Complete point spectrum in smooth Dirichlet towers with prescribed rooms

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Luminary Oddities

Abstract

For every dimension $d \geq 2$, every prescribed positive unbounded sequence of ball radii, and every prescribed sequence of positive passage lengths, we construct a connected domain with C^∞ boundary whose ordinary Dirichlet Laplacian has a complete orthonormal eigenbasis and spectrum and essential spectrum equal to $[0, \infty)$. The domain contains the prescribed balls at their specified axial separations, with cylindrical passages of the specified lengths and positive radii tending to zero. Only the passage radii are selected. An explicit smooth outward join gives nested finite domains. On a fixed ambient space their resolvents converge in norm as a single passage closes, allowing the continuation of whole finite spectral clusters. Strict domain monotonicity of every ordered finite-domain eigenvalue permits aperture choices that avoid the next prescribed room spectrum. Dense targets on entire finite prefixes, including passage volume, make the limiting reducing subspaces complete for the actual infinite-domain operator. This gives a smooth positive-length-passage realization of complete point spectrum, complementing the existing window-based construction. The finite reducing blocks also satisfy an explicit spatial-prefix tail bound.

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1 Introduction

An open set in $\mathbb{R}^d$ is quasi-conical if it contains balls of arbitrarily large radius. Its Dirichlet Laplacian has the nonnegative half-line as its spectrum, by cutoff plane waves supported in those balls. This spectral set is compatible with different spectral types. In particular, complete point spectrum means that eigenfunctions span the Hilbert space; the eigenvalue set itself is then countable, with the half-line as its closure.

Krejčiřík and Lotoreichik [2] constructed connected quasi-conical domains with prescribed positive embedded eigenvalues, using enlarging cells coupled through small openings. Their Remark 3.3 also discusses other cells and thin-tube connections. Kriebel [3] subsequently proved complete point spectrum for an enlarging cube tower with shrinking windows. His argument continues whole finite spectral clusters under successive attachments and uses refreshed dense targets to prove completeness. Thus the complete window-based spectral realization is an existing result.

We obtain a smooth-boundary realization with preassigned ball radii and positive-volume cylindrical passages of preassigned positive lengths. Only unboundedness is required of the radii sequence. Finite shrinking-handle decoupling is classical domain-perturbation theory: Daners [1] gives bounded-ambient norm-resolvent convergence and, in Example 8.2, the model of two balls connected by a shrinking handle. Spectral cluster continuation is likewise a classical consequence of norm-resolvent convergence; see [1, Corollaries 4.2 and 4.7].

The additional geometric and completeness requirements are addressed explicitly here. A smooth maximum produces nested outward joins with an exact cylindrical segment of each specified length. The passage volume changes the Hilbert spaces, so all finite comparisons use zero-extended resolvents on a

fixed ambient space. Strict aperture monotonicity lets us avoid the next prescribed room spectrum through a one-room look-ahead selection. Thus both the room radii and passage lengths remain fixed, replacing the adaptive room scaling used in the window construction. Bases on the complete finite domains, rather than on the rooms alone, capture the passage states. The resulting projection limits reduce the physical infinite-domain resolvent, and their ranges exhaust its entire Hilbert space. We include all these arguments, as well as the spectral-set and tail proofs.

2 The domain and spectral statement

Write points in $\mathbb{R}^d$ as $(t, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and denote by $B_a(ce_1)$ the open ball of radius a centered at ce_1. For any open set U, its ordinary Dirichlet Laplacian H_U is the nonnegative self-adjoint operator associated with the closed form

$$h_U(f, g) = \int_U \nabla f \cdot \overline{\nabla g}, \qquad \operatorname{dom} h_U = H_0^1(U). \tag{1}$$

Here $H_0^1(U)$ is the H^1 closure of $C_c^\infty(U)$. Sobolev functions and L^2 operators are extended by zero when domains are compared.

Theorem 2.1. *Fix an integer $d \geq 2$, a sequence of positive radii $(a_n)_{n\geq 1}$ with $\sup_n a_n = \infty$, and any sequence of finite positive lengths $(\ell_n)_{n\geq 1}$. There exist passage radii δ_n satisfying*

$$0 < \delta_n < \frac{1}{4} \min\{a_n, a_{n+1}, 2^{-n}\},$$

and centers defined by

$$c_1 = a_1, \qquad c_{n+1} = c_n + a_n + \ell_n + a_{n+1},$$

with the following properties. There is a connected open set $\Omega \subset \mathbb{R}^d$ with C^∞ boundary containing every ball $B_{a_n}(c_n e_1)$. In each gap between consecutive balls,

$$\Omega \cap \{c_n + a_n < t < c_n + a_n + \ell_n\} = \{(t, y) : c_n + a_n < t < c_n + a_n + \ell_n,\ |y| < \delta_n\}. \tag{2}$$

The operator $H = H_\Omega$ has a complete orthonormal eigenbasis and

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty). \tag{3}$$

Both continuous spectral subspaces are zero. All eigenvalues are positive, and their set is a countable dense subset of $[0, \infty)$.

