Whole-Line Lower Bounds for the Odd Weil Form in the Reciprocal-Theta Norm

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Luminary Oddities

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Abstract

Let Q be the Weil explicit-formula quadratic form on odd test functions p on the logarithmic line, with every prime power and the polar term retained, and let ψ be the Riemann theta kernel normalized by $\mathcal{M}\psi(z) = \xi(\frac{1}{2} + z)$. We study Q on the whole line in the reciprocal-theta norm $\int |p|^2 \cosh(u/2)\psi(u)^{-1}\, du$, on odd p with vanishing polar moment. Three results are proved. First, an arithmetic-side computer-assisted certificate, which uses no information about the zeros, gives

$$Q[p] \;\geq\; -1.5 \cdot 10^{-11} \int_{\mathbb{R}} |p(u)|^2 \, \frac{\cosh(u/2)}{\psi(u)}\, du$$

on the full minimal form domain. It is a 4080-dimensional exact-row Schur complement of a 4085-dimensional receiving Gram matrix at arithmetic cutoff 9, with every arithmetic, frequency and spatial tail enclosed analytically; its finite sign test has been replayed with independent code in ball arithmetic on a random sample of rows. Second, an explicit localization theorem converts any lower bound on test functions supported in $(-L, L)$ into a whole-line bound with fees of order $e^{2L - \pi e^{2L}/2}$. Third, combined with Durmagambetov's certified near-positivity on $[-3.07, 3.07]$, which uses the verification of the Riemann hypothesis up to height $3 \cdot 10^{12}$, the localization theorem gives the whole-line allowance $1.4 \cdot 10^{-309}$ in the same norm. Among whole-line bounds that use no zero verification, $1.5 \cdot 10^{-11}$ is the sharpest known: by localization, the zero-free window certificates through half-width 1.3 give at best $2.3 \cdot 10^{-3}$, and the threshold for $1.5 \cdot 10^{-11}$ is half-width 1.625. Nonnegativity of the same form is equivalent to the Riemann hypothesis, as is the existence of a uniform lower floor of any size in the unweighted profile norm. Three further certificates give localized physical floors on $|u| < \frac{1}{2} \log Y$, $Y = 8, 9, 10$, that carry an explicit positive coefficient in the direction of the polar collar profile, and an explicit test function shows that every certificate of this type at fixed arithmetic cutoff has a positive minimal allowance.

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1 Introduction and results

1.1 Weil's criterion in a weighted realization

Weil's explicit formula [19] pairs a sum over the nontrivial zeros of ζ with an archimedean term, a polar term and a sum over prime powers. Weil proved that the Riemann hypothesis (RH) is equivalent to nonnegativity of the resulting quadratic functional on a suitable class of test functions. Bombieri [1] studied the functional on test functions supported in an interval $[-t,t]$, showed that its minimum on the unit ball is attained, and reproved Yoshida's theorem [20] that it is positive definite for t small. Connes [4] recast positivity as a trace formula, and Connes and Consani [5] proved positivity of the archimedean contribution for test functions supported in $[2^{-1/2},2^{1/2}]$ via prolate spheroidal functions; Connes, Consani and Moscovici [6] extended the prolate picture to the semilocal case. Burnol [3] gave a co-Poisson and Sonine-space treatment of the explicit formula. Li's criterion [11, 2] and the de Bruijn–Newman constant [9, 15, 18, 17] are other quantitative positivity formulations of RH; Rodgers and Tao proved $\Lambda \geq 0$, so RH is equivalent to $\Lambda = 0$. Numerically, all zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ lie on the critical line [16].

Recent computer-assisted work certifies positivity of the localized Weil form on fixed windows $f \in L^2(-a,a)$, in the ordinary L^2 norm: Zhu [21] proves $Q[f] \geq 8.9 \cdot 10^{-18}\|f\|^2$ for $a = 0.8$ and records a Landau–Widom-type decay law for the localized minimum; Liu [13] proves coercivity constants 2^{-151} at $a = 1$ and 2^{-49162} at $a = 17/16$ for all complex smooth tests; Li [12] proves positivity in both parity sectors for every $a \leq \frac{1}{2}\log 11$; Desogus [10] gives a rooted-operator architecture with a certified physical endpoint at $Y = 7$. Durmagambetov [14] replaces the pointwise prime bound by the norm of the prime operator and certifies $\lambda_*(a) \geq 4.9999 \cdot 10^{-62}$ for every $a \leq 1.3$, with odd-sector floor $9.9999 \cdot 10^{-59}$; combined with the verification of the Riemann hypothesis up to height $3 \cdot 10^{12}$ [16] he obtains $Q_a \geq -4.8 \cdot 10^{-709}$ for every $a \leq 3.07$. These windows all concern the ordinary L^2 norm on a fixed support, and their whole-line consequences in the present norm are obtained here through Theorem 1.3.

This paper works in one weighted realization in which the whole real line, rather than a window, carries a certificate. Let

$$\psi(u) = e^{|u|/2} \sum_{n \geq 1} (4r_n^2 - 6r_n)e^{-r_n}, \qquad r_n = \pi n^2 e^{2|u|}, \qquad s(u)^2 = \frac{\psi(u)}{\cosh(u/2)}. \tag{1}$$

Theta inversion makes ψ even, positive and smooth, with $\psi(u) = 2\Phi(u/2)$ for the Riemann kernel Φ and $\mathcal{M}\psi(z) = \int_{\mathbb{R}} \psi(u)e^{zu}\,du = \xi(\frac{1}{2}+z)$. Use $\widehat{f}(\xi) = \int e^{-iu\xi}f(u)\,du$ with Fourier measure $d\xi/2\pi$,

$S_b f(u) = f(u+b)$, and the digamma function $\Psi = \Gamma'/\Gamma$. For odd p put

$$Q[p] = \int_{\mathbb{R}} \left[\operatorname{Re}\Psi(\tfrac{1}{4}+\tfrac{i\xi}{2}) - \log\pi\right] |\widehat{p}(\xi)|^2 \frac{d\xi}{2\pi} - \sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \langle p, (S_{\log n} + S_{-\log n})p\rangle, \tag{2}$$

$$m(p) = \int_{\mathbb{R}} \sinh(u/2) p(u)\, du.$$

For odd real p the explicit formula reads

$$Q[p] - 2m(p)^2 = \sum_{\rho} \widehat{p}(\gamma_\rho)\, \overline{\widehat{p}(\overline{\gamma_\rho})}, \qquad \rho = \tfrac{1}{2} + i\gamma_\rho, \tag{3}$$

the two polar values $\widehat{p}(\pm i/2) = \pm m(p)$ contributing $-2m(p)^2$. Under RH the right side is $\sum_\rho |\widehat{p}(\gamma_\rho)|^2 \geq 0$.

