Rank-one interface replacement and strict edge correlations for random forests

Scott Schubr
Luminary Oddities

Abstract

For the unrooted spanning-forest measure, we show that a weighted background graph attached through a rank-one interface, with activities $s_h r_v$ from retained vertices h to background vertices v, can be replaced exactly by finitely many isolated sites. The site weights are the negatives of the reciprocal roots of the background's normalized one-hub partition polynomial; they are positive by the one-star real-rootedness of our earlier work. The replacement holds for an arbitrary retained graph and preserves jointly every labeled retained-edge indicator and every retained vertex's interface degree, and equality of one-hub polynomials characterizes equality of all such responses. The proof is an exact elimination in the Grassmann representation of unrooted forests, in which the interface response is linear in the one-hub polynomial at fixed total profile mass. As an application, two disjoint edges whose four endpoints form a rank-one clique attached to an arbitrary weighted background are strictly negatively correlated for all positive activities of the two edges, with an explicit lower bound on the Rayleigh determinant. Together with our earlier incident-strictness theorem, this shows that every pair of distinct edges of K_n, $n \geq 3$, with vertex-factor activities $w_{ij} = t_i t_j > 0$ is strictly negatively correlated; constant vertex factors recover the common-fugacity case, with a new short proof for disjoint pairs.

Keywords. Unrooted spanning forest; arboreal gas; negative correlation; rank-one interface; graph replacement; real stable polynomial; Grassmann integral.
2020 Mathematics Subject Classification. 05C05; 05C31; 60C05; 60K35.

1 Introduction

A graph transformation can preserve selected observables without preserving the entire random subgraph. For the ordinary weighted forest measure, an eliminated graph generally leaves connectivity information at its attachment boundary. We identify a factorized interface, of rank one in its activities, for which all that information is captured by one univariate polynomial. The resulting replacement is simultaneous in every edge activity of the retained graph, so it preserves a whole system of labeled edge observables rather than a single partition value.

Throughout, a forest is an acyclic spanning edge set, including isolated vertices. For a finite loopless weighted graph G, write

$$Z_G = \sum_{F \in \mathcal{F}(G)} w(F), \qquad w(F) = \prod_{e \in F} w_e, \qquad \mathbb{P}_G(F) = w(F)/Z_G. \tag{1}$$

Present edge activities are positive; zero activities can equivalently be omitted. Parallel edges may be kept as labeled alternatives, of which a forest selects at most one in each parallel class. There is no choice of roots and no component-size factor.

Fix a background graph H on V and a profile $r = (r_v)_{v\in V} \geq 0$. Adjoin one new vertex with legs of activities qr_v, and denote its partition polynomial by $f(q)$. Put

$$Z_0 = Z_H, \qquad P(q) = f(q)/Z_0, \qquad R = \sum_{v\in V} r_v. \tag{2}$$

The one-star stability result recalled in Section 2 gives

$$P(q) = \prod_{i=1}^{m}(1 + \lambda_i q), \qquad \lambda_i > 0, \qquad \sum_i \lambda_i = R, \tag{3}$$

where $m = |\{v : r_v > 0\}|$. The empty product is one. We call λ_i the *root site weights*; they are determined by the roots of P.

Let Q be any retained graph on a disjoint vertex set U. Its internal labeled edges have activities b_e. For scales $s = (s_h)_{h\in U} \geq 0$, form $G(H, Q; r, s)$ by adding the cross edges

$$w_{hv} = s_h r_v \qquad (h \in U,\ v \in V). \tag{4}$$

Thus every retained vertex sees the same background profile, up to its own scale. Form $G_*(Q; \lambda, s)$ by keeping Q unchanged and replacing H by m isolated sites, that is, new vertices with no edges among themselves, with activities $s_h\lambda_i$ from h to site i.

Theorem 1.1 (Rank-one environment replacement)**.** *With the preceding definitions, for every retained graph Q,*

$$Z_{G(H,Q;r,s)} = Z_0\, Z_{G_*(Q;\lambda,s)}. \tag{5}$$

The identity is polynomial in all retained activities b_e and all scales s_h. Under the corresponding forest probability laws, it preserves jointly all labeled Q-edge indicators and all cross-degree counts at vertices of Q. Hence it also preserves the total degree of every vertex of Q, jointly with these indicators.

The theorem gives a positive graph realization of the entire retained response. Equality of normalized one-hub polynomials is sufficient for equality of all these attachment laws. Section 4 makes the converse precise using a one-hub probe. The replacement uses exactly one site per positive profile coordinate; its purpose is exact observable equivalence.

The main calculation uses the classical non-Gaussian forest integral of Caracciolo, Jacobsen, Saleur, Sokal and Sportiello [1], and the component-marked framework of Caracciolo, Sokal and Sportiello [2]. These representations already encode unrooted forests and support partial integration. The new step is to eliminate a rank-one family of sites and prove that its response is linear in P at fixed R. One-star real-rootedness then realizes the averaged response by positive isolated sites. The stability input is our optional-component-mark theorem [13, Theorem 2.1], built on the marked-forest identity [2, Theorem 5.4] and on classical matching stability, formal inversion and differential contraction [3, 4, 5]. Its univariate consequence is the one-star degree law of [13, Corollary 1.4]. A finite-terminal analogue of the replacement is the three-terminal compression of [13, Section 4], where a background meeting a retained network in three vertices is replaced by a positive network on at most four vertices with proportional terminal-partition weights. Here the interface may have arbitrarily many attachment vertices, provided its activities factor through one profile.

