixed source terms.

Set $L_0 = 1 - 4\partial_u^2$, $B(D) = 8L_0^{-1}$ and

$$\mathcal{C}_Q = \frac{8}{\pi} Q M_s L_0^{-1} M_s Q, \qquad \|f\|_{1,4}^2 = \int_{\mathbb{R}} (|f|^2 + 4|f'|^2).$$

The inverse-form interpretation and explicit numerical reference bounds are proved in Appendices B–C. The following exact polar cancellation is the ingredient needed for the actual arithmetic inverse.

Lemma 4.1 (Exact polar Green cancellation)**.** *On the original odd polar space let* $B(D) = 8(1 + 4D^2)^{-1}$ *and* $K_a = M_s B(D)\cos(aD)M_s$. *For every* $a \geq 0$,

$$\|Q_b K_a Q_b\|_{\mathrm{HS}} \leq 12 e^{2a} e^{-\pi(e^a - 1)}. \tag{15}$$

Consequently, for either sign of the cosine and any further original orthogonal constraint projection Q *inside the polar space,*

$$\|Q M_s[B(D)(\pm\cos(UD))_+]M_s Q - \mathcal{C}_Q\|_{\mathrm{HS}} \leq 12 M^2 e^{-\pi(M-1)}. \tag{16}$$

Proof. The full kernel of K_a is $s(u)s(v)[e^{-|u-v-a|/2} + e^{-|u-v+a|/2}]$. Subtract the bounded rank-two kernel $2e^{-a/2}s(u)s(v)\cosh((u-v)/2)$ on the whole plane. Its even component vanishes on the odd space; its odd component is $-2e^{-a/2}|b\rangle\langle b|$ and is killed by Q_b. The exact remainder is

$$-2s(u)s(v)\sinh((|u-v| - a)/2)1_{|u-v|\geq a}.$$

Put $A = e^a$. On its support $r_u + r_v \geq 2\pi A$ by AM–GM. Its square is at most $A^{-1}s(u)^2 s(v)^2 e^{|u|+|v|}$. Split $e^{-(r_u+r_v)} \leq e^{-2\pi(A-1)}e^{-(r_u+r_v)/A}$. After enlarging the remaining integration to the whole plane, each factor is bounded by

$$\int_{\mathbb{R}} 16 r^2 e^{-r/A} e^{|u|} du \leq \frac{16}{\sqrt{\pi}} A^{5/2}\Gamma(5/2) = 12A^{5/2}.$$

Taking square roots proves (15), including every atom and both infinite arms. The nonconstant real cosine coefficients of $\cos_+$ have absolute sum $1/2 + 1/\pi < 1$. The right side of (15) decreases with $A \geq 1$; apply it to each harmonic $a = nU$. The constant coefficient is $1/\pi$, giving (16). Further original projections contract the bound. □

5 An actual inverse estimate

The exact polar phase cancellation also pays a positive lower of the *actual* averaged arithmetic receiver. This gives a whole-profile receiving theorem in the operator used by the signed criterion. Write $\overline{P}_M = P_{M,1}$, $\overline{n}_M = n_{M,1}$, $U = \log M$. For $0 < \Omega \leq 1$, let E_Ω be the complete source map of Section 4 on $[0, \Omega]$, with the same true profile metric.

Theorem 5.1 (Uniform actual-positive origin bound)**.** *There are constants* $C > 0$, M_0 *such that for every integer* $M \geq M_0$, *every* $0 < \Omega \leq 1$ *and every finite exact constraint space inside the original odd polar space,*

$$E_\Omega^*(\overline{P}_{M,Q} + M^{-1}I)^{-1}E_\Omega \preceq C\Omega^3(1 + \log\log M)^3 I. \tag{17}$$

Here $\overline{P}_{M,Q}$ *is the closed-form restriction of the same actual positive operator and the source uses its exact orthogonal projection. In particular, for* $\Omega = 1/\log M$, *the entire origin cap is* $O((\log\log M/\log M)^3)$ *and tends to zero. Every schedule* $\Omega = o(1/\log\log M)$ *gives a vanishing actual cap. For any fixed* $a > 0$ *the same conclusion holds at ridge* M^{-a}, *with constants depending on* a.

Proof. First use only the original polar constraint Q_b. Normalize one arbitrary complex profile by $D_\phi = 1$, and put $A_\Omega = 16\sqrt{M}\Omega^3/(3\pi)$. Common-profile Cauchy–Schwarz gives

$$|g(u)| \le \sqrt{A_\Omega}|u|, \qquad |g^{(j)}(u)| \le \sqrt{A_\Omega}\Omega^{j-1} \quad (j \ge 1). \tag{18}$$

All constants below are independent of this profile and Ω.

A phase-safe actual positive lower. For $E_\Lambda(t) = \Psi_\Lambda(t) - t$ define the complete Abel error

$$\overline{e}_M = \frac{3}{\sqrt{M}} + \frac{1}{\sqrt{M}}\int_1^M |E_\Lambda(t)|t^{-3/2}dt + \sqrt{M}\int_M^\infty |E_\Lambda(t)|t^{-5/2}dt.$$

Split the weighted prime integral at M and integrate by parts; the common boundary cancels because $\min(1, M/t)$ is continuous. The derivative moduli are at most $(1/2 + F)t^{-3/2}$ and $(3/2 + F)Mt^{-5/2}$. Together with the lower endpoint this gives

$$\sup_{|\xi|\le F}\left|\frac{S_{M,1}(\xi)}{\sqrt{M}} - \frac{e^{iU\xi}}{1/4+\xi^2}\right| \le (2+F)\overline{e}_M.$$

The unconditional quantitative PNT [25, Theorem 1.1] implies $|E_\Lambda(t)| \le C_N t/(\log t)^N$ for every fixed N. Splitting the first integral at $\sqrt{M}$ gives $\overline{e}_M = O_N(U^{-N})$; the second normalized integral has mass2. Take $F = U^5$ and $N = 20$. With $B(\xi) = 8/(1+4\xi^2)$,

$$\frac{m_{M,1}(\xi)}{\sqrt{M}B(\xi)} \ge -\cos(U\xi) - \varepsilon_M, \quad \varepsilon_M = \frac{3(1+4F^2)}{4\sqrt{M}} + \frac{(1+4F^2)(2+F)}{4}\overline{e}_M \longrightarrow 0.$$

We used $a \ge 0$ and $m_0 \ge -6$. Eventually $\varepsilon_M \le 1/2$. The scalar positive-part inequality therefore gives the closed-form order $\overline{P}_M \succeq A_F$, where

$$A_F = \sqrt{M}Q_b M_s B(D) w(UD) 1_{|D|\le F} M_s Q_b, \quad w(t) = (-\cos t - 1/2)_+.$$

This is a bounded positive operator. Its phase mean is $\mu = (\sqrt{3} - \pi/3)/(2\pi) > 0$.

A compact smooth preimage. Choose a fixed Gevrey-2 step χ, equal to one to the left of0 and zero to the right of1, with $|\chi^{(j)}| \le C_0 C_1^j (j!)^2$. Such a step is obtained from $\eta(t) = e^{-1/t}$ for $t > 0$ and0 otherwise, by $\chi(t) = \eta(1-t)/(\eta(t) + \eta(1-t))$. Cauchy's estimate on a disk of radius ct bounds derivatives of η by $j!(C/t)^j e^{-c'/t}$; maximization in t gives $C^j(j!)^2$. The denominator is bounded below near each endpoint, and the reciprocal Leibniz recurrence preserves this bound; on the remaining compact middle use a uniform analytic disk.

Put $R = \frac{1}{2}\log(2U/\pi)$ and $f(u) = \chi(|u| - R)g(u)$. For large U, $R \ge 1$, the cutoff is constant near zero and is smooth and even. Its transition has fixed physical width. Leibniz and (18), with $\Omega \le 1$, give

$$\|f\|_{1,4}^2 \le C A_\Omega (R+1)^3, \qquad \|f^{(j)}\|_2 \le C\sqrt{A_\Omega}(R+1)^{3/2} C_1^j (j!)^2. \tag{19}$$

Indeed the undifferentiated g is bounded by $\sqrt{A_\Omega}(R+1)$, each differentiated g by $\sqrt{A_\Omega}$, and the factorial ratios in the Leibniz sum are absorbed by one fixed exponential constant. Define

$$Y = \frac{L_0 f}{8\mu\sqrt{M}\, s}.$$

It is compact smooth odd and exactly polar legal: $\int \sinh(u/2) L_0 f = 0$ by two integrations by parts. On its support $r \le 2e^2 U$; the original lower theta bound gives

$$\|Y\|^2 \le C M^{2e^2-1} A_\Omega (R+1)^3. \tag{20}$$

Complete residual and energy. Let A_∞ omit the frequency truncation in A_F. The cosine coefficients of w are absolutely summable, because its first derivative has bounded variation. Lemma 4.1 then gives

$$A_\infty = \mu\sqrt{M} Q_b M_s B(D) M_s Q_b + H_M, \qquad \|H_M\| \le h_M = C M^{5/2} e^{-\pi(M-1)}.$$

Since $B(D)L_0 = 8$ and $Q_b Y = Y$, its mean acts on Y as $Q_b(sf)$, and the exact truncation action is

$$(A_\infty - A_F)Y = \mu^{-1} Q_b M_s w(UD) 1_{|D|>F} f.$$

Thus the complete residual $e - A_F Y$ is the sum of $Q_b[s(g-f)]$, $-H_M Y$ and this Fourier exterior. All mixed terms are paid by the squared triangle bound with factor3. The full two-arm theta estimate gives

$$\|Q_b[s(g-f)]\|^2 \le C A_\Omega \int_{2U}^{\infty} r e^{-r} (\log r)^2 dr \le C A_\Omega U e^{-2U} (\log U)^2.$$

For $j = \lceil U \rceil$, (19) gives

$$\|1_{|D|>F} f\| \le F^{-j} \|f^{(j)}\| \le C\sqrt{A_\Omega} (R+1)^{3/2} e^{-3U \log U + O(U)}.$$

Consequently $M\|e - A_F Y\|^2$ is Ω^3 times a negligible term; the polynomial trial cost (20) is dominated by h_M. Also $A_F \preceq A_\infty$ gives

$$\langle Y, A_F Y \rangle \le \frac{\langle L_0 f, f \rangle}{8\mu\sqrt{M}} + h_M \|Y\|^2 \le C\Omega^3 (R+1)^3 + \text{smaller terms}.$$

For complex f, integration by parts gives $\langle L_0 f, f \rangle = \|f\|_{1,4}^2$.

For any bounded positive A and $\kappa > 0$, least squares gives

$$\langle e, (A+\kappa)^{-1} e \rangle \le \langle Y, AY \rangle + \kappa^{-1} \|e - AY\|^2. \tag{21}$$

One obtains this by choosing $A^{1/2}Y$ as the coefficient in the regularized preimage formula. It is valid when A has a kernel. Apply it to A_F, $\kappa = M^{-1}$, and use inverse order from $A_F \preceq \overline{P}_M$. Equations above prove (17) uniformly on the entire profile unit ball.

For any additional finite constraint space Z, the variational supremum for the restricted inverse is taken over $V \cap Z^\perp$, a subset of the original trial space. For the exactly projected source its value is therefore at most the original inverse energy. This gives the same constant for every such constraint space. The closed restriction is densely defined: approximate in the dense form domain and correct the finitely many row evaluations using fixed form-domain dual vectors. Finally, for fixed ridge exponent $a > 0$, replace $r_R = 2U$ by cU, where $c > a + 1/2$. The source residual costs $C\Omega^3 M^{a+1/2-c} U (\log U)^2$; the other errors stay negligible. This proves the last assertion. □

6 Complete covariance at prescribed polynomial ridges

The complete source admits an unconditional norm estimate at the ridge M^{-1} used in the signed return. More generally, the following holds at every fixed polynomial ridge. Its proof combines the original hard-cutoff realization [24] with the actual arithmetic preimage and the Vinogradov–Korobov prime-number error [25].

Theorem 6.1 (Complete covariance at polynomial ridges)**.** *For every fixed $b > 0$, there are C_b, M_b such that, for $M \geq M_b$,*

$$\sup_K \left| \|E^*_{M,K}(P_{M,K} + M^{-b}I)^{-1}E_{M,K}\| - 1 \right| \leq \frac{C_b\sqrt{\log M}\exp(48\sqrt{b\log M})}{\sqrt{M}}. \tag{22}$$

The supremum is over all finite genuine operator-null spaces, including moving and growing choices, with same-form restriction to $K^\perp$. The upper bound holds for every exact finite restriction with dense restricted form domain. Every negative source frequency is included.

Proof. Write $U = \log M$ and $\kappa = M^{-b}$. Use a second arithmetic cutoff m, distinct from the averaged-receiver parameter M. The original minimal realization has

$$H = Q_b s(a + b_m)(D)sQ_b + V_m, \qquad \|V_m\| \leq d_m := \frac{640(m+1)^2}{20^{m+1}}, \qquad |b_m| \leq 18\sqrt{m}.$$

Here $b_m(\xi) = m_0 - 2\sum_{2\leq n\leq m}\Lambda(n)n^{-1/2}\cos(\xi\log n)$, while $a \geq 0$ is increasing on the positive half-line and $a(L) \geq \frac{1}{2}\log(2L)$ for $L \geq \frac{1}{2}$. These are complete paired form identities; the raw remainder is never separated into its edge pieces. Put

$$k = \max\left\{4, \left\lceil \frac{\log(2560/\kappa)}{\log 10} \right\rceil \right\}, \quad m = k - 1, \quad D_0 = 18\sqrt{m}, \quad L = e^{2D_0}.$$

Since $k^2 \leq 2^k$, $d_m \leq \kappa/4$. The multiplier $v_m = (-(a + b_m))_+$ lies in $[0, D_0]$ and is supported in $[-L, L]$. On the unitary odd Fourier profile space put $T_m = v_m(D)^{1/2}$ and $E = Q_b sT_m$. Scalar sign decomposition gives $H \succeq -EE^* - d_m I$ on the original form domain.

Let $S = \|s\|_\infty < 3$, and choose an even Gevrey-2 cutoff χ_R equal to one on $[-R, R]$ and zero off $[-R - 1, R + 1]$, with a fixed-width transition and bounds $\|\chi_R^{(j)}\|_\infty \leq C_0C_1^j(j!)^2$. Set

$$X_0 = \log\frac{104SD_0}{\kappa}, \qquad R = \tfrac{1}{2}\log(2X_0), \qquad E_0 = Q_b s\chi_R T_m.$$

The complete theta envelope gives

$$\sup_{|u|\geq R} s(u) \leq 26X_0e^{-3X_0} \leq 13e^{-X_0} = \frac{\kappa}{8SD_0}.$$

Consequently $\|EE^* - E_0E_0^*\| \leq \kappa/4$ and

$$H \succeq -E_0E_0^* - (\kappa/2)I. \tag{23}$$

This constructs a source majorant of the negative sign of H without applying a non-monotone negative-part operation to an operator inequality.

We next bound the actual inverse on the entire coefficient unit ball of E_0. Theorem 1.4 of [25] gives $|\Psi(x) - x| \leq Cx(\log x)^{1.801}\exp[-0.1853(\log x)^{3/5}/(\log\log x)^{1/5}]$ for $x \geq 23$. In the complete Abel error, split $M^{-1/2}\int_1^M |\Psi(t) - t|t^{-3/2}\,dt$ at $\sqrt{M}$; its lower part is $O(M^{-1/4})$, and its upper part and $M^{1/2}\int_M^\infty |\Psi(t) - t|t^{-5/2}\,dt$ obey the displayed stretched-exponential error. Thus, for every fixed $A > 0$,

$$\bar{e}_M = O_A(e^{-A\sqrt{U}}).$$

Take $T = \exp[(50\sqrt{b} + 2)\sqrt{U}]$. The exact arithmetic phase debit

$$\varepsilon_M(T) = \frac{3(1 + 4T^2)}{4\sqrt{M}} + \frac{(1 + 4T^2)(T + 2)}{4}\bar{e}_M$$

tends to zero. Hence, for large M, the actual positive form dominates

$$A_T = \sqrt{M} Q_b s B(D) w(UD) 1_{|D|\le T} s Q_b, \quad B(\xi) = \frac{8}{1+4\xi^2}, \quad w(t) = (-\cos t - 1/2)_+.$$

If μ is the positive mean of w, its full-frequency continuation is $A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M$, with $\|J_M\| \le C M^{5/2} e^{-\pi(M-1)}$.

For $\|\phi\| \le 1$ put $g = T_m \phi$, $f = \chi_R g$ and $e = E_0 \phi$. Plancherel and product differentiation give, uniformly over this entire ball,

$$\|f\|_{1,4}^2 \le C D_0 (1 + L^2), \qquad \|f^{(j)}\|_2 \le C\sqrt{D_0}(L + Cj^2)^j.$$

Here $\|f\|_{1,4}^2 = \|f\|_2^2 + 4\|f'\|_2^2$. Since $m/U \to b/\log 10$, eventually $D_0 \le 12\sqrt{bU}$ and $L \le e^{24\sqrt{bU}}$. Choosing $j = \lceil U \rceil$ yields

$$\|1_{|D|>T} f\|_2 \le C\sqrt{D_0} e^{-c_b U^{3/2}}.$$

The compact smooth odd trial $Y = (1 - 4\partial_u^2) f / (8\mu\sqrt{M} s)$ is exactly polar legal by integration by parts against $\sinh(u/2)$. On its support the lower theta bound $s(u)^2 \ge 2r^2 e^{-r}$, $r = \pi e^{2|u|}$, gives

$$\|Y\|^2 \le C M^{-1} e^{C X_0} D_0 (1 + L^4).$$

This pays division by s on the moving support. Since $X_0 = bU + O_b(\log U)$, this growth is harmless against the full harmonic error. The exact mean action and residual are

$$\mu\sqrt{M} Q_b s B(D) s Q_b Y = e, \qquad e - A_T Y = -J_M Y + \mu^{-1} Q_b s w(UD) 1_{|D|>T} f.$$

Their complete squared residual, including the cross term, is smaller than every fixed inverse power of M after multiplication by κ^{-1}. Also

$$\langle Y, A_T Y \rangle \le C M^{-1/2} \|f\|_{1,4}^2 + \|J_M\| \|Y\|^2.$$

The kernel-safe trial inequality (21) therefore proves

$$\|E_0^* R_M(\kappa) E_0\| \le \frac{C_b \sqrt{U} e^{48\sqrt{bU}}}{\sqrt{M}}. \tag{24}$$

The proof is uniform over the full coefficient unit ball, with no finite-column or dimension factor.

Finally $N_M = P_M + W_M - H$ and (23) give

$$R_M^{1/2} N_M R_M^{1/2} \preceq I + R_M^{1/2} E_0 E_0^* R_M^{1/2} + (\delta_M - \kappa/2) R_M.$$

The last term is nonpositive for large M. Equality of the source and physical operator norms gives the required complete upper bound. For any exact same-form restriction its inverse, extended by zero, is at most the base inverse. Restricting the form inequality thus proves the upper uniformly without commuting projections with an inverse.

For the lower take one nonzero $v \in \mathrm{Dom}\, H \cap (\ker H)^\perp$. Such a vector exists since the archimedean form is unbounded above on compact high-frequency packets, whereas its arithmetic perturbation is bounded. It belongs to every genuine-null complement. For completeness, the averaged quadratic mean used here follows directly from the complete Abel formula. On each fixed frequency band, $n_{M,1}/\sqrt{M} - B(\xi)(\cos(U\xi))_+ \to 0$ uniformly. The phase mean of $(\cos)_+$ is $1/\pi$, and its absolutely

convergent Fourier series gives weak convergence against every integrable test on that band. The uniform complete bound $n_{M,1} \le 16\sqrt{M}$ pays the exterior of $|\widehat{sv}|^2$ by Plancherel. Hence

$$N_M[v]/\sqrt{M} \longrightarrow \langle v, \mathcal{C}_{Q_b} v\rangle =: c_v > 0.$$

Strict positivity follows from $s > 0$ and $B(\xi) > 0$. This is the averaged order-two mean; the hard-cutoff order-one mean in Appendix F is a separate result. Therefore

$$\|G_{M,K}(\kappa)\| \ge \frac{N_M[v]}{N_M[v] + q_H[v] - \langle v, W_M v\rangle + \kappa\|v\|^2} \ge 1 - C_v M^{-1/2}$$

uniformly in K and $0 < \kappa \le 1$. This completes the proof. □

At $\kappa = M^{-1}$ the complete largest covariance norm is therefore $1 + O(M^{-1/2+o(1)})$. A signed covariance excess $o(M^{-1/2})$ gives the original-domain positivity criterion in Corollary 6.2. The complete null/collar certificates in Sections 7–9 retain exact null cancellation in this formulation.

Corollary 6.2 (Critical covariance characterization)**.** *Let K_M be any sequence of finite genuine operator-null spaces and put $\gamma_M = \|E^*_{M,K_M}(P_{M,K_M} + M^{-1}I)^{-1}E_{M,K_M}\|$. Then*

$$H \succeq 0 \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M - 1)_+ = 0.$$

Thus the critical covariance condition is equivalent to the original Weil positivity criterion and hence to RH in the realization of Theorem 2.2.

