Real-section permanent-derivative compensation and quadratic phase control

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Luminary Oddities

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Abstract

We prove a uniform lower bound for a compensated homogeneous permanent derivative on the entire real centered positive semidefinite section of order six. If r is the trace, the bound is strictly greater than $5000r^4$ for nonzero matrices, with explicit positive reserves from two actual omission variances. The proof cancels two signed contractions by subtracting fifty-eight times the fourth-degree omission variance and pays the ordinary side with the third-degree omission variance. No spectral gap, rank or entry-sign restriction is needed. Conjugation symmetry makes the loss along an imaginary perturbation quadratic; explicit coefficient and multilinear norm budgets then prove the lower bound $219r^4$ on the closed complex tube $\|\operatorname{Im} K\|_{\mathrm{F}} \le r/1000$. We also record the unrestricted complex cancellation identity and its quadratic-state correction. Its remaining coupled remainder is not paid globally.

1 The real section and the new compensation mechanism

All matrices have order six. Put $J = \mathbf{1}\mathbf{1}^*$ and $W = 6I - J$. Positive semidefinite means Hermitian positive semidefinite; centered means $K\mathbf{1} = 0$. A bar means entrywise conjugation and $\circ$ denotes entrywise multiplication. With entrywise absolute square, set

$$f_K(v) = \operatorname{per}(K + vJ), \qquad \mathcal{P}(K) = f_K(1) + f_K'(1), \qquad D(K) = \mathcal{P}(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2,$$

and define the actual coefficients

$$E_j(K) = [x^j]\, \partial_v D(xK + vW)|_{v=0}\,, \qquad F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{1}$$

The additive direction is vW after xK. An extra x in that direction would give a different coefficient. Put

$$\alpha_j(K) = [x^j] \operatorname{per}(J + xK), \qquad \delta_j(K) = \partial_v \alpha_j(K + vW)|_{v=0} \quad (j = 4, 5).$$

The quantities F_4, δ_5, δ_4 have respective homogeneous degrees four, four and three over nonnegative real scalars.

The weighted expression $f(1) + f'(1)$ occurs in the border identity of Li–Zhao–Luo [5, Proposition3.2(iii)]. A complete permanent comparison does not imply positivity of a signed homogeneous derivative. Earlier papers [2, 1] established compensation for centered complex matrices satisfying $K\overline{K} = 0$, first in rank one and then on the complete rank-at-most-two locus. The real section is different: for nonzero real K, its overlap $\operatorname{tr}(K\overline{K}) = \operatorname{tr}(K^2)$ is positive. The null-locus theorem therefore does not cover this section.

The comparison-matrix-positive class studied by Knowledge Sense [6] provides a different sufficient condition for Chollet's full inequality at arbitrary order. Its graph-factorization and pencil argument concerns that class, not the homogeneous functional F_4 on the entire real centered section.

The new proof retains $p_2 = \sum_i g_i^2$ in the Gram forms and uses two actual omission reserves. The cancellation multiplier is 58 rather than the null-locus multiplier 10: it removes both a signed cubic contraction and a diagonal mixed contraction. The remaining ordinary cubic term is paid by δ_4, not by independently maximizing its moments.

Theorem 1.1 (Uniform real-section compensation)**.** *For every real symmetric centered $K \succeq 0$, with $r = \operatorname{tr} K$,*

$$\begin{aligned} F_4(K) \geq (20160 - 464\sqrt{2})\delta_5(K) + 14800r\delta_4(K) \\ + (39744 - 10368\sqrt{2})r^2 \operatorname{tr}(K^2). \end{aligned} \tag{2}$$

Both omission variances are nonnegative. For $K \neq 0$, $F_4(K) > 5000r^4$.

All real centered ranks one through five, zero rows and eigenvalue degenerations are included. A centered matrix cannot have rank six. Positive anchors $K + sJ$ have rank six when the centered residual has rank five; (2) concerns the derivative of that residual.

Theorem 1.2 (Quadratic phase control)**.** *For centered complex $K \succeq 0$ with trace $r > 0$, set $R = \operatorname{Re} K$ and $s = \|\operatorname{Im} K\|_{\mathrm{F}}/r$. There are explicit constants*

$$\begin{aligned} A_2 &= 4454218368 + 228096000\sqrt{2} + 412128\sqrt{6} < 4780000000, \\ A_4 &= 740689920 + 38016000\sqrt{2} < 795000000 \end{aligned}$$

such that

$$|F_4(K) - F_4(R)| \leq r^4(A_2 s^2 + A_4 s^4).$$

Consequently $\|\operatorname{Im} K\|_{\mathrm{F}} \leq r/1000$ implies $F_4(K) > 219r^4$. The tube endpoint is closed.