The two-hub specialization has a useful alternative route to its covariance sign: its replacement graph is $K_{2,m}$, so classical series-parallel forest negative correlation applies [7]. Section 5 gives a direct strict certificate and the proportional splitting identity. These supply a finite positive

approximation for a stronger application in Section 6. There, two disjoint internal edges are strictly negatively correlated when their four endpoints form a rank-one clique and attach with one common profile to an arbitrary weighted background. A positive normalized weak-site limit creates the clique, while a uniform root gap preserves strictness. Theorem 6.1 includes an explicit Rayleigh lower bound. Its main consequence concerns complete graphs with vertex-factor activities.

Theorem 1.2 (Vertex-factor complete graphs)**.** *Let $n \geq 3$ and $t_1, \dots, t_n > 0$. In the unrooted forest law on K_n with activities $w_{ij} = t_i t_j$, every two distinct edges are strictly negatively correlated. For a disjoint pair ab, cd the conclusion persists when the two target activities are arbitrary positive numbers and the edges among the vertices outside $\{a, b, c, d\}$ carry arbitrary nonnegative activities.*

It is proved in Section 6.4. For constant vertex factors $t_i = \beta^{1/2}$, Theorem 1.2 is the complete-graph case of our earlier common-fugacity theorem [13, Theorem 1.2], proved there by Hermite counts, continued fractions and exact crossing barriers. For disjoint pairs the endpoint-clique theorem gives an independent short proof, and it extends the result to arbitrary positive vertex factors; incident pairs follow from [13, Theorem 3.1].

The complete-graph comparison concerns vertex inhomogeneity. Uniform forests on K_n have strict pairwise negative correlation by Gupta [8]; common-activity regimes and electrical large-activity methods are treated in [9]. The weighted random-cluster results of [9, Theorem 1.9 and Proposition 5.7] likewise constrain only edges at the four endpoints, in the regime of activities bounded below; Theorem 6.1 treats the forest scaling, where those activities tend to zero. Tang and Zhang's arbitrary-weight correlation bound has forest-limit ratio $2(1 - 1/M)$, with M the number of edges, parallel edges counted [10, Section 5.3, Eq. (5.5)]; Theorem 6.1 gives strict negative correlation, a correlation ratio below one, on the endpoint-clique family. Rooted p-forest inequalities [12] concern a different normalization. Summing parent-weighted orientations over roots multiplies the ordinary rank-one edge-product weight by $\prod_{B \in \mathcal{C}(F)} \sum_{v \in B} p_v$, up to a graph-wide factor, and this component-dependent multiplier is absent from (1). Our endpoint theorem allows an arbitrary background and constrains only the edges incident to the four specified endpoints. Its empty-background weak sign lies within the finite-host theorem for graphs on at most seven vertices [6, Section 5.3] and within rank-three matroid results, including the recent Z-Rayleigh theorem of Zhao and Zhou [14]; the arbitrary weighted background of Theorem 6.1 carries no rank or size restriction.

2 One-star real-rootedness

We first recall the positivity input in (3). A real polynomial is *real stable* if it has no zero when all variables have positive imaginary parts. Polynomial limits, real specialization and positive affine substitution preserve stability, allowing the zero polynomial as a limit; these follow from Hurwitz's theorem [5].

Let $C = (C_{ij})$ be the symmetric weighted adjacency matrix of a finite simple graph G_C, with $C_{ij} \geq 0$ and $C_{ii} = 0$. Define

$$\Phi_C(a) = \sum_{J \in \mathcal{F}(G_C)} w(J) \prod_{B \in \mathcal{C}(J)} \left(1 + \sum_{i \in B} a_i\right). \tag{6}$$

Every component, including a singleton, is either unmarked or has one marked vertex. The unmarked contribution is one. This is the optional-component specialization of the classical marked-forest identity [2, Theorem 5.4].

Proposition 2.1 ([13, Theorem 2.1])**.** *The polynomial Φ_C is real stable.*

The proof in [13] identifies $\Phi_C(a)$ with $D_C(\partial_\alpha)M_C(\alpha)$ at $\alpha = 1 + \deg_C + a$, where $D_C(t) = \det(I - C\,\mathrm{diag}\,t)$ and M_C is the signed unmatched-vertex matching polynomial, by Lagrange–Good inversion [4]. Matching stability [3], Hermitian determinant stability and the Lieb–Sokal contraction [5, Lemma 3.2] then give stability.