Proof. If $H \succeq 0$, the signed identity gives $R_M^{1/2} N_M R_M^{1/2} \preceq I + (\delta_M - M^{-1})R_M \preceq I$ for all large M, also after any same-form restriction. Conversely, if H is not nonnegative, choose a fixed nonzero $v \in \operatorname{Dom} H \cap (\ker H)^\perp$ with $q_H[v] < 0$ by spectral projection onto a bounded negative interval. This vector belongs to every $K_M^\perp$. Its same-state quotient gives

$$\gamma_M - 1 \ge \frac{-q_H[v] + \langle v, W_M v\rangle - M^{-1}\|v\|^2}{P_M[v] + M^{-1}\|v\|^2} \ge c_v M^{-1/2}.$$

The last inequality uses the fixed positive numerator for large M and $P_M[v] = N_M[v] + q_H[v] - \langle v, W_M v\rangle = O_v(\sqrt{M})$. This contradicts the stated liminf condition, which therefore establishes $H \succeq 0$ on the original domain. The original Weil equivalence in Lemma A.1 now applies. Hence, under the stated critical covariance condition, the Riemann hypothesis holds. □

7 Genuine nulls and complete source certificates

Genuine null vectors provide coordinates adapted to the signed form. We first establish completeness at each fixed arithmetic cutoff, then quantify the full covariance error of a finite null head.

Lemma 7.1 (Genuine derivative nulls)**.** *For every $j \ge 0$, $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$ belongs to* $\operatorname{Dom} H$ *and satisfies $Hz_j = 0$. Every finite initial family is linearly independent.*

Proof. The theta Mellin identity of Appendix A gives

$$\mathcal{M}(sz_j)(z) = (z^2 - 1/4)(-z)^{2j+1}\xi(1/2 + z).$$

It vanishes at both polar points and at every centered nontrivial zero. To verify the original domain, use compact even cutoffs χ_R and the polar-preserving approximants $(\partial_u^2 - 1/4)(\chi_R\psi^{(2j+1)})/s$. The

complete derivative tails, also quantified in Proposition 8.2, give convergence in L^2 after division by s and convergence of the weighted physical derivative in L^2. The bound $a(\xi) \le 17\xi^2$ proves convergence in V. The same complete tails and integration by parts give arbitrary vertical inverse powers for the Mellin transforms on $|\Re z| \le 1/2$. The zero-count bound therefore passes the polarized explicit formula through these approximants as in Appendix A. It yields $q_H[z_j, h] = 0$ first on the compact core and then on all V. Form representation gives the asserted operator nullity. Finally a vanishing finite combination would make a nonzero polynomial times $(z^2 - 1/4)\xi(1/2+z)$ identically zero. Analyticity excludes this. □

Theorem 7.2 (Complete null-generated source)**.** *Let*

$$z_i = \frac{\psi^{(2i+3)} - \psi^{(2i+1)}/4}{s}, \qquad i \ge 0,$$

be the genuine operator nulls of the original realization. For every fixed M with nonzero negative source,

$$\overline{E_M^* \operatorname{span}\{z_i : i \ge 0\}} = \mathcal{Y}_M.$$

Moreover E_M is Hilbert–Schmidt. If Π_r projects onto $\operatorname{span}\{E_M^* z_0, \dots, E_M^* z_{r-1}\}$, then, for each fixed M and $\kappa > 0$,

$$\|(I - \Pi_r) G_M(\kappa)(I - \Pi_r)\| \longrightarrow 0.$$

Proof. The complete averaged prime multiplier is bounded for fixed M, whereas the archimedean multiplier tends to infinity. Hence S_M lies in a bounded interval $[0, F_M]$. The derivative rule gives, up to one fixed nonzero normalization factor,

$$E_M^* z_i(\xi) = v_M(\xi)(-1)^i \xi^{2i}, \qquad v_M(\xi) = \sqrt{n_{M,1}(\xi)}\, \xi(\xi^2 + 1/4)\widehat{\psi}(\xi).$$

The nonzero entire function $\widehat{\psi}$ has discrete real zeros, so $v_M \ne 0$ almost everywhere on S_M. It belongs to $L^2(S_M)$. If $f \in \mathcal{Y}_M$ is orthogonal to every displayed function, the finite complex measure $\overline{f} v_M \, d\xi/\pi$ annihilates every polynomial in ξ^2. Such polynomials are uniformly dense in $C([0, F_M])$, by polynomial approximation after the homeomorphism $\xi \mapsto \xi^2$. The measure is therefore zero, and $f = 0$ almost everywhere.

The source kernel satisfies

$$\|E_M\|_{\mathrm{HS}}^2 = \int_{S_M} n_{M,1}(\xi) \|Q_b[s \sin(\xi\,\cdot)]\|^2 \frac{d\xi}{\pi} < \infty.$$

Thus $G_M(\kappa)$ is positive trace class. The proved density gives $\Pi_r \to I$ strongly, and compactness gives the asserted norm limit. □

The theorem concerns the source images of known nulls. It does not identify the full physical kernel of H. Its density assertion is on the base polar recipient; imposing further physical constraints can alter that assertion.

Theorem 7.3 (Signed remainder and complete trace certificate)**.** *Fix M, $\kappa > 0$ and finitely many independent derivative-null columns Z. Define*

$$K = Z^* N_M Z, \quad V_Z = Z K^{-1/2}, \quad J = E_M^* V_Z, \quad Q = I - JJ^*, \quad \ell = \|V_Z^* V_Z\|.$$

Then $K \succ 0$, $J^*J = I$, and

$$J^* G_M J \preceq \big[1 + (\delta_M + \delta_M^2/\kappa)\ell\big] I, \tag{25}$$

$$\tau := \|E_M Q\|_{\mathrm{HS}}^2 = \operatorname{Tr} N_M - \operatorname{Tr}(K^{-1} Z^* N_M^2 Z), \tag{26}$$

$$\|G_M\| \le 1 + (\delta_M + \delta_M^2/\kappa)\ell + \tau/\kappa. \tag{27}$$

For an increasing initial derivative family, $\tau \to 0$ at fixed M. If Z has r columns, then

$$\ell \ge \lambda_r(N_M)^{-1} \ge r/\operatorname{Tr} N_M,$$

where the positive eigenvalues are arranged in decreasing order.

Proof. The nonvanishing common factor in Theorem 7.2 shows that every finite source polynomial family is independent, hence $K \succ 0$. Suppress M and write $A = P + \kappa I$, $R = A^{-1}$, $F = (W - \kappa I)V_Z$. Genuine operator nullity gives $NV_Z = AV_Z + F$, so exact expansion gives

$$GJ = J + E^* RF, \qquad J^* GJ = I + V_Z^*(W - \kappa I)V_Z + F^* RF.$$

Using $R \preceq \kappa^{-1} I$ only after combining these terms yields

$$V_Z^*(W - \kappa I)V_Z + F^* RF \preceq V_Z^*(-W + W^2/\kappa)V_Z \preceq (\delta + \delta^2/\kappa)V_Z^* V_Z.$$

This proves (25). In particular the ridge does not produce a separate $\kappa\ell$ charge.

Expanding Q and using cyclicity of the trace gives (26); all operators under the trace are trace class or finite rank. Also $\|QGQ\| \le \kappa^{-1}\|EQ\|_{\mathrm{HS}}^2 = \tau/\kappa$. For a positive two-block operator with diagonal norms a, b, positive-form Cauchy–Schwarz bounds its quadratic form by $(\sqrt{a}\|x\| + \sqrt{b}\|y\|)^2$, and consequently its norm by $a + b$. This proves (27), including the mixed block. Hilbert–Schmidt approximation and source density give $\tau \to 0$. Finally $V_Z^* NV_Z = I$ makes the Rayleigh quotient of N on $\operatorname{ran} V_Z$ at least $1/\ell$. Min–max and $r\lambda_r(N) \le \operatorname{Tr} N$ prove the last two inequalities. □

Proposition 7.4 (Directional Schur alternative)**.** *With the same notation, put $b = \|QG_M Q\|$, $r_F = \|F^* R_M F\|$ and $\varepsilon = (\delta_M + \delta_M^2/\kappa)\ell$. For $\lambda = 1 + \eta > b$, the condition*

$$\varepsilon + \frac{b r_F}{\lambda - b} \le \eta$$

implies $G_M \preceq \lambda I$.

Proof. The mixed block is $QG_M J = QE_M^* R_M F$ and its squared norm is at most br_F. Complete the square in the Q coordinate of $\lambda I - G_M$, using $\lambda I - QG_M Q \succeq (\lambda - b)I$, and then apply (25). □

8 Regularized null certificates and physical lift budgets

The null-source construction can avoid an unregularized source-Gram inverse. The price becomes an explicit approximation error and the physical norm of its constructed lift. In this section write $d = \|W_M\|$ and $\mathcal{D} = d + d^2/\kappa$.

Theorem 8.1 (Regularized complete certificate)**.** *Let $Z : \mathbb{C}^r \to \mathcal{X}$ be a physical isometry with range in $\operatorname{Dom} H \cap \ker H$. Put $A = E_M^* Z$, $K = A^* A$, and, for $t > 0$,*

$$J = A(K + tI)^{-1} A^*, \qquad T = Z(K + tI)^{-1} A^*.$$

Then $0 \preceq J \preceq I$, $J = E_M^* T$, and

$$\|G_M\| \le 1 + \frac{\mathcal{D}}{4t} + \frac{\tau_t}{\kappa}, \qquad \tau_t = \operatorname{Tr} N_M - \operatorname{Tr}\big((K + tI)^{-1} K (K + tI)^{-1} Z^* N_M^2 Z\big). \tag{28}$$

If a Hilbert–Schmidt map $X : \mathcal{X} \to \mathbb{C}^r$ satisfies $\|E_M^* - AX\|_{\mathrm{HS}} \le \epsilon$ and $\|X\|_{\mathrm{HS}} \le L_0$, then

$$\|G_M\| \le 1 + \frac{2\epsilon^2}{\kappa} + \sqrt{\frac{2\mathcal{D}}{\kappa}} L_0. \tag{29}$$

Every trace and approximation here uses the complete source metric.

Proof. The singular-value multipliers of T are $\sigma/(\sigma^2 + t)$; hence $\|T\|^2 \le 1/(4t)$. The exact signed-null identity, applied to Tf, gives

$$\langle Jf, G_M Jf\rangle \le \|Jf\|^2 + \mathcal{D}\|Tf\|^2 \le (1 + \mathcal{D}/(4t))\|f\|^2.$$

Use the positive partition $N_M = E_M J^2 E_M^* + E_M(I - J^2)E_M^*$. After inverse congruence its first term has norm $\|JG_M J\|$, and its second is at most $\kappa^{-1}\operatorname{Tr}(E_M(I - J^2)E_M^*)$. Expanding J^2 and using cyclicity proves (28). Completing the finite-dimensional square gives

$$\operatorname{Tr}(E_M(I - J)E_M^*) = \inf_X \{\|E_M^* - AX\|_{\mathrm{HS}}^2 + t\|X\|_{\mathrm{HS}}^2\}.$$

Since $I - J^2 \preceq 2(I - J)$, the excess in (28) is at most $\mathcal{D}/(4t) + 2(\epsilon^2 + tL_0^2)/\kappa$. For $\mathcal{D}, L_0 > 0$ optimize at $t = \sqrt{\mathcal{D}\kappa/(8L_0^2)}$; the zero cases follow by limits. □

For independent raw columns Z_0, replace $K + tI$ by $Z_0^* N_M Z_0 + tZ_0^* Z_0$. In (29) the physical lift norm is then $L_0 = \|Z_0 X_0\|_{\mathrm{HS}}$. This preserves the original physical metric without requiring explicit formation of its inverse square root.

The collar estimate of Theorem 9.1 also survives regularization. With P_I denoting source multiplication by $I \subset [0, (\log M)^4]$, suppose

$$\|(I - P_I)E_M^* - AX\|_{\mathrm{HS}} \le \epsilon_{\mathrm{out}}, \qquad \|X\|_{\mathrm{HS}} \le L_{\mathrm{out}}.$$

Completing the same square with $(I - P_I)E_M^*$ and using the triangle inequality in inverse energy gives

$$\|G_M\| \le 1 + \frac{\mathcal{D}}{4t} + \left(\sqrt{c_I} + \sqrt{\frac{2(\epsilon_{\mathrm{out}}^2 + tL_{\mathrm{out}}^2)}{\kappa}}\right)^2, \quad c_I = C|I|(1 + \log\log M). \tag{30}$$

No projection is commuted with J. The square includes the mixed term, and the exterior includes every frequency outside I.

Proposition 8.2 (Order-uniform original-norm null budget)**.** *For $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$,*

$$\|z_j\|_2 \le 100\, 64^{2j+3}(2j + 3)!, \qquad j \ge 0.$$

Consequently the raw map $Z_0 = (z_0, \dots, z_{r-1})$ satisfies

$$\|Z_0\|_{\mathrm{HS}}^2 \le B_r := 10000r\, 64^{4r+2}[(2r + 1)!]^2, \qquad \|Z_0 X_0\|_{\mathrm{HS}} \le \sqrt{B_r}\|X_0\|_{\mathrm{HS}}. \tag{31}$$

Proof. Set $\rho = 1/64$ and $r_u = \pi e^{2|u|}$. The complete theta series is holomorphic in $|\operatorname{Im} z| < \pi/4$ and even there. On $|z - u| = \rho$, reflect to the right half-plane if needed. Then $\operatorname{Re}(\pi e^{2z}) > 3r_u/4$ and $|\pi e^{2z}| \le e^{2\rho} r_u$. Its full series gives

$$|\psi(z)| \le 24 e^{|u|/2} r_u^2 e^{-3r_u/4} \sum_{n \ge 1} n^4 e^{-3(n^2-1)\pi/4} < 48 e^{|u|/2} r_u^2 e^{-3r_u/4}.$$

Indeed the $n = 2$ summand is below $1/4$ and subsequent ratios are below $1/256$. Cauchy's estimate yields $|\psi^{(k)}(u)| \le 48k!64^k e^{|u|/2} r_u^2 e^{-3r_u/4}$. Divide by the original lower envelope $s^2 \ge 2r_u^2 e^{-r_u}$ and integrate both arms using $du = dr_u/r_u$ and $e^{|u|} = \sqrt{r_u/\pi}$. Thus

$$\|\psi^{(k)}/s\|_2^2 \le \frac{1152}{\sqrt{\pi}}(k!64^k)^2 \int_\pi^\infty r^{3/2} e^{-r/2}\, dr \le 3456\sqrt{2}(k!64^k)^2 < 80^2(k!64^k)^2.$$

The triangle inequality at $k = 2j + 3$ and $k - 2$ proves the first bound. Summing its squares and using the largest summand proves the second. □

Since $\log B_r = O(r \log(r+1))$, this gives a useful cofinal lift window. At $\kappa = M^{-1}$ the complete averaged remainder has $d_M \le \mathrm{poly}(M) e^{-\pi M}$. If $r_M = o(M/\log M)$ and $\log(1+\|X_{0,M}\|_{\mathrm{HS}}) = o(M)$, then

$$\sqrt{M\mathcal{D}_M}\, \|Z_{0,M} X_{0,M}\|_{\mathrm{HS}} \le e^{-\pi M/2 + o(M)}.$$

Thus the lift part of (29) is paid at every fixed polynomial scale. A complete-source approximation with $\epsilon_M^2 = o(M^{-a-1})$, $1/2 < a < 1$, would then give the excess $o(M^{-a})$ and the vanishing-allowance endpoint. The approximation norm must include the entire source exterior; a polynomial approximation on a bounded or polylogarithmic frequency interval alone does not establish that condition.

9 Uniform covariance on small sets in a growing band

The null-source representation can be poorly conditioned near zeros of $\widehat{\psi}$. The actual positive inverse controls small frequency sets there without division by that transform.

Theorem 9.1 (Growing-band observation bound)**.** *There are constants C, M_0 such that for every integer $M \ge M_0$ and every measurable $I \subset [0, (\log M)^4]$,*

$$\|E_I^*(P_M + M^{-1}I)^{-1} E_I\| \le C|I|(1 + \log\log M).$$

The constant is independent of the number and locations of the components of I. The upper bound survives additional finite exact constraints, with the same-form restricted receiver and exactly projected source.

Proof. Put $U = \log M$, $T = U^5$ and $e_\xi = Q_b[s \sin(\xi\,\cdot)]$. We first prove, uniformly for $0 \le \xi \le U^4$,

$$\langle e_\xi, (P_M + M^{-1})^{-1} e_\xi \rangle \le CM^{-1/2}(1 + \xi^2)(1 + \log U). \tag{32}$$

Use the actual positive lower A_T from the proof of Theorem 5.1, with phase $w(t) = (-\cos t - 1/2)_+$ and mean $\mu > 0$. Put $R_0 = \frac{1}{2}\log(2U/\pi)$, $f_\xi(u) = \chi(|u| - R_0)\sin(\xi u)$, using the same fixed-width Gevrey-2 cutoff, and set

$$Y_\xi = \frac{(1 - 4\partial_u^2) f_\xi}{8\mu\sqrt{M}\, s}.$$

It is compact smooth, odd and polar legal by integration by parts. Uniformly in the stated frequency range,

$$\|f_\xi\|_{1,4}^2 \le C(1 + \xi^2)(1 + \log U), \qquad \|s(\sin(\xi\,\cdot) - f_\xi)\|^2 \le CUe^{-2U}.$$

The first inequality uses the fixed transition width and support length $O(1 + \log U)$; the second uses the complete two-arm theta envelope and $|\sin| \le 1$.

For $j = \lceil U \rceil$, Leibniz and $|\chi^{(l)}| \le C_0 C_1^l (l!)^2$ give

$$\|f_\xi^{(j)}\|_2 \le C\sqrt{1 + \log U}\, [C(\xi + j^2)]^j.$$

Indeed $(l!)^2 \le j^{2l}$ in the Leibniz sum. Division by T^j bounds the complete Fourier exterior by $C\sqrt{1+\log U}\exp[-U\log U+O(U)]$, uniformly for $\xi \le U^4$. The lower theta envelope on the support bounds $\|Y_\xi\|^2$ by a fixed power of M times a fixed polynomial in U; the factor $\xi^4 \le U^{16}$ is harmless here.

The exact residual identity from the actual-origin proof has three terms: the physical cut-off error, the complete harmonic error, and the Fourier exterior. Its harmonic operator norm is $O(M^{5/2}e^{-\pi(M-1)})$, which absorbs the trial norm. A squared triangle estimate pays all three mixed residual terms. At ridge M^{-1}, the physical error costs $O(U/M)$, while the Fourier and harmonic errors are smaller than every fixed inverse power of M. The kernel-safe least-squares estimate (21) and inverse order $A_T \preceq P_M$ therefore prove (32).

The same actual scalar admission used in Theorem 5.1 gives, throughout $[0, U^5]$ and for all sufficiently large M,

$$n_{M,1}(\xi) \le \frac{12\sqrt{M}}{1+4\xi^2}.$$

Consequently the Hilbert–Schmidt trace of $R_M^{1/2}E_I$ is at most

$$\int_I n_{M,1}(\xi)\langle e_\xi, R_M e_\xi\rangle \frac{d\xi}{\pi} \le C|I|(1+\log U).$$

This trace bounds the full operator norm, retaining every profile combination. Variational restriction gives the additional-constraint statement. Constants and the admission threshold are independent of I. □

Corollary 9.2 (Complete collar and null-source certificate)**.** *Use the null head of Theorem 7.3, set $\kappa = M^{-1}$, and let P_I be the source multiplication projection onto $I \subset [0,(\log M)^4]$. Define*

$$N_{\mathrm{out}} = E_M(I-P_I)E_M^*, \qquad \tau_{\mathrm{out}} = \operatorname{Tr} N_{\mathrm{out}} - \operatorname{Tr}(V_Z^* N_{\mathrm{out}}^2 V_Z), \qquad c_I = C|I|(1+\log\log M).$$

If

$$(\delta_M + \delta_M^2/\kappa)\ell + \left(\sqrt{c_I} + \sqrt{\tau_{\mathrm{out}}/\kappa}\right)^2 \le \eta, \tag{33}$$

then $G_M(\kappa) \preceq (1+\eta)I$, and on the entire original domain

$$q_H[h] \ge -\left[\delta_M + \kappa + \frac{\eta B_{M,1}}{1+\eta}\right]\|h\|^2.$$

In particular, a cofinal family satisfying (33) *with $\eta = o(M^{-1/2})$ gives vanishing original-norm allowances and RH.*

Proof. Let $Q = I - JJ^*$ as above. Without commuting Q and P_I, cyclicity gives

$$\|E_M(I-P_I)Q\|_{\mathrm{HS}}^2 = \operatorname{Tr} N_{\mathrm{out}} - \operatorname{Tr}(V_Z^* N_{\mathrm{out}}^2 V_Z).$$

Here V_Z uses the full N_M normalization. The triangle inequality in actual inverse energy gives

$$\|QG_MQ\| \le \left(\|R_M^{1/2}E_MP_IQ\| + \|R_M^{1/2}E_M(I-P_I)Q\|\right)^2 \le \left(\sqrt{c_I} + \sqrt{\tau_{\mathrm{out}}/\kappa}\right)^2.$$

Combine this with (25) and the positive-block estimate in Theorem 7.3. The original-domain lower bound follows from Corollary 3.2, with complete negative source and no finite-source omission. Since $B_{M,1} = O(\sqrt{M})$ and $\delta_M \to 0$, the stated rate gives the endpoint. □

All frequencies outside I, including those above $(\log M)^4$, occur in τ_{out}. The mixed collar/exterior cost is the cross term in the square in (33). Thus the displayed criterion specifies a complete quantitative construction. Fixed-M source density provides neither its growing-rank conditioning bound nor a cofinal rate for its complete trace remainder.