The general complex identity in Appendix C is an additional organizing tool, not a global complex sign theorem. Even global positivity of F_4 would leave the other homogeneous blocks and the full Chollet conjecture to be proved.

2 Two actual omission variances

The zero matrix is immediate. Unless stated otherwise, Sections 2–5 normalize $\operatorname{tr} K = 1$; the general-trace conclusions follow by homogeneity. For real K write

$$t = \operatorname{diag} K, \quad b = \operatorname{diag}(K^2), \quad e = \operatorname{diag}(K^3), \quad d_j = \sum_i t_i^j, \quad \beta_j = \operatorname{tr}(K^j), \quad S_j = \sum_{i,k} K_{ik}^j,$$

and put

$$U = t^T b, \qquad C = t^T (K \circ K \circ K)\mathbf{1}, \qquad X = \operatorname{tr}(K^2(K \circ K)).$$

Powers inside tr or diag are ordinary matrix products.

Take real centered Gram linear forms g_i, with $\langle g_i, g_j \rangle = K_{ij}$. Use the Fischer inner product $\|z^\gamma\|^2 = \gamma!$; products of linear forms pair by their Gram permanent [4]. Let $p_j = \sum_i g_i^j$. Define

$$\xi_i = g_i^4 - \tfrac{1}{2} g_i^2 p_2 - \tfrac{1}{3} g_i p_3, \qquad \psi_i = g_i^3 - \tfrac{1}{2} g_i p_2.$$

Newton's identities with $p_1 = 0$ show that the omitted elementary fourth products differ from ξ_i by a common term, and the omitted third products equal $p_3/3 - \psi_i$. Differentiating their Gram permanents along W cancels common terms. The residual-slot factors are $(6-5)! = 1$ and $(6-4)! = 2$, so

$$\delta_5 = 6\sum_i \|\xi_i\|^2 - \left\|\sum_i \xi_i\right\|^2 \geq 0, \tag{3}$$

$$\delta_4 = 2\left[6\sum_i \|\psi_i\|^2 - \left\|\sum_i \psi_i\right\|^2\right] \geq 0. \tag{4}$$

Each bracket is the sum of squared pairwise differences. In particular the factor two in (4) is essential.

Three-factor matching gives

$$\|\psi_i\|^2 = 6t_i^3 + \tfrac{1}{2}t_i\beta_2 + e_i - 6t_i b_i, \qquad \sum_i \psi_i = p_3, \qquad \|p_3\|^2 = 6S_3.$$

Therefore

$$\delta_4 = 72d_3 - 12S_3 - 72U + 6\beta_2 + 12\beta_3. \tag{5}$$

The corresponding unrestricted four-factor matching gives

$$\begin{aligned}\sum_i \|\xi_i\|^2 &= 24d_4 + \|b\|^2 + 4t^T e + \beta_2 d_2 + \tfrac{2}{3}S_3 + 6X \\ &\quad - 24(t^{\circ 2})^T b - 16C + 4t^T K b, \\ \left\|\sum_i \xi_i\right\|^2 &= 24S_4 - 24\|b\|^2 + 2\beta_2^2 + 4\beta_4.\end{aligned}$$

For example the squared $g_i^2 p_2/2$ term contributes $b_i^2 + 4t_i e_i + t_i^2\beta_2$, and the squared $g_i p_3/3$ term contributes $2t_i S_3/3 + 2(K(K\circ K)K)_{ii}$. The cross terms with g_i^4 give $-24t_i^2 b_i - 16t_i c_i$, where $c = (K\circ K\circ K)\mathbf{1}$; the remaining cross term gives $4\sum_{j,k} K_{ik}^2 K_{ji}K_{jk} + 4t_i(Kb)_i$. Their summed matrix contractions contribute $6X$. Also $\sum_i \xi_i = p_4 - p_2^2/2$. Substituting into (3) proves

$$\begin{aligned}\delta_5 = {} & 144d_4 - 144(t^{\circ 2})^T b + 30\|b\|^2 + 6\beta_2 d_2 + 24t^T e + 24t^T K b \\ & + 4S_3 + 36X - 96C - 24S_4 - 2\beta_2^2 - 4\beta_4.\end{aligned} \tag{6}$$

No null-overlap relation has been imposed; the quadratic state p_2 is retained in every matching.