Corollary 2.2 ([13, Corollary 1.4])**.** *The normalized one-hub polynomial in* (2) *has the factorization* (3)*.*

Proof. Parallel classes of H may be merged by summing their activities; this leaves Z_H and f unchanged, so H may be taken simple. For a fixed background forest J, the hub can send no leg or exactly one leg into each component; two legs into the same component would close a cycle. Thus

$$f(q) = \sum_{J \in \mathcal{F}(H)} w(J) P_J(q), \qquad P_J(q) = \prod_{B \in \mathcal{C}(J)} (1 + q a_B), \quad a_B = \sum_{v \in B} r_v. \tag{7}$$

It is obtained from (6) by the real specialization $a_v = 0$ when $r_v = 0$, followed by the positive substitution $a_v = q r_v$ for $r_v > 0$. Proposition 2.1 therefore gives univariate stability; real coefficients then make every zero real. Its constant is $Z_0 > 0$ and its coefficients are nonnegative, so all roots are strictly negative. Its degree is m: no forest uses more than m hub legs, and the empty background forest permits all m simultaneously with positive weight. This gives the m positive factors in (3). Finally each J has $\sum_B a_B = R$, so $P_J'(0) = R$ and $P'(0) = R$. □

3 A linear response for rank-one attachments

The replacement theorem separates into a formal linear-response identity and a positive graph realization. Only the latter uses Corollary 2.2.

3.1 The exact unrooted Grassmann identity

Use anticommuting generators $\bar\psi_v, \psi_v$ and set $\tau_v = \bar\psi_v \psi_v$. The integral is the coefficient functional normalized by $\int \prod_v \tau_v = 1$. Even elements commute; every series below is a finite exterior polynomial.

For a loopless graph G, put

$$\mathcal{A}_G = \sum_v \tau_v + \sum_{uv \in E(G)} w_{uv}(\tau_u + \tau_v - \bar\psi_u \psi_v - \bar\psi_v \psi_u - \tau_u \tau_v).$$

Parallel labeled alternatives are summed in the action before their individual markers are substituted.

Lemma 3.1 (Classical unrooted-forest identity)**.** *One has $Z_G = \int \exp(\mathcal{A}_G)$.*

Proof. This is the $t = 1, u = -1$ case of [1, Eq. (13)]. Its normalization follows from the principal-minor matrix-tree expansion, as follows. Split the action as $\bar\psi L \psi + \sum_v \tau_v - \sum_{uv} w_{uv}\tau_u\tau_v$, where L is the weighted Laplacian. The last two sums consist of even commuting elements, so

$$\exp\Big(\sum_v \tau_v - \sum_{uv} w_{uv}\tau_u\tau_v\Big) = \prod_v (1 + \tau_v) \prod_{uv} (1 - w_{uv}\tau_u\tau_v),$$

and nilpotency leaves vertex-disjoint marker families. With $\int \tau_v = 1$ one has $\int \prod_{v \in A} \tau_v\, e^{\bar\psi L \psi} = \det L[V \setminus A]$. An insertion $\prod_{v \in A} \tau_v$ into the Laplacian Gaussian gives forests with exactly one root

from A in each tree, by the principal-minor matrix-tree theorem. Adjoin every marked edge to the two rooted trees at its endpoints. Each resulting component is a tree marked by one vertex, with coefficient $+1$, or by one edge, with coefficient -1. Conversely deleting its marked edge recovers those two rooted trees, so this accounts for every choice and its multiplicity. The component multiplier is $|V(T)|-|E(T)|=1$. Summing markers therefore gives exactly (1), including singleton components. The negative quartic term is what removes the root-count multiplier. □

3.2 Eliminating isolated sites

Fix Q, scales s_h, and masses $a_1,\dots,a_c \geq 0$ with $\sum_j a_j = R$. Join hub h to isolated site j with activity $s_h a_j$. Retain the hub generators and abbreviate

$$\begin{aligned} S &= \sum_h s_h, \quad T = \sum_h s_h \tau_h, \quad A = S - T, \\ B &= \sum_h s_h \bar{\psi}_h, \quad C = \sum_h s_h \psi_h, \quad D = BC. \end{aligned} \tag{8}$$

In particular $B^2 = C^2 = 0$ and $D^2 = 0$, for any number of hubs. The even hub factor is

$$E_{Q,R} = \exp\left[\mathcal{A}_Q + R\sum_h s_h \tau_h\right]. \tag{9}$$

At a site of mass a with generators $\bar{\eta}, \eta$, the remaining action is

$$(1 + aS - aT)\bar{\eta}\eta - aB\eta - a\bar{\eta}C.$$

The coefficient of $\bar{\eta}\eta$ in its exponential is

$$1 + aA - a^2 D. \tag{10}$$

Indeed the product of the two even source terms is $(-aB\eta)(-a\bar{\eta}C) = -a^2 BC\bar{\eta}\eta$; terms containing $\bar{\eta}\eta$ and either source vanish. This verifies the sign without a matrix inverse.

Let $p(q) = \prod_j (1 + a_j q)$. Since $D^2 = 0$, integrating all sites gives

$$\begin{aligned} \prod_j (1 + a_j A - a_j^2 D) &= p(A) - D\sum_j a_j^2 \prod_{\ell \neq j}(1 + a_\ell A) \\ &= p(A) - D\,\mathcal{H}_R p(A), \end{aligned} \tag{11}$$

where the linear polynomial operation is

$$(\mathcal{H}_R p)(q) = \frac{Rp(q) - p'(q)}{q}. \tag{12}$$

The quotient is a polynomial because $p'(0) = Rp(0)$. More explicitly,

$$Rp(q) - p'(q) = q\sum_j a_j^2 \prod_{\ell \neq j}(1 + a_\ell q).$$

Thus (11) requires no division by the exterior element A, by S, or by a mass.