10 Feasibility and complete exterior control

The preceding certificates retain exact signed information. This section identifies the costs of two simpler bounds and locates the earlier scalar complementary criterion within the complete construction.

Proposition 10.1 (Cost of the positive majorant)**.** *For the explicit E_0 in the proof above, with $m \sim b \log M/\log 10$,*

$$\|E_0^* R_M(M^{-b}) E_0\| \geq \frac{c_b\sqrt{m}}{\sqrt{M}(\log m)^3}$$

for all sufficiently large M. Thus this positive-majorant inverse cost is not $o(M^{-1/2})$.

Proof. Choose compact smooth odd f with $\int f(u)\sinh(u/2)\,du = 0$ but $\int uf(u)\,du \neq 0$. Two odd bump pairs construct such an f, since u and $\sinh(u/2)$ are linearly independent on a positive interval. Then $y = f/s$ is legal, $|\widehat{f}(\xi)| \geq c|\xi|$ near zero, and for large M, $E_0^* y = T_m f$. On $|\xi| \leq 1/(2\log m)$ every $\cos(\xi \log n) > 1/2$ for $n \leq m$. The prime number theorem gives $\sum_{n\leq m} \Lambda(n)/\sqrt{n} \geq c\sqrt{m}$; boundedness of a near zero therefore yields $v_m(\xi) \geq c\sqrt{m}$ on this interval. Plancherel gives $\|E_0^* y\|^2 \geq c\sqrt{m}/(\log m)^3$. The full bound $\|N_M\| \leq 144\sqrt{M}$ and the signed identity imply $P_M[y] + M^{-b}\|y\|^2 = O_y(\sqrt{M})$. The inverse variational quotient proves the assertion. □

This concerns the particular positive majorant, not the signed covariance excess. It explains why the next construction retains the exact null remainder rather than trying to sharpen this majorant by constants alone.

Why a low-band raw trace does not pay the exterior. For every fixed $p > 0$, set $F = (\log M)^p$. The actual source satisfies

$$\mathrm{Tr}(E_M 1_{[F,2F]} E_M^*) \geq c\sqrt{M}/F$$

for all sufficiently large M. Indeed the complete arithmetic formula gives, uniformly on this band,

$$\left| \frac{m_M(\xi)}{\sqrt{M}B(\xi)} + \cos((\log M)\xi) \right| \leq C\{F^2 \log(2+F)/\sqrt{M} + F^3 \bar{e}_M\} \longrightarrow 0.$$

The subset with cosine at least $1/2$ has measure at least $F/4$ for large M, and there $n_M \geq \sqrt{M}B/4 \geq \sqrt{M}/(12F^2)$. For $e_\xi = Q_b(s\sin(\xi u))$, Riemann–Lebesgue applied to s^2 and bs gives

$$\|e_\xi\|^2 = \|s\|_2^2/2 - \tfrac{1}{2}\int s^2\cos(2\xi u)\,du - |\langle s\sin(\xi u), b\rangle|^2/\|b\|^2 \longrightarrow \|s\|_2^2/2 > 0.$$

Integrating in the original measure $d\xi/\pi$ proves the trace lower bound. Thus a head supported below F leaves a raw trace fee at ridge M^{-1} at least $cM^{3/2}/(\log M)^p$. This rules out that particular exterior payment, not the actual inverse covariance. Global null-polynomial heads are not supported below F; their full exterior or an actual-inverse exterior estimate must remain in the construction.

10.1 The scalar mixed-band criterion

Proposition 10.2 (Complete mixed-band cap)**.** *For the same positive inverse $R_P = (P + \kappa I)^{-1}$, split the exact observation source as $E = [E_o, E_c]$. If its two diagonal caps are $E_o^* R_P E_o \preceq aI$ and $E_c^* R_P E_c \preceq bI$, and the full mixed block has norm at most ζ, then*

$$\|E^* R_P E\| \le \tfrac{1}{2}[a + b + \sqrt{(a-b)^2 + 4\zeta^2}] \le a + b \quad \textit{when } \zeta \le \sqrt{ab}.$$

Proof. Positive-form Cauchy–Schwarz pays the mixed block by $\sqrt{ab}$. More generally bound each two-vector quadratic form by the real matrix $\left[\begin{smallmatrix} a & \zeta \\ \zeta & b \end{smallmatrix}\right]$ and compute its largest eigenvalue. All mixed observation terms are retained. □

Corollary 10.3 (A complementary-band cofinal criterion for RH)**.** *Let a_M be the bound in* (17) *at $\Omega = 1/\log M$, and use any finite genuine-null recipient. If, along an unbounded sequence, the complete complementary observation for the same $\overline{P}_M + M^{-1} I$ has cap*

$$b_M \le 1 - a_M + M^{-1},$$

then on the original domain

$$q_H[h] \ge -\left[d_{M,1} + \frac{1}{M} + \frac{B_{M,1}}{M+1}\right] \|h\|^2,$$

and RH holds. A paid mixed block can instead use the sharper condition in Proposition 10.2 at target $1 + 1/M$.

Proof. The proposition gives the complete cap $1 + 1/M$ including all mixed terms. Apply the bounded-negative absorption of Corollary 3.2 with $k = 1$, $\tau = 0$, $\eta = \kappa = 1/M$. Here $B_{M,1} = O(\sqrt{M})$, so the displayed allowance tends to zero; Theorem 2.2 applies. □

10.2 Why bounded null rank does not fund the scalar criterion

The sufficient scalar comparison in Corollary 10.3 loses directional covariance information. The following result gives its precise rank limitation at the displayed schedule.

Proposition 10.4 (Bounded-rank scalar obstruction)**.** *Put $\Omega = 1/\log M$, $\kappa = M^{-1}$ and split the complete source into E_o on $[0, \Omega]$ and its entire complement E_c. For the same-form recipient obtained by removing a genuine-null space K, put*

$$a = \|E_o^* R_M E_o\|, \qquad b = \|E_c^* R_M E_c\|, \qquad \chi = \|E_o^* R_M E_c\|.$$

For every fixed integer $D \ge 0$ there are positive constants c_D, C_D and a finite threshold, uniform over all K with $\dim K \le D$, such that

$$\chi^2 \ge c_D \Omega^{4D+3}, \qquad b \ge 1 - C_D \Omega^{4D+7} - C_D M^{-3/2}.$$

Consequently the scalar matrix $\left[\begin{smallmatrix} a & \chi \\ \chi & b \end{smallmatrix}\right]$ is not bounded by $(1+1/M)I$ for all sufficiently large M. Any cofinal realization of that scalar target requires the null dimension to eventually exceed every fixed D.

Proof. Let $L_D = \operatorname{span}\{z_0, \dots, z_D\}$, with the genuine nulls of Lemma 7.1. Dimension gives a unit $z = \sum_{i=0}^{D} \alpha_i z_i \in L_D \cap K^{\perp}$. The derivative Fourier rule and $\widehat{\psi}(0) > 0$ show, near zero,

$$|\widehat{sz}(\xi)| \geq c\xi \left| \sum_{i=0}^{D} (-1)^i \alpha_i \xi^{2i} \right|.$$

The complete Abel formula gives $n_{M,1} \geq c\sqrt{M}$ on $[0, \Omega]$ for large M, since $\cos((\log M)\xi) \geq \cos 1 > 0$. The positive moment matrix $G_D = (1/(2i + 2j + 3))_{0 \leq i,j \leq D}$ gives

$$\int_0^{\Omega} \xi^2 \left| \sum_{i=0}^{D} (-1)^i \alpha_i \xi^{2i} \right|^2 d\xi \geq \lambda_{\min}(G_D) \Omega^{4D+3} \|\alpha\|^2.$$

The fixed physical Gram of L_D bounds $\|\alpha\|$ below. Therefore $N_o[z] \geq c_D \sqrt{M} \Omega^{4D+3}$, uniformly in the moving K. All constants here are for fixed D; no uniform conditioning in D is used.

Write $A_o = E_o^* R_M E_o$ and $C_{oc} = E_o^* R_M E_c$. The exact null identity gives

$$C_{oc} E_c^* z = (I - A_o) E_o^* z + E_o^* R_M (W_M - \kappa I) z.$$

Since $R_M^2 \preceq \kappa^{-1} R_M$,

$$\chi \sqrt{N_c[z]} \geq (1 - a) \sqrt{N_o[z]} - (d_{M,1} + \kappa) \sqrt{a/\kappa}.$$

The actual origin upper is $a \leq C\Omega^3 (1 + \log\log M)^3$. Dividing its error term by the lower bound for $\sqrt{N_o[z]}$ gives at most $C_D M^{-3/4} \Omega^{-2D} (1 + \log\log M)^{3/2} + o(1)$, which tends to zero. The complete bound $N_c[z] \leq 144\sqrt{M}$ then proves the stated lower for χ^2. If $N_c[z] = 0$, the displayed identity already contradicts its positive right side for large M.

For the complementary lower choose a unit $z' \in \operatorname{span}\{z_{D+1}, \dots, z_{2D+1}\} \cap K^{\perp}$. Its transform vanishes to order at least $2D + 3$ at zero, so $N_o[z'] \leq C_D \sqrt{M} \Omega^{4D+7}$. The averaged quadratic mean proved in Theorem 6.1 is positive definite on this fixed finite-dimensional span. Polarization makes its convergence uniform on that span's unit sphere; hence $N_M[z'] \geq c'_D \sqrt{M}$ for every allowed moving choice. The exact null quotient yields

$$b \geq \frac{N_c[z']}{P_M[z'] + \kappa} \geq 1 - C_D \Omega^{4D+7} - C_D M^{-3/2}.$$

Finally put $t = 1 + 1/M$. If $b > t$ the scalar target fails immediately. Otherwise its necessary determinant inequality would imply

$$\chi^2 \leq (t - a)(t - b) \leq 2\{M^{-1} + C_D \Omega^{4D+7} + C_D M^{-3/2}\} = o(\Omega^{4D+3}),$$

contradicting the proved lower. □

The proposition concerns scalarization of the blocks. It neither rejects the operator-valued Schur criterion nor establishes sufficiency of growing null rank. The regularized whole-source construction retains the signed correlations that this scalar comparison discards.

A The odd original-domain criterion

We give the needed realization-specific reduction to the classical Weil criterion [5, 6]. For $F(z) = \int f(u) e^{zu} \, du$, the complete explicit formula has zero-side form

$$W[f, g] = \sum_{\rho} F(\rho - 1/2) \overline{G(1/2 - \overline{\rho})}. \tag{34}$$

The sum counts multiplicities. On odd real tests $W[f] = -\sum_\rho F(\rho - 1/2)^2$. The physical form in (5) equals $W[f] + 2|F(1/2)|^2$ on odd tests; the polar term vanishes on $\mathcal{K}$.

Lemma A.1 (Odd original-domain criterion)**.** *Nonnegativity of q_H on V is equivalent to RH.*

Proof. The theta Mellin identity, in the normalization (3), is

$$M\psi(z) = \xi(1/2 + z), \qquad M\psi'(z) = -z\xi(1/2 + z), \qquad M\psi'(1/2) = -1/4.$$

Indeed, with $t = e^{2u}$ and $A(t) = \sum_{n\geq 1} e^{-\pi n^2 t}$, one has $\psi = e^{u/2}(4t^2A'' + 6tA')$. Two integrations by parts on $0 < t < \infty$, initially for $\Re s > 1$, give

$$M\psi(s - 1/2) = \frac{s(s-1)}{2} \int_0^\infty A(t)t^{s/2-1}\,\mathrm{d}t = \xi(s).$$

Theta inversion gives the entire continuation. For any compact smooth real odd f, put $g = f + 4F(1/2)\psi'$. Its polar moment is zero and its transform agrees with F at every nontrivial zero, so $W[g] = W[f]$.

Here g/s belongs to the original minimal domain. To see this directly, take even smooth cutoffs χ_R and a fixed compact real odd b with $Mb(1/2) = 1$. Set $g_R = \chi_R g - M(\chi_R g)(1/2)b$. Then g_R is in the exact polar compact core. Complete theta derivative envelopes give

$$\|(g_R - g)/s\|_2 \to 0, \qquad \|(g_R - g)'\|_2 \to 0.$$

For example $|\psi'| \leq 40e^{|u|/2}r^3e^{-r}$ and the lower bound (6) give an integrable upper $800e^{|u|}r^4e^{-r}$ for $|\psi'/s|^2$. The derivative atom polynomials are $-8r^3 + 30r^2 - 15r$ and $16r^4 - 112r^3 + 165r^2 - 75r/2$. Their absolute coefficient bounds for $r \geq \pi > 3$, and consecutive atom ratios, give $|\psi''| \leq 160e^{|u|/2}r^4e^{-r}$. These complete tails pay both limits. Bounded cutoff derivatives and the vanishing compact moment correction preserve them. The bound $a \leq 17\xi^2$ now gives convergence in (4).

The same tails give exponential-weighted L^1 convergence of $g_R - g$ and its first derivative. Integration by parts therefore gives

$$\sup_{|\Re z|\leq 1/2} (1 + |\Im z|)|M(g_R - g)(z)| \longrightarrow 0.$$

The zero count implies $\sum_\rho (1 + |\Im\rho|)^{-2} < \infty$, so the complete quadratic zero sum converges as well. Passing the compact explicit formula to the limit yields $q_H[g/s] = W[g] = W[f]$. Thus positivity on V implies $W[f] \geq 0$ for every real odd compact smooth f.

Suppose an off-line centered zero $\sigma_0 = \delta + i\gamma$ exists, with $\delta, \gamma > 0$ by symmetry. Nontrivial real zeros are absent because the alternating eta series gives $\zeta(s) < 0$ for real $0 < s < 1$. Fix $\Gamma > \gamma + 1$. An even real polynomial P annihilates every zero orbit in $|\Im\sigma| \leq \Gamma$ except the quartet of σ_0, and $P(\sigma_0) \neq 0$. Choose real a, b with $iP(\sigma_0)(a + b\sigma_0^2) = 1$, possible because $\Im\sigma_0^2 \neq 0$. Put

$$F_T(z) = iP(z)(a + bz^2)[e^{T(z-i\gamma)^2} - e^{T(z+i\gamma)^2}], \quad T \geq 1.$$

This entire function is odd and real type. It is the bilateral Laplace transform of a real odd Gaussian multiplied by polynomial and trigonometric factors. At each fixed T, even compact cutoffs of that physical function converge in exponential Schwartz seminorms. Repeated integration by parts gives transform convergence with every polynomial vertical decay uniformly on $|\Re z| \leq 1/2$. The zero count $N_\zeta(Y) = O(Y \log(Y + 2))$ therefore permits dominated convergence in (34), so $W[f_T] \geq 0$.

At σ_0,

$$e^{-T\delta^2}F_T(\sigma_0) = 1 - e^{-4T\gamma^2+4iT\delta\gamma} \longrightarrow 1.$$

The target quartet, of multiplicity m, contributes $-4m\Re F_T(\sigma_0)^2 \sim -4me^{2T\delta^2}$. All other zeros in the finite window are annihilated. For exterior zeros put $d_\rho = ||\Im\sigma_\rho| - \gamma| > 1$. A fixed polynomial envelope bounds their squared transform by

$$C(1+|\Im\sigma_\rho|)^{2D}e^{2T(1/4-d_\rho^2)} \le Ce^{-3T/2}(1+|\Im\sigma_\rho|)^{2D}e^{-(d_\rho^2-1)}.$$

The full remaining zero sum is $O(e^{-3T/2})$ by the same zero-count bound. The negative quartet dominates, a contradiction. This proves RH. Conversely, RH makes (34) a sum of squared absolute transforms on the polar core; original form closure extends positivity to V. □

B Local inverse covariance for the averaged reference

For $k = 1$ the continuous prime integral is exactly

$$\int_1^\infty t^{-1/2+i\xi}\min(1, M/t)\,dt = \frac{M^{1/2+i\xi}}{1/4+\xi^2} - \frac{1}{1/2+i\xi}.$$

Its phase-mean negative receiver, divided by $\sqrt{M}$, is

$$\mathcal{C}_Q = \alpha Q M_s L_0^{-1} M_s Q, \qquad \alpha = 8/\pi, \quad L_0 = 1 - 4d^2/du^2. \tag{35}$$

Here Q is an original orthogonal projection on the odd space, with a finite constraint space $Z = \ker Q$. Write $\|g\|_{1,4}^2 = \int(|g|^2 + 4|g'|^2)$.

Theorem B.1 (Full inverse-form source covariance)**.** *For $h \in \operatorname{ran} Q$,*

$$\|\mathcal{C}_Q^{-1/2}h\|^2 = \frac{\pi}{8}\inf_{z\in Z:\,(h+z)/s\in H^1}\|(h+z)/s\|_{1,4}^2, \tag{36}$$

with value infinity when the set is empty. For any finite odd H^1 profile family g_i and $e_i = Q(sg_i)$ this yields the simultaneous Gram inequality

$$[\langle \mathcal{C}_Q^{-1/2}e_i, \mathcal{C}_Q^{-1/2}e_j\rangle] \preceq \frac{\pi}{8}[\langle g_i, g_j\rangle + 4\langle g_i', g_j'\rangle].$$

Equality holds when Z is any finite span of the original polar row and canonical derivative-null rows. All statements allow arbitrary finite source and row counts.

Proof. Factor $\mathcal{C}_Q = \alpha AA^*$ with $A = QM_sL_0^{-1/2}$. The actual-range identity $\operatorname{ran}(AA^*)^{1/2} = \operatorname{ran} A$ and polar decomposition identify the inverse form with the minimum squared preimage norm [18]. Writing $g = L_0^{-1/2}a$ translates $Aa = h$ to $sg - h \in Z$, and $\|a\|^2 = \|g\|_{1,4}^2$. The chosen profiles give the matrix upper on every common coefficient vector.

More exactly, project the profiles orthogonally in H^1, with this norm, off $\{g \in H^1 : sg \in Z\}$. Their projected Gram gives equality in (36). This kernel is zero for the canonical rows. Indeed $b/s = \sinh(u/2)$, and for $z_{2j+1} = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$,

$$\frac{z_{2j+1}(u)}{s(u)} \sim \frac{(-2\pi)^{2j+3}}{2}e^{(4j+13/2)u} \quad (u \to +\infty).$$

To obtain this, the first theta atom derivative polynomial has leading coefficient $4(-2)^m$ at order m, by the recurrence $P_{m+1} = (1/2 - 2r)P_m + 2rP_m'$. At fixed order the higher atoms contribute a polynomial times e^{-3r} relative to the first. The exponents of these rows and b/s are distinct, so a nonzero finite combination is not in L^2. This proves the stated equality. □

If the full profile Gram is bounded by KD and the full profile-error Gram by $\eta^2 D$ in this same H^1 metric, the resulting inverse-form Grams differ in either order by at most $(\pi/8)(2\sqrt{K}\eta + \eta^2)D$. Expand each common coefficient combination and apply Cauchy–Schwarz before passing to coordinates. The exact profile projection in the proof is a contraction, so the same error transport applies to every finite constraint space.

Lemma B.2 (Regularized preimage identity)**.** *For $a, \kappa > 0$ and $e = Q(sg)$, $g \in H^1_{\mathrm{odd}}$,*

$$\langle e, (a\mathcal{C}_Q + \kappa I)^{-1} e\rangle = \inf_{f \in H^1_{\mathrm{odd}}} \left\{ \frac{\|f\|^2_{1,4}}{a\alpha} + \frac{\|Q[s(g-f)]\|^2}{\kappa} \right\}. \tag{37}$$

Proof. For $B = \sqrt{a\alpha} Q M_s L_0^{-1/2}$, minimize $\|z\|^2 + \kappa^{-1}\|e - Bz\|^2$. Its value is $\langle e, (BB^* + \kappa I)^{-1} e\rangle$. Substitute $f = \sqrt{a\alpha} L_0^{-1/2} z$. This is the classical approximate-preimage regularization identity in the present weighted and projected space [19]. □

C Explicit reference origin bounds

Let $U = \log M$, $\Omega = L/U$, and use the actual smoothed negative symbol $n_{M,1}$. On the *entire* profile space $\mathcal{Y}^{\mathrm{org}}_{M,L} = L^2([0,\Omega], n_{M,1}(\xi)\, d\xi/\pi)$ put

$$g_\phi(u) = \int_0^\Omega n_{M,1}(\xi)\phi(\xi)\sin(\xi u)\frac{d\xi}{\pi}, \quad E\phi = Q(sg_\phi), \quad D_\phi = \|\phi\|^2_{\mathcal{Y}^{\mathrm{org}}_{M,L}}.$$

The reference $\mathcal{C}_Q$ is exactly (35).