3 The real cancellation and its global payment

Define the actual paired derivative

$$T_4 = [x^4]\,\partial_v \operatorname{per}|\sqrt{2}J + xK + vW|^2\big|_{v=0}, \qquad P_4 = \frac{T_4}{8\sqrt{2}}.$$

The unrestricted mixed partition calculation in Appendix A gives, for real centered K,

$$\begin{aligned}P_4 = {} & 8208d_4 - 8352(t^{\circ 2})^T b + 1848\|b\|^2 + 744\beta_2 d_2 + 960t^T e \\ & - 5568C + 1824t^T K b + 496S_3 + 1440X - 1368S_4 - 200\beta_2^2 - 160\beta_4.\end{aligned} \tag{7}$$

Since $5568 = 58 \cdot 96$, subtracting $58\delta_5$ cancels both C and $(t^{\circ 2})^T b$ exactly:

$$\begin{aligned} P_4 &= 58\delta_5 + R_{58}, \\ R_{58} &= -144d_4 + 108\|b\|^2 + 396\beta_2 d_2 - 432 t^T e + 432 t^T K b \\ &\qquad + 264 S_3 - 648 X + 24 S_4 - 84\beta_2^2 + 72\beta_4. \end{aligned} \tag{8}$$

Lemma 3.1. *For every normalized real centered $K \succeq 0$, $R_{58} \le 1296\beta_2$.*

Proof. At trace one, $0 \preceq K \preceq I$ and

$$d_2 \le \beta_2 \le 1, \qquad \|b\|^2 \le \beta_4 \le \beta_2^2 \le \beta_2.$$

Positive semidefinite diagonal entries give $t^T e \ge 0$, and the Schur product theorem gives $X \ge 0$ and $S_3 = \operatorname{tr}(K(K \circ K)) \ge 0$. Moreover

$$S_3 \le \|K\|_{\mathrm{op}} d_2 \le \beta_2, \qquad S_4 \le \sum_{i,k} t_i^2 t_k^2 = d_2^2 \le \beta_2,$$

$$|t^T K b| \le \|K\|_{\mathrm{op}} \sqrt{d_2} \|b\| \le \beta_2^{3/2} \le \beta_2.$$

Discard the negative terms in (8) and bound every remaining contraction by its coefficient times β_2. Their coefficient sum is $108 + 396 + 432 + 264 + 24 + 72 = 1296$. No inverse or lower-eigenvalue assumption is used. □

4 The ordinary cubic reserve completes the proof

The actual ordinary coefficient identities, recalled with their normalizations in Appendix B, give

$$\begin{aligned} F_4 &= O - 8\sqrt{2} P_4, \\ O &= 20160\delta_5 + 1067904 d_3 - 154944 S_3 - 1065600 U \\ &\qquad + 128544\beta_2 + 179904\beta_3 + 5760\beta_2(6d_2 - \beta_2). \end{aligned} \tag{9}$$

Subtracting $14800\delta_4$ using (5) yields the exact identity

$$\begin{aligned} O &= 20160\delta_5 + 14800\delta_4 + 2304 d_3 + 22656 S_3 + 39744\beta_2 \\ &\qquad + 2304\beta_3 + 5760\beta_2(6d_2 - \beta_2). \end{aligned} \tag{10}$$

Every additional term is nonnegative: $d_3, \beta_3 \ge 0$, $S_3 \ge 0$ by the Schur product theorem, and $6d_2 \ge 1 \ge \beta_2$ since $\sum_i t_i = 1$. Combining (8), Lemma 3.1 and (10) proves (2) at trace one. Centering puts K on a five-dimensional space, so $\beta_2 \ge 1/5$ by Cauchy–Schwarz on its eigenvalues. The rational bound $\sqrt{2} < 71/50$ gives

$$(39744 - 10368\sqrt{2})/5 > 5000.$$

The two variance coefficients are positive. This proves the strict claim for nonzero K. Scaling restores r: δ_5 has degree four, δ_4 degree three and β_2 degree two. Zero rows, all singular limits and $K = 0$ require no separate approximation.

5 Quadratic phase loss and the closed complex tube

Let $K \succeq 0$ be complex, centered and trace one, and write $K = R + iS$, where R is real symmetric and S is real antisymmetric. Then $R = (K + \overline{K})/2$ is real positive semidefinite and has trace one and Frobenius norm at most one. The pencil $R + ixS$ is positive semidefinite for $|x| \le 1$ by convexity between K and $\overline{K}$. The original permanent definitions imply $F_4(\overline{K}) = F_4(K)$; hence

$$F_4(R + ixS) = a + bx^2 + cx^4. \tag{11}$$

Neither sign of b, c nor monotonicity in x is assumed.