Moreover, Ω admits a nested exhaustion by bounded connected C^∞ domains D_n, containing the first n balls and their preceding passages, and nested finite-rank reducing projections $P_j \to I$ strongly such that

$$\|\mathbf{1}_{\Omega\setminus D_n} P_j\| \leq 2^{-n} \qquad (n \geq j + 1). \tag{4}$$

The eigenbasis can be grouped into finite blocks so that a normalized eigenvector in the jth block has squared L^2 mass outside D_n at most 4^{-n} for every $n \geq j + 1$.

The prescribed lengths may be bounded, unbounded, or tend to zero. Their strict positivity separates the closures of the two domains in every local disconnected comparison. The ball radii can vary non-monotonically. The one-prefix offset in the block-tail bound comes from joining the first two balls before protecting spectral projections. Smoothness in the theorem is a local boundary property.

3 Smooth nested attachment geometry

Lemma 3.1 (A local smooth maximum)**.** *For every $\varepsilon > 0$ there is a smooth function $M_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\max\{a, b\} \le M_\varepsilon(a, b) \le \max\{a, b\} + \varepsilon/2, \tag{5}$$

$$M_\varepsilon(a, b) = \max\{a, b\} \quad \text{if } |a - b| \ge \varepsilon. \tag{6}$$

Proof. Choose an even nonnegative $\eta \in C_c^\infty((-1, 1))$ with integral one, and put

$$\phi_\varepsilon(s) = \int_{-1}^{1} |s - \varepsilon z| \eta(z)\, dz, \qquad M_\varepsilon(a, b) = \frac{a + b + \phi_\varepsilon(a - b)}{2}.$$

Convolution makes ϕ_ε smooth. Jensen's inequality and the zero first moment of η give $\phi_\varepsilon(s) \ge |s|$, while the triangle inequality gives $\phi_\varepsilon(s) \le |s| + \varepsilon$. When $|s| \ge \varepsilon$, the sign in the integrand is fixed, so its integral equals $|s|$ exactly. These statements prove the lemma. □

For a ball write

$$g_{a,c}(t) = a^2 - (t - c)^2.$$

The ball is the region $|y|^2 < g_{a,c}(t)$ between its two poles. We maintain finite domains of the form

$$D_n = \{(t, y) : 0 < t < b_n,\ |y|^2 < h_n(t)\}, \qquad b_n = c_n + a_n, \tag{7}$$

where h_n is smooth and positive on $(0, b_n)$, equals the first ball profile near zero, and equals g_{a_n,c_n} on the entire right half $[c_n, b_n]$. Initially $D_1 = B_{a_1}(a_1 e_1)$.

Lemma 3.2 (Outward smooth join)**.** *Given such D_n, fix $a > 0$, $\ell > 0$, and $0 < \delta < \min\{a_n, a, 1\}/4$. There is a bounded connected C^∞ domain $D_{n+1}(\delta)$ containing both D_n and the full ball $B_a(ce_1)$, where $c = b_n + \ell + a$. Its profile agrees with h_n for $t \le c_n$, agrees with $g_{a,c}$ for $t \ge c$, and equals δ^2 throughout the gap $[b_n, b_n + \ell]$.*

Proof. Set $m = b_n + \ell/2$. Define a profile on $(0, c + a)$ by

$$h_{n+1,\delta}(t) = \begin{cases} h_n(t), & 0 < t \le c_n, \\ M_{\delta^2}(g_{a_n,c_n}(t), \delta^2), & c_n \le t \le m, \\ M_{\delta^2}(g_{a,c}(t), \delta^2), & m \le t \le c, \\ g_{a,c}(t), & c \le t < c + a. \end{cases} \tag{8}$$

Near c_n the new branch equals g_{a_n,c_n} by (6). Since h_n is smooth and agrees with this quadratic on $[c_n, b_n]$, its full jet at c_n is the same, so the two pieces glue smoothly. Near c the adjacent formulas agree identically. Near m, both ball profiles are negative, and both adjacent formulas are identically δ^2. Thus the profile is smooth at every displayed join.