Theorem C.1 (Uniform ridged origin cap)**.** *For every $L > 0$, every integer $M \geq 64$ and every finite original constraint space,*

$$E^*(\sqrt{M}\,\mathcal{C}_Q + M^{-1} I)^{-1} E \preceq K(M,L) I_{\mathcal{Y}^{\mathrm{org}}_{M,L}}, \tag{38}$$

where, with $R = \frac{1}{2}\log(2\log M)$,

$$K(M,L) = \frac{160}{3} L^3 \left(\frac{R}{\log M}\right)^3 + \frac{4096\pi^2}{3} L^3 M^{3/2-2\pi}.$$

In particular $K(M,L) = O_L((\log\log M/\log M)^3)$, and $K(M,1) < 1/10$ for every integer $M \geq 2^{20}$.

Proof. The elementary prime bound and $\mathbb{E}e^{v/2} = 2$ give $n_{M,1} \leq 6 + 16(\log 2)\sqrt{M} < 16\sqrt{M}$. Thus, for $A_M = \int_0^\Omega n_{M,1}(\xi)\xi^2 d\xi/\pi$,

$$A_M \leq \frac{16\sqrt{M}L^3}{3\pi U^3}, \qquad |g_\phi(u)|^2 \leq D_\phi A_M u^2, \quad |g'_\phi(u)|^2 \leq D_\phi A_M.$$

These common-profile estimates are Cauchy–Schwarz and $|\sin(\xi u)| \leq |\xi u|$. The finite frequency band puts g_ϕ in H^1. Choose even χ_R equal to one on $[-R,R]$, zero outside $[-2R,2R]$, and linear on each intervening interval. Then $|\chi'_R| \leq 1/R$, and direct integration gives

$$\|\chi_R g_\phi\|^2_{1,4} \leq 80 D_\phi A_M R^3, \quad \|Q[s(g_\phi - \chi_R g_\phi)]\|^2 \leq D_\phi A_M T_R, \quad T_R = \int_{|u|>R} s(u)^2 u^2 du.$$

Indeed $\int_{-2R}^{2R} u^2 du = 16R^3/3$ and the derivative square is at most $(56/3)D_\phi A_M R$; multiplication by four gives the constant80 for $R \geq 1$.

The complete theta estimate $s^2 \le 16r^2e^{-r}$, $r = \pi e^{2|u|}$, and the two-arm Jacobian $du = dr/r$ give

$$T_R \le 16\int_{r_R}^{\infty} r^3 e^{-r}dr \le 32 r_R^3 e^{-r_R}, \qquad r_R = \pi e^{2R} \ge 6.$$

The last integral has logarithmic slope at most $-1/2$. Use the trial $f = \chi_R g_\phi$ in (37), with $a = \sqrt{M}$, $\kappa = M^{-1}$ and $r_R = 2\pi U$. The two costs are exactly bounded by the two terms of $K(M,L)D_\phi$. This proves the inequality on every profile, hence as a full observation operator and for every common finite source combination. Projection contraction makes it uniform in all finite original constraints. Both terms decrease on $M \ge 64$. Directed scalar evaluation gives $K(2^{20},1) < 0.091767 < 1/10$, completing the explicit threshold. □

Corollary C.2 (Complete positive continuum-model cap)**.** *Define the positive continuum receiver*

$$P_M^{\mathrm{cont}} = QM_s\left[\frac{8\sqrt{M}}{1+4D^2}(-\cos(UD))_+\right]M_sQ.$$

For the same actual origin observation E,

$$E^*(P_M^{\mathrm{cont}} + M^{-1}I)^{-1}E \preceq 2K(M,L)I.$$

For $L = 1$ this cap is below $1/5$ for every $M \ge 2^{20}$.

Proof. Lemma 4.1 gives error $e_M = 12M^{5/2}e^{-\pi(M-1)} \le 1/(2M)$ on $M \ge 64$. The gate holds at64 using $\pi > 3$, $e > 2$, and the majorant decreases thereafter. Hence $P_M^{\mathrm{cont}} + M^{-1}I \succeq \frac{1}{2}(\sqrt{M}\mathcal{C}_Q + M^{-1}I)$. Inverse order and Theorem C.1 finish the proof. □

D Source-aware positive receiving

The polynomial construction controls complete positive complements in well-conditioned coordinates. A second, compatible finite reduction retains the actual positive and negative parts of the signed symbol. It prices the error on its finite source rather than on the whole inverse.

Fix $M \ge 2$, put $p_M = (m_M)_+$ and $n_M = (-m_M)_+$, and let X_0 be the odd Hilbert space with the polar row and any specified finite exact rows imposed. On this same recipient define the closed positive form

$$P[v] = \int_0^\infty p_M(\xi)|\widehat{sv}(\xi)|^2\frac{\mathrm{d}\xi}{\pi}, \qquad N[v] = \int_0^\infty n_M(\xi)|\widehat{sv}(\xi)|^2\frac{\mathrm{d}\xi}{\pi}.$$

Thus $H_M = P - N$ as forms. In this section P denotes this positive operator, rather than the range projection of Section G. Every inverse and observation below uses the original Hilbert norm and the same exact-row recipient.

Theorem D.1 (Complete inverse exhaustion and finite-source convergence)**.** *Let $\kappa > 0$, $R = (P+\kappa I)^{-1}$, and let P_F restrict the complete positive observation to $[0,F]$. Then $R_F = (P_F+\kappa I)^{-1}$ converges strongly to R, while*

$$\|R_F - R\| = \kappa^{-1}$$

for every finite F. The same norm equality holds for every finite-rank positive lower $P_f \preceq P$. For each fixed finite column map E, however,

$$\|E^*(R_F - R)E\| \longrightarrow 0.$$

Proof. The original graph closure is $\{v : sv \in H_{\log}\}$, with the exact rows imposed, where $H_{\log}$ has Fourier weight $1+\log(2+|\xi|)$. Indeed the digamma series makes $1+a$ comparable to this weight, and $p_M - a$ is bounded. The square root of the weight is submultiplicative up to a constant. Weighted Young's inequality therefore bounds multiplication by a smooth cutoff by its weighted Fourier L^1 norm. Dilated even cutoffs have uniformly bounded such norms. Schwartz density, followed by cutoff convergence in H^1, proves cutoff convergence in $H_{\log}$. On each compact interval division by the smooth positive s, then even mollification, supplies odd smooth compact graph approximants. Multiplication back by s is bounded in $H_{\log}$ because s and its derivatives decrease rapidly. Correct the finitely many row moments by fixed compact odd dual functions; their coefficients tend to zero in the graph norm. This proves the asserted minimal-domain identification. Increasing closed-form convergence now gives strong resolvent convergence [14].

For the norm assertion first use the ambient odd space. Choose a nonzero even smooth compactly supported χ and unit odd packets $h_k = \chi \sin(ku)/\|\chi \sin(k\cdot)\|_2$. These are weakly zero. The finite-band observation is Hilbert–Schmidt, since its squared norm is bounded by $\|s\|_2^2 \int_0^F p_M \, \mathrm{d}\xi/\pi$; hence P_F is compact. Consequently $R_F h_k - \kappa^{-1} h_k \to 0$. The digamma asymptotic [15] gives $p_M(\xi) = \log|\xi| + O_M(1)$. Translating a Schwartz Fourier transform and using the integrable logarithmic majorant shows, for each smooth compact g,

$$\|(\log k)^{-1} p_M(D)(g e^{iku}) - g e^{iku}\|_2 \to 0.$$

Apply this to both sine arms of $v_k = (\log k)^{-1} s^{-2} h_k$. These compact smooth functions lie in the operator domain and satisfy $v_k \to 0$ and $(P+\kappa)v_k - h_k \to 0$. Thus $R h_k \to 0$. Inverse order gives $0 \preceq R_F - R \preceq \kappa^{-1} I$, establishing equality. A finite-rank P_f has the same compactness argument.

For independent exact rows Z_0, the constrained inverse extended by zero is

$$R_X = R_Y - R_Y Z_0 (Z_0^* R_Y Z_0)^{-1} Z_0^* R_Y.$$

Its correction is finite rank. Projecting the weakly zero packets onto X_0 changes them by $o(1)$, so the norm argument survives. Redundant rows are first removed. Finally strong convergence is uniform on the unit sphere of a fixed finite-dimensional source range, proving the last assertion. □

D.1 A full mixed observation residual

Let $P_f \preceq P$ be a bounded finite positive lower, $R_f = (P_f + \kappa I)^{-1}$, and $V = R_f E$. Its actual trial columns are assumed in the original positive form domain. The variational principle gives the full matrix estimate

$$0 \preceq E^* R_f E - E^* R E \preceq V^* (P - P_f) V. \tag{39}$$

To prove the upper, evaluate the complete variational formula for $\langle Ez, REz\rangle$ at Vz; its value is $\langle Ez, R_f Ez\rangle - (P - P_f)[Vz]$. This proves the matrix order on every coefficient vector.

Here is a computable price including all cross terms. Write $P = T^*T$, where $Tv(\xi) = \sqrt{p_M(\xi)}\widehat{sv}(\xi)$ in $L^2((0,\infty), \mathrm{d}\xi/\pi)$. Suppose

$$P_f = T^* Z (D + \rho I)^{-1} Z^* T, \qquad D = Z^* Z, \quad \rho > 0,$$

and Z is supported in $[0, F_0]$. For $F \geq \max(F_0, 1)$, a trial coefficient matrix B, and $r \geq 1$, define

$$A_{\mathrm{in}} = \int_0^F (TV - ZB)^* (TV - ZB) \frac{\mathrm{d}\xi}{\pi}, \quad D_r = [\langle D^r(sv_i), D^r(sv_j)\rangle],$$

and $C_M = 4 + 2\sum_{n\le M}\Lambda(n)/\sqrt{n}$. Then

$$V^*(P - P_f)V \preceq A_{\mathrm{in}} + \rho B^*B + \frac{C_M + \log(3F)}{F^{2r}} D_r. \tag{40}$$

Ridge-square completion gives the first two terms and the full exterior $\int_F^\infty (TV)^* TV \,\mathrm{d}\xi/\pi$. The digamma series, compared with its integral, gives $a(\xi) \le 4 + \frac{1}{2}\log(1 + 4\xi^2)$, hence $p_M(\xi) \le C_M + \log(3\xi)$ for $\xi \ge 1$. The quotient by ξ^{2r} decreases there. Parseval proves the exterior bound with the displayed complete derivative Gram. Actual finite-band columns and row corrections have finite D_r: their weighted profiles are s^2 times bounded-band integrals, complete theta derivatives, and $s^2\sinh(u/2)$.

The finite inverse is explicit. For $P_f = WD_p^{-1}W^*$, with its actual metric $D_p \succ 0$, set $L = W^*W + \kappa D_p$. Then

$$V = \kappa^{-1}\{E - WL^{-1}W^*E\}. \tag{41}$$

On an exactly constrained recipient use the finite inverse-row correction in the proof of Theorem D.1. This retains the row/source, row/positive and all mixed covariances.

Theorem D.2 (Paid source-aware cofinal reduction)**.** *For each j let $\kappa_j > 0$ tend to zero and let $\tau_j, \delta_j \ge 0$. Take the recipient to be the original polar-free space with a finite span of genuine operator nulls removed, and on this recipient let*

$$N_{*,j} = E_j D_{n,j}^{-1} E_j^*, \qquad 0 \preceq N_j - N_{*,j} \preceq \tau_j I, \qquad D_{n,j} \succ 0.$$

Suppose $P_{f,j} \preceq P_j$ is a finite positive lower and, for $0 < \gamma_j < 1/2$,

$$E_j^*(P_j + \kappa_j)^{-1}E_j \preceq (1 - 2\gamma_j)D_{n,j}, \qquad V_j^*(P_j - P_{f,j})V_j \preceq \gamma_j D_{n,j}.$$

The second inequality may be certified by (40). If the complete raw arithmetic error is δ_j and $\tau_j + \delta_j \to 0$, then

$$q_H[h] \ge -(\kappa_j + \tau_j + \delta_j)\|h\|_2^2$$

on the corresponding genuine-null complement. These certificates, with genuine operator nulls restored, imply RH.

Proof. Equation (39) gives $E_j^*(P_{f,j}+\kappa_j)^{-1}E_j \preceq (1-\gamma_j)D_{n,j} \preceq D_{n,j}$. The positive Schur complement is equivalent to $N_{*,j} \preceq P_{f,j} + \kappa_j I$. Add the negative approximation and complete raw errors. The original closed-form domain and exact null transport in Theorem 2.2 prove the conclusion. □

Corollary D.3 (Polynomial excess above one)**.** *On the same original polar-free recipient with finitely many genuine operator nulls removed, suppose instead that $\eta_M, \tau_M, \delta_M \ge 0$, $\kappa_M > 0$, $0 \preceq P_{f,M} \preceq P_M$, and the full finite-source cap satisfies*

$$E_M^*(P_{f,M} + \kappa_M I)^{-1}E_M \preceq (1 + \eta_M)D_{n,M}, \qquad 0 \preceq N_M - E_M D_{n,M}^{-1}E_M^* \preceq \tau_M I.$$

Let B_M be an upper bound for $\|s\|_\infty^2(|m_0| + 2\sum_{n\le M}\Lambda(n)/\sqrt{n})$ and let δ_M pay the complete raw tail. Then, on the entire original domain,

$$q_H[h] \ge -\left\{\delta_M + \kappa_M + \frac{\tau_M + \eta_M B_M}{1 + \eta_M}\right\}\|h\|_2^2. \tag{42}$$

In particular complete caps on a cofinal sequence with $\kappa_M = \eta_M = \tau_M = 1/M$ give original allowances $O(M^{-1/2})$ tending to zero, and hence RH.

Proof. The cap is equivalent to $N_* \preceq (1+\eta)(P_f+\kappa I)$, hence $N \preceq (1+\eta)P+tI$, where $t=(1+\eta)\kappa+\tau$. The exact symbol gives $0 \preceq N \preceq BI$. Writing $q = P - N + W$, $W \succeq -\delta I$, the following identity of forms proves the absorption without bounding the unbounded P:

$$(1+\eta)(q+\varepsilon I) = [(1+\eta)P + tI - N] + \eta[BI - N] + (1+\eta)[W+\delta I], \quad \varepsilon = \delta + \frac{t+\eta B}{1+\eta}.$$

Each summand is nonnegative. Genuine-null transport returns the bound to the whole original domain. The elementary prime-power estimate (78) and $\|s\|_\infty < 3$ permit the choice $B_M = 9R_M = O(\sqrt{M})$, with R_M from (46). Substitution yields (42) and its stated rate. All finite-source, ridge and continuous errors are included in the cap or in τ_M. □

The cap in this alternative may lie above one; its excess is paid by the complete bounded negative part. It uses a different regularization/cutoff schedule from Theorem H.3. The source-residual enclosure (39) can fund the finite positive receiver against either target.

D.2 Complete strip price and integration collar

For $f_i = sv_i$ put $A_\eta = [\langle f_i(\cdot - i\eta), f_j(\cdot - i\eta)\rangle]$, with $\eta = 1/8$. For actual finite observation and genuine-row columns these analytic weighted profiles give

$$\int_F^\infty (TV)^* TV \frac{\mathrm{d}\xi}{\pi} \preceq 2[C_M + \log(3F)]e^{-F/4} A_\eta, \qquad F \geq 4. \tag{43}$$

Contour shifting gives $\widehat{f(\cdot - i\eta)} = e^{\eta\xi}\widehat{f}$; full Plancherel and the positive-half measure pay the factor two. The scalar multiplier $[C_M + \log(3\xi)]e^{-\xi/4}$ decreases for $\xi \geq 4$, proving the full matrix order.

For completeness, usable whole-source constants are

$$\|s^2(\cdot \pm i/8)\|_2 < 20, \quad \|\psi^{(m)}(\cdot \pm i/8)\|_2 \leq 32\,16^m m!, \quad \|s^2(\cdot \pm i/8)\sinh((\cdot \pm i/8)/2)\|_2 \leq 32.$$

Here is a direct complete-tail proof. Put $x = |\Re z|$, $r = \pi e^{2x}$. On $|\Im z| \leq 1/4$, the theta atom series and reflection give $|\psi(z)| \leq 7e^{x/2}r^2e^{-7r/8}$: each atom is at most $6e^{x/2}(\ell^2 r)^2 e^{-7\ell^2 r/8}$ and successive ratios are below $1/8$. On the $1/8$ line, the sharper cosine bound $31/32$ gives $|s^2(z)| \leq 28r^2e^{-31r/32}$. The squared two-arm integral is at most $4704(16/31)^4 < 400$. Cauchy's circle of radius $1/16$ bounds the derivative by $14\,16^m m! e^{x/2}r^2e^{-3r/4}$; its squared two-arm integral is at most $196\int_0^\infty r^4e^{-3r/2}\,\mathrm{d}r < 1024$. The polar bound follows from $|\sinh(z/2)| \leq e^{x/2}$ and the same integral comparison.

An observation feature $e = s\int_0^B a(\xi)\sin(\xi u)\,\mathrm{d}\xi/\pi$ therefore has weighted strip norm at most $20e^{B/8}\int_0^B |a|\,\mathrm{d}\xi/\pi$. A genuine derivative-null feature has weighted profile $\psi^{(2i+3)} - \psi^{(2i+1)}/4$, so the displayed derivative bounds apply. If $sV = gA$ uses b such features and their strip norms are bounded by k_ℓ, then

$$A_\eta \preceq bA^* \operatorname{diag}(k_\ell^2)A.$$

This finite matrix includes the actual inverse metrics and row coefficients. Under the explicit hypothesis $A_\eta \preceq e^{B_j/4+L_j}D_{n,j}$, the exterior is at most

$$2[C_{M_j} + \log(3F_j)]e^{-(F_j-B_j)/4+L_j}D_{n,j}.$$

For $\gamma_j = c\kappa_j^\rho$, $0 < \rho < 1$, it suffices that

$$\frac{F_j - B_j}{4} \geq L_j + \rho\log(1/\kappa_j) + \log\frac{8[C_{M_j} + \log(3F_j)]}{c}.$$

If $L_j = O(U_j)$, $\log B_j = O(U_j)$, $B_j \geq 1$ and $\log C_{M_j} = O(\log U_j)$, with $U_j = \log(1/\kappa_j)$, a sufficiently large fixed C permits $F_j = B_j + CU_j$. This is an integration collar for the chosen receiver. The interior residual in (40) and the complete cap in Theorem D.2 retain their stated hypotheses.

D.3 Null leakage in the same positive metric

Let T now be an isometry onto finitely many genuine operator nulls inside a base recipient X_0, before their removal; write $Q_T = I - TT^*$, $C = P + \kappa I$, $S = T^*CT$. This use of T is distinct from the observation map above. Suppose $N_* = ED_n^{-1}E^*$, $0 \preceq N - N_* \preceq \tau I$, and the raw tail has norm at most $\delta < \kappa$. Put

$$A_0 = D_n^{-1}E^*T, \quad B_* = Q_T E A_0, \quad s_0 = \kappa - \delta + \lambda_{\min}(T^*N_*T) > 0.$$

For a finite positive lower $P_f = WD_p^{-1}W^* \preceq P$ on the *base* recipient, define

$$J_B = B_*^*B_* - B_*^*W(W^*W + \kappa D_p)^{-1}W^*B_* \succeq 0.$$

Then, for every $t > 0$,

$$\kappa\|T^*C^{-1}T - S^{-1}\| \le s_0^{-2}\|(1+t)J_B + (1+1/t)(\tau+\delta)^2 I\|. \tag{44}$$

Indeed the exact null identity $HT = 0$ gives $PT = NT - W_{\text{tail}}T$. Thus $B = Q_T PT$ differs from B_* by at most $\tau + \delta$, and $S \succeq s_0 I$. The exact trial-residual identity is

$$T^*C^{-1}T - S^{-1} = S^{-1}B^*C^{-1}BS^{-1}.$$

Inverse order, quadratic Young's inequality and (41) give $\kappa B^*C^{-1}B \preceq \widetilde{J}$, where $\widetilde{J} = (1+t)J_B + (1+1/t)(\tau+\delta)^2 I$. This proves (44). More sharply the matrix-relative target $\widetilde{J} \preceq \eta S^2$ yields debit at most η by congruence with S^{-1}; no commutation is assumed. The finite lower must be certified before removing T, so all its mixed row covariances remain present.

These null-source tests are compatible with the full-source cap but have distinct recipients. For example $P = \operatorname{diag}(1,1)$, $N = \operatorname{diag}(1,2)$, $T = e_1$ gives $HT = 0$ and zero null debit for every κ, while the second direction has negative signed form. The complete cap in Theorem D.2 keeps that direction in its test.

Finally a trial inverse from a different positive reference has a paid exact bridge. For $A \succeq 0$, bounded self-adjoint $K = P - A$, $V_A = (A+\kappa)^{-1}Z$ and $G_A = Z^*V_A$, square completion gives

$$Z^*(P+\kappa)^{-1}Z \preceq G_A - V_A^*KV_A + \kappa^{-1}V_A^*K^2V_A.$$

The identity uses residual $(P+\kappa)V_A - Z = KV_A$, followed by $(P+\kappa)^{-1} \preceq \kappa^{-1}I$. It preserves the full covariance of the changed reference.