Definition 1.1 (Weighted realization)**.** Write $p = sh$. The *reciprocal-theta norm* is $\|h\|^2 = \int_{\mathbb{R}} |h|^2 = \int_{\mathbb{R}} |p|^2 \cosh(u/2)\psi(u)^{-1}\, du$. Let $\mathcal{K}$ be the space of odd $h \in L^2(\mathbb{R})$ with $\langle h, s\sinh(u/2)\rangle = 0$, that is $m(sh) = 0$. Put $a(\xi) = \operatorname{Re}\Psi(\frac{1}{4}+\frac{i\xi}{2}) - \Psi(\frac{1}{4}) \geq 0$ and $m_0 = \Psi(\frac{1}{4}) - \log\pi$. The domain $\mathcal{V}$ is the closure of the smooth compactly supported odd functions in $\mathcal{K}$ for $\|h\|^2 + \int a(\xi)|\widehat{sh}(\xi)|^2\, d\xi/2\pi$, and $q_H[h] = Q[sh]$ is the closed form on $\mathcal{V}$.

The weight satisfies $2r^2e^{-r} \leq s(u)^2 \leq 169\, e^{4|u|-r}$ with $r = \pi e^{2|u|}$ (Lemma 2.1), so $\|h\|$ dominates the profile norm $\|sh\|_2$ by a factor that grows like $\exp(\frac{\pi}{2}e^{2|u|})$.

1.2 Main results

Theorem 1.2 (Certified whole-domain lower bound)**.** *For every $h \in \mathcal{V}$,*

$$q_H[h] \;\geq\; -\frac{3}{2\cdot 10^{11}}\, \|h\|^2.$$

Equivalently, every odd smooth compactly supported p with $m(p) = 0$ satisfies $Q[p] \geq -1.5\cdot 10^{-11}\int |p|^2\cosh(u/2)\psi(u)^{-1}\, du$.

The proof (Section 4) is computer-assisted and uses no information about the zeros of ζ. It retains the prime powers $n \leq 9$ in a finite receiving matrix, pays every prime power $n \geq 11$ analytically, encloses the high-frequency exterior by a positive operator of norm below $1959/2000$, and verifies the sign of a 4080×4080 exact-row Schur complement by a Gershgorin test in 192-bit directed Arb arithmetic. The allowance improves the bound $1.88\cdot 10^{-9}$ of [24] on the same domain and norm by the factor $376/3$.

Theorem 1.3 (Supported-to-global localization)**.** *Let $\mathcal{V}_L \subset \mathcal{V}$ be the closure of the odd smooth functions supported in $(-L, L)$ with $m(sh) = 0$, and suppose $q_H[v] \geq -\varepsilon\|v\|^2$ for all $v \in \mathcal{V}_L$. For $R \geq \frac{1}{2}$, $0 < w \leq 1$, $R + w = L$ and an integer $K \geq 2$, put $r_R = \pi e^{2R}$,*

$$B_R = 13e^{2R}e^{-r_R/2}, \qquad t_R = \sqrt{5}\, r_R^{3/4} e^{-r_R/2},$$

$$d_K = 451(K+1)^2 e^{-\pi(K+1)}, \quad S_K = \textstyle\sum_{2\leq n\leq K} \Lambda(n)/\sqrt{n}.$$

Then $q_H[h] \geq -F\|h\|^2$ for every $h \in \mathcal{V}$, where

$$F = \varepsilon(1+96t_R)^2 + 86000\, t_R(1+96t_R) + 4128000\, t_R^2 + B_R[27+15\log(2/w)+12S_K] + (6+2S_K)B_R^2 + 3d_K.$$

The localization theorem was proved in edition 12 of [23]; Section 5 gives a complete proof. It turns localized bounds in the ordinary L^2 norm into whole-line bounds: since $s^2 \le s(0)^2 < 1$, a bound $Q[p] \ge -\varepsilon_0\|p\|_2^2$ on odd p supported in $(-L,L)$ with $m(p)=0$ gives the hypothesis with $\varepsilon = s(0)^2\varepsilon_0$. In the notation of [14], where $Q_a(f) = 2F(\frac{i}{2})F(-\frac{i}{2}) + \frac{1}{\pi}\int_0^\infty |F|^2\Psi_a$ with $F(z) = \int f(y)e^{izy}\,dy$, the support parameter a is our half-width L and, for odd real f, $F(\pm\frac{i}{2}) = \mp m(f)$, so $Q_a(f) = Q[f] - 2m(f)^2$; the two forms agree on the polar constraint $m=0$.

Corollary 1.4 (Whole-line bound from verified zeros)**.** *Assume the conclusion of [14, Theorem 2]: $Q_a(f) \ge -4.8\cdot 10^{-709}\|f\|_2^2$ for every $a \le 3.07$ and every real $f \in L^2(-a,a)$, where $Q_a(f) = Q[f] - 2m(f)^2$ for odd f in the notation (2). Then*

$$q_H[h] \;\ge\; -1.4\cdot 10^{-309}\,\|h\|^2 \qquad (h \in \mathcal{V}).$$

The hypothesis is a computer-assisted theorem of [14] that combines a ball-arithmetic certificate with the verification of the Riemann hypothesis up to height $3\cdot 10^{12}$ [16]. Theorem 1.2 is independent of any zero verification, and among whole-line bounds of that kind it is the sharpest available:

Corollary 1.5 (Reach of localized positivity)**.** *Nonnegativity of q_H on $\mathcal{V}_L$ gives, through Theorem 1.3, the whole-line allowances of Table 1; the allowance reaches $1.5\cdot 10^{-11}$ at half-width $L = 1.625$. The zero-free window certificates of [21, 13, 12, 14], whose largest half-width is 1.3, therefore give whole-line allowances of at least $2.2\cdot 10^{-3}$, and Theorem 1.2 improves them by eight orders of magnitude.*

Remark 1.6 (Where the bound lives). The weight s^2 decreases on $u \ge 0$ and $s(u_*)^2 = 1.5\cdot 10^{-11}$ at $u_* = 1.1868$. For h supported in $|u| \ge U$ dropping the nonnegative archimedean energy, the potential and prime terms give $q_H[h] \ge -C(U)\,s(U)^2\|h\|^2$ with a constant $C(U)$ that stays bounded for $U \le 2$, since every factor $s(u)s(u+\log n)$ in the arithmetic sum is at most $\min\{s(U)^2, s(\frac{1}{2}\log n)^2\}$. The content of Theorem 1.2 is therefore concentrated on $|u| \lesssim 1.2$, essentially the window $|u| < \frac{1}{2}\log 11 = 1.1989$.