It is positive on $(0, c + a)$. It dominates each ball profile wherever that profile is used, and equals the old profile to the left of c_n. Hence the associated domain contains D_n and the entire new ball. On the gap the relevant ball quadratic is nonpositive, so (6) makes the profile exactly δ^2. On a facing polar cap, modification occurs only where the ball quadratic is below $2\delta^2$.

At every interior boundary point, positivity of the profile gives $y \neq 0$. The defining function $h_{n+1,\delta}(t) - |y|^2$ therefore has a nonzero y gradient. At the two global tips the profile equals a sphere profile, with a simple zero and nonzero t derivative. The regular-level theorem proves C^∞ boundary everywhere, including the axis tips. Positive cross-sections give connectedness. The last right half-ball is unchanged, preserving the induction property. □

4 The local resolvent comparison on a fixed space

For an open set U, let $i_U : L^2(U) \to L^2(\mathbb{R}^d)$ be zero extension and write

$$R_U = i_U(H_U + 1)^{-1} i_U^*. \tag{9}$$

This is a positive self-adjoint contraction. If U is bounded, the resolvent is compact: zero extension embeds $H_0^1(U)$ into $H_0^1(G)$ for a surrounding bounded box G, and Rellich compactness applies. The box Poincaré inequality also gives a positive lower Dirichlet spectral bound. Thus the nonzero eigenvalues of R_U lie in $(0, 1)$, with zero as their only possible accumulation point.

The following is the finite shrinking-handle convergence needed below. It is a specific instance of the domain-perturbation results in [1]; we give the proof to fix the ambient space and the role of the collapsed passage.

Lemma 4.1 (Thin-handle norm convergence)**.** *Fix D_n, a, and ℓ in Lemma 3.2. Put $U_\delta = D_{n+1}(\delta)$ and $U_0 = D_n \cup B_a(ce_1)$. Then*

$$\|R_{U_\delta} - R_{U_0}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \longrightarrow 0 \qquad (\delta \downarrow 0). \tag{10}$$

On $L^2(U_0)$ the limiting resolvent is the direct sum of the old-domain and new-ball resolvents.

Proof. All U_δ for small δ lie in one bounded box G. The profile (8) shows that outside the old and new domains the added neck and caps collapse to the axial segment between the facing poles. Consequently, away from that segment and from the old and new boundaries, membership in U_δ converges pointwise to membership in U_0. The exceptional sets have Lebesgue measure zero.

Suppose $\delta_l \to 0$, $u_l \in H_0^1(U_{\delta_l})$, and their zero extensions are bounded and weakly convergent in $H_0^1(G)$ to u. Rellich gives a strongly L^2 convergent subsequence, and a further subsequence converges almost everywhere. At every nonexceptional point outside U_0, the functions u_l are eventually zero. Thus $u = 0$ almost everywhere outside U_0.

The closures of D_n and the new ball are separated by the positive gap ℓ. Their union is a bounded Lipschitz open set with two components. The zero-extension characterization for Lipschitz domains therefore gives $u \in H_0^1(U_0)$. Indeed, the interior and exterior traces of a global H^1 function agree, and its zero exterior restriction forces zero boundary trace. Conversely, $U_0 \subset U_\delta$, so every member of $H_0^1(U_0)$ is itself an admissible recovery sequence. These are the weak-limit and recovery conditions for the Dirichlet forms. Only the Lebesgue-null property of the collapsed segment was used.

For fixed $f \in L^2(\mathbb{R}^d)$, the variational solutions $R_{U_\delta} f$ are uniformly bounded in $H_0^1(G)$. Each L^2 cluster limit belongs to $H_0^1(U_0)$ by the preceding argument. Testing against $v \in H_0^1(U_0)$, admissible for every δ, identifies it uniquely as $R_{U_0} f$. This proves strong convergence on each fixed input.

To prove norm convergence, suppose it fails and choose $\delta_l \to 0$ and unit inputs f_l with $\|(R_{U_{\delta_l}} - R_{U_0}) f_l\|$ bounded below. Pass to $f_l \rightharpoonup f$. The varying solutions are bounded in $H_0^1(G)$ and hence relatively compact in $L^2(G)$. Every cluster limit lies in $H_0^1(U_0)$, and their variational equations tested against that space have right sides $\langle f_l, v\rangle \to \langle f, v\rangle$. Thus every limit is $R_{U_0} f$, giving $R_{U_{\delta_l}} f_l \to R_{U_0} f$ strongly. Compactness of R_{U_0} also gives $R_{U_0} f_l \to R_{U_0} f$, a contradiction. The direct-sum assertion follows from the disconnected open union. □

Remark 4.2. In dimension two a line segment need not have zero H^1 capacity. The proof uses its null volume, global H^1 compactness, and the Lipschitz trace characterization of the separated limiting domains.