D.4 A complete 547-source inverse-action example

The source-aware viewpoint also gives a quantitative action certificate for a fixed finite positive reference. We use the cutoff-thirteen head tuple comprising its first 46 negative observations of the signed-frequency companion [4]: 547 negative columns E, 1680 positive columns W, and the original polar row with eleven genuine complete-operator null rows. Every column is projected through the full original twelve-row Gram. Let $D_n, D_p \succ 0$ be the actual midpoint frequency metrics plus their rational ridges and put

$$\kappa = 5 \cdot 10^{-19}, \quad B = \kappa I + WD_p^{-1}W^*, \quad C = E^*W, \quad L = W^*W + \kappa D_p.$$

The full frequency quadrature uncertainty is funded within half each ridge, so $B \preceq \kappa I + P_{13}$ in the original metric. Use the exact 1680-by-547 dyadic proposal F in the included head data and set

$$R = C^* - LF, \quad Y = \kappa^{-1}(E - WF), \quad Q_F = \kappa^{-1}(E^*E - CF - F^*C^* + F^*W^*WF + \kappa F^*D_pF).$$

Proposition D.4 (Full finite-reference covariance and action errors)**.** *For this complete tuple there is a saved full mixed covariance upper $H_R \succeq R^* D_p^{-1} R$ satisfying $H_R \prec 9.36 \cdot 10^{-96} D_n$. In the same coefficient metric the following strict bounds hold:*

$$\begin{aligned}
0 \preceq Q_F - E^* B^{-1} E &\prec 3.745 \cdot 10^{-59} D_n, \\
(Y - B^{-1}E)^* B (Y - B^{-1}E) &\prec 3.745 \cdot 10^{-59} D_n, \\
(BY - E)^* (BY - E) &\prec 1.685 \cdot 10^{-57} D_n, \\
(Y - B^{-1}E)^* (Y - B^{-1}E) &\prec 7.490 \cdot 10^{-41} D_n.
\end{aligned}$$

The reference is the stated bounded finite B; inverse order gives $E^*(P_{13} + \kappa)^{-1} E \preceq E^* B^{-1} E$.

Proof. Woodbury square completion, with the actual rectangular F, gives

$$Q_F - E^* B^{-1} E = \kappa^{-1} R^* L^{-1} R \preceq \kappa^{-2} R^* D_p^{-1} R, \qquad BY - E = \kappa^{-1} W D_p^{-1} R.$$

Moreover $W^* B^{-1} W \preceq D_p$. Thus, for $e = Y - B^{-1}E$,

$$e^* B e \preceq \kappa^{-2} R^* D_p^{-1} R, \qquad e^* e \preceq \kappa^{-3} R^* D_p^{-1} R.$$

The complete positive-frame trace bounds $\|D_p^{-1/2} W^* W D_p^{-1/2}\| < 44.991$, which pays the action residual separately.

Here is the complete numerical error contract. Reconstruct all 2227 ambient sources and all 2227-by-12 original row crosses, then short by the actual enclosed 12-by-12 row Gram. The source construction in [4] gives a joint error $\delta < 4.129 \cdot 10^{-58}$ in metric $\mathrm{diag}(D_n, D_p)$, including both sine images and the complete kernel, root, reflected and local errors. Let R_0 be the modeled residual and $H_0 = R_0^* D_p^{-1} R_0$. Applying that joint error to the full coefficient map $[I; -F]$ proves

$$H_R = 2H_0 + 2\delta^2 (D_n + F^* D_p F) \succeq R^* D_p^{-1} R.$$

All mixed entries are formed before taking the positive normalized trace. The public saved matrix and actual D_n blocks give, by directed arithmetic,

$$\mathrm{Tr}(D_n^{-1} H_R) < 9.36 \cdot 10^{-96}.$$

Since $H_R \succeq 0$, this proves its full relative upper. The preceding identities, the frame trace and the exact powers of κ prove the table.

The source reconstruction used 768-bit directed arithmetic; its separate sufficient head-cap congruence used 2048 bits. The new source reconstruction and residual assembly took approximately 448 seconds. The reader's saved-array verification reparses all 598418 entries of the two stored 547-square matrices, verifies finite symmetric enclosures and recomputes the actual D_n-normalized residual trace; it is a saved-array replay, rather than another source reconstruction. Its slightly widened durable decimal bounds are those stated above. The public supplement contains the full arrays, true frequency-metric blocks, portable readback and exact algebra controls. Full source regeneration uses the included scientific input subset, analytic source/error supplement and portable producer. □

The distinction between the finite reference and the complete positive observation is retained throughout. A two-sided complete-positive inverse comparison uses the additional source-observation residual of (39); the finite-reference action estimates above supply their own bounded application of the method.

D.5 A necessary full-cap margin

The full cap also has a quantitative necessary test independent of null-source conditioning. Let $\mathcal{L}$ be a $(D+1)$-dimensional genuine operator-null space, and let the exact recipient have codimension at most D in the polar space. Suppose $N[v] \geq b_D\|v\|^2$ on $\mathcal{L}$, with $b_D > 0$. If

$$N_* \preceq (1-\gamma)(P+\kappa I), \qquad 0 \preceq N - N_* \preceq \tau I,$$

on that recipient, and the complete raw tail has norm at most δ, then necessarily

$$\gamma \leq \frac{\kappa+\tau+\delta}{b_D+\kappa+\delta}. \tag{45}$$

Indeed dimension gives a unit $v \in \mathcal{L}$ in the recipient. The exact null identity gives $P[v] \leq N[v]+\delta$. Together with $N_* \succeq N - \tau I$, the cap gives

$$N[v] - \tau \leq (1-\gamma)(N[v]+\delta+\kappa),$$

which proves (45). The lower observation floors developed for genuine derivative nulls in [1] can be inserted without changing this argument. The estimate concerns the entire signed cap, whereas (44) prices the inverse error on its specified null source.

E Complete observation under moving arithmetic cutoffs

The finite-source criteria above admit a complementary reduction on the *entire* negative observation space. Its cutoff is an operator energy, rather than a Fourier window or a finite-rank projection. The useful new input is a complete-positive source estimate uniform in the arithmetic cutoff, the number of source profiles and the number of genuine null rows.

Let X_M be the original odd polar space with any finite subspace K_M of $\ker H$ removed. All projections and norms in this section are original Hilbert ones. Let $P = P_{M,K_M}$ and $N = N_{M,K_M}$ be the restrictions of the complete positive and negative forms to this recipient. Put

$$\begin{aligned} R_M &= 6 + 8(\log 2)\sqrt{M}, \qquad & B_M &= 9R_M, \\ J_M &= 3(84R_M + 310 + 2^{27}), \qquad & C_M^{\mathrm{src}} &= R_M J_M^2. \end{aligned} \tag{46}$$

Here $\|s\|_\infty < 3$, $0 \leq n_M \leq R_M$ and $\|N\| \leq B_M$. Appendix L proves these constants from the complete theta series and an elementary prime-power bound.

Theorem E.1 (Uniform complete-positive source moment)**.** *Let $\phi_1, \dots, \phi_r$ be any finite family in*

$$\mathcal{Y}_M = L^2(\{\xi > 0 : n_M(\xi) > 0\}, n_M(\xi)\,\mathrm{d}\xi/\pi),$$

and let $D \succ 0$ dominate their true joint Gram matrix in this space. Writing Q_M for the original orthogonal projection onto X_M, define

$$g_i(u) = \int_0^\infty n_M(\xi)\phi_i(\xi)\sin(\xi u)\frac{\mathrm{d}\xi}{\pi}, \qquad E_i = Q_M(sg_i).$$

Then every E_i belongs to $\operatorname{Dom} P$*, and*

$$E^*E \preceq B_M D, \qquad (PE)^*(PE) \preceq C_M^{\mathrm{src}} D. \tag{47}$$

The constants are independent of r and $\dim K_M$*. In particular, for $\kappa, \Lambda > 0$,*

$$0 \preceq E^* 1_{(\Lambda,\infty)}(P)(P+\kappa)^{-1}E \preceq \frac{C_M^{\mathrm{src}}}{\Lambda^2(\Lambda+\kappa)} D. \tag{48}$$

Proof. Appendix L proves, for an odd F-bandlimited g,

$$\|Q_M(sg)\| \le 3\|g\|, \qquad \|PQ_M(sg)\| \le 3[51R_M + 33\log(2+F) + 210 + 2^{27} + \delta_M]\|g\|.$$

The complete negative support lies in $[-\Omega_M, \Omega_M]$, with $\Omega_M = e^{R_M+1}$. For every common coefficient vector z, Plancherel gives $\|g_z\|^2 \le R_M z^* D z$ and $\log(2+\Omega_M) < R_M + 3$. Since $\delta_M < 1$, the displayed graph constant is below J_M. This proves both full matrix inequalities, including all mixed terms. The spectral theorem gives

$$\frac{1_{\lambda>\Lambda}}{\lambda+\kappa} \le \frac{\lambda^2}{\Lambda^2(\Lambda+\kappa)},$$

which proves (48). The second moment uses $E_i \in \operatorname{Dom} P$ and does not require $E_i \in \operatorname{Dom} P^2$. □

Define the complete observation $T_M : X_M \to \mathcal{Y}_M$ by

$$(T_M h)(\xi) = \int_{\mathbb{R}} s(u)h(u)\sin(\xi u)\,\mathrm{d}u.$$

Thus $T_M^*\phi = Q_M(sg_\phi)$ and $N = T_M^* T_M$. Let

$$\kappa_M = M^{-1}, \quad \Lambda_M = (4MC_M^{\mathrm{src}})^{1/3}, \quad L_M = T_M 1_{[0,\Lambda_M]}(P)(P+\kappa_M)^{-1}T_M^*. \tag{49}$$

Theorem E.2 (Complete-observation energy criterion)**.** *For every $M \ge 2$ and every such finite genuine-null choice,*

$$0 \preceq T_M 1_{(\Lambda_M,\infty)}(P)(P+\kappa_M)^{-1}T_M^* \preceq \frac{1}{4M} I_{\mathcal{Y}_M}. \tag{50}$$

If on an unbounded sequence of integers M the complete low-energy cap

$$L_M \preceq \left(1 + \frac{3}{4M}\right) I_{\mathcal{Y}_M} \tag{51}$$

holds, then on the entire original domain

$$q_H[h] \ge -\left[\delta_M + \frac{1}{M} + \frac{B_M}{M+1}\right]\|h\|_2^2. \tag{52}$$

Conversely, $H \succeq 0$ implies (51) *for every $M \ge 2$ and every finite genuine-null choice. The cofinal energy criterion is therefore equivalent to positivity of this original H and, by Theorem 2.2, to RH. Its original allowance is $O(M^{-1/2})$, and $\Lambda_M = O(M^{5/6})$.*

Proof. Theorem E.1 applies to every finite tuple of profiles. On a dense profile core it bounds both $T_M^*\phi$ and $PT_M^*\phi$ by fixed multiples of $\|\phi\|_{\mathcal{Y}_M}$. For a convergent profile sequence, both images are Cauchy. Closedness of P gives graph-norm completion to every $\phi \in \mathcal{Y}_M$, with the same bounds. Thus (48) extends to the entire observation space. The choice of Λ_M gives (50); its high and low pieces are the exact functional-calculus decomposition of the same resolvent.

Adding (51) gives $T_M(P+\kappa_M)^{-1}T_M^* \preceq c_M I$, $c_M = 1 + 1/M$. The equivalent form inequality is $N \preceq c_M(P + \kappa_M I)$. Rearrange using the bounded N, rather than introducing an adverse multiple of unbounded P:

$$P - N \succeq -\kappa_M I - (1 - c_M^{-1})N \succeq -\left[\frac{1}{M} + \frac{B_M}{M+1}\right] I.$$

Add the complete original-norm raw tail. Exact genuine-null transport from Theorem 2.2 proves (52) on all original V. Its allowance tends to zero.

For the converse, $H = P - N + W_M \succeq 0$ gives $N \preceq P + \delta_M I$. The bound (8) satisfies $\delta_M < 1/M$ for all $M \geq 2$. A rational admission at $M = 2$ is

$$2 \cdot 451 \cdot 9(10/27)^9(10/13) < 1,$$

using $\pi > 31/10$, $e > 27/10$ and $e^{3/10} > 13/10$. The consecutive majorant ratio is at most $(8/3)e^{-\pi} < (8/3)(3/8)^3 < 1$. Hence the complete observation cap is at most I, and so is its low-energy summand. □

The spectral space $1_{[0,\Lambda_M]}(P)X_M$ may be infinite dimensional. Theorem E.2 specifies a complete operator comparison; the finite-source Schur criterion and source-residual enclosures of Section D supply separate constructive interfaces. The energy scale does not replace the Fourier support Ω_M or the rank of a chosen positive receiving frame.

Corollary E.3 (A finite cutoff interval from the M16 certificate)**.** *If a complete original-domain certificate gives $H \succeq -aI$ with $a = 5.289 \cdot 10^{-19}$, as in the signed-frequency companion [4], then for every integer $2 \leq M \leq 10^{18}$ and every finite genuine-null choice,*

$$T_M(P + M^{-1}I)^{-1}T_M^* \preceq I, \qquad L_M \preceq I.$$

Proof. At every arithmetic cutoff the same original bound gives $N_M \preceq P_M + (a + \delta_M)I$. It suffices that $M(a+\delta_M) < 1$. For $2 \leq M \leq 15$, the decreasing majorant used above gives $M\delta_M < 819/1000$, while $Ma < 1/1000$. For $M \geq 16$,

$$M\delta_M \leq 451 \cdot 16 \cdot 17^2(3/8)^{51} < 1/4, \qquad Ma \leq 5289/10000 \quad (M \leq 10^{18}).$$

These rational bounds prove the assertion. The bridge is the complete original-domain inequality, rather than an identification of the M16 finite matrices with matrices at another cutoff. □

This finite-interval consequence and the unbounded-sequence hypothesis in Theorem E.2 have different quantifiers. The former follows from one fixed global allowance; the latter is the complete low-energy comparison that would make the allowance vanish.

E.1 True observation compression and a paid omitted trace

The complete energy criterion has a finite observation-space interface. It retains the true source columns in its finite head and pays the entire remaining covariance, including mixed blocks, by positivity. Keep $\mathcal{Y}_M, T_M, P, \kappa = M^{-1}$ and $\Omega = e^{R_M+1}$ from Section E, and write $E = T_M^*$ and $\nu = n_M(\xi)\,\mathrm{d}\xi/\pi$ on $[0, \Omega]$. Put

$$W = R_M\Omega/\pi, \quad a_* = 31/32, \quad t(L) = 28e^{-a_* r_L}(r_L/a_* + a_*^{-2}), \quad r_L = \pi e^{2L}.$$

The complete two-arm bound in Appendix L gives

$$\nu([0,\Omega]) \leq W, \qquad \|s\|_2^2 \leq 28/a_*^2 < 32, \qquad \int_{|u|>L} s(u)^2\,\mathrm{d}u \leq t(L).$$

The last integral follows by substituting $r = \pi e^{2|u|}$ in $s^2 \leq 28r^2e^{-a_* r}$; both arms together have Jacobian $\mathrm{d}r/r$.

Theorem E.4 (True finite head and complete observation return)**.** *Choose $L \geq 1$ with $Wt(L) \leq 1/(8M^2)$ and an integer $J \geq 1$ such that*

$$N_J = 2J + 1 \geq 2e\Omega L, \qquad N_J \geq \log_2(16M\sqrt{W}). \tag{53}$$

Let Π be the true orthogonal projection in $\mathcal{Y}_M$ onto $\operatorname{span}\{\xi, \xi^3, \ldots, \xi^{2J-1}\}$*, and let $U : \mathbb{C}^J \to \mathcal{Y}_M$ be an isometry onto that span. Then*

$$\|E(I - \Pi)\|_{\mathrm{HS}}^2 \leq \frac{1}{4M^2}. \tag{54}$$

For $R_{\mathrm{low}} = 1_{[0,\Lambda_M]}(P)(P + \kappa)^{-1}$ define the true *finite matrix $G_M = (EU)^* R_{\mathrm{low}}(EU)$. The finite-head condition*

$$G_M \preceq \left(1 + \frac{1}{2M}\right) I_J \tag{55}$$

implies the complete low-energy cap (51)*. Its validity along an unbounded sequence of admitted cut-offs and finite genuine-null complements is equivalent to original H positivity, with allowance* (52)*. Sufficient choices have $J = O(\Omega \log M) = \exp(O(\sqrt{M}))$.*

Proof. The ambient sine-source kernel is Hilbert–Schmidt, with square at most $W\|s\|_2^2$. Applying the original physical orthogonal projections preserves this property. Replace its sine on $|u| \leq L$ by $p_J(\xi u) = \sum_{j=0}^{J-1} (-1)^j (\xi u)^{2j+1}/(2j+1)!$ and set the approximant to zero outside. The resulting source map E_J has rank at most J and

$$\|E - E_J\|_{\mathrm{HS}}^2 \leq W \left[t(L) + 32 \left\{ \frac{(\Omega L)^{N_J}}{N_J!} \right\}^2 \right]. \tag{56}$$

Indeed p_J is also the sine Taylor polynomial through degree $2J$, so its real remainder is bounded by $|\xi u|^{N_J}/N_J!$. Orthogonal polar and genuine-null projections contract the whole error. The observation dependence of E_J lies in $\operatorname{ran}\Pi$; thus $E_J(I - \Pi) = 0$. The elementary bound $N! \geq (N/e)^N$ and (53) make the remainder at most $2^{-N_J} \leq 1/(16M\sqrt{W})$. Equation (56) is then at most $1/(4M^2)$, proving (54).

For any positive $0 \preceq R \preceq \kappa^{-1} I$, the operator $\mathcal{L} = E^* R E$ is positive trace class. Positivity bounds its mixed quadratic term between $x \in \operatorname{ran}\Pi$ and $y \in \ker\Pi$ by the geometric mean of its diagonal terms. Scalar Cauchy–Schwarz consequently gives

$$\|\mathcal{L}\| \leq \|\Pi\mathcal{L}\Pi\| + \operatorname{tr}((I - \Pi)\mathcal{L}(I - \Pi)) \leq \|\Pi\mathcal{L}\Pi\| + \kappa^{-1}\|E(I - \Pi)\|_{\mathrm{HS}}^2. \tag{57}$$

This pays all mixed blocks without declaring them zero. Use $R = R_{\mathrm{low}}$: the omitted trace is at most $1/(4M)$, and (55) gives $\|\mathcal{L}\| \leq 1 + 3/(4M)$. Theorem E.2 now returns the original allowance. Conversely $H \succeq 0$ gives complete observation cap at most I, so every such true compression satisfies (55).

The choices terminate because $t(L) \to 0$. Since $\log W = O(\sqrt{M})$, a polynomially growing r_L suffices, giving $L = O(\log M)$; the two requirements on N_J then give the asserted sufficient rank. The monomial Gram is positive definite because $m_M(0) < 0$ and hence ν is positive on an interval near zero. This proves existence of U; numerical evaluation must certify its actual observation metric. □

The finite matrix uses EU, not the Taylor-surrogate $E_J U$. This true compression is what permits a quadratic omitted-error payment. The rank bound concerns the observation compression

and makes no rank assertion about $1_{[0,\Lambda_M]}(P)$. For upper certification one may instead bound the larger full-resolvent head $V^*(P+\kappa)^{-1}V$, $V = EU$. If trial columns Y lie in $\operatorname{Dom} P$ and $\mathcal{A} = V - (P+\kappa)Y$ is their *complete* action residual, then

$$\|V^*(P+\kappa)^{-1}V - \operatorname{Herm}(V^*Y)\| \le \kappa^{-1}\|V\|\,\|\mathcal{A}\|.$$

This follows from $(P+\kappa)^{-1}V - Y = (P+\kappa)^{-1}\mathcal{A}$. It supplies a paid upper in the true source coordinates. A residual $V - BY$ for a different finite reference still leaves $(P+\kappa-B)Y$ to be enclosed; all sources, norms and operators in the complete head estimate must match.

F The hard-cutoff mean and intrinsic receiver rank

The original hard-cutoff receivers have a useful independent description. On the polar space set

$$\mathcal{C}_0 = \frac{4}{\pi} Q_b M_s \beta(D) M_s Q_b, \quad \beta(\xi) = (1+4\xi^2)^{-1/2}, \quad \mathcal{R} = (\ker H)^{\perp}.$$

The two references $\mathcal{C}_0$ and $\mathcal{C}_Q$ have Fourier orders one and two, respectively. The next estimates concern the hard cutoff. For the right-continuous prime-power sum $\Psi_\Lambda(x) = \sum_{n\le x}\Lambda(n)$ write $E_\Lambda(x) = \Psi_\Lambda(x) - x$ and

$$e_\Lambda(M) = \frac{|E_\Lambda(M)|}{M} + \frac{1}{\sqrt{M}}\int_1^M |E_\Lambda(x)| x^{-3/2}dx.$$

Theorem F.1 (Complete common mean and one-sided rate)**.** *The operators $N_M/\sqrt{M}$ converge strongly to $\mathcal{C}_0$. On $\operatorname{Dom} H$, $P_M h/\sqrt{M} \to \mathcal{C}_0 h$, and the selfadjoint positive operators have the same strong-resolvent limit. For $F \ge 1$, $0 < \eta < 1/2$ and $M^\eta \ge \sqrt{(1+2\eta)/(1-2\eta)}$,*

$$\frac{N_M}{\sqrt{M}} \succeq \mathcal{C}_0 - \rho_M I, \tag{58}$$

$$\rho_M = 18(1+F)e_\Lambda(M) + \frac{9(12+\log(3F))}{\sqrt{M}} + \frac{36}{\sqrt{1+4F^2}} + \frac{30M^{-\eta}}{1-2\eta}.$$

The same one-sided estimate compresses to every finite row complement. Classical PNT permits $F \to \infty$ with $\rho_M \to 0$.