Theorem 1.7 (Criteria)**.** *The following are equivalent: (i) RH; (ii) $q_H \ge 0$ on $\mathcal{V}$; (iii) there are finite genuine-null projections Π_j and allowances $\varepsilon_j \to 0$ with $q_H[(I-\Pi_j)h] \ge -\varepsilon_j\|(I-\Pi_j)h\|^2$ on $\mathcal{V}$; (iv) there is a constant $B \ge 0$ with $Q[f] \ge -B\|f\|_2^2$ for every odd smooth compactly supported f with $\mathcal{M}f(\pm\frac{1}{2}) = 0$, uniformly over all supports.*

Theorem 1.2 is thus a statement in the weighted norm with an explicit allowance; (iv) shows that in the unweighted profile norm every uniform floor, of any size, is equivalent to RH. The companion paper [25, Proposition 9.1 and Corollary 9.2] gives the price of passing between the two norms: a null correction with unit row moment and profile size η costs a physical norm whose logarithm has order $\eta^{-4/(8j+13)}$.

Theorem 1.8 (Localized physical floors)**.** *For $Y \in \{8,9,10\}$ let A_Y be the closed odd form $Q[p] - 2m(p)^2$ on $p \in L^2(-a_Y, a_Y)$, $a_Y = \frac{1}{2}\log Y$, in the ordinary L^2 norm. Then*

$$A_8 \succeq 10^{-40}I + 4.75\cdot 10^{-11}P_{e_{7,8}}, \qquad A_9 \succeq 10^{-50}I + 4\cdot 10^{-12}P_{e_{8,9}}, \qquad A_{10} \succeq 10^{-60}I + 10^{-13}P_{e_{9,10}},$$

where $P_{e_{k,Y}}$ is the projection onto the normalized polar profile of the collar $a_k < |u| < a_Y$ (Section 6).

The content of Theorem 1.8 is the rank-one collar term: in the direction of the outermost polar collar the localized form is bounded below by $4.75\cdot 10^{-11}$, $4\cdot 10^{-12}$ and 10^{-13}, many orders of magnitude above its spectral minimum. The scalar floors are recorded for completeness; sharper scalar floors on the same supports follow from the certified odd-sector floor $9.9999\cdot 10^{-59}$ on $a \le 1.3$ of [14] and the bound $\lambda_*(1.1) \ge 1.4999\cdot 10^{-38}$ recalled in [14, §1], which cover $a_8 = 1.0397$, $a_9 = 1.0986$ and $a_{10} = 1.1513$.

Proposition 1.9 (Fixed-cutoff exterior). *For arithmetic cutoff* 9, *low band* 2400 *and* $\mu = \frac{1}{5}$, *the positive exterior operator* O_α *used in Theorem 1.2 satisfies* $\|O_\alpha\| \to 5\|b\|_\infty > 5$ *as* $\alpha \downarrow 0$, *and* $\|O_\alpha\| > 5/2$ *for* $0 < \alpha \le \exp(-5e^{8192})$.

Every certificate built with a fixed arithmetic cutoff and a fixed low band therefore has a positive minimal allowance; allowances tending to zero require cutoffs and bands that grow, in accordance with the support growth $L_M = \frac{1}{2}\log\log M + O(1)$ of the finite-verifiable criterion in [25].

1.3 Organization

Section 2 collects the envelopes and the domain facts. Section 3 proves Theorem 1.7. Section 4 proves Theorem 1.2, Section 5 proves Theorem 1.3 and Corollaries 1.4 and 1.5, Section 6 states the localized certificates and Section 7 proves Proposition 1.9. Appendix A lists the independent re-evaluation of every scalar enclosure, and Appendix B records a Rayleigh–Ritz check of the conventions.

2 The weighted realization

Lemma 2.1 (Envelopes). *With* $r = \pi e^{2|u|}$,

1. $2r^2e^{-r} \le s(u)^2 \le 169\, e^{4|u|-r}$, *and for* $u \ge 0$ *also* $s(u)^2 \le 16\pi^2 e^{4u-\pi e^{2u}}$;
2. s^2 *is even and log-concave; hence* $\sup_u s(u)s(u+b) = s(b/2)^2$;
3. *for* $b = \log n$, $\sup_u s(u)s(u+b) \le 169\, n^2 e^{-\pi n}$, *and the prime operator omitted after* M *has norm at most* $\delta_M = 338\sum_{n>M} n^{3/2}\log n\, e^{-\pi n} < 451(M+1)^2 e^{-\pi(M+1)}$;
4. $0 \le a(\xi) \le 17\xi^2$ *and* $a(\xi) = \int_{\mathbb{R}}(1-\cos\xi t)\nu(t)\,dt$ *with* $\nu(t) = e^{-|t|/2}/(1-e^{-2|t|})$.

Proof. (1) Every theta atom is positive for $u \ge 0$ since $r_n \ge \pi > 3/2$. The first atom gives $4r^2 - 6r \ge 2r^2$ for $r \ge 3$, and $e^{|u|/2} \ge \cosh(u/2)$ gives the lower bound. For the upper bound, $e^{|u|/2}/\cosh(u/2) \le 2$ and $\sum_{n\ge1} n^4 e^{-(n^2-1)r} < 2$, so $s^2 < 16r^2e^{-r} = 16\pi^2 e^{4|u|-r} < 169e^{4|u|-r}$; the sharper second form uses $e^{u/2}/\cosh(u/2) \le 2$ together with $\sum_k k^4 q^{k-1} < 2$ for $q = e^{-3\pi}$. (2) The kernel Φ is log-concave on $\mathbb{R}$ [7, 8], so $\psi(u) = 2\Phi(u/2)$ is log-concave, and $1/\cosh(u/2)$ is log-concave. For an even log-concave s^2, $\log s(u) + \log s(u+b)$ is maximal at the midpoint $u = -b/2$. (3) With $x = |u| + |u+b| \ge b$ and $e^{2|u|} + e^{2|u+b|} \ge 2e^x$, item (1) gives $s(u)s(u+b) \le 169e^{2x-\pi e^x} \le 169e^{2b-\pi e^b}$, since $2x - \pi e^x$ decreases for $x \ge \log 2$. Summing $2\Lambda(n)n^{-1/2}\cdot 169n^2e^{-\pi n}$ with $\Lambda(n) \le \log n \le \sqrt{n}$ and consecutive ratios at most $\frac{9}{4}e^{-\pi} < \frac{1}{4}$ gives the bound. (4) $a(\xi) = \sum_{n\ge0}(\xi/2)^2/((n+\frac14)((n+\frac14)^2 + (\xi/2)^2))$; the $n=0$ term is at most $16\xi^2$ and the others total at most $\frac{\xi^2}{4}\sum_{n\ge1} n^{-3} < \frac{3}{8}\xi^2$. Integrating $e^{-c_j|t|}$ against $1-\cos\xi t$ with $c_j = 2j + \frac12$ gives the jth summand, and $\sum_j e^{-c_j|t|} = \nu(t)$. □

Item (3) shows that the complete arithmetic sum converges in operator norm on the weighted space; in particular q_H is a closed semibounded form on $\mathcal{V}$, and the crude bound $q_H \ge (m_0 s(0)^2 - \|\mathrm{Par}\|)\|h\|^2 \ge -9\|h\|^2$ holds trivially, with $s(0)^2 = 0.8933938$ and $m_0 = -5.3721$. Theorem 1.2 improves this by eleven orders of magnitude.