For fixed Q, s, R, define on the linear space $p'(0) = Rp(0)$

$$\mathcal{L}_{Q,s,R}[p] = \int_U E_{Q,R}\{p(A) - D\,\mathcal{H}_R p(A)\}. \tag{13}$$

This is linear in p. The site factors are even, so the integral may be taken over the sites first and then over U. Lemma 3.1 and (11) therefore prove that $\mathcal{L}_{Q,s,R}[p]$ is precisely the partition of Q joined to those isolated sites. The factor $E_{Q,R}$ depends on their mass list only through R.

3.3 Averaging the actual background

Fix a background forest J. The forests F with $F \cap E(H) = J$ are the sets $J \cup X$ with X disjoint from $E(H)$, so the remaining edges of H are deleted. By the contraction rule for the graphic matroid [11, Section 3.1], $J \cup X$ is a forest of $G(H, Q; r, s)$ exactly when X is a forest of the contraction G/J, in which parallel edges are kept. Contract each component B of J to one site of mass a_B from (7). The legs from a hub h into B become one parallel class of total activity $s_h a_B$; several hubs may attach to the same site, exactly as several hubs attach to B. No retained Q edge is contracted, so no loops arise, and the unrooted law carries no vertex or component weights, so the contracted vertex needs no extra factor. Hence the conditional attachment partition is the isolated-site partition just calculated. Components of zero mass are still present; they contribute isolated sites of factor one.

Every J has the same total mass R. Equations (7) and (13) therefore give

$$\frac{Z_{G(H,Q;r,s)}}{Z_0} = \mathbb{E}_J \mathcal{L}_{Q,s,R}[P_J] = \mathcal{L}_{Q,s,R}[P]. \tag{14}$$

All sums and coefficient extractions are finite, and this exchange uses only the linearity of $\mathcal{L}_{Q,s,R}$ established above.

Proof of Theorem 1.1. By Corollary 2.2, $P = \prod_i (1 + \lambda_i q)$ with $\sum_i \lambda_i = R$. Apply the same isolated-site calculation to the mass list $\lambda_1, \ldots, \lambda_m$. Its Q, s, R functional is unchanged, so (14) equals $Z_{G_*(Q;\lambda,s)}$. This proves (5). The calculation is polynomial in b_e, s_h and covers zero scales directly.

For labeled retained-edge markers y_e and cross-degree markers z_h, replace b_e by $b_e y_e$ and s_h by $s_h z_h$ in that polynomial identity. The root site weights remain fixed by H, r. Dividing by its value at $y = z = 1$ proves equality of the complete joint probability generating functions. Denominators are positive because the empty forest is allowed. Total degree at h is its cross-degree plus the sum of incident retained-edge indicators, which proves the final assertion. □

4 Environment equivalence and an example

The polynomial P is a complete invariant for the normalized response under the specified interfaces. More precisely, say two background/profile pairs have the same response if, for every retained Q and scales s, their normalized attachment partitions Z_G/Z_0 agree as polynomials in the retained activities and scales.

Corollary 4.1. *Two pairs have the same response if and only if their normalized one-hub polynomials P are equal. Equality of P also gives equality of every joint retained-edge/cross-degree law under all common probes.*

Proof. Equality of P includes equality of $R = P'(0)$, so (14) proves sufficiency. Necessity follows by taking Q to be one isolated hub. Its normalized attachment partition is exactly $P(s)$. The joint-law assertion follows by the marker argument in Theorem 1.1. □

For unnormalized partitions the additional multiplier Z_0 must be kept. There is also a probability-law formulation of necessity: for a single hub at any fixed $s > 0$, its degree PGF is $P(sz)/P(s)$. Equality of these PGFs implies equality of $P(s)$ by setting $z = 0$, since $P(0) = 1$, and then equality of the polynomials P.

For an explicit background, take two vertices joined by an edge of activity $w > 0$ and set $r = (1, 1)$. Directly from (7),

$$Z_0 = 1 + w, \qquad P(q) = 1 + 2q + \frac{q^2}{1 + w}, \qquad \lambda_\pm = 1 \pm \sqrt{\frac{w}{1 + w}}.$$

Both root site weights are positive. For every retained graph Q, replace this two-vertex background by two isolated sites with cross activities $s_h\lambda_+$ and $s_h\lambda_-$, and multiply its partition by $1+w$. The internal edge has disappeared, while the complete retained response is unchanged. At $w=0$ the same formula yields two weights equal to one. The roots describe the polynomial P; they are distinct from the Laplacian eigenvalues $0, 2w$ of the background.