Proof. Put $z = 1/2 + i\xi$, $U = \log M$. Abel summation, with its lower endpoint retained, gives

$$\frac{1}{\sqrt{M}}\sum_{n\le M}\Lambda(n)n^{-1/2+i\xi} = \frac{e^{i\xi U}}{z} + \frac{z-1}{z\sqrt{M}} + e^{i\xi U}\frac{E_\Lambda(M)}{M} - \frac{z-1}{\sqrt{M}}\int_1^M E_\Lambda(x)x^{-3/2+i\xi}dx.$$

As $|z-1| = |z|$, the difference from $e^{i\xi U}/z$ on $|\xi| \le F$ is at most $M^{-1/2} + (1+F)e_\Lambda(M)$. The positive-part map is 1-Lipschitz. Using $|m_0| < 6$ and $a(\xi) \le 4 + \log(3F)$, the normalized negative symbol differs on this interval by at most $d = 2(1+F)e_\Lambda(M) + (12+\log(3F))/\sqrt{M}$ from

$$v_M(\xi) = 4\beta(\xi)(\cos(\xi U - \arctan(2\xi)))_+.$$

The absolutely convergent Fourier series of $\cos_+$ has constant coefficient $1/\pi$ and nonconstant complex coefficient sum $1/2 + 1/\pi$. Riemann–Lebesgue gives the local weak multiplier mean $4\beta/\pi$. The

normalized symbols are uniformly bounded by (46); their local weak means therefore extend against every L^1 frequency test. For fixed input, truncate its Fourier profile $\widehat{sh}$ to $[-F,F]$. Multiplication by s after a finite frequency projection is Hilbert–Schmidt, so it converts the weak convergence on that band to strong convergence. The input exterior tends to zero uniformly in M. This proves the negative strong limit.

For a quantitative continuum bound, the positive lth harmonic continues analytically on $|\Im z| < 1/2$ as

$$e^{ilUz}(1-2iz)^{(l-1)/2}(1+2iz)^{-(l+1)/2}.$$

Shift to $\Im z = \eta$ on compact smooth physical tests. The multiplier bound is $(1-4\eta^2)^{-1/2}q^l$, $q = M^{-\eta}\sqrt{(1+2\eta)/(1-2\eta)} \le 1$. The two physical factors acquire opposite exponential weights and $\|e^{\eta u}s\|_\infty \|e^{-\eta u}s\|_\infty < 9$. Indeed the complete envelope bounds their squares by $169x^{2+\eta}e^{-\pi x} \le 169e^{-\pi} < 9$, $x = e^{2|u|} \ge 1$. Vertical contour ends vanish by smooth compact integration by parts; density extends the resulting bounded operators to all inputs. Summing the harmonics costs at most $30M^{-\eta}/(1-2\eta)$. Discard the actual positive negative-symbol exterior beyond F and pay the omitted continuum by $36/\sqrt{1+4F^2}$. The retained symbol error costs $9d$. This proves (58).

Finally $P_M = N_M + H - W_M$ on $\operatorname{Dom} H$ gives the positive limit there. The resolvent identity at $(\mathcal{C}_0 + i)h$ and the uniform resolvent norm bound extend it by density to strong resolvents. PNT makes $e_\Lambda(M) \to 0$ by splitting its integral at a fixed large threshold; for example take $F = \min(M^{1/4}, e_\Lambda(M)^{-1/2})$. □

Theorem F.2 (Intrinsic hard-cutoff receiver rank)**.** *There is $c_* > 0$ such that*

$$\lambda_j(P_{\mathcal{R}}\mathcal{C}_0|_{\mathcal{R}}) \ge c_*/j, \qquad j \ge 1.$$

Consequently, if a rank-J positive receiver B_M on the complement of any *finite genuine-null space K_M satisfies $N_{M,K_M} \preceq B_M + \kappa_M I$, then*

$$J + 1 \ge \frac{c_*}{\rho_M + \kappa_M/\sqrt{M}}. \tag{59}$$

With a classical PNT remainder one may choose $\rho_M = O(e^{-c\sqrt{\log M}})$. Along $\kappa_M \to 0$, the required receiver rank is therefore at least $c_1 e^{c_2\sqrt{\log M}}$ for sufficiently large M.

Proof. The operator $\mathcal{C}_0$ is compact: its finite-frequency sandwiches are Hilbert–Schmidt, and the omitted multiplier has norm at most $(36/\pi)\sup_{|\xi|>F}\beta(\xi) \to 0$. Appendix M proves the eigenvalue floor using the original-domain critical-zero representers and complete Schwartz-weight Gram estimates. On its $(J+1)$-dimensional protected span choose a unit vector in $\ker B_M$. It belongs to every genuine-null complement. Combining the floor with (58) gives (59). The classical error $|E_\Lambda(x)| \le Axe^{-c\sqrt{\log x}}$ gives $e_\Lambda(M) = O(e^{-c_0\sqrt{\log M}})$ by splitting at $\sqrt{M}$. Choose $F = e^{c_0\sqrt{\log M}/2}$ and fixed $\eta = 1/4$. □

For arbitrary finite, possibly nongenuine row spaces of dimension d, an elementary version is also useful. If $\sigma_R = \min_{|u|\le R} s^2$ and $a_R = 4\sigma_R/[\pi\sqrt{1+4\pi^2/R^2}]$, then $\lambda_j(\mathcal{C}_0) \ge a_R/(j+1)$ and $J + d + 2 \ge a_R/(\rho_M + \kappa_M/\sqrt{M})$. Take the first $j+1$ odd Dirichlet sine modes f on $[-R,R]$, set $h = f/s$ and impose the single polar moment. Fourier Jensen gives

$$\langle f, (1+4D^2)^{-1/2}f\rangle \ge \frac{\|f\|^2}{\sqrt{1+4\|f'\|^2/\|f\|^2}},$$

and $\|f'\| \le \pi(j+1)\|f\|/R$. Minmax and codimension interlacing give the displayed assertions. These are representation costs for the hard-cutoff split; the smoothed reference of Section B has its own source geometry.

G The full residual covariance and the signed return

The finite receiving gate must retain the source covariance between the phase space, the exact rows and the complementary carriers. The following exact Schur identity gives the interface. It is classical finite-rank elimination specialized to the original constrained recipient [12, 13].

Theorem G.1 (Residual-covariance signed gate)**.** *Let $Z : \mathbb{C}^d \to \mathcal{H}$ have finite-dimensional range containing the exact row map $\mathcal{R}$. Let P be its range projection and $Q = I - P$. For a finite carrier map U, define*

$$G = Z^*Z, \qquad F = U^*Z, \qquad J = U^*QU.$$

Let $C = C^*$ and $a > \|J\|$. Then

$$aI - ZCZ^* - UU^* \succeq 0 \quad \text{on } \ker \mathcal{R}^* \tag{60}$$

if and only if

$$aG - GCG - aF^*(aI - J)^{-1}F \succeq 0 \quad \text{on } \{c : \mathcal{R}^*Zc = 0\}. \tag{61}$$

The columns of Z may be dependent. If $J \preceq \overline{J} \prec aI$, substitution of $\overline{J}$ gives a sufficient gate. For any finite matrix V of trial coefficients,

$$J \preceq (U - ZV)^*(U - ZV). \tag{62}$$

Proof. The Q block of (60) is $aQ - QUU^*Q \succ 0$. Its exact elimination adds

$$PUU^*P + PUU^*Q(aQ - QUU^*Q)^{-1}QUU^*P = aPU(aI - U^*QU)^{-1}U^*P$$

to the adverse finite block. Every $x \in P\mathcal{H}$ has the form Zc; evaluation gives (61). Since $\operatorname{ran}\mathcal{R} \subseteq \operatorname{ran} Z$, the entire Q space satisfies the exact rows, so this elimination holds on the stated kernel. Inverse operator monotonicity proves the upper-covariance substitution. Finally $Q(U - ZV) = QU$, and the orthogonal P, Q split of the trial residual proves (62). □

For injective Z, $J = U^*U - FG^{-1}F^*$. Formula (62) supplies a complete positive upper without subtracting nearly equal numerical matrices or assuming a projected row Gram floor. It keeps the full residual covariance rather than replacing it by unrelated diagonal carrier bounds.

G.1 Paid common-source errors

Theorem J.1 supplies the joint source enclosure, its exact-row shorting and the full congruence payment. The numerical application in Section K retains every phase/row/carrier block.

A positive padded covariance $\Gamma \succeq \delta \operatorname{diag}(D_p, D_r)$ has shorted phase block at least δD_p: minimize its quadratic form over the row coefficients. This supplies a lower bound for the padded elimination without requiring a lower bound on the angle between unpadded phase and row spans. The padding is charged in the signed inequality. An original full-row Gram floor applies to its stated receiving map; a complementary projection requires its own analysis or this paid residual construction.

G.2 Return to the original form

For a finite arithmetic head and $x = \sqrt{\alpha + s^2/5}\, h$, the exact identity is

$$q_{H_M}[h] + \alpha\|h\|_2^2 = \|x\|_2^2 - \int_{\mathbb{R}} (1/5 - m_M(\xi))|\widehat{t_\alpha x}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}. \tag{63}$$

The original polar row becomes $t_\alpha \sinh(u/2)$ in this variable. To separate the favorable unbounded part, fix a finite Λ and write $d_M = 1/5 - m_M$ as

$$d_M = d_{\mathrm{ret}} + b - g, \quad d_{\mathrm{ret}} = 1_{|\xi|\le\Lambda} d_M, \quad b = 1_{|\xi|>\Lambda}(d_M)_+, \quad g = 1_{|\xi|>\Lambda}(-d_M)_+ \ge 0.$$

The original graph domain pays each integral. Dropping the favorable $-g$ gives a form upper bound; only d_{ret} and b receive bounded-operator approximations. Suppose these supply the complete upper domination

$$M_{t_\alpha}(1/5 - m_M)(D)M_{t_\alpha} \preceq ZCZ^* + UU^* + \eta I, \qquad \eta < 1,$$

as forms on the transformed original core. For example, a finite signed source approximation, a positive defect approximation and complete norm errors supply this domination after retaining the favorable exterior by its sign. Set $a = 1 - \eta$ and apply Theorem G.1 on the exact transformed rows. A successful gate gives

$$q_{H_M}[h] \ge -\alpha \|h\|_2^2, \qquad q_H[h] \ge -(\alpha + \delta_M)\|h\|_2^2,$$

where (7) supplies a full raw arithmetic payment. Finite genuine-null deflation is returned as in Theorem 2.2. All identities begin on the exact polar core and extend through the original closure; bounded retained perturbations preserve its domain.

Thus, for $\alpha_j = 2^{-j}$ and any cofinal $M_j \to \infty$, complete signed gates returning (63) yield the vanishing allowances required by Theorem 2.2. Theorem H.1 supplies finite receiving spaces, complete positive complements and certified continuous coordinates for every such index. The exact residual gate specifies the signed comparison to be certified in those coordinates.

The coordinated schedule of Theorem H.3 supplies the explicit original allowance $(1 + 2^{-37})2^{-j}$. The direct return adds the raw norm error to α_j; any stronger inverse-normalized error requirement used to establish a chosen finite gate is paid within that gate.

H A complete conditioned construction

The receiving space will approximate every frequency source at once. Put

$$\alpha_j = 2^{-j}, \qquad t_j = \frac{s}{\sqrt{\alpha_j + s^2/5}}, \qquad j \ge 16.$$

Let ν_j be a positive finite measure supported in $[0, \Omega_j]$, with mass A_j, and define the positive operator on $\mathcal{H} = L^2_{\mathrm{odd}}(\mathbb{R})$ by

$$O_j = \frac{1}{\pi} \int |t_j \sin(\xi\,\cdot)\rangle\langle t_j \sin(\xi\,\cdot)|\, \mathrm{d}\nu_j(\xi). \tag{64}$$

For an even nonnegative multiplier b, take $\mathrm{d}\nu = b(\xi)\,\mathrm{d}\xi$ on the positive half-line. The identity $\cos(x-y) - \cos(x+y) = 2\sin x \sin y$ gives exactly (64) in the Fourier convention of Section 2. Positive atoms are also allowed.

Theorem H.1 (Conditioned complete complements)**.** *Choose a computable integer* $B_j \ge \max(1, A_j)$ *and put*

$$L_j = j + \log(1 + B_j),$$
$$T_j = j \log 2 + \log(1 + B_j) + 4 \log L_j + 20, \qquad R_j = \tfrac{1}{2}\log(T_j/\pi),$$

$$E_j = 12\Omega_j R_j + 2\log(1+B_j) + 4\log L_j + 40.$$

Choose an even integer N_j with $E_j \le N_j < E_j + 4$. Let V_j be t_j times the odd polynomials of degree less than N_j, restricted to $[-R_j, R_j]$, and let Q_j project onto its entire odd orthogonal complement. Then

$$\|Q_j O_j Q_j\| < \frac{1}{2\log L_j}, \qquad \dim V_j \le 6\Omega_j R_j + \log(1+B_j) + 2\log L_j + 22. \tag{65}$$

For computable supplied parameters, a terminating certified construction gives a dyadic basis transformation whose actual continuous receiving Gram lies between $3I/5$ and $5I/3$. Its condition number is at most $25/9 < 3$.

Proof. The elementary inequalities

$$L/2 < T < 3L, \qquad 0 < R < \tfrac{1}{2}\log L$$

hold for $L \ge 16$. For the upper bound use $j\log 2 + \log(1+B) \le L$ and $4\log L + 20 < 2L$; the latter ratio decreases from its strict endpoint value at 16. For $0 \le \xi \le \Omega$, real Taylor approximation supplies an odd polynomial p_ξ of degree less than N with

$$|\sin(\xi u) - p_\xi(u)| \le \frac{(\Omega R)^N}{N!} < 4^{-N} \quad (|u| \le R),$$

since $N \ge 12\Omega R$ and $e < 3$. Thus the squared interior error after multiplication by t, where $t^2 < 5$, is at most $10R16^{-N}$. The complete theta upper bound (6), with $x = e^{2|u|}$, gives the full exterior estimate

$$\begin{aligned}\int_{|u|>R} t^2\,\mathrm{d}u &\le 169\,2^j \int_{T/\pi}^{\infty} x e^{-\pi x}\,\mathrm{d}x \\ &= \frac{169e^{-20}}{(1+B)L^4}\frac{T+1}{\pi^2}.\end{aligned} \tag{66}$$

Orthogonal projection is at least as accurate as this explicit approximation. Integrating the squared error against the whole positive source gives

$$\|QOQ\| \le \mathrm{Tr}(QOQ) \le \frac{A}{\pi}\left(10R16^{-N} + \frac{169e^{-20}(T+1)}{\pi^2(1+B)L^4}\right). \tag{67}$$

This trace includes every covariance combination and both infinite spatial tails. Because $N \ge 2\log(1+B) + 4\log L + 40$ and $\log 16 > 2$, $16^{-N} < e^{-80}(1+B)^{-4}L^{-8}$. Multiplying the first term in (67) by $\log L$ bounds it by $(5/3)e^{-80}(\log L)^2 L^{-8} < 1/4$. The second term times $\log L$ is at most $(676/27)e^{-20}L^{-2} < 1/4$, using $A/(1+B) \le 1$, $T+1 < 4L$ and $\pi > 3$. This proves (65). Upward rounding an outward enclosure of E of width less than one to the next even integer gives the required N without a transcendental equality decision.

For conditioning, the first theta atom and the decrease of r^2e^{-r} for $r > 2$ give, throughout $[-R, R]$,

$$\frac{s^2}{\alpha_j} \ge c_j = \frac{2T^2e^{-20}}{(1+B)L^4}, \qquad \frac{e^{-20}}{2(1+B)L^2} < c_j < 1.$$

Consequently

$$t_j^2 \ge \frac{c_j}{1 + c_j/5} > \frac{e^{-20}}{4(1+B)L^2} > \ell_j := \frac{2^{-32}}{(1+B)L^2}. \tag{68}$$

The final comparison follows from $30\log 2 > 20$. In an ordinary $L^2([-R,R])$-orthonormal odd Legendre basis the true receiving Gram satisfies $\ell_j I \preceq G_j \preceq 5I$.

Choose a positive rational $\ell_j' \le \ell_j$ by outward evaluation. Compute a symmetric rational moment enclosure $\widetilde{G}_j$ with $\|G_j - \widetilde{G}_j\| \le \ell_j'/4$; entrywise error $\ell_j'/(4\dim V_j)$ suffices. On this compact interval the complete theta series has effective uniform derivative tails and its denominator is positive. Certified partitions and increasing precision therefore terminate. The moments here are physical receiving moments, determined by s, t_j, R_j; computability of ν_j is needed only when evaluating the source operator or its mass rather than using its supplied bounds.

Since $\widetilde{G}_j \succeq 3\ell_j' I/4$, increasingly accurate inverse-Cholesky proposals give a dyadic invertible W_j passing the finite check

$$\tfrac{3}{4} I \preceq W_j^* \widetilde{G}_j W_j \preceq \tfrac{5}{4} I.$$

The error bound and $G_j \succeq \ell_j' I$ imply $(3/4)G_j \preceq \widetilde{G}_j \preceq (5/4)G_j$. Hence

$$\tfrac{3}{5} I \preceq W_j^* G_j W_j \preceq \tfrac{5}{3} I.$$

Only the basis changes; the space, its entire complement and the trace bound are unchanged. □

The condition estimate concerns the polynomial phase Gram on V_j. Adjoined exact rows enter the joint signed gate with their actual mixed Gram, as specified in Section G.

For fixed A, Ω, a fixed B yields $\dim V_j = O(\log j)$ and a complementary norm tending to zero. The next section proves necessity of this order for a nonzero fixed density. Polynomial approximation, finite-band concentration and whitening are classical mechanisms [9, 10, 11]; the explicit theta-dependent wall, whole-tail payment, funded Gram floor and original signed interface specify their use here.

H.1 Moving arithmetic heads

The construction also applies when the complete arithmetic head grows. Put

$$m_M(\xi) = \Re\Psi(1/4 + i\xi/2) - \log\pi - 2\sum_{2\le n\le M} \frac{\Lambda(n)}{\sqrt{n}} \cos(\xi \log n), \qquad b_M = (1/5 - m_M)_+.$$

Every prime power is included. The following elementary bounds, also used in the signed-frequency companion [4], give concrete inputs.

Lemma H.2. *For $M \ge 2$, the positive-half support and mass of b_M are bounded by*

$$\Omega_M = e^{7+8(\log 2)\sqrt{M}}, \qquad A_M \le \Omega_M[7 + 8(\log 2)\sqrt{M}].$$

The coordinated choice below turns these inputs into explicit original-domain tail and receiving-space budgets.

Proof. Write $\vartheta_*(N) = \sum_{n\le N} \Lambda(n)$. Prime-power valuation of binomial coefficients gives

$$\vartheta_*(2m) \le \vartheta_*(m) + 2m\log 2, \qquad \vartheta_*(2m+1) \le \vartheta_*(m+1) + 2m\log 2.$$

For the odd step, the two equal central binomial entries each have size at most 2^{2m}. Induction gives $\vartheta_*(N) \le 2N\log 2$. Abel summation now gives

$$S_M = \sum_{n\le M} \frac{\Lambda(n)}{\sqrt{n}} \le 4(\log 2)\sqrt{M} - 2\log 2.$$

The digamma series and comparison with its decreasing positive summand's integral imply

$$\Re\Psi(1/4+i\xi/2)-\log\pi\geq\Psi(1/4)-\log\pi+\tfrac{1}{2}\log(1+4\xi^2)>-6+\log(2|\xi|).$$

Here $\Psi(1/4)=-\gamma-\pi/2-3\log 2$ and $\Psi(1/4)-\log\pi>-6$. Thus outside Ω_M, $m_M>1+\log 2>1/5$. Everywhere the defect is bounded by $7+2S_M<7+8(\log 2)\sqrt{M}$. The asserted mass follows. □

Theorem H.3 (Coordinated arithmetic and receiving budgets)**.** *For $j\geq 16$, put $\kappa_j=2^{-j}$ and choose an integer M_j with*

$$E_j\leq M_j<E_j+2,\qquad E_j=\frac{j\log 2+3\log(j+2)+40}{\pi}. \tag{69}$$

Then the complete original arithmetic remainder satisfies

$$\delta_{M_j}<2^{-37}\kappa_j. \tag{70}$$

For the moving positive source b_{M_j}, Theorem H.1 gives condition number at most $25/9$ and

$$\log\dim V_j\leq 8\log 2\sqrt{\frac{\log 2}{\pi}}\,\sqrt{j}+O(\log j). \tag{71}$$

The leading coefficient is $2.60467320417428\ldots$. If complete signed head gates return $q_{H_{M_j}}[h]\geq-\kappa_j\|h\|_2^2$ on the original recipient, then

$$q_H[h]\geq-(1+2^{-37})2^{-j}\|h\|_2^2. \tag{72}$$

Consequently such a cofinal family satisfies Theorem 2.2.