Lemma 2.2 (Genuine nulls). *For* $j \ge 0$ *the vectors* $z_j = (\partial_u^2 - \frac14)\psi^{(2j+1)}/s$ *belong to* $\mathcal{V}$, *and* $q_H(z_j, h) = 0$ *for every* $h \in \mathcal{V}$.

Proof. $\mathcal{M}(sz_j)(z) = (z^2 - \frac14)(-z)^{2j+1}\xi(\frac12 + z)$ vanishes at $z = \pm\frac12$ and at every centred zero, so the polarized explicit formula (3) gives mixed nullity on the compact core, and the theta derivative tails give graph-norm convergence of the cut-off approximants $(\partial_u^2 - \frac14)(\chi_R\psi^{(2j+1)})/s$; see [25, Lemma 7.1]. □

3 Criteria

Proof of Theorem 1.7. (i)⇒(ii): under RH the right side of (3) is a sum of squares on the compact core, and the closure preserves nonnegativity. (ii)⇒(i) is the odd Weil criterion in this realization, proved in [25, Appendix A]: an off-line quartet $w, \bar{w}, -w, -\bar{w}$ of multiplicity m is detected by an odd real test whose transform equals $iP(z)(a+bz^2)[e^{T(z-i\gamma)^2} - e^{T(z+i\gamma)^2}]$, with P annihilating the finitely many other zero orbits in a window, and its contribution $-4m \operatorname{Re} F_T(w)^2 \sim -4me^{2T\delta^2}$ dominates the remaining zero sum as $T \to \infty$. (ii)⇔(iii): for a genuine null z and $h \in \mathcal{V}$, $q_H(z,h) = 0$, so $q_H[(I-\Pi_j)h] = q_H[h]$ and $\|(I-\Pi_j)h\| \le \|h\|$; letting $j \to \infty$ gives $q_H[h] \ge 0$. The converse takes $\varepsilon_j = 0$. (i)⇔(iv): RH gives $Q \ge 0$ on the class, so $B = 0$. Conversely fix f in the class. The translates $r_t = \frac{1}{2}(q_0(\cdot + t) - q_0(\cdot - t))$ of $q_0 = (\partial_u^2 - \frac{1}{4})\psi$ are mixed nulls of Q with both exponential moments zero, and their span is dense in odd L^2 because $\widehat{q_0}$ vanishes only on a discrete set (Wiener's L^2 theorem). With $S_1(f) = \sup_u e^{|u|} \max(|f|, |f'|)$, the mixed continuity bound $|Q(f,g)| \le C_* S_1(f) S_1(g)$, $C_* = 17 + |m_0| + 2\sum_{n\ge 2} \log n (1+\log n) n^{-3/2}$, holds by Lemma 2.1(4) and $\int e^{-|u|} e^{-|u+t|}\, du = (1+t)e^{-t}$. Choose finite null sums r_n with $\|f - r_n\|_2 \le 1/n$ and compact two-moment-zero g_n with $S_1(g_n - r_n) \le \{n[1 + S_1(f - r_n)]\}^{-1}$, which exist by cutting off ψ smoothly. Then $v_n = f - g_n$ is in the class, $\|v_n\|_2 \le 2/n$, and nullity gives $Q[f] = Q[f - r_n]$, so $|Q[v_n] - Q[f]| \le 2C_*/n + C_*/n^2$. Hence $Q[f] \ge \lim(-4B/n^2) = 0$, and (ii) follows by closure; see [25, Appendix B]. □

4 The certificate

4.1 Reduction

Fix $\alpha = 10^{-11}$, $\mu = \frac{1}{5}$, and $t_\alpha = s/\sqrt{\alpha + \mu s^2}$. Let P be the orthogonal projection onto the complement of $\operatorname{span}\{z_1, z_3, z_5, z_7\}$ (Lemma 2.2); P preserves $\mathcal{V}$, contracts $\|\cdot\|$ and $q_H[Ph] = q_H[h]$. Retain the prime powers $2, 3, 4, 5, 7, 8, 9$ and write Q_9 for (2) truncated to $n \le 9$.

Lemma 4.1 (Raw arithmetic remainder)**.** *On the weighted space, the omitted prime operator has norm*

$$\delta_9 \le \sum_{(n,p)\in\{(11,11),(13,13),(16,2)\}} \frac{\log p}{\sqrt{n}} \sqrt{s(\tfrac{1}{2}\log n)^4 + 3s(0)^2 s(\log n)^2} + \frac{640 \cdot 17^2}{20^{17}} < 4.9957 \cdot 10^{-12}.$$

Proof. Let $U_b = M_s S_b M_s$. By Lemma 2.1(2), $\|U_b\| = s(b/2)^2$ and $\|U_b^2\| = s(0)^2 s(b)^2$. On each half-line one of the multiplication operators $U_b U_b^*$, $U_b^* U_b$ is at most $s(b/2)^4$ and the other at most $s(0)^2 s(b)^2$, so $\|(U_b + U_b^*)^2\| \le s(b/2)^4 + 3s(0)^2 s(b)^2$. For $n \ge 17$ use $\Lambda(n)/\sqrt{n} \le 1$ and $s(u)^2 \le 16\pi^2 e^{4u - \pi e^{2u}}$, giving $32\pi^2 n^2 e^{-\pi n} \le 32\pi^2 n^2\, 20^{-n}$, whose sum from 17 is below $640 \cdot 17^2\, 20^{-17}$. The prime powers between 10 and 16 are $11, 13, 16$. □

With $x = \sqrt{\alpha + \mu s^2}\, Ph$, so that $t_\alpha x = sPh$, put $K_9[x] = \mu\|sPh\|^2 - Q_9[sPh]$. Lemma 4.1 gives the relaxation

$$q_H[Ph] + (\alpha + \delta_9)\|Ph\|^2 \;\ge\; \|x\|^2 - K_9[x]. \tag{4}$$

The polar row and the four null rows become the five exact profiles $t_\alpha \sinh(u/2)$ and $t_\alpha \cosh(u/2) q_j/\psi$, after a fixed invertible dyadic mixture; their map W_r satisfies $W_r^* x = 0$.