5 Finite spectral clusters and the induction

Lemma 5.1 (Strict aperture monotonicity)**.** *For a fixed finite attachment in Lemma 3.2, the domains $D_{n+1}(\delta)$ are increasing with δ. For $0 < \delta_1 < \delta_2$ within the geometric cap, every ordered Dirichlet eigenvalue, counted with multiplicity, satisfies*

$$\lambda_j(D_{n+1}(\delta_2)) < \lambda_j(D_{n+1}(\delta_1)) \qquad (j \geq 1).$$

Proof. If Z has the even probability density used in Lemma 3.1, then its mean is zero and

$$M_\varepsilon(g,\varepsilon) = \frac{g+\varepsilon+\mathbb{E}|g-\varepsilon(1+Z)|}{2} = \mathbb{E}\max\{g,\varepsilon(1+Z)\}.$$

Since $1+Z>0$, this is nondecreasing in ε. All breakpoints in (8) are fixed in this comparison. Thus $D_{n+1}(\delta_1)\subset D_{n+1}(\delta_2)$. The inclusion adds a nonempty open annulus in the positive-length cylindrical gap.

We record why this implies strictness for every ordered eigenvalue, including multiplicities. Let $U\subset V$ be bounded domains, with V connected and $V\setminus\overline{U}$ containing a nonempty open set. Minmax gives $\lambda_j(V)\le\lambda_j(U)$. Suppose equality at a value λ. Let F be the j-dimensional span of the first j U-eigenfunctions, zero-extended into $H_0^1(V)$. Every nonzero member of F has Rayleigh quotient at most λ. The V spectral space strictly below λ has dimension at most $j-1$. Choose a nonzero $u\in F$ orthogonal to that space. Its V spectral expansion has Rayleigh quotient at least λ, so equality holds and all its spectral mass lies at λ. Thus u is a V-eigenfunction. It vanishes on the open set $V\setminus\overline{U}$. Interior Helmholtz eigenfunctions are real analytic, so unique continuation on connected V gives $u=0$, a contradiction. Apply this argument to the two aperture domains. □

Lemma 5.2 (Avoidance by passage radius)**.** *Fix an old prefix, the next ball, and its positive gap length. For any countable set $S\subset(0,\infty)$, the set of positive admissible apertures δ for which*

$$\sigma(H_{D_{n+1}(\delta)})\cap S\neq\varnothing$$

is countable. A nonresonant aperture can therefore be chosen below any additional positive cap.

Proof. For each ordered index j and each $s\in S$, Lemma 5.1 shows that $\lambda_j(D_{n+1}(\delta))=s$ has at most one solution. The union over countably many pairs (j,s) is countable and cannot fill any positive aperture interval. □

Lemma 5.3 (Simultaneous continuation)**.** *Let $P_1\le\cdots\le P_s$ be finite-rank spectral projections of R_{D_n}, selecting whole nonzero eigenspaces. Suppose the prescribed next ball has no Dirichlet eigenvalue in common with D_n. For every $\eta>0$, all sufficiently small positive δ give nested spectral projections $P_1'\le\cdots\le P_s'$ of $R_{D_{n+1}(\delta)}$ with*

$$\operatorname{rank}P_j' = \operatorname{rank}P_j, \qquad \|P_j'-P_j\|<\eta \quad (1\le j\le s).$$

An additional prescribed positive upper bound on δ can be imposed.

Proof. For the disconnected limiting resolvent in Lemma 4.1, each selected nonzero value has exactly its old eigenspace by the nonresonance assumption and the injectivity of $\lambda\mapsto(\lambda+1)^{-1}$. Around the finitely many distinct values selected by P_s, choose disjoint small circles avoiding zero and all unselected spectrum. Each old P_j is a sum over a nested subcollection of these circles. Norm convergence of the operators and the resolvent identity give uniform norm convergence of their resolvents on the circles. Their Riesz integrals consequently converge in norm. They are orthogonal projections, since the operators are self-adjoint. Projections at norm distance below one have injective restrictions in both directions and hence equal finite ranks. Summing over the same nested circle subcollections preserves nestedness. A multiple eigenvalue may split inside its circle; its full cluster is retained. Finitely many strict bounds and the aperture cap can be met together. □

We now construct the geometry and projections with the prescribed radii. The first two balls are joined before any projection is protected. Apply Lemma 5.2 to this attachment with $S=\sigma(H_{B_{a_3}(c_3e_1)})$, a countable positive set. Choose $0<\delta_1<\frac{1}{4}\min\{a_1,a_2,2^{-1}\}$ so that the resulting D_2 has spectrum disjoint from that of the third ball. This step allows arbitrary initial resonances between the first two balls.