Proof. An outward enclosure of E_j of width less than one, followed by upward integer rounding, supplies the stated integer. For $j\geq 16$, $\log(j+2)\leq j/2$, $\log 2<1$ and $\pi>3$ imply

$$M_j+1<E_j+3<\frac{5j+98}{6}\leq 3(j+2).$$

Insert this into (8) and use $M_j\geq E_j$:

$$\delta_{M_j}<\frac{4059e^{-40}}{j+2}2^{-j}\leq\frac{4059}{18}\left(\frac{3}{8}\right)^{40}2^{-j}<2^{-37}2^{-j}.$$

Here $e>8/3$; the last comparison is rational. Moreover $M_j=(\log 2/\pi)j+O(\log j)$. Choose B_j by upward enclosure of the mass bound in Lemma H.2. Then $\log(1+B_j)=8(\log 2)\sqrt{M_j}+O(\log M_j)$, $L_j=j+O(\sqrt{j})$, and $R_j=O(\log j)$. Substitution into (65) proves (71). Finally the bounded raw remainder costs at most $\delta_{M_j}\|h\|_2^2$, proving (72). Finite genuine-null deflations return by Theorem 2.2. □

The constructed rank in (71) prices the positive receiving space at the final original-norm scale. With $\epsilon_j=(1+2^{-37})2^{-j}$, the same estimate reads

$$\log\dim V_j\leq\frac{8\log 2}{\sqrt{\pi}}\sqrt{\log(1/\epsilon_j)}+O\big(\log\log(1/\epsilon_j)\big).$$

The arithmetic and positive-complement estimates hold at every index. The signed hypothesis in Theorem H.3 is the finite gate of Section G, including its complete approximation and residual-covariance payments.

I The necessary fixed-source dimension

This lower bound counts the whole retained odd span, including any adjoined receiving directions.

Theorem I.1 (Sharp dimension order)**.** *Let $b \ge 0$ be a nonzero bounded density on positive frequencies supported in a fixed $[a, \Omega]$ with $a > 0$, extended evenly as a Fourier multiplier. Fix one integer $B \ge \max(1, A)$ and use the corresponding R_j of Theorem H.1. Let $O_j = M_{t_j} b(D) M_{t_j}$ on the odd space. Set*

$$A = \int_0^\infty b, \qquad B_2 = \int_0^\infty b^2, \qquad B_\infty = \|b\|_\infty.$$

For any finite spaces W_j with $\|Q_{W_j} O_j Q_{W_j}\| \le \tau_j \to 0$,

$$\liminf_{j\to\infty} \frac{\dim W_j}{R_j - 1} \ge \frac{B_2}{\pi B_\infty^2} > 0.$$

Together with Theorem H.1, the optimal dimension order for obtaining a complementary norm tending to zero is $\Theta(\log j)$; the constructed rate is the one in (65)*.*

Proof. Take $L = R_j - 1$. On $|u| \le L$, the first-atom lower bound gives

$$s^2/\alpha_j \ge 2\pi^2 \exp(j \log 2 - T_j/e^2) \longrightarrow \infty.$$

Thus $t_j^2 \ge 4$ there for all sufficiently large j. Compare the full source covariance with the source covariance obtained by replacing t_j by $1_{[-L,L]}$. The former is at least four times the latter in frequency-source space. By equality of the nonzero spectra of U^*U and UU^*, and min–max,

$$\lambda_n(O_j) \ge 4\lambda_n(T_L), \qquad T_L = 1_{[-L,L]} b(D) 1_{[-L,L]} \tag{73}$$

on odd functions. This comparison is made in the common source space.

Put $C_L(\omega) = 2\sin(L\omega)/\omega$, continuously defined at zero. The complete sine-carrier formulas give

$$\begin{aligned}
\operatorname{Tr} T_L &= \frac{1}{2\pi} \int_0^\infty b(\xi)[2L - C_L(2\xi)]\, \mathrm{d}\xi, \\
\operatorname{Tr} T_L^2 &= \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty b(\xi) b(\eta) [C_L(\xi - \eta) - C_L(\xi + \eta)]^2 \, \mathrm{d}\xi \, \mathrm{d}\eta.
\end{aligned}$$

Consequently $\operatorname{Tr} T_L / L \to A/\pi$. The measures $C_L(\omega)^2 \, \mathrm{d}\omega/(4\pi L)$ are a positive approximate identity of mass one. Applied to the continuous autocorrelation of the zero-extended b, this gives the difference-term limit B_2/π in $\operatorname{Tr} T_L^2/L$. Since $\xi + \eta \ge 2a$, its square term is $O(1)$ and the mixed term is $O(\sqrt{L})$ by Cauchy–Schwarz. Therefore

$$\operatorname{Tr} T_L^2 / L \longrightarrow B_2/\pi, \qquad \|T_L\| \le B_\infty.$$

If $N_j = \dim W_j$, min–max and (73) give $\lambda_{N_j+1}(T_L) \le \tau_j/4$. Hence

$$\operatorname{Tr} T_L^2 \le N_j B_\infty^2 + (\tau_j/4) \operatorname{Tr} T_L.$$

Divide by L and take the lower limit. Finally $R_j \sim \frac{1}{2} \log j$ for fixed B. □

The first-two-trace argument belongs to classical time–frequency concentration. Its present use matches the explicit complete-theta upper construction, including arbitrary competing odd receiving spans. Compression by the fixed polar row changes a dimension count by at most one; additional growing constraint dimensions are counted with their actual span.

J Complete joint sources and stable row elimination

The phase construction controls a positive complement. To evaluate its signed recipient, collect the actual phase, exact-row and carrier profiles in one source map. Put $q = s^2$, $w_\alpha = q/(\alpha + q/5) = t_\alpha^2$. On a finite interval I, write

$$Yc = \sqrt{w_\alpha} \sum_a c_a f_a, \qquad \Gamma = Y^* Y.$$

Supported phase profiles are polynomials on their support cells, the polar profile is $\sinh(u/2)$, and a derivative-row profile is

$$f_k = (\psi^{(k+2)} - \psi^{(k)}/4)/q \quad (k \text{ odd}).$$

Literal carriers have profile $\sqrt{\nu_a/\pi}\sin(\xi_a u)$. An exact column transformation is applied to the entire derivative polynomial before evaluation. The original preimages determine genuine-null status; the transformed rows are receiving coordinates.

Theorem J.1 (Common-weight full covariance)**.** *Let the finitely many measurable profiles belong to $L^2(I, w_\alpha)$, and let a real positive function p satisfy $|p - w_\alpha| \le \eta w_\alpha$ on I, where $0 < \eta < 1$. Define $\Gamma_p = \int_I f(u)^* f(u) p(u)\,\mathrm{d}u$. Then*

$$(1-\eta)\Gamma \preceq \Gamma_p \preceq (1+\eta)\Gamma. \tag{74}$$

For a partition into retained and row coefficients, define the shorted form $S[x] = \inf_z (x,z)^* \Gamma (x,z)$, and similarly S_p. Then

$$(1-\eta)S \preceq S_p \preceq (1+\eta)S.$$

These assertions allow dependent columns and redundant rows.

Let $\widehat{Y}_p$ approximate $Y_p = \sqrt{p}\, f$ and let $D \succ 0$ be a specified coefficient metric. If

$$\|(Y_p - \widehat{Y}_p) D^{-1/2}\| \le e, \qquad \|\widehat{Y}_p D^{-1/2}\| \le M,$$

then, with $\delta = 2Me + e^2$,

$$-\delta D \preceq \Gamma_p - \widehat{Y}_p^* \widehat{Y}_p \preceq \delta D, \qquad \Gamma \preceq \frac{\widehat{Y}_p^* \widehat{Y}_p + \delta D}{1-\eta}. \tag{75}$$

Shorting this entire upper gives an upper for S.

Proof. Integrate the pointwise weight inequalities against $|\sum c_a f_a|^2$. Taking the infimum over the same row coefficients proves the shorted inequalities. For the profile change put $E = Y_p - \widehat{Y}_p$ and expand $Y_p^* Y_p$. The two cross terms have total D-normalized norm at most $2Me$, and E^*E at most e^2. Combining with the left inequality of (74) proves (75). Quadratic-form order is preserved by the defining infimum. □

This is a source-specific use of classical positive integration and variational Schur order [12, 11]. In particular, the relative-weight step preserves the exact kernel. A profile approximation is priced separately by (75). No lower angle between the phase and row spans enters these bounds. For any exact coefficient map W, an additive enclosure $-\delta D \preceq \Gamma - \widehat{\Gamma} \preceq \delta D$ becomes the enclosure $\delta W^* D W$ after congruence. This is the payment used for enrichment or row changes.

J.1 Positive polynomial integration

Here is a fully specified realization of the weight step. Suppose w_α is analytic on each closed disk of radius d centered in $[-R, R]$, with $|w_\alpha| \le H$, and $w_\alpha \ge w_0 > 0$ on the real interval. These are checkable source hypotheses. Split the interval into cells of half-width $h \le d/8$, also splitting at every supported-profile endpoint. On a cell centered at c, let $L \ge m \ge 1$ and $\zeta_\ell = e^{2\pi i\ell/L}$. Use

$$a_k = \left(\frac{2h}{d}\right)^k \frac{1}{L}\sum_{\ell=0}^{L-1} w_\alpha(c + (d/2)\zeta_\ell)\zeta_\ell^{-k}, \qquad p(c+hx) = \sum_{k<m} a_k x^k.$$

Exact conjugate-paired sampling makes p real. Cauchy's coefficient bound and root-of-unity aliasing give

$$\|w_\alpha - p\|_\infty \le \frac{H(h/d)^m}{1-h/d} + \frac{H\,2^{-L}}{(1-2^{-L})(1-h/d)}. \tag{76}$$

Indeed the discrete coefficient at index $k < L$ contains only Taylor indices $k + vL$; summing their bounds proves the second term. Literal sample and real dyadic coefficient errors add their full uniform payment. Choosing their sum with (76) below ηw_0 supplies the positive approximant in Theorem J.1.

For polynomial profile approximants of degree below N, a positive Gauss–Legendre rule of order at least $N+m$ on each cell integrates $p\widehat{f}^*\widehat{f}$ exactly: every entry has degree below $2N+m$. Certified node, weight and evaluation errors are included when the rule is represented by literal numbers. Thus its matrix is a single positive common-factor sum. The rule uses

$$O((R/d+1)(N+m))$$

points on a uniform grid, plus the support-boundary subdivisions. This argument establishes a constructive rule under its stated strip and profile hypotheses.

J.2 Complete growing-row tails

A full-line covariance also needs physical tails. Write a premixed row profile on $u \ge 0$ as

$$f_a(u) = c_a \sinh(u/2) + Q_a(u)/q(u), \quad Q_a(u) = e^{u/2}\sum_{n\ge1} P_a(\pi n^2 e^{2u}) e^{-\pi n^2 e^{2u}}.$$

Use odd reflection on the other arm. Let $D \ge 2$ bound all polynomial degrees, K_a bound each polynomial's absolute coefficient sum, and $F_0 = 1 + \sum_a(|c_a|^2 + K_a^2)$.

Proposition J.2 (Premixed complete row tails)**.** *For $\alpha = 2^{-j}$ and $T = \pi e^{2R} \ge \max(\pi, 4D)$,*

$$\sum_a \int_{|u|>R} w_\alpha |f_a|^2 \,\mathrm{d}u \le 676\, 2^j F_0 T^{2D} e^{-T}. \tag{77}$$

For $0 < \varepsilon < 1$, a sufficient wall for this trace to be below ε^2 is

$$T \ge 4D, \qquad T \ge 4\{j\log 2 + \log(676F_0) + 2D\log(8D) + 2\log(1/\varepsilon) + 10\}.$$

Proof. Put $r = \pi e^{2u}$. For $r \ge 4D$, comparison of successive terms gives $|\sum_n P_a(rn^2)e^{-rn^2}| \le 2K_a r^D e^{-r}$. For example $n^{2D}e^{-r(n^2-1)}$ is decreasing with a geometrically bounded ratio, using $\log n \le (n^2-1)/2$. The first positive theta atom gives $q \ge 2r^2e^{-r}$, while (6) gives $q \le 169e^{4u-r}$ and $w_\alpha \le 2^j q$. Apply $|x+y|^2 \le 2|x|^2 + 2|y|^2$, $\sinh^2(u/2) \le e^u$, $\mathrm{d}u = \mathrm{d}r/(2r)$ and both arms. Enlarging powers to r^{2D} and constants bounds the trace by $338\,2^j F_0 \int_T^\infty r^{2D}e^{-r}\,\mathrm{d}r$. Since $T \ge 4D$, integration by parts or a geometric series gives an upper $2T^{2D}e^{-T}$. Finally $2D\log T \le T/4 + 2D\log(8D)$ proves the sufficient wall. □

In particular $D = O(j/\log(j+2))$, $\log F_0 = O(j)$ and $\log(1/\varepsilon) = O(j)$ permit $T = O(j)$ and $R = O(\log j)$. If a common strip has $d^{-1} = O(j)$, logarithmic profile bounds $O(j)$, and $\log(w_0^{-1}) = O(j)$, the construction above reaches exponential accuracy with $O(j)$ scaled coefficient precision. For the raw identity metric this also requires logarithmic column count $O(j)$; a different metric additionally pays its inverse norm. High-frequency sine profiles use degree $O(|\xi|h+j)$ on each cell. These statements specify source precision and node counts under the displayed hypotheses.

Complete mixed tails follow from the full source factor: if the row and carrier tail traces are E_r, E_c, their joint tail trace is $E_r + E_c$ and the cross block has norm at most $\sqrt{E_r E_c}$. Supported phase tails vanish outside their support. The full covariance therefore remains a single error-controlled input to Theorem G.1.

K A complete 4473-column source certificate

We give a whole-line application of the common-source construction at $\alpha = 3/10^{12}$. The supplied data specify exact dyadic row coordinates and literal frequency carriers. The polynomial-weight construction of Section J is one general realization; this application uses a directly enclosed positive Gaussian rule for the actual theta sources.

Theorem K.1 (Full mixed physical Gram)**.** *Let Y have the 4080 supported phase columns, 41 complete receiving rows and 352 literal carriers specified below and in the scientific data. Let $\widehat{\Gamma}$ be the exact 2^{-100}-dyadic block-arrow matrix in* `data/joint_centers.npz`*. Then*

$$\|Y^*Y - \widehat{\Gamma}\| < 1.384607318 \cdot 10^{-20} < 2^{-65}.$$

The norm is the raw Euclidean coefficient operator norm. All phase/row, phase/carrier and row/carrier terms, the entire global block, quadrature errors and both infinite physical tails are included. In particular, putting $\delta = 2^{-65}$,

$$Y^*Y \preceq \widehat{\Gamma} + \delta I, \qquad \widehat{\Gamma} + 2\delta I \succeq \delta I.$$

K.1 Exact columns and a portable construction

The negative half-line is split into 100 cells of half-width $r = 1/80$, centered at $c_i = (-199+2i)/80$, $0 \le i < 100$. Odd reflection supplies the positive half-line. On the last 68 negative cells ($32 \le i < 100$), take the 60 local columns

$$t_\alpha(u)\sqrt{\frac{2k+1}{2r}}\,\mathrm{P}_k((u-c_i)/r), \qquad 0 \le k < 60,$$

where P_k is the Legendre polynomial; outside the paired cell supports they vanish. Their unweighted odd Gram is $2I_{4080}$. Disjoint cell interiors give exact zero off-cell blocks.

The 41 global rows are an exact dyadic triangular mixture of the polar profile and the profiles with odd derivative indices $1, 3, \dots, 79$. The file `row_coordinates.json` gives the literal coefficients and polynomial normalizations. To reconstruct them, start with

$$P_0(x) = 4x^2 - 6x, \qquad P_{\ell+1}(x) = (1/2 - 2x)P_\ell(x) + 2xP'_\ell(x).$$

Then $\psi^{(\ell)}(u) = e^{u/2}\sum_n P_\ell(\pi n^2 e^{2u})e^{-\pi n^2 e^{2u}}$ on $u \ge 0$. Normalize $P_{k+2} - P_k/4$ by its stated largest absolute coefficient and premix the exact polynomials before evaluation. The file

`literal_rule.json` gives 352 rational frequencies $2400 < \xi_a < 6000$ and positive dyadic weights ν_a with $\sum_a \nu_a < 2$. Their columns are exactly $t_\alpha\sqrt{\nu_a/\pi}\sin(\xi_a u)$.

The script `generate_rows.py` evaluates all 80,000 negative-arm Gaussian nodes at 512-bit ball precision using 16 theta atoms and a complete polynomial-dependent remainder. It produces double-double source arrays and certified point errors. The global assembly uses those source arrays, computes carrier phases by directed ball arithmetic, and pays all point and fixed-point errors jointly. The exact row coordinates are mathematical inputs; no numerical QR is part of the certificate.

K.2 Whole quadrature and tail payments

The following explicit bounds specify the analytic contract implemented by `row_budget.py` and `joint_gram.py`. They also permit reproduction without a sampled Gram hypothesis. On each Gaussian cell use the Bernstein ellipse $\rho = 7/5$. Its imaginary height is

$$h = \frac{r}{2}(\rho - \rho^{-1}) < 1/200.$$

The actual theta denominator is bounded in this strip by first-atom comparison on $|\Re u| \le 7/5$ and by regularization outside. For $v = \pi e^{2z}$ the first scaled atom satisfies $|P_0(v)| \ge 2|v|^2$. The other atoms have relative sum at most $3\sum_{n\ge 2} n^4 e^{-(n^2-1)\Re v} < 1/10$ for $\Re v > 2.99$, giving the lower denominator modulus $9X_-^2/5$ below. Inner-strip phase comparison gives $\Re\psi > 0$; the outer envelope and $\Re\cosh(z/2) > .99$ give receiving denominator real part above $\alpha/2$. The complete strip envelope for ψ is below 100, which also funds the displayed bound $4\cdot 10^{19}$ for t_α^2. The finite checks are

$$13/400 + \pi e^{14/5}/100 < 1, \qquad 120e^{63/10 - \pi e^{14/5}\cos(1/100)} < \alpha/4.$$

Use even reflection $z \mapsto -z$ for a negative-real portion of the first positive-arm ellipse; the weight is even and the receiving rows are odd. For a positive-arm cell let $x_- = \max(0, c - r(\rho+\rho^{-1})/2)$ and $x_+ = c + r(\rho+\rho^{-1})/2$ be its extreme real coordinates on that ellipse, $X_\pm = \pi e^{2x_\pm}$, and $c_h = \cos(2h)$. A valid majorant is

$$t_2 = \min\{4\cdot 10^{19},\ (240/\alpha)e^{9x_+/2 - X_- c_h}\}, \qquad d_0^{-1} = 5/(9X_-^2).$$

For a premixed derivative polynomial P_a, let A_a be its absolute-coefficient polynomial. Its complete atom numerator is bounded by

$$B_a = \sum_{n=1}^{16} A_a(n^2 X_+)e^{-(n^2-1)X_- c_h} + 2A_a(289X_+)e^{-288X_- c_h}.$$

The script verifies that the remaining atom ratio is below 1/2. The corresponding row bound is $\sqrt{t_2}\, e^{x_+/2}(|S_{0a}| + d_0^{-1}B_a)$. The local Legendre factor is bounded by $70\rho^{59}$ and the carrier sine factor by e^{6000h}. Positive Gaussian integration of an analytic scalar product has error at most

$$\frac{16r\rho^{-1600}}{1-\rho^{-1}} \times (\text{its ellipse supremum}).$$

Trace bounds for row blocks and Frobenius bounds for mixed blocks give a complete summed quadrature error below $5.6\cdot 10^{-185}$.

For the whole physical tail put $Z = \pi e^5$, and define

$$I_0 = e^{-Z}, \qquad I_k = kI_{k-1} + Z^k e^{-Z} = \int_Z^\infty x^k e^{-x}\,\mathrm{d}x.$$

If $A_a(x)^2 = \sum b_{a,k} x^k$, the complete row-tail trace is bounded by

$$E_r = \sum_a \left\{ \frac{4}{\alpha} \sum_{k\ge 2} b_{a,k} I_{k-2} + \frac{240}{\alpha} |S_{0a}|^2 I_2 \right\} < 1.384607235 \cdot 10^{-20}.$$

The full exterior atom sum is bounded by twice its first absolute-polynomial term; the explicit ratio tests at Z in the script verify this for every included degree. The carrier tail has bound

$$E_c = \frac{\sum \nu_a}{\pi} \frac{240}{\alpha} (e^5/3 + 1/9) e^{-3e^5}.$$

The row/carrier mixed tail is at most $\sqrt{E_r E_c}$. Supported phase columns vanish outside their own wall. Adding $E_r + E_c + 2\sqrt{E_r E_c}$ is therefore a conservative full-block tail payment.