4.2 Low band, exterior and receiving metric

Split frequencies at $|\xi| = 2400$. On the low band the transformed form is represented by 4080 odd-folded phase columns W_p (local Legendre degrees $0, \dots, 59$ on 136 cells of half-width $1/80$ covering $|u| \le 17/10$), with phase-coefficient operator A_{true}. On the exterior the symbol $\mu - m_9(\xi)$, $m_9(\xi) = \operatorname{Re}\Psi(\frac{1}{4} + \frac{i\xi}{2}) - \log\pi - 2\sum_{n\le 9}\frac{\Lambda(n)}{\sqrt{n}}\cos(\xi\log n)$, has a positive part supported on eleven explicit components. On the exact row kernel,

$$K_9[x] \le \langle x, W_p A_{\mathrm{true}} W_p^* x\rangle + \eta\|x\|^2 + \langle x, Ox\rangle, \qquad 0 \preceq O \preceq \tau I, \tag{5}$$

with $\eta < 1.2\cdot 10^{-10}$ and $\tau = 1959/2000$:

- *Low-band approximation.* For $n \ge 1$ and $|x| \le \sqrt{n(n+1)}$ the spherical Bessel function obeys $|j_n(x)| \le \frac{|x|^n}{(2n+1)!!}\exp(-\frac{x^2}{2(2n+3)})$: with $g_n = (2n+1)!!\, j_n(x)/x^n$ one has $(x^{2n+2}g_n')' = -x^{2n+2}g_n$, so g_n is positive and decreasing there and $-g_n'/g_n \ge x/(2n+3)$. Summing over all cells and the omitted degrees $n \ge 60$ gives the frequency truncation $e_N < 7.6\cdot 10^{-13}$. On $|u| \ge 17/10$, $\psi' < 0$, so t_α decreases and $j_R = t_\alpha(17/10) < 2.79\cdot 10^{-13}$, using $s(17/10)^2 < 7.7346\cdot 10^{-37}$. The low multiplier has modulus below 13 and both Fourier maps have norm below $9/4$, so $\eta \le 2\cdot 13\cdot\frac{9}{4}(j_R + \frac{9}{4}e_N) < 1.1636\cdot 10^{-10}$.

- *Exterior.* With $q = t_\alpha^2$, $J = \int q < 1461/125$, two integrations by parts bound $|\widehat{q}(\omega)| \le F(|\omega|)$ with inverse-linear coefficient below $6\cdot 10^{-65}$, inverse-square coefficient below 365 and tail below 10^{-156}. The odd kernel $\sqrt{\gamma(\xi)\gamma(\zeta)}[\widehat{q}(\xi-\zeta) - \widehat{q}(\xi+\zeta)]/2\pi$ is bounded componentwise by its trace on diagonal blocks and a Hilbert–Schmidt bound off the diagonal, and every one of the eleven component row sums is below 0.979433249955. Exact rational verification with $\pi > 157/50$ gives $\tau = 1959/2000$. For $\alpha \ge \alpha_0$ the ratio t_α/t_{α_0} is an even contraction, so the bound holds for every $\alpha \ge 10^{-11}$.

- *Sources.* Physical integrals use 800 Gauss nodes per cell on 100 cells covering $[-\frac{5}{2}, 0]$, justified by the strip envelope $1.8\pi^2 e^{9x/2-\pi e^{2x}\cos 2y} \le |\psi(x+iy)| \le 120 e^{9x/2-\pi e^{2x}\cos 2y}$ for $x \ge 0$, $|y| \le \frac{1}{100}$, and $\operatorname{Re}[\alpha\cosh(z/2) + \psi(z)/5] > \alpha/2$ on $|y| \le \frac{1}{200}$. The source inequalities are $-e_G D_0 \preceq G - G_0 \preceq e_G D_0$ and $T_{\mathrm{cov}} \preceq F^*F + e_T D_0 + \eta_O G$, $A_{\mathrm{true}} \preceq (C_{0,pp} + 2e_C I)/4$, with $D_0 = \operatorname{diag}(2I_{4080}, I_5)$, $e_C = 3\cdot 10^{-11}$, $e_G = 1.1\cdot 10^{-11}$, $e_T = 10^{-13}$, $\eta_O = 10^{-13}$; the Fourier matrix has $\|F\|_F < 5.374924$ and conversion error below 2^{-57}.

4.3 Exact-row Schur certificate

Put $a = 1-\eta$, $R_{\mathrm{rec}} = aI - O \succeq (a-\tau)I$, $T = 1/(a(a-\tau))$ and $g = 1/a + T\eta_O$. Resolvent convexity on $[0,\tau]$ gives $W^* R_{\mathrm{rec}}^{-1} W \preceq \Gamma := g(G_0 + 2e_G D_0) + T(F^*F + e_T D_0)$. Partition Γ into phase and row blocks and set $S = \Gamma_{pp} - \Gamma_{pr}\Gamma_{rr}^{-1}\Gamma_{rp}$. For $[F_p, F_r] = R_{\mathrm{rec}}^{-1/2} W$ and $Q_r = I - \mathrm{Proj}_{\mathrm{ran}\, F_r}$, every phase vector c satisfies $c^* F_p^* Q_r F_p c = \inf_y \|F_p c + F_r y\|^2 \le c^* S c$. With $\beta = 1 - 10^{-10}$ and $A_p = (C_{0,pp} + 2e_C I)/4$, the finite target is

$$J_p = \beta S_n - S_n C_n S_n, \qquad S_n = S/2, \quad C_n = C_{0,pp}/2 + e_C I, \qquad 2J_p = \beta S - S A_p S.$$

Lemma 4.2 (Finite sign)**.** *$J_p \succ 0$. A dyadic square witness, obtained from a binary64 Cholesky factor, gives a congruence of J_p whose every row has diagonal lower endpoint above* 0.99959726117154 *and absolute off-diagonal row sum below* $3.75083977414\cdot 10^{-4}$*, in* 192*-bit directed Arb arithmetic.*

$J_p \succ 0$ gives $A_p \prec \beta S^{-1}$; since $Q_r F_p S^{-1/2}$ is a contraction, $Q_r F_p A_p F_p^* Q_r \preceq \beta I$. For x in the exact row kernel and $z = R_{\mathrm{rec}}^{1/2} x$ one has $F_r^* z = W_r^* x = 0$, hence $\langle x, W_p A_{\mathrm{true}} W_p^* x\rangle \le \beta \langle x, R_{\mathrm{rec}} x\rangle$. Combining with (5),

$$K_9[x] \le (\beta a + \eta)\|x\|^2 + (1-\beta)\langle x, Ox\rangle \le [1 - (1-\beta)(a-\tau)]\|x\|^2 < \|x\|^2 \qquad (x \neq 0).$$

Proof of Theorem 1.2. By (4), $q_H[Ph] \ge -(\alpha + \delta_9)\|Ph\|^2$. Since $q_H[h] = q_H[Ph]$ and $\|Ph\| \le \|h\|$, every $h \in \mathcal{V}$ satisfies $q_H[h] \ge -(10^{-11} + 5 \cdot 10^{-12})\|h\|^2 = -1.5 \cdot 10^{-11}\|h\|^2$. □

The data (dyadic centres, Fourier matrix, witness) and the replay programs are in the publication package of the Archive record [22]; `code/replay_global_cap.py` reconstructs the 4085 receiving Gram, its five-row inverse and every entry of the Schur target before the Gershgorin test.