The following proof-level budgets come from the actual monomial and multilinear estimates in [3, Lemma5.1]. They are not an inference of a second derivative bound from a Lipschitz constant. At fixed trace one the quartic ordinary terms are $20160\delta_5 + 5760\eta(6d_2 - \eta)$, and the remaining ordinary term is $23040S_3$. Their entrywise budgets, together with the paired term, are

term	degree	absolute coefficient budget
δ_5	4	36000
S_3	3	36
$\eta(6d_2 - \eta)$	4	2592
T_4	4	$38016000\sqrt{2}$.

For transparency, the differentiated-slot weight for δ_5 is $5! \sum_{i,j} |W_{ij}| = 7200$; choosing four of the five untouched slots gives 36000. The S_3 budget is its 36 entry monomials. The factors in $\eta(6d_2 - \eta)$ have budgets 36 and 72. The paired derivative is majorized coefficientwise in absolute value by $14400(\sqrt{2} + x)^{11}$; its fourth coefficient is $38016000\sqrt{2}$.

The exact cubic coefficient, printed in Appendix B, has a real trilinear representation with Frobenius norm at most

$$L_3 = 5060352 + 274752\sqrt{6}.$$

Its entry and diagonal contractions have trilinear norm at most one: bound one entry by the Frobenius norm and apply Cauchy–Schwarz to the other two. Trace contractions satisfy $|\operatorname{tr} ABC| \le \|A\|_{\mathrm{F}}\|B\|_{\mathrm{F}}\|C\|_{\mathrm{F}}$, and the two terms with a trace factor have the additional bound $|\operatorname{tr} A| \le \sqrt{6}\|A\|_{\mathrm{F}}$. Summing the actual coefficients gives L_3.

Set $s = \|S\|_{\mathrm{F}}$. Expand each entry or conjugate entry of $R + ixS$. Every R entry is bounded by one and every S entry by s. There are at most six ways to choose two S slots in a quartic monomial, three ways in a cubic, and one way to choose four in a quartic. The real trilinear bound applies to the Hermitian arguments R, iS, iS. Odd phase terms cancel by (11). Therefore

$$\begin{aligned} |F_4(K) - F_4(R)| &\le A_2 s^2 + A_4 s^4, \\ A_4 &= 20160 \cdot 36000 + 5760 \cdot 2592 + 38016000\sqrt{2}, \\ A_2 &= 6A_4 + 3 \cdot 23040 \cdot 36 + \tfrac{3}{2} L_3. \end{aligned}$$

These expressions equal the constants in Theorem 1.2. The rational inequalities $\sqrt{2} < 99/70$ and $\sqrt{6} < 49/20$ prove their stated strict upper bounds. If $s \le 10^{-3}$, the real theorem gives

$$F_4(K) > 5000 - 4780 - 0.000795 > 219.$$

This proves the closed tube, including its boundary. General trace follows by scaling. No rank, condition-number or phase-pattern restriction was introduced.

The closed tube complements the earlier uniform small-conjugate-overlap region of [1]. Complex matrices away from their union still require control of the actual quadratic and quartic phase coefficients or a different coupled certificate.

6 A coefficient-sign shortcut does not replace compensation

The following obstruction, implicit in the trine formula of [2], explains why a signed polynomial should be paid as a whole. It is not a separate new positivity or counterexample theorem. Let $\omega = (-1+\sqrt{-3})/2$, and let U, V be trace-one orthogonal projectors onto $(1, \omega, \omega^2, 0, 0, 0)$ and $(0, 0, 0, 1, \omega, \omega^2)$. Their ranges and conjugate ranges are mutually orthogonal, so $K = \lambda U + \mu V$ is an actual centered positive semidefinite null-locus matrix for $\lambda, \mu \geq 0$. The earlier exact trine polynomial, with squared Gram norms $\lambda/3, \mu/3$ and zero cross overlap, gives

$$\begin{aligned} F_4(K) = (66624 - 448\sqrt{2}/3)(\lambda^4 + \mu^4) \\ + (89088 - 512\sqrt{2}/3)(\lambda^3\mu + \lambda\mu^3) - (8832 + 1600\sqrt{2})\lambda^2\mu^2. \end{aligned}$$

The negative middle coefficient defeats coefficientwise eigenweight positivity, but the actual functional remains positive. Likewise, (11) alone supplies no sign for its phase coefficients.