Proof of Theorem K.1. Use the common 800-point positive Gauss–Legendre rule on each of the 100 cells. Complete theta source intervals pay the point uncertainty. Directed 256-bit ball products compute the sampled matrix; an alternative supplied route quantizes the common factors to 2^{-120} and uses exact integer products. Directed products retain the double-double uncertainties in their full matrix intervals. In the fixed-point alternative, the common-factor inequality (75) pays source uncertainty and factor quantization together. For this factor payment let the phase/global errors be e_p, e_g, and let the represented global trace be T_g. The phase factor norm is at most $\sqrt{10}$ before perturbation, since $t_\alpha^2 < 5$. Thus

$$e = (e_p^2 + e_g^2)^{1/2}, \quad M \le \{(\sqrt{10} + e_p)^2 + T_g\}^{1/2}, \quad E_{\mathrm{pt}} \le 2Me + e^2.$$

Center the exact products at 2^{-100}. The block-arrow center-rounding error is at most

$$E_{\mathrm{ctr}} \le \max(E_p, E_g) + E_{pg},$$

where E_p, E_g are maximal directed row sums of the phase/global center radii and E_{pg} is the Frobenius norm of all mixed radii. This uses the block-diagonal phase norm, the global block norm, and the complete rectangular cross-block Frobenius norm; disjoint-support zeros remain exact. For exact fixed-point products the simpler bound $\{\max(60, 393) + \sqrt{4080 \cdot 393}\} 2^{-100}$ suffices. The portable assembly verifies $E_{\mathrm{pt}} + E_{\mathrm{ctr}} < 8.3 \cdot 10^{-28}$. Add the whole quadrature and tail bounds above to obtain the stated radius. The final two inequalities follow from positive semidefiniteness of Y^*Y and its two-sided enclosure. □

All uncertainty in this theorem concerns the stated literal carriers. A continuous frequency measure is supplied to the original signed return with its own operator-approximation debit. The raw covariance error transports through an enrichment W as $\delta W^* W$. The scientific supplement contains the exact columns, complete source arrays, centers and portable generators, together with a short verification command and controls that reject underfunded error caps.

L Complete source bounds with uniform null cost

This appendix proves the actual-source estimates used in Section E. The Fourier convention, theta weight, original polar row and minimal realization are those of Section 2. No auxiliary positive reference replaces $P_{M,K}$.

L.1 Elementary arithmetic and logarithmic symbol bounds

Write $\vartheta_C(N) = \log \operatorname{lcm}(1, \dots, N)$. The quotient for $\vartheta_C(2m) - \vartheta_C(m)$ divides $\binom{2m}{m}$: a prime power in $(m, 2m]$ supplies a positive binomial valuation, and there is at most one such new power per prime. Similarly the quotient for $\vartheta_C(2m+1) - \vartheta_C(m+1)$ divides $\binom{2m+1}{m}$. The two equal odd central binomials together are at most 2^{2m+1}, so either is at most 2^{2m}. Induction, starting at $N = 1$, gives

$$\vartheta_C(N) \le 2N \log 2.$$

Partial summation, with the bound extended to real arguments by flooring, therefore gives

$$S_M := \sum_{2 \le n \le M} \frac{\Lambda(n)}{\sqrt{n}} \le 4(\log 2)\sqrt{M} - 2\log 2. \tag{78}$$

This estimate needs no prime number theorem.

The identity $m_0 = -\gamma - \pi/2 - 3\log 2 - \log \pi$ gives $-6 < m_0 < 0$ and $m_0 + 4 + \log 2 < 0$. Here are elementary admissions for these strict constants. The bounds $\gamma < 1$, $\pi < 22/7$, $\log 2 < 7/10$ and $\log \pi < 7/6$ give $m_0 > -6$; the logarithmic bounds follow from positive exponential Taylor sums. Also $\gamma > H_8 - \log 9 > H_8 - 11/5 = 29/56 > 1/2$, since the first six Taylor terms give $e^{11/10} > 3$. Together with $\pi > 3$, $\log 2 > 2/3$ and $\log \pi > 1$, this gives $m_0 + 4 + \log 2 < -1/3$.

For $t = |\xi|/2$, the summand $f_t(q) = t^2/[q(q^2 + t^2)]$ is nonnegative and decreasing for $q > 0$. Integral comparison with the digamma series gives

$$\tfrac{1}{2}\log(1 + 4\xi^2) \le a(\xi) \le 4 + \tfrac{1}{2}\log(1 + 4\xi^2). \tag{79}$$

Using the negative m_0, (78) and $\frac{1}{2}\log(1 + 4\xi^2) \le \log 2 + \log(2 + |\xi|)$ yields

$$0 \le n_M \le R_M, \quad |p_M - a| \le R_M, \quad p_M(\xi) \le R_M + \log(2 + |\xi|). \tag{80}$$

The middle bound also holds where $m_M < 0$, since there $a < |m_0| + 2S_M$. For $|\xi| \ge \Omega_M = e^{R_M + 1}$, the lower bound $a(\xi) \ge \log(2|\xi|)$ proves $m_M(\xi) > 0$. Thus the full negative support is bounded without importing a sampled sign pattern at another cutoff. The raw tail satisfies $\delta_M < 1$ for $M \ge 2$, since its decreasing majorant at $M = 2$ is below $4059(3/8)^9 < 1$.

L.2 Complete theta constants and convolution

We first record the complete strip estimates

$$\|s^2(\cdot \pm i/8)\|_2 < 20, \qquad \|s^2(\cdot \pm i/8)\sinh((\cdot \pm i/8)/2)\|_2 < 32. \tag{81}$$

For $z = x + iy$, $x \ge 0$, $|y| \le 1/8$ and $r = \pi e^{2x}$, $\cos(2y) > 31/32$ and $4r_\ell^2 + 6r_\ell < 6r_\ell^2$. The consecutive ratios of the majorized theta atoms are below $16e^{-3(31/32)\pi} < 1/8$. Summing *all* atoms gives

$$|\psi(z)| \le 7e^{x/2} r^2 e^{-31r/32}, \qquad |s^2(z)| \le 28 r^2 e^{-31r/32}.$$

The second estimate uses $|\cosh(z/2)| \ge \cos(1/16)\cosh(x/2) > \cosh(x/2)/2$. Evenness covers the other real arm. The squared two-arm integral for the first norm in (81) is at most $4704(16/31)^4 < 400$. For the second use $|\sinh((x + i/8)/2)|^2 \le e^{|x|} \le r/\pi$; the squared norm is at most $(784 \cdot 24/\pi)(16/31)^5 < 1024$. On the real line the stronger atom-ratio bound gives $s^2 \le 16 r^2 e^{-r}$; its maximum for $r \ge \pi > 3$ is less than $144e^{-3} < 9$, so $\|s\|_\infty < 3$.

Put $w = s^2$. Analyticity and complete double-exponential decay justify shifting the two Fourier contours, including the vertical end pieces. Plancherel and (81) give

$$\int_{\mathbb{R}} e^{|\xi|/4} |\widehat{w}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} < 800.$$

Weighted Cauchy–Schwarz implies

$$\int_{\mathbb{R}} |\widehat{w}| \frac{\mathrm{d}\xi}{2\pi} < 33, \qquad \int_{\mathbb{R}} \log(2+|\xi|) |\widehat{w}| \frac{\mathrm{d}\xi}{2\pi} < 210. \tag{82}$$

Indeed the reciprocal-weight integrals are $4/\pi$ and at most $164/\pi$, using $\log(2+x) \le 1+x$; then $3200/3 < 33^2$ and $131200/3 < 210^2$ suffice. For an odd g with Fourier support in $[-F, F]$, the inequality $2 + |\xi| \le (2 + |\xi - \zeta|)(2 + |\zeta|)$ and weighted Young's inequality now prove

$$\|p_M(D)(wg)\|_2 \le [33(R_M + \log(2+F)) + 210] \|g\|_2. \tag{83}$$

The logarithmic multiplier is used only on these paid sources, rather than asserted bounded on all L^2.

L.3 Original polar return and count-independent null return

On the ambient odd space let $\overline{P}_M = M_s p_M(D) M_s$ and $h_0 = sg$. Bandlimited L^2 functions have every derivative in L^2; complete theta multiplication and smooth compact cutoffs put h_0 in the minimal form domain. Equation (83) and form representation then put it in the ambient operator domain and give

$$\|\overline{P}_M h_0\| \le 3[33(R_M + \log(2+F)) + 210] \|g\|. \tag{84}$$

For example the compact approximants converge in L^2 and their weighted physical derivatives converge in L^2; $a \le 17\xi^2$ pays convergence in the specified minimal form norm.

The original odd polar row is $b = s \sinh(u/2)$. Put $f_b = sb = s^2 \sinh(u/2)$. From (81), its exponentially weighted Fourier norm is at most $32\sqrt{2}$. Since $\sup_{x \ge 0} x^2 e^{-x/8} = 256/e^2 < 36$,

$$\|f_b''\| < 1728, \qquad \|\overline{A} b\| < 88128.$$

On each interval $1/4 \le |u| \le 1/2$, the first complete theta atom gives $s^2 > 18 \cdot 3^{-10}$ and $\sinh^2(u/2) \ge 1/64$. Consequently

$$\|b\|^2 > \frac{9}{64 \cdot 3^{10}} > 2^{-20}, \qquad \frac{\|\overline{A} b\|}{\|b\|} < 2^{27}. \tag{85}$$

Here $r = \pi e^{2|u|} < \pi e < 10$, $r > 3$ and $e < 3$ suffice for the lower bound. The same compact form approximation and representation justify b's operator membership. From (80), $\overline{P}_M - \overline{A}$ is bounded by B_M. Writing Q_b for the polar projection and $x = Q_b h_0$, therefore,

$$\|P_M x\| \le \|\overline{P}_M h_0\| + (2^{27} + B_M) \|h_0\|.$$

The constrained action is $P_M x = Q_b \overline{P}_M x$; preservation of the polar condition by the ambient action is not assumed.

Now let T be an original Hilbert isometry onto any finite-dimensional $K \subseteq \ker H$. Bounded perturbation in the same minimal realization gives $\operatorname{Dom} H = \operatorname{Dom} P_M$ and

$$P_M T = N_M T - W_M T, \qquad \|P_M T\| \le B_M + \delta_M.$$

For $e = Q_K x$, the operator on the further constrained form domain acts as $P_{M,K} e = Q_K P_M e$, whence

$$\|P_{M,K} e\| \le \|P_M x\| + (B_M + \delta_M)\|x\|.$$

Combining the last displays, $\|x\| \le \|h_0\| \le 3\|g\|$ and (84) proves the promised count-independent bound

$$\|P_{M,K} Q_K Q_b(sg)\| \le J(M,F)\|g\|, \quad J(M,F) = 3[51 R_M + 33\log(2+F) + 210 + 2^{27} + \delta_M]. \quad (86)$$

This argument uses actual operator-nullity. An arbitrary additional row requires its own graph cost. Independence of $\dim K$ is an analytic norm estimate and does not assert that numerical projection is free.

For arbitrary negative profiles and a common complex coefficient vector,

$$\|g_z\|^2 = \int_0^\infty n_M^2 |\phi_z|^2 \frac{\mathrm{d}\xi}{\pi} \le R_M z^* D z.$$

Their support is bounded by Ω_M, and $J(M, \Omega_M) < J_M$. Equation (86) proves the complete joint moment in Theorem E.1. Applying the same inequalities to profile differences and using closedness of $P_{M,K}$ proves the full observation-space graph completion used in Theorem E.2.

The same graph estimate also pays finite inverse-action trials. If E_0, W_0 are actual negative and positive source families, the latter supported below F_0, and D_n, D_p dominate their true frequency metrics, put $V = \kappa^{-1} Q_M (E_0 - W_0 F_c)$ and $F_b = \max(\Omega_M, F_0)$. Before multiplication by s, the two profiles have disjoint Fourier sign supports. Thus

$$(PV)^*(PV) \preceq \kappa^{-2} J(M, F_b)^2 [R_M D_n + (R_M + \log(2 + F_0)) F_c^* D_p F_c].$$

This supports complete source-action residual accounting. Physical mixed covariance after multiplication by s remains present; disjoint raw frequency sign supports do not make the receiving cross block vanish.

M Protected critical representers and their spectral floor

We supply the original-domain and uniform Gram arguments used in Theorem F.2. The complete polarized formula is

$$q_H(h_1, h_2) = \sum_z m_z \, \mathcal{M}(sh_2)(z) \overline{\mathcal{M}(sh_1)(-\bar{z})}, \qquad \mathcal{M} f(z) = \int f(u) e^{zu} du.$$

All centered nontrivial zeros occur with multiplicity.

Lemma M.1 (Critical evaluation vectors in the operator range)**.** *For every positive critical ordinate* γ, $v_\gamma = Q_b(s \sin(\gamma u))$ *belongs to* $\operatorname{ran} H$.

Proof. Put $\sigma = i\gamma$ of exact multiplicity m and $P_0 = d^2/du^2 - 1/4$. For a function whose first r transform jets vanish at σ, define

$$R_\sigma^r f(u) = \int_u^\infty \frac{(t-u)^{r-1}}{(r-1)!} e^{\sigma(t-u)} f(t) dt.$$

Its canceled lower-tail expression is $(-1)^r \int_{-\infty}^u (u-t)^{r-1} e^{\sigma(t-u)} f(t) dt/(r-1)!$. The vanishing jets prove equality, and direct integration gives $\mathcal{M}(R_\sigma^r f)(z) = \mathcal{M} f(z)/(z-\sigma)^r$. With $\eta = R_\sigma^m \psi$, the

detector $h_\sigma = P_0\eta/s$ therefore has transform $(z^2-1/4)\xi(1/2+z)/(z-\sigma)^m$. It evaluates only at σ, where its value A is nonzero.

Here is its original minimal-domain return. Each fixed-order theta derivative is bounded on either arm by a polynomial in $r = \pi e^{2|u|}$ times $e^{|u|/2-r}$; the first atom gives the matching positive lower for s^2. In the upper primitive use $t = u + v$ and $e^{2v} \geq 1 + 2v$; its complete tail is bounded by a polynomial times $e^{u/2-r}e^{-rv}v^{m-1}$ beyond a fixed onset. Use the canceled lower primitive on the other arm. Differentiation gives the corresponding fixed-order estimates. For compact even cutoffs, $P_0(\chi_R\eta)$ has both pole moments zero exactly, converges after division by s in L^2, and its derivative converges in ordinary L^2. The bound $a \leq 17\xi^2$ gives convergence in V. Weighted integration by parts gives arbitrary vertical inverse powers, so the $O(T\log T)$ zero count passes the full zero formula to the limit, as in Appendix A.

Oddify by $h_o(u) = h_\sigma(u) - h_\sigma(-u)$. Its only evaluations are A and $-A$ at $\pm i\gamma$. The polarized formula gives, in the conjugate-linear-first convention, $q_H(h_o, h) = 2mi\bar{A}\langle v_\gamma, h\rangle$. This is a bounded Hilbert functional; form representation puts $h_o \in \operatorname{Dom} H$ and $Hh_o = -2miAv_\gamma$. In particular $v_\gamma \perp \ker H$. □

Classical positive critical-line proportion [17] and the local total zero bound $N(T+1)-N(T) = O(\log(T+3))$ give a sequence of distinct critical ordinates with any fixed large separation G and $\gamma_j \leq Dj$. To see this, greedily retain a point and discard the following interval of length G. Each discarded cluster has total multiplicity $O_G(\log T)$ below height T, whereas the critical-line count is at least $cT\log T$. Thus the number retained below T is at least $c_G T$, even when multiplicities are charged to clusters. Discard any fixed initial interval and enlarge D if necessary.

For an even positive Schwartz weight F, its sine Gram is

$$\tfrac{1}{2}[\widehat{F}(\gamma_i - \gamma_j) - \widehat{F}(\gamma_i + \gamma_j)].$$

Fourier decay bounds the off-diagonal difference row sums by $C_N \sum_{l\neq 0}(1 + G|l|)^{-N}$; the sum-frequency row sums tend to zero with the first retained height. Schur's test therefore gives uniform upper and lower Riesz bounds, simultaneously for $F = s^2$ and s^4. The polar coefficients $c_j = \langle b, s\sin(\gamma_j u)\rangle/\|b\|^2$ have an arbitrarily small *whole* ℓ^2 norm after increasing that height. Hence for $h = \sum_{i\leq j} a_i v_{\gamma_i}$ and $f = sh$,

$$l_h\|a\|^2 \leq \|h\|^2 \leq u_h\|a\|^2, \qquad \|f\|^2 \geq l_f\|a\|^2.$$

Differentiating $f = \sum a_i s^2 \sin(\gamma_i u) - sb\sum a_i c_i$ and applying the upper cosine/sine Gram bounds with weights s^4 and $|(s^2)'|^2$ gives $\|f'\|^2 \leq C\gamma_j^2\|a\|^2$ uniformly in j. Fourier Jensen, as displayed after Theorem F.2, now gives $\langle \mathcal{C}_0 h, h\rangle \geq c_*/j\|h\|^2$ on this j-dimensional span. Lemma M.1 puts it in $\mathcal{R}$. Compact minmax proves the required floor. Complete theta decay supplies all Schwartz seminorms used here; no constant depends on the number of subsequently removed genuine nulls.

N Origin compression and fixed-window separation

This appendix records an additional constraint on the original hard-cutoff origin layer. Put $U = \log M$, $0 < L < \pi/2$ and fix $B < \infty$, $C \geq 0$. On the original polar-only space let N_M^{org} use the negative window $[0, L/U]$, and let $P_M^{[B]}$ use the actual positive window $[0, B/U]$.

Proposition N.1 (A bounded scaled positive window)**.** *There are compact smooth original polar tests h_M and constants $c, c' > 0$, with an integer k fixed before M, such that eventually*

$$(N_M^{\mathrm{org}} - CP_M^{[B]})[h_M] \geq c\sqrt{M}/U, \quad \|h_M\|^2 \leq c'U^{4k+2}.$$

Thus this fixed positive-window budget has an adverse Rayleigh depth at least a constant times $\sqrt{M}/U^{4k+3}$.

Proof. PNT and Abel summation give uniformly on every fixed scaled interval $m_M(t/U)/\sqrt{M} \to -4\cos t$, with positive/negative parts converging to $4(-\cos t)_+$ and $4(\cos t)_+$. Choose $b > \max(B, L, \pi/2)$ and $p_k(t) = t(1 - t^2/b^2)^k$. The positive integral on $[0, B]$ is exponentially small in k, since its support starts at $\pi/2$. The negative integral on $[0, L]$ is at least a positive multiple of k^{-3} by restricting to $[0, 1/(2k)]$. Fix k so the negative integral exceeds C times the positive one by a strict amount.

On a fixed compact interval the odd functions $u, u^3, \dots, u^{2k+1}, \sinh(u/2)$ are independent. A positive even bump and their Gram inverse supply compact smooth odd moment duals. Writing $p_k(t) = \sum_{j \le k} a_j t^{2j+1}$, choose f_M with exact polar moment zero and $\int f_M u^{2j+1} = (-1)^j (2j+1)! U^{2j+1} a_j$. Then $h_M = f_M/s$ is in the original compact polar core, $\|h_M\| = O(U^{2k+1})$, and its sine transform at t/U is $p_k(t) + O(U^{-2})$ uniformly on the two retained intervals. Substitute in the actual observations and use their scaled limits. The fixed strict polynomial gap gives the asserted inequalities. □

There is also a positive finite-frequency compression for any original projection Q. With $Th(\xi) = \langle Q(s\sin(\xi u)), h\rangle$ in the true origin measure, its monomial projection Π_d gives $0 \preceq T^*\Pi_d T \preceq T^*T$. This is a true source projection and keeps its full monomial Gram. A sine Taylor approximation and the complete theta moments give the bound used in the portable controls:

$$\|T^*T - T^*\Pi_d T\| \le 64\beta_M \frac{[\Omega \log(4m)/2]^{2m}}{(m!)^2}, \quad m = 2d+1, \quad \beta_M = \int_0^\Omega n_M \, d\xi/\pi.$$

For completeness put $r_m = \log(4m)/2$. For $x = |u| \le r_m$, $x^{2m} \le r_m^{2m}$. For $x \ge r_m \ge \log 2$, monotonicity of xe^{-2x} gives

$$2m \log(x/r_m) \le 2me^{-2r_m} e^{2x} = \tfrac{1}{2} e^{2x} < \tfrac{\pi}{2} e^{2x}.$$

Thus $x^{2m} \le r_m^{2m} e^{r/2}$ everywhere, and both real arms give

$$\|su^m\|^2 \le 16 r_m^{2m} \int_\pi^\infty re^{-r/2} dr = 16(2\pi + 4) e^{-\pi/2} r_m^{2m} < 64 r_m^{2m}.$$

The real Taylor remainder is $|\xi u|^m/m!$; the squared observation-map error integrates to the displayed bound. The exact projection has no larger error. At $\Omega = L/\log M$, for any fixed $c > (p + 1/2)/8$, taking $d = \lceil c \log M / \log\log M \rceil$ pays M^{-p} eventually, by Stirling. This is a compression estimate in the true observation metric, complementary to the complete ridged comparison of Theorem C.1.

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