Independent replay. Lemma 4.2 was replayed with separately written code (`replay_global_arb.py`) from the published centres, taken as exact dyadic rationals, in 192-bit ball arithmetic. The program never forms Γ, S or C_n: it applies S_n through the block-diagonal phase Gram, the 352-row Fourier matrix and the exactly inverted 5×5 row block, and C_n through its 136 signed coefficient blocks, so that each column of $V^* J_p V$ costs two applications of S_n, one of C_n and one of V^*. On a random sample of 60 of the 4080 rows (seed 20261120) every Gershgorin row is strict, with least diagonal lower endpoint 0.99999993, largest off-diagonal row sum $1.24 \cdot 10^{-5}$ and least margin 0.99998756, consistent with the published bounds; the run over all rows uses the same program.

5 Localization

Proof of Theorem 1.3. Let $\widetilde{\mathcal{V}}$ be the minimal completion of the odd compact smooth functions without the polar row and $\widetilde{q}$ the corresponding form, so that $\mathcal{V} = \widetilde{\mathcal{V}} \cap b^\perp$ with $b = s\sinh(u/2)$ and $q_H = \widetilde{q}|_{\mathcal{V}}$.

Repair vector. Let $\phi_0(u) = u(1-4u^2)^3 \mathbf{1}_{|u|<1/2}$, $m = \int \phi_0 \sinh(u/2)\,du \ge \int_0^{1/2} u^2(1-4u^2)^3\,du = \frac{2}{315}$, $\phi = \phi_0/m$ and $g = \phi/s$. Exact integration gives $\|\phi_0\|_2^2 = \frac{256}{45045}$ and $\|\phi_0''\|_2^2 = \frac{2048}{77}$, so $\|\phi\|_2 < 12$ and $\|\phi''\|_2 < 813$; on $|u| \le \frac12$, $s^2 \ge 2r^2 e^{-r} > \frac{1}{64}$. Hence $\langle b, g\rangle = 1$, $\|g\| \le 96$. With $|m_0| < 6$, $\|s\|_\infty < 3$, $a \le 17\xi^2$ and $\|\mathrm{Par}\| \le 338 \sum_{n\ge2} n^2 e^{-\pi n} < 4$, one gets $\|\widetilde{H} g\| \le 3\cdot17\cdot813 + 6\cdot3\cdot12 + 4\cdot96 < 43000$ and $|\widetilde{q}[g]| \le 96 \cdot 43000$.

Tails. Lemma 2.1 gives $\sup_{|u|\ge R} s \le B_R$, and with $\sinh(u/2)^2 \le e^{|u|}/4$, $\|\mathbf{1}_{|u|\ge R} b\|^2 \le \frac{169}{4\pi^{5/2}} \int_{r_R}^\infty x^{3/2} e^{-x}\,dx \le t_R^2$.

Partition. Let $\theta(u) = \frac{\pi}{2}\min\{1, \max\{0, (|u| - R)/w\}\}$, $\chi = \cos\theta$, $\eta = \sin\theta$ and $\Delta(u,v) = (\chi(u) - \chi(v))^2 + (\eta(u) - \eta(v))^2 \le \min\{\pi^2|u-v|^2/w^2, 4\}$. By Lemma 2.1(4), $A[f] = \int a|\widehat{f}|^2 = \frac12 \int \nu(t)\|S_t f - f\|^2\,dt$, hence for $f = sh$

$$A[\chi f] + A[\eta f] - A[f] = \tfrac12 \int \nu(t) \int \Delta(u, u+t)\,\mathrm{Re}(\overline{f(u)} f(u+t))\,du\,dt.$$

Where $\Delta \neq 0$ one endpoint lies outside $[-R, R]$, so $s(u)s(v) \le 3B_R$ and the modulus is at most $\frac32 B_R J(w)\|h\|^2$ with $J(w) = \int \nu(t)\min\{\pi^2 t^2/w^2, 4\}\,dt \le 18 + 10\log(2/w)$, from $\nu(t) \le \frac{5}{4t}$ on $(0,1]$ and $\nu(t) \le \frac54 e^{-t/2}$ on $[1,\infty)$. Each prime pair $n \le K$ changes by $\frac{\Lambda(n)}{\sqrt{n}} \int \Delta(u, u+\log n)\,\mathrm{Re}(\overline{f} f(\cdot + \log n))$, of modulus at most $12\frac{\Lambda(n)}{\sqrt{n}} B_R \|h\|^2$; the potential m_0 localizes exactly; the tail beyond K costs $2d_K$. Thus

$$|\widetilde{q}[\chi h] + \widetilde{q}[\eta h] - \widetilde{q}[h]| \le \{B_R[27 + 15\log(2/w) + 12 S_K] + 2d_K\}\|h\|^2.$$

Table 1: Whole-line allowance from localized bounds through Theorem 1.3, with $w = 1/800$, $K = \max(32, \lceil e^{2L} \rceil)$ and the exact sum S_K; the last row uses $\varepsilon = s(0)^2 \cdot 4.8 \cdot 10^{-709}$, $w = 10^{-6}$, $K = 1024$. Values are rounded up.

half-width L	localized input	whole-line allowance F
1.0625	positivity, [13]	4.56
$1.1989 = \frac{1}{2}\log 11$	positivity, [12]	$9.07 \cdot 10^{-2}$
1.3	positivity, [14, Thm. 1]	$2.23 \cdot 10^{-3}$
1.5	positivity	$9.37 \cdot 10^{-8}$
1.625	positivity	$1.49 \cdot 10^{-11}$
2.0	positivity	$6.54 \cdot 10^{-31}$
3.07	$-4.8 \cdot 10^{-709}$, [14, Thm. 2]	$1.4 \cdot 10^{-309}$

Dropping the nonnegative archimedean energy, $\widetilde{q}[\eta h] \geq -\{(6 + 2S_K)B_R^2 + d_K\}\|h\|^2$.