A Universal mixed slots and the real paired coefficient

For general real centered symmetric A and symmetric B, put

$$b_B = B\mathbf{1}, \quad q = (A \circ A)\mathbf{1}, \quad c = (A \circ A \circ A)\mathbf{1}, \quad h = (A \circ B)\mathbf{1}, \quad s_j = \sum A_{ik}^j, \quad H = \sum A_{ik}B_{ik}.$$

The universal mixed formulas used in [2, 1] are

$$\begin{aligned} V_4 := [x^3y]\alpha_4(xA + yB) &= 12\sum A_{ik}^3 B_{ik} - 8c^T b_B - 12q^T h + \tfrac{4}{3}s_3 \sum B_{ik} \\ &\quad + 4q^T A b_B + s_2 H + 2\sum B_{ik}(A^3)_{ik}, \\ V_3 := [xy^2]\alpha_3(xA + yB) &= 12\sum A_{ik}B_{ik}^2 - 24h^T b_B + 6H\sum B_{ik} + 6b_B^T A b_B. \end{aligned}$$

Here B need not be centered. The ordered residual-slot prefactor for α_j is $(6-j)!/j!$. Partition inclusion–exclusion has block weight $(-1)^{m-1}(m-1)!$ for a block of size m. A singleton A slot vanishes by centering. For A^3B, the surviving row partitions are a four-block U of weight -6, an AAA block and B singleton V of weight 2, and three pairings P of B with one A, of weight 1. Column partitions are identical. Their grouped contractions, before the factor $4 \cdot 2!/4! = 1/3$, are

(U, U)	$36\sum A_{ik}^3 B_{ik}$
$(U, V) + (V, U)$	$-24c^T b_B$
$(U, P) + (P, U)$	$-36q^T h$
(V, V)	$4s_3 \sum B_{ik}$
$(V, P) + (P, V)$	$12q^T A b_B$
(P, P), same partner	$3s_2 H$
(P, P), different partners	$6\sum B_{ik}(A^3)_{ik}$.

For AB^2, the surviving triple block has weight 2 and the two pairings of A with B have weight -1. Their grouped contractions are $4\sum A_{ik}B_{ik}^2$, $-8h^T b_B$, $2H\sum B_{ik}$ and $2b_B^T A b_B$; multiply by $3 \cdot 3!/3! = 3$.

For an arbitrary centered Hermitian K, take $A = K + \overline{K}$, $B = K \circ \overline{K}$. The degree-five homogeneous part of the paired permanent is $8\sqrt{2}[\alpha_5(A) + V_4 + V_3]$. Its slot patterns are five A, three A and one B, or one A and two B. Under $K + vW$,

$$\dot{A} = 12I - 2J, \qquad \dot{B} = 12\,\mathrm{diag}(t) - A, \qquad P_4 = 2\delta_5(A) + \dot{V}_4 + \dot{V}_3.$$

Differentiation is performed before any real or null-locus specialization. At a real basepoint $A = 2K$, $B = K \circ K$, one has $2\delta_5(A) = 32\delta_5(K)$. The complete derivative table is

term	derivative at real K
$12\sum A_{ik}^3 B_{ik}$	$2880d_4 - 480S_4$
$-8c^T b_B$	$-1152(t^{\circ 2})^T b + 192\|b\|^2 - 768C$
$-12q^T h$	$-1728(t^{\circ 2})^T b + 288\|b\|^2 - 1152C$
$\frac{4}{3}s_3 \sum B_{ik}$	$192\beta_2 d_2 - 32\beta_2^2 + 128S_3$
$4q^T A b_B$	$768t^T K b + 192\|b\|^2 - 32\beta_2^2$
$s_2 H$	$96S_3 + 144\beta_2 d_2 - 24\beta_2^2$
$2\sum B_{ik}(A^3)_{ik}$	$192t^T e - 32\beta_4 + 288X$
$12\sum A_{ik} B_{ik}^2$	$720d_4 - 120S_4$
$-24h^T b_B$	$-864(t^{\circ 2})^T b + 144\|b\|^2 - 576C$
$6H\sum B_{ik}$	$216\beta_2 d_2 - 36\beta_2^2 + 144S_3$
$6b_B^T A b_B$	$288t^T K b + 72\|b\|^2 - 12\beta_2^2$.