5 Two hubs and proportional splitting

The arbitrary-Q theorem includes a particularly explicit two-hub case. Let Q consist of two isolated hubs with positive scales s, t. Put $q = s+t$, $p = st$, and let $F(s,t)$ be the partition with both hubs. By Theorem 1.1 the replacement graph is $K_{2,m}$, and f below denotes the one-hub polynomial of H. A forest on this graph has at most one site attached to both hubs, so

$$F(s,t) = f(q) + p\,\frac{Rf(q) - f'(q)}{q} = f(q)\{1 + p\mathsf{B}(q)\}, \quad \mathsf{B}(q) = \sum_i \frac{\lambda_i^2}{1+\lambda_i q}. \tag{15}$$

The numerator in the quotient is polynomially divisible by q.

Corollary 5.1 (Strict two-hub degree correlation)**.** *The selected degrees D_1, D_2 of these hubs satisfy $\mathrm{Cov}(D_1, D_2) < 0$ when $m \geq 2$. For $m \leq 1$ they are independent.*

Proof. Set $\mathsf{C}(q) = \sum_i \lambda_i^2/(1+\lambda_i q)^2$. Then $\mathsf{B} + q\mathsf{B}' = \mathsf{C}$ and $(\log f)'' = -\mathsf{C}$. Differentiating (15) gives

$$\partial_s\partial_t \log F = \frac{p(\mathsf{B}'' - 2\mathsf{B}\,\mathsf{C}) + p^2(\mathsf{B}\,\mathsf{B}'' - (\mathsf{B}')^2 - \mathsf{B}^2\mathsf{C})}{(1+p\mathsf{B})^2}. \tag{16}$$

To record the cancellation, if $K = 1 + p\mathsf{B}$ then $K_s = t\mathsf{B} + p\mathsf{B}'$, $K_t = s\mathsf{B} + p\mathsf{B}'$ and $K_{st} = \mathsf{C} + p\mathsf{B}''$; substitution into $-\mathsf{C} + (KK_{st} - K_sK_t)/K^2$ yields (16).

For $m \geq 1$, write $b_i = \lambda_i^2/(1+\lambda_i q) > 0$ and $z_i = \lambda_i/(1+\lambda_i q) > 0$. Thus

$$\mathsf{B} = \sum_i b_i, \quad \mathsf{C} = \sum_i z_i^2, \quad \mathsf{B}' = -\sum_i b_i z_i, \quad \mathsf{B}'' = 2\sum_i b_i z_i^2.$$

In particular $2\mathsf{B}\,\mathsf{C} - \mathsf{B}'' = 2\sum_{i\neq j} b_i z_j^2$, strictly positive for $m \geq 2$. For the second coefficient put $\rho_i = b_i/\mathsf{B}$, $\mu = \sum_i \rho_i z_i$, $M = \max_i z_i$. Since $\mathbb{E}_\rho z^2 \leq M\mu$,

$$\frac{\mathsf{B}\,\mathsf{B}'' - (\mathsf{B}')^2}{\mathsf{B}^2} = 2\mathbb{E}_\rho z^2 - \mu^2 \leq 2M\mu - \mu^2 \leq M^2 \leq \mathsf{C}.$$

Both numerator coefficients in (16) are nonpositive, the first strictly so for $m \geq 2$. The degree covariance is $st\,\partial_s\partial_t \log F$, proving strictness. For $m = 1$, the exact partition is $F = Z_0(1+\lambda_1 s)(1+\lambda_1 t)$; for $m = 0$ it is Z_0. These factorizations give independence. □

The weak sign also follows from the classical arbitrary-weight series-parallel theorem [7]: $K_{2,m}$ is a parallel composition of paths of length two, and summing its cross-star edge covariances gives the degree covariance. Formula (16) provides a direct strict certificate within the representation.

5.1 Two proportional split pairs

Attach two disjoint pairs u_ℓ, v_ℓ to H, with no edges between the pairs. Pair ℓ has an internal edge of activity $\theta_\ell > 0$ and boundary legs $x_\ell r_v, y_\ell r_v$, where $x_\ell, y_\ell > 0$. Set $s_\ell = x_\ell + y_\ell$ and $k_\ell = x_\ell y_\ell / s_\ell$. Both pairs use the same profile r.

For one pair and a fixed background forest with masses a_B, if the internal edge is selected, no component attaches to both endpoints. If it is absent, at most one component can attach to both; two such components would create a cycle. The total partition is therefore

$$(1+\theta)f(s) + xy\sum_J w(J)\sum_B a_B^2 \prod_{B'\neq B}(1+sa_{B'}) = (1+\theta+kR)f(s) - kf'(s).$$

All components are retained, and the square a_B^2 includes both equal and unequal attachment vertices in the same component.