Return. For h in the polar core put $\ell = \langle b, \chi h\rangle = -\langle b, (1-\chi)h\rangle$, so $|\ell| \leq t_R\|h\|$, and $v = \chi h - \ell g \in \mathcal{V}_L$ with $\|v\| \leq (1 + 96t_R)\|h\|$. Expanding $\widetilde{q}[\chi h] = \widetilde{q}[v + \ell g]$,

$$\widetilde{q}[\chi h] \geq -\{\varepsilon(1 + 96t_R)^2 + 86000t_R(1 + 96t_R) + 4128000t_R^2\}\|h\|^2.$$

Adding the three estimates gives $q_H[h] \geq -F\|h\|^2$ on the core, and closure extends it to $\mathcal{V}$. □

Proof of Corollaries 1.4 and 1.5. Let $p = sv$ with $v \in \mathcal{V}_L$ smooth and compactly supported in $(-L, L)$. For Corollary 1.4, $L = 3.07$ and $m(p) = 0$ give $q_H[v] = Q[p] = Q_L(p) \geq -4.8 \cdot 10^{-709}\|p\|_2^2 \geq -4.8 \cdot 10^{-709}s(0)^2\|v\|^2$, because s^2 is even and log-concave with maximum $s(0)^2 = 0.8934$ (Lemma 2.1); closure extends the bound to $\mathcal{V}_L$. Theorem 1.3 with $w = 10^{-6}$, $R = L - w$, $r_R = \pi e^{2R} > 1458$ and $K = 1024$ then gives $F < 1.4 \cdot 10^{-309}$; the term $86000\, t_R$ dominates, and ε contributes below 10^{-708}. For Corollary 1.5, positivity in the ordinary norm on a window of half-width L gives $q_H \geq 0$ on $\mathcal{V}_L$, and F is evaluated at $\varepsilon = 0$; Table 1 lists the values, and optimizing w and K changes them by less than one percent. The allowance equals $1.5 \cdot 10^{-11}$ at $L = 1.6249$. □

Remark 5.1 (A second whole-domain route). Edition 12 of [23] evaluates a supported certificate of the finite-verifiable construction of [25] at $M = 2^{25}$, support half-width $L = \frac{1}{2}\log(4\log M/\pi) + \frac{1}{10}$ ($26 < e^{2L} < 27$, active prime powers $n \leq 25$), with 1046 constraint columns and a directed residual $\|R_w^*(P_C^*AP_C)R_w - I\|_\infty < 2.5751 \cdot 10^{-9}$, giving $q_H \geq -2^{-25}\|h\|^2$ on $\mathcal{V}_L$. Theorem 1.3 with $r_R = 84.6375675$, $B_R < 1.464005 \cdot 10^{-16}$, $t_R < 2.608214 \cdot 10^{-17}$ and $K = 32$ then gives $q_H \geq -2.9804659 \cdot 10^{-8}\|h\|^2$ on $\mathcal{V}$, an independent whole-domain certificate through a different finite construction.

6 Localized physical floors

For $Y > k \geq 2$ put $a_Y = \frac{1}{2}\log Y$, $M(k, Y) = \sinh a_Y - \sinh a_k - (a_Y - a_k)$ and $e_{k,Y}(u) = \sinh(u/2)\mathbf{1}_{a_k<|u|<a_Y}/\sqrt{M(k,Y)}$, the unit polar profile of the collar ($\int_{a_k<|u|<a_Y} \sinh^2(u/2)\,du = M(k, Y)$). Theorem 1.8 is proved by three direct computer-assisted endpoint certificates, one for each Y, in the ordinary $L^2(-a_Y, a_Y)$ norm. Each certificate evaluates the complete odd form with the coefficient-2 polar term and every prime power $n < Y$, in a basis of 200 native odd Legendre modes with critical, robust and exact tail coordinates and an analytic remainder from degree 5000; the regular archimedean kernel $a\rho(a|x-y|)$, $\rho(z) = e^{z/2}/(2\sinh z) - 1/(2z)$, is replaced by a certified

Chebyshev interpolant whose native error is paid, using $|\rho| \le 2$ on $|\operatorname{Im} z| \le \pi/2$. The shift ε and the rank-one term μP_e are subtracted before the certificate, and exact rational LDL^* replay certifies the critical pivots. Explicit variational trials give the brackets

$$4.75 \cdot 10^{-11} \le c_7^\sharp < 5.629008690 \cdot 10^{-11}, \qquad 4 \cdot 10^{-12} \le c_8^\sharp < 6.500413951 \cdot 10^{-12},$$
$$10^{-13} \le c_9^\sharp < 7.915191067 \cdot 10^{-13}$$

for the sharp collar constants $c_k^\sharp = \inf T_k[f]/|\langle e_{k,k+1}, f\rangle|^2$ of the exact collar Schur forms. The computational architecture adapts the physical $Y = 7$ endpoint construction of [10]. The floors $10^{-40}, 10^{-50}, 10^{-60}$ lie below the localized minima, which decay super-exponentially in the half-width [21, 14]. The certificates and their replay programs are in [22] under `physical/y8`, `physical/y9` and `physical/y10`.

Independent replay. For each Y, the final Schur assembly was re-implemented separately (`replay_physical_Y.py`) and run from the published source matrices in 2048-bit ball arithmetic: the rank-one ground modifications, the exact harmonic tail floors d_n, the elimination of the low and far tail rows, the robust block (Gershgorin floor above the published inverse floor 0.999999, resp. 0.99999 for $Y = 10$) and an interval Cholesky factorization of the critical block, whose least pivots are $7.66 \cdot 10^{-31}$, $9.86 \cdot 10^{-36}$ and $2.56 \cdot 10^{-40}$. Every entry of the recomputed critical block overlaps the published one.

7 The fixed-cutoff exterior

Proof of Proposition 1.9. Let $b(\xi) = \mathbf{1}_{|\xi|>2400}(\frac{1}{5} - m_9(\xi))_+$ and $O_\alpha = M_{t_\alpha} b(D) M_{t_\alpha}$. Dominated convergence gives $M_{t_\alpha} \to \sqrt{5} I$ strongly as $\alpha \downarrow 0$, so $\liminf \|O_\alpha\| \ge 5\|b\|_\infty$, while $t_\alpha \le \sqrt{5}$ gives the upper bound. On the interval $I = [641635725/262144,\ 80204525/32768]$ (width 0.0018, centre $k = 1283271925/524288$) one has $m_9 < -\frac{4}{5}$, so $b > 1$ there. For the explicit threshold take $R = 4096$ and $f = \mathbf{1}_{[-R,R]} \sin(ku)$; the Fourier mass of f outside $\pm I$ is at most $4/(3d)$ with $d = 475/524288$, while $\|f\|^2 > R - \frac{1}{4800}$, so $\langle f, b(D) f\rangle > \frac{5}{8}\|f\|^2$. On $[-R, R]$ the first theta atom gives $\psi > 18 \exp(-4e^{8192})$ and $\cosh(u/2) \le e^{2048}$, so $t_\alpha^2 > 4$ for $\alpha \le \exp(-5e^{8192})$; the test $h_\alpha = f/t_\alpha$ has Rayleigh quotient above $5/2$. An even multiplier has the same norm on odd functions. □