Summing these eleven rows with 32 times (6) proves (7), and therefore the cancellation identity.

B Ordinary normalization and coefficient budgets

For general centered Hermitian K set

$$\eta = \operatorname{tr}(K\overline{K}), \quad Z_0 = \operatorname{Re}\sum_{i,k} t_i K_{ik}^2, \quad T_{21} = \operatorname{tr}(K^2\overline{K}), \quad S_{21} = \sum K_{ik}|K_{ik}|^2, \quad U = \sum t_i |K_{ik}|^2.$$

The ordered residual-slot rule gives $\alpha_0 = 720$, $\alpha_1 = 0$, $\alpha_2 = 12\eta$, $\alpha_3 = 4S_3$, and

$$\alpha_4 = 3\sum K_{ik}^4 - 3\operatorname{Re}\sum_i (\sum_k K_{ik}^2)^2 + \tfrac{1}{4}\eta^2 + \tfrac{1}{2}\operatorname{tr}((K\overline{K})^2).$$

The surviving four-slot partition types are 4 and $2+2$. Differentiating gives $\delta_4 = 72d_3 - 12S_3 - 72Z_0 + 6r\eta + 12T_{21}$. Since $\mathcal{P}(K) = \sum_j (7-j)\alpha_j(K)$, the ordinary quartic derivative is

$$R_4 = 20160\delta_5 + 23040 r S_3 + 5760\eta(6d_2 - \eta).$$

The degree-four paired coefficient before differentiation is

$$\begin{aligned} 16\alpha_4(A) + 192\sum A_{ik}^2 B_{ik} - 192 q^T b_B + 48 s_2 \sum B_{ik} \\ + 192[(\sum B_{ik})^2 - 2\|b_B\|^2 + \|B\|_{\mathrm{F}}^2]. \end{aligned}$$

Differentiating along the actual $\dot{A}, \dot{B}$ and subtracting from the ordinary cubic derivative gives

$$\begin{aligned} E_3 = 2135808 d_3 - 361728 S_3 - 2163456 Z_0 + 265920 r\eta \\ + 360576 T_{21} + 5760 S_{21} + 32256 U - 8832 r\beta_2 - 768\beta_3. \end{aligned}$$

These are inherited actual coefficient identities from [3, Section2 and AppendixA], recalled here to expose their normalization and the phase budgets. For real K, $\eta = \beta_2$, $Z_0 = U$, $T_{21} = \beta_3$ and $S_{21} = S_3$. Thus $F_4 = R_4 - T_4 + E_3/2$ gives exactly (9) at trace one. The seven cubic contractions

without a trace factor have coefficient sum 5060352 in absolute value; the two with a trace factor have sum 274752. Their norm bounds give the L_3 used in Section 5.

For portable exact checking, put $N_{ij}(x) = 2 + \sqrt{2}xA_{ij} + x^2B_{ij}$. Direct differentiation gives

$$T_4 = [x^4] \sum_{\pi \in S_6} \sum_i (2\sqrt{2}W_{i,\pi(i)} + xW_{i,\pi(i)}A_{i,\pi(i)}) \prod_{j \neq i} N_{j,\pi(j)}(x).$$

An alternative subset recurrence extends a column set S of size i by an unused j with $P_{S\cup\{j\}} \mathrel{+}= P_S n_{ij}$ and $\dot{P}_{S\cup\{j\}} \mathrel{+}= \dot{P}_S n_{ij} + P_S \dot{n}_{ij}$, starting with $P_\varnothing = 1$, $\dot{P}_\varnothing = 0$. Ordinary slots are $1 + xK_{ij}$ with direction W_{ij}; paired slots and directions are the displayed N_{ij} and derivative. Truncation at degree four is exact for the desired derivatives. These procedures check arithmetic in rational quadratic fields.

C The unrestricted complex cancellation

This appendix records the nonzero-p_2 correction underlying the real specialization. It is not needed to infer a global complex sign. All matrices here are centered Hermitian PSD with trace one. Keep $A = K + \overline{K}$, $B = |K|^2$, $t = \operatorname{diag} K$, $b = \operatorname{diag}(K^2)$, $e = \operatorname{diag}(K^3)$ and the generic quantities of Appendix A. Put

$$C_2 = K \circ K, \quad z = C_2\mathbf{1}, \quad \eta = \operatorname{tr}(K\overline{K}), \quad \eta_2 = \operatorname{tr}((K\overline{K})^2), \quad \gamma = \operatorname{diag}(K\overline{K}K).$$