Let $F(s_1, s_2)$ be the merged two-hub polynomial and Z the split partition. The one-pair derivation uses H only as a background whose profile has total mass R. Apply it first to pair 1, with background H together with pair 2 and the profile extended by zero on pair 2; then apply it to pair 2, with background H together with a hub of the first scale and the profile extended by zero on that hub, differentiating this second identity in the first scale where the first step requires a derivative. This gives

$$Z = A_1A_2F - k_1A_2F_1 - k_2A_1F_2 + k_1k_2F_{12}, \qquad A_\ell = 1 + \theta_\ell + k_\ell R.$$

The derivatives act on F before evaluation at $s_\ell = x_\ell + y_\ell$; A_ℓ, k_ℓ are held fixed. Differentiating in the internal activities and cancelling gives

$$Z_{\theta_1}Z_{\theta_2} - ZZ_{\theta_1\theta_2} = k_1k_2(F_1F_2 - FF_{12}).$$

Consequently

$$\operatorname{Cov}(\mathbf{1}_{u_1v_1}, \mathbf{1}_{u_2v_2}) = \frac{\theta_1\theta_2k_1k_2F^2}{s_1s_2Z^2}\operatorname{Cov}(D_1, D_2). \tag{17}$$

The multiplier is positive. Corollary 5.1 thus gives strict negative correlation precisely when the common profile has at least two positive coordinates, and zero correlation otherwise.

For example, take a clique on $n \geq 2$ background vertices, two disjoint internal edges, and join all four endpoints to every clique vertex. This is $K_n \vee (2K_2)$. The two internal-edge indicators are strictly negatively correlated at every common activity $\beta > 0$: use $r_v = 1$, $x_\ell = y_\ell = \theta_\ell = \beta$, and give the background its activity β. More generally (17) permits any positive background and any common nonnegative profile with at least two positive coordinates.

This example also illustrates a simultaneous-zero electrical case in the large-activity framework of [9]. In the unit-conductance graph, the potential for unit current across the first internal edge is $\pm 1/(n+2)$ at its endpoints and zero elsewhere; the analogous potential is supported on the other pair. Each Laplacian equation follows by summing the n core legs and the internal edge. The opposite-edge transfer current is zero, and for every edge ab the alternating product of these two potentials, $\phi(a)\psi(b) - \phi(b)\psi(a)$, is zero. Thus both the leading transfer-current square and the next electrical sum of squares in [9, Theorem 1.5] vanish for this family, in contrast with the complete joins of [9, Example 4.11], where the two target pairs are joined to each other. The finite-activity strict sign above comes instead from the proportional-profile representation.

6 Strict correlation across an endpoint clique

Let H be any finite positively weighted background on V, where $V = \varnothing$ is allowed, and let $r_v \geq 0$. Add four distinct vertices a, b, c, d with positive scales t_a, t_b, t_c, t_d and all legs hv of activities $t_h r_v$. Fix $\gamma > 0$. Give ac, ad, bc, bd activities $\gamma t_h t_j$, and give the two target edges $e = ab, f = cd$ arbitrary positive activities W_e, W_f. These are all edges incident to the four new vertices. Denote this graph's partition by Z and its target Rayleigh determinant by

$$\Delta = Z_e Z_f - Z Z_{ef},$$

where subscripts denote derivatives in W_e, W_f. Write

$$s_1 = t_a + t_b, \quad s_2 = t_c + t_d, \quad q_0 = s_1 + s_2, \quad p = s_1 s_2.$$

Let f_0 be the one-hub polynomial for H, r, and use Corollary 2.2 to set

$$\frac{f_0(q)}{Z_H} = \prod_i (1 + \rho_i q), \qquad \mathsf{B}_0 = \sum_i \frac{\rho_i^2}{1 + \rho_i q_0}.$$

An empty profile, in particular an empty background, gives an empty sum $\mathsf{B}_0 = 0$. Finally let F_* be the unrooted partition on H plus two merged vertices of scales s_1, s_2, their legs of weights $s_\ell r_v$, and one mutual edge of weight $\gamma s_1 s_2$.

Theorem 6.1 (Strict correlation across a rank-one endpoint clique)**.** *For the preceding graph,*

$$\Delta \;\geq\; \frac{2\gamma^2 t_a t_b t_c t_d \, F_*^2}{[1 + p(\mathsf{B}_0 + \gamma)]^2} > 0. \tag{18}$$

Here $F_* = Z_{ef}$ is the twice-contracted derivative partition, as shown in Section 6.3, and is independent of the target activities. In particular

$$\mathrm{Cov}(\mathbf{1}_e, \mathbf{1}_f) = -\frac{W_e W_f \Delta}{Z^2} < 0$$

for every $W_e, W_f > 0$, including when r vanishes identically.

6.1 A finite weak-site approximation and the clique limit

Temporarily keep only the two internal target edges with fixed positive labeled activities b_e, b_f. Adjoin $m \geq 2$ weak sites, with no edges among themselves, each joined to the four endpoints with activities $t_h \epsilon_m$, where $\epsilon_m = \sqrt{\gamma/m}$. These sites have no edges to H. The background is the disjoint union of H and the weak sites; its common profile is r on H and ϵ_m on every weak site. Thus Section 5 applies to the two internal marked edges for every finite m.