For every subsequent physical prefix $n\ge 2$ we maintain the invariant

$$\sigma(H_{D_n})\cap\sigma(H_{B_{a_{n+1}}(c_{n+1}e_1)})=\varnothing. \tag{11}$$

Only the following ball is used in this condition. Once D_k, $k \geq 2$, is fixed, choose an orthonormal Dirichlet eigenbasis $\{b_{k,m} : m \geq 1\}$ of $L^2(D_k)$, extended by zero to $L^2(\mathbb{R}^d)$. These vectors are approximation targets. Families born at different stages need not be orthogonal to each other.

At stage $n \geq 2$ maintain nested finite-rank spectral projections

$$P_{1,n} \leq \cdots \leq P_{n-1,n}$$

of $R_n := R_{D_n}$, selecting whole nonzero eigenspaces. Earlier projections have been continued to this stage. Choose the new one as

$$Q_n = P_{n-1,n} = \mathbf{1}_{[\tau_n,\infty)}(R_n), \qquad \tau_n > 0, \quad \tau_n \notin \sigma(R_n), \tag{12}$$

with positive rank, containing $P_{n-2,n}$ when $n > 2$, and satisfying

$$\|(I - Q_n)b_{k,m}\| \leq 2^{-n} \qquad (k \geq 2,\ k + m \leq n). \tag{13}$$

There are finitely many targets, all supported in D_n. Positive cutoffs of its compact injective resolvent tend strongly to identity on $L^2(D_n)$, so these requirements can be met together with (12).

The next radius and center are already prescribed. Invariant (11) and Lemma 5.3 give an interval of sufficiently small positive apertures on which all historical clusters continue with the required ranks and nestedness. Inside that interval, Lemma 5.2 permits avoiding the spectrum of the following ball, numbered $n + 2$. Choose δ_n there, also imposing

$$\delta_n < \tfrac{1}{4}\min\{a_n, a_{n+1}, 2^{-n}\}, \qquad \|P_{j,n+1} - P_{j,n}\| \leq 2^{-(n+1)} \quad (1 \leq j \leq n-1). \tag{14}$$

This continues the protected projections and establishes (11) at $n + 1$. Then choose Q_{n+1} at the next stage. The finite projection constraints and the countable resonance exclusion are compatible in every sufficiently small interval. The geometric choices use one-room look-ahead and leave all prescribed radii and lengths unchanged.

6 The infinite smooth domain and its physical operator

Let $\Omega = \bigcup_n D_n$. The domains are open, connected and nested. Each contains its first n full balls, whose prescribed radii are unbounded. Thus Ω is connected and quasi-conical. Their left poles are

$$c_n - a_n = 2\sum_{r<n} a_r + \sum_{r<n} \ell_r \longrightarrow \infty,$$

because a positive unbounded radii sequence has divergent sum. In particular the centers tend to infinity, even when the radii oscillate.

The profile construction changes h_n only to the right of c_n. Every bounded axial interval is therefore unchanged after finitely many steps. The limiting profile h_∞ is smooth and positive on $(0, \infty)$ and agrees with the first sphere profile near zero, with a simple zero at zero. The defining function $h_\infty(t) - |y|^2$ has nonzero gradient on the boundary: away from the left tip, $y \neq 0$, and at the tip its t derivative is nonzero. Consequently $\partial\Omega$ is C^∞. The prescribed cylindrical gap of stage n is unchanged by subsequent steps, proving (2).

Use the final physical Hilbert space $\mathcal{H} = L^2(\Omega)$ and regard all finite operators and projections as zero-extended operators on it. This restriction does not change any earlier norm comparison. In particular, the passage volume belongs to $\mathcal{H}$.

Lemma 6.1 (Physical strong resolvent limit)**.** *On $\mathcal{H}$,*

$$R_n \longrightarrow R := (H_\Omega + 1)^{-1} \quad \textit{strongly}. \tag{15}$$

Proof. The zero-extended