The product $K\overline{K}$ is complex symmetric, not in general Hermitian. Define

$$X = \operatorname{Re}\sum (K^2)_{ik}K_{ik}^2, \quad Y = \operatorname{Re}\sum \overline{(K^2)_{ik}}K_{ik}^2, \quad M = \sum B_{ik}(A^2)_{ik}, \quad Q = \operatorname{Re}\sum K_{ik}^2 B_{ik},$$
$$C_3 = \operatorname{Re}\sum t_i K_{ik}^3, \qquad \delta_0 = 144d_4 + 4S_3 + 12X - 96C_3 - 24S_4.$$

For the actual Gram forms let $a_i = g_i^4 - g_i p_3/3$. Then $\delta_0 = 6\sum \|a_i\|^2 - \|p_4\|^2$, whereas the actual omitted states are $\xi_i = a_i - g_i^2 p_2/2$. The exact difference $\delta_5 = \delta_0 + \Delta$ is

$$\begin{aligned} \Delta = & -144\operatorname{Re}((t^{\circ 2})^T z) + 24\operatorname{Re}[(Kt)^T z + \operatorname{tr}(K\overline{K}C_2)] \\ & + 24\operatorname{Re}\sum_i z_i^2 + 6d_2\eta + 24t^T\gamma + 6\|z\|^2 - 2\eta^2 - 4\eta_2. \end{aligned} \tag{12}$$

Indeed the required full Fischer contractions are

$$\begin{aligned} \sum_i \langle g_i^4, g_i^2 p_2\rangle &= 24\sum_i t_i^2 z_i, \\ \sum_i \langle g_i p_3, g_i^2 p_2\rangle &= 12[(Kt)^T z + \operatorname{tr}(K\overline{K}C_2)], \\ \langle p_4, p_2^2\rangle &= 24\sum_i z_i^2, \\ \sum_i \|g_i^2 p_2\|^2 &= 4d_2\eta + 16t^T\gamma + 4\|z\|^2, \\ \|p_2^2\|^2 &= 8\eta^2 + 16\eta_2. \end{aligned}$$

In the second line the singled-out g_i pairs with one of two g_i columns or one of two g_k columns; each case has twelve matchings. In the fourth, the numbers of distinguished-factor matches are

two, one and zero, contributing the displayed three terms. The last line has eight disconnected and sixteen connected matchings. Substituting these into

$$\Delta = -6\,\mathrm{Re}\sum_i \langle a_i, g_i^2 p_2\rangle + \mathrm{Re}\langle p_4, p_2^2\rangle + \tfrac{3}{2}\sum_i \|g_i^2 p_2\|^2 - \tfrac{1}{4}\|p_2^2\|^2$$

proves (12) without suppressing mixed terms.

Let $R_0(K)$ denote the following algebraic expression, evaluated without a null-locus hypothesis:

$$\begin{aligned} R_0 = {} & 6768 d_4 - 5184 (t^{\circ 2})^T b + 696 \|b\|^2 + 576 \beta_2 d_2 + 240 t^T e \\ & - 2304 t^T h + 624 t^T A b + 24 S_3 + 432 S_{21} + 24 X + 144 Y + 216 M \\ & + 120 S_4 - 672 Q - 576 \|B\|_{\mathrm{F}}^2 - 92 \beta_2^2 - 20 \beta_4. \end{aligned}$$

Set

$$Z = q - 2b = 2\,\mathrm{Re}\, z, \quad E = \mathrm{diag}(A^3) - 2e, \quad \chi = \mathrm{tr}(A^4) - 2\beta_4,$$
$$N_c = \sum_{i,k} (A_{ik}^2 - 2B_{ik})(K\overline{K} + \overline{K}K)_{ik},$$
$$\begin{aligned} D_A = {} & -1584 (t^{\circ 2})^T Z + 432 b^T Z + 72 \|Z\|^2 + 120 t^T E + 168 \eta d_2 \\ & + 144 t^T A Z - 88 \beta_2 \eta - 20 \eta^2 - 10 \chi + 72 N_c. \end{aligned}$$

Then the universal identity is

$$\frac{T_4}{8\sqrt{2}} - 10\delta_5 = R_0 + D_A - 10\Delta. \tag{13}$$

The subtraction $2B_{ik}$ in N_c avoids counting its cross contraction twice, since M already contains the full A^2.