To identify their limiting law, retain Q_0 consisting of H, its endpoint legs and the two marked internal edges. Give its four endpoints scales t_h and every vertex of H scale zero. In (8), $S = q_0$, while T, B, C, D, A use these scales on the fixed finite retained algebra. Write $E_{Q_0} = \exp(\mathcal{A}_{Q_0})$. The exact weak-site partition is

$$Z_m = \int E_{Q_0} e^{m\epsilon_m T} (1 + \epsilon_m A)^m \left(1 - \frac{\gamma D}{1 + \epsilon_m A}\right). \tag{19}$$

Indeed (11) has $m\epsilon_m^2 = \gamma$. The inverse exists as a finite nilpotent inverse with positive scalar part $1 + \epsilon_m S$. Normalize by the positive scalar

$$c_m = (1 + \epsilon_m S)^m.$$

Set $\alpha_m = \epsilon_m/(1+\epsilon_m S)$. In the finite retained algebra,

$$m\epsilon_m T + m\log(1-\alpha_m T) = (m\epsilon_m - m\alpha_m)T - \sum_{j=2}^{N} \frac{m\alpha_m^j}{j} T^j,$$

$$m\epsilon_m - m\alpha_m = \frac{\gamma S}{1+\epsilon_m S} \longrightarrow \gamma S,$$

$$m\alpha_m^2 \longrightarrow \gamma, \qquad m\alpha_m^j \longrightarrow 0 \quad (j \geq 3).$$

Here $N = |V(Q_0)|$ suffices because $T^{N+1} = 0$. Exponentiation is a finite polynomial operation on nilpotent coefficients. Also the last factor in (19) tends to $1 - \gamma D = e^{-\gamma D}$. Consequently

$$\frac{Z_m}{c_m} \longrightarrow Z_* := \int E_{Q_0} \exp\{\gamma(ST - T^2/2 - D)\}. \tag{20}$$

The new action is exactly

$$\gamma(ST - T^2/2 - D) = \sum_{h<j} \gamma t_h t_j (\tau_h + \tau_j - \bar{\psi}_h \psi_j - \bar{\psi}_j \psi_h - \tau_h \tau_j).$$

The diagonal $t_h^2 \tau_h$ terms cancel between ST and D, and $\tau_h^2 = 0$. The zero scales on H generate no new edges touching H. Lemma 3.1 therefore identifies Z_* with the ordinary unrooted partition of the desired endpoint clique, with the two extra labeled edges b_e, b_f in parallel with its ab, cd edges.

This convergence holds coefficientwise in independent markers on b_e, b_f: their action remains a fixed finite polynomial, and c_m is independent of those markers. Dividing by the positive physical limit Z_* proves convergence of their joint indicator law and covariance. Strictness will follow from a uniform estimate, rather than from convergence alone.

6.2 A uniform strict gap from the weak roots

For the disjoint-union background of the weak-site approximation,

$$\frac{f_m(q)}{f_m(0)} = \frac{f_0(q)}{Z_H}(1+\epsilon_m q)^m. \tag{21}$$

Thus its root list is the fixed ρ_i together with m copies of ϵ_m. At q_0 the quantity in (15) satisfies

$$\mathsf{B}_m = \mathsf{B}_0 + \frac{m\epsilon_m^2}{1+\epsilon_m q_0} \leq \mathsf{B}_0 + \gamma.$$

From the positive-sum gap in the proof of Corollary 5.1, retaining only ordered pairs of distinct weak roots gives

$$2\mathsf{B}_m \mathsf{C}_m - \mathsf{B}_m'' \geq g_m := \frac{2m(m-1)\epsilon_m^4}{(1+\epsilon_m q_0)^3} \longrightarrow 2\gamma^2.$$

The second numerator coefficient in (16) is nonpositive. Hence the merged hub degrees obey the exact uniform estimate

$$\mathrm{Cov}_m(D_1, D_2) \leq -\frac{p^2 g_m}{[1+p(\mathsf{B}_0+\gamma)]^2}. \tag{22}$$

No limiting law of these unbounded degrees is required.

Let F_m be the twice-merged partition and Z_m the split partition in (19). Put $\kappa_1 = t_a t_b/s_1$, $\kappa_2 = t_c t_d/s_2$. Equation (17) gives

$$\mathrm{Cov}_m(\mathbf{1}_{b_e}, \mathbf{1}_{b_f}) = K_m \,\mathrm{Cov}_m(D_1, D_2), \qquad K_m = \frac{b_e b_f \kappa_1 \kappa_2 F_m^2}{p Z_m^2} > 0.$$

Merging endpoints replaces their scales by sums, so their total scale is still $S = s_1 + s_2$. The same c_m therefore normalizes both partitions. Applying (20) to the merged retained graph gives $F_m/c_m \to F_* > 0$; merging the four cross-pair alternatives yields the mutual activity $\gamma s_1 s_2$. Meanwhile $Z_m/c_m \to Z_* > 0$. Thus

$$K_m \longrightarrow K_* := \frac{b_e b_f \kappa_1 \kappa_2 F_*^2}{p Z_*^2} > 0.$$

The label covariance itself converges, and (22) implies

$$\mathrm{Cov}_*(\mathbf{1}_{b_e}, \mathbf{1}_{b_f}) \le -\frac{2p^2\gamma^2 K_*}{[1 + p(\mathsf{B}_0 + \gamma)]^2} < 0. \tag{23}$$

This proves strictness before passing to the simple target graph.