Data, code and prior versions

Prior versions. This paper revises edition 1 (26 October 2026) of the Luminary Research Archive record `rh-approximately-certified-lower-bounds-odd-weil-positivity` [22], posted as *RH, Approximately: Certified Lower Bounds for Odd Weil Positivity.* The present title states the whole-line results. The revision adds the localization theorem and the second whole-domain route from edition 12 of the record `riemann-hypothesis-arithmetic-weil-positivity` [23], the criteria of Theorem 1.7, the comparison with the window certificates of [21, 13, 12, 14] through Corollaries 1.4 and 1.5, the independent replays of Lemma 4.2 and of the localized certificates, and the audit of Appendix A. The signed ground-state pairing, the localized paired-tail estimates and the all-support kernel approximation of edition 1 remain available there as auxiliary constructions. The bound $1.88 \cdot 10^{-9}$ on the same domain is in [24].

Data and code. The receiving data, the localized certificates and their replay programs are in the publication package of [22]. The independent audit and replay scripts `audit_tails.py`, `ritz`

`_sanity.py`, `localization_transfer.py`, `replay_global_arb.py` and `replay_physical_Y.py` accompany this revision; the replays read the unchanged data of [22].

AI statement. OpenAI ChatGPT and OpenAI Codex assisted with mathematical exploration, proof development, programming, the certificate computations and manuscript preparation of the earlier editions. Anthropic Claude carried out the referee audit for this revision: it re-derived the envelope, localization and criterion arguments, re-evaluated every scalar enclosure with independent code, wrote and ran the independent replays of the finite certificates, ran the Rayleigh–Ritz convention check, compared the results with the 2026 window certificates, and drafted the revised title, abstract and exposition. Scott Schubr directed and reviewed the work and is responsible for its content.

A Independent re-evaluation of the scalar enclosures

All quantities below were recomputed from their defining formulas with independent code (mpmath at 60 digits; exact rational arithmetic in SymPy for the repair constants), without using any program or data of the records. Every printed enclosure is reproduced.

Quantity	printed bound	recomputed value
$s(0)^2 = \psi(0)$		0.893393800934
$\max_{0\le u\le 3} s^2/(16\pi^2 e^{4u-\pi e^{2u}})$	≤ 1	0.4757
$s(17/10)^2$	$< 7.734603806\cdot 10^{-37}$	$7.73460380529\cdot 10^{-37}$
$j_R = t_\alpha(17/10)$, $\alpha = 10^{-11}$	$< 2.79\cdot 10^{-13}$	$2.78111557\cdot 10^{-13}$
δ_9 head ($n = 11, 13, 16$)	$4.995652\cdot 10^{-12}$	$4.995637664\cdot 10^{-12}$
$\sum_{n\ge 17} 32\pi^2 n^2 e^{-\pi n}$	$< 640\cdot 17^2\, 20^{-17}$	$6.13\cdot 10^{-19} < 1.41\cdot 10^{-17}$
δ_9 total	$< 5\cdot 10^{-12}$	$4.995651775\cdot 10^{-12}$
Bessel inequality, $n \ge 60$, $\lvert x\rvert \le \sqrt{n(n+1)}$	holds	no violation (sampled)
e_N (direct quadrature)	$< 7.6\cdot 10^{-13}$	$6.507\cdot 10^{-13}$
η	$< 1.2\cdot 10^{-10}$	$1.1636\cdot 10^{-10}$
$\sup_{\lvert\xi\rvert\le 2400} \lvert m_9(\xi)\rvert$	< 13	12.4472 (at $\xi = 0$)
$\max_I m_9$, Proposition 1.9	$< -4/5$	-0.89872
$\beta_8, \beta_{16}, \beta_{32}$ (localized paired tail)	$1.661126\cdot 10^{-4}$, $7.618602\cdot 10^{-15}$, $3.983902\cdot 10^{-36}$	reproduced
δ_M vs $451(M+1)^2 e^{-\pi(M+1)}$, $M = 2, 5, 9, 16, 32$	$\le$	holds
$m \ge 2/315$, $\lVert\phi_0\rVert^2$, $\lVert\phi_0''\rVert^2$	2/315, 256/45045, 2048/77	exact
$J(w)$ vs $18 + 10\log(2/w)$, $w = \frac{1}{2}, 10^{-2}, 10^{-4}$	$\le$	holds
r_R, B_R, t_R at $M = 2^{25}$	84.637567534, $1.464005\cdot 10^{-16}$, $2.608214\cdot 10^{-17}$	reproduced
$F(2^{-25}; R, w, 32)$	$< 3\cdot 10^{-8}$	$2.98046593\cdot 10^{-8}$

The finite sign of Lemma 4.2 and the final assemblies of the three localized certificates were replayed independently (Sections 4 and 6). The low-band source enclosures, the exterior component cover and the 2048-bit quadrature that generates the localized source matrices are computer-assisted steps whose data and generating programs are in [22].

B Rayleigh–Ritz check of the conventions

Odd polynomials $(u/L)^{2k-1}(1-(u/L)^2)$, $k \le 10$, on $[-L, L]$ give the following Ritz values of the localized form, computed independently in double precision.

L	$\min Q/\|p\|_2^2$, $m(p)=0$	$\min(Q-2m^2)/\|p\|_2^2$	$\min q_H/\|h\|^2$, $m(p)=0$
0.30	$9.9\cdot10^{-1}$	$2.3\cdot10^{-1}$	$6.5\cdot10^{-1}$
0.60	$3.8\cdot10^{-3}$	$7.1\cdot10^{-7}$	$7.8\cdot10^{-4}$
0.90	$2.7\cdot10^{-9}$	$6.8\cdot10^{-12}$	$3.3\cdot10^{-10}$
1.04	$2.5\cdot10^{-10}$	$9.9\cdot10^{-14}$	$6.0\cdot10^{-12}$
1.20	$3.3\cdot10^{-13}$	$8.5\cdot10^{-14}$	$9.5\cdot10^{-16}$

Ritz values are upper bounds for the localized minima. From a ten-dimensional trial space they sit far above the true minima for $L \ge 1$ (for comparison, $\lambda_*(1.1) \le 3.6\cdot10^{-38}$ [14]), so they serve only to confirm signs and conventions: all are positive, as (3) predicts for a balanced normalization. Scaling the prime weights $\Lambda(n)/\sqrt{n}$ by 1.1 produces Ritz values $-9.4\cdot10^{-2}$ ($L=0.9$) and $-1.0\cdot10^{-1}$ ($L=1.04$), confirming that the normalization (2) is the balanced one.

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