For a complete derivation, differentiating the universal mixed formulas gives

$$\dot{b}_B = 12t, \quad \dot{q} = 48t, \quad \dot{c} = 144 t^{\circ 2} - 6q, \quad \dot{h} = 36 t^{\circ 2} - 2b - q,$$
$$\dot{s}_2 = 48, \quad \dot{s}_3 = 144 d_2 - 6 s_2, \quad \dot{H} = 36 d_2 - 2\beta_2 - s_2, \quad (\textstyle\sum B_{ik})^{\cdot} = 12, \quad (A^3)^{\cdot} = 36 A^2.$$

The unrestricted real-Gram matching, as in Section 2 with K replaced by A, gives

$$\begin{aligned} 2\delta_5(A) = {} & 4608 d_4 + 60 \|q\|^2 + 96 t^T \mathrm{diag}(A^3) + 48 s_2 d_2 + 16 s_3 + 72 N \\ & - 1152 (t^{\circ 2})^T q - 384 t^T c + 96 t^T A q - 48 s_4 - 4 s_2^2 - 8\,\mathrm{tr}(A^4), \end{aligned}$$

where $N = \sum A_{ik}^2 (A^2)_{ik}$. Adding the mixed derivatives then yields, with $G = \sum A_{ik}^2 B_{ik}$,

$$\begin{aligned} \frac{T_4}{8\sqrt{2}} = {} & 8208 d_4 - 2016 (t^{\circ 2})^T b - 1584 (t^{\circ 2})^T q + 120 \|b\|^2 + 72 \|q\|^2 \\ & + 144 q^T b + 408 \beta_2 d_2 + 84 s_2 d_2 + 120 t^T \mathrm{diag}(A^3) - 480 t^T c \\ & - 864 t^T h + 336 t^T A b + 144 t^T A q + 32 s_3 + 72 N - 60 s_4 \\ & - 5 s_2^2 - 24 s_2 \beta_2 - 24 \beta_2^2 - 10\,\mathrm{tr}(A^4) + 120 H + 72 M - 96 G - 24 \|B\|_{\mathrm{F}}^2. \end{aligned}$$

Use $q = 2b + Z$, $s_2 = 2\beta_2 + 2\eta$ and the entrywise identities

$$t^T c = 2 C_3 + 3 t^T h, \quad s_3 = 2 S_3 + 6 S_{21}, \quad H = 2 S_{21},$$
$$N = 2X + 2Y + 2M + N_c, \quad G = 2Q + 2\|B\|_{\mathrm{F}}^2, \quad s_4 = 2 S_4 + 8 Q + 6 \|B\|_{\mathrm{F}}^2.$$

Subtracting $10\delta_0$ cancels C_3 and leaves $R_0 + D_A$; then (12) proves (13).

There is also the coarse explicit estimate

$$D_A - 10\Delta \le 5760\sqrt{\eta} + 480\eta + 60\eta^2. \tag{14}$$

To verify it, use $K \preceq I - J/6$, $d_2 \le 5/6$, $\sqrt{d_4 d_2} \le 25/36$, and the actual PSD matrix C_2 to obtain

$$\|z\|^2 \le d_2\eta, \quad \|K\overline{K}\|_{\mathrm{F}} \le \sqrt{\eta}, \quad \|E\| \le 6\sqrt{\eta}, \quad \chi = 8\,\mathrm{tr}(K^3\overline{K}) + 4\,\mathrm{tr}(K^2\overline{K}^2) + 2\eta_2 \ge 0.$$

The six mixed cubic words give the bound for E. PSD trace pairings give the bound for χ; positivity of the anticommutator is not used. Thus

$$D_A \le [3168\sqrt{d_4 d_2} + 1728 d_2 + 720\sqrt{d_2}]\sqrt{\eta} + 456 d_2\eta \le 4360\sqrt{\eta} + 380\eta.$$

In (12), bound each of $(Kt)^T z$ and $\mathrm{tr}(K\overline{K}C_2)$ by $d_2\sqrt{\eta}$ in absolute value, use $\gamma \ge 0$, $\eta_2 \le \eta^2$, and $24\,\mathrm{Re}\sum z_i^2 + 6\|z\|^2 \ge -18\|z\|^2$. This gives

$$-10\Delta \le 1440\sqrt{d_4 d_2 \eta} + 480 d_2\sqrt{\eta} + 120 d_2\eta + 60\eta^2 \le 1400\sqrt{\eta} + 100\eta + 60\eta^2,$$

and proves (14). The remainder R_0 still lacks a global off-locus payment. In particular the rank-two support relations from [1] cannot be imposed here.

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