6.3 Parallel thinning and the determinant bound

Merge each parallel class at e, f. Its activities become $W_e' = \gamma t_a t_b + b_e$ and $W_f' = \gamma t_c t_d + b_f$. Conditional on a merged forest, the retained alternative in a selected class is chosen with probability b_e/W_e' or b_f/W_f', independently between classes, because the forest condition involves only which class is selected. Therefore

$$\mathrm{Cov}_*(\mathbf{1}_{b_e}, \mathbf{1}_{b_f}) = \frac{b_e}{W_e'}\frac{b_f}{W_f'}\,\mathrm{Cov}(\mathbf{1}_e, \mathbf{1}_f) = -\frac{b_e b_f \Delta}{Z_*^2}.$$

Substitute this and K_* into (23) and cancel the positive factors. Since $\kappa_1\kappa_2 p = t_a t_b t_c t_d$, the result is (18) at W_e', W_f'.

For fixed remaining activities, multiaffinity writes

$$Z = A_0 + W_e A_e + W_f A_f + W_e W_f A_{ef}, \qquad \Delta = A_e A_f - A_0 A_{ef}.$$

Thus the determinant is independent of both target activities. The same quantitative bound, not only its sign, holds at all $W_e, W_f > 0$. Contracting both targets identifies $A_{ef} = Z_{ef} = F_*$: their endpoint legs sum to $s_\ell r_v$, and the four cross-pair edges sum to $\gamma s_1 s_2$. This completes the proof of Theorem 6.1.

6.4 Complete graphs

Incident pairs are covered by our incident-strictness theorem [13, Theorem 3.1]: two positive incident edges of a finite simple weighted graph are strictly negatively correlated exactly when some cycle contains both, with an explicit path lower bound.

Proof of Theorem 1.2. Two incident edges vi, vj of K_n lie on the triangle vij, so [13, Theorem 3.1] gives strictness at every positive activity. For a disjoint pair ab, cd, take H to be the graph on the other $n-4$ vertices, possibly empty, with its given activities and with edges of activity zero omitted, and put $r_v = t_v$ and $\gamma = 1$ in Theorem 6.1. The legs then have activities $t_h t_v$, the four cross-pair edges have activities $t_h t_j$, and the two target activities are arbitrary positive numbers. □

Data availability. All mathematical arguments needed for the results are contained in this article and in the cited works.

Use of AI tools. AI tools assisted with literature discovery, proof exploration, drafting and verification code. Separate AI-assisted internal checks were performed.

References

[1] S. Caracciolo, J. L. Jacobsen, H. Saleur, A. D. Sokal and A. Sportiello, Fermionic field theory for trees and forests, *Phys. Rev. Lett.* **93** (2004), 080601. doi:10.1103/PhysRevLett.93.080601.

[2] S. Caracciolo, A. D. Sokal and A. Sportiello, Grassmann integral representation for spanning hyperforests, *J. Phys. A: Math. Theor.* **40** (2007), 13799–13835. doi:10.1088/1751-8113/40/46/001.

[3] O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Commun. Math. Phys.* **25** (1972), 190–232. doi:10.1007/BF01877590.

[4] I. J. Good, Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes, *Proc. Cambridge Philos. Soc.* **56** (1960), 367–380. doi:10.1017/S0305004100034666.

[5] D. G. Wagner, Multivariate stable polynomials: theory and applications, *Bull. Amer. Math. Soc.* **48** (2011), 53–84. `https://arxiv.org/abs/0911.3569`.

[6] A. Erickson, *Negative Correlation Properties for Matroids*, M.Math. thesis, University of Waterloo, 2008. University of Waterloo repository.

[7] A. Erickson, Sums of squares and negative correlation for spanning forests of series parallel graphs, *Australas. J. Combin.* **52** (2012), 75–87. `https://ajc.maths.uq.edu.au/pdf/52/ajc_v52_p075.pdf`.

[8] A. Gupta, Edges of the uniform random forest of K_n are pairwise negatively correlated for every n, arXiv:2609.26134v1 (2026). `https://arxiv.org/abs/2609.26134`.

[9] R. A. Çiçeksiz and M. Ravichandran, Negative correlation for forests and the $q < 1$ random-cluster model, arXiv:2610.00549v1 (2026). `https://arxiv.org/abs/2610.00549`.

[10] P. Tang and Z. Zhang, Pairwise correlation bounds for the random-cluster model with cluster weight at most one, arXiv:2611.11993v1 (2026). `https://arxiv.org/abs/2611.11993`.

[11] J. Oxley, *Matroid Theory*, 2nd ed., Oxford University Press, Oxford, 2011.

[12] J. Pitman, Random mappings, forests, and subsets associated with Abel–Cayley–Hurwitz multinomial expansions, *Sém. Lothar. Combin.* **46** (2001), Article B46h. `https://www.mat.univie.ac.at/~slc/wpapers/s46pitman.pdf`.

[13] S. Schubr, Stable star marginals and strict edge correlations for random forests, Luminary Research Archive, edition 4 (2026). `https://research.luminaryoddities.org/degree-laws-and-edge-correlations-for-random-forests/v/4`.

[14] Q. Zhao and X. Zhou, Every rank-three matroid is Z-Rayleigh, arXiv:2611.15512v1 (2026). `https://arxiv.org/abs/2611.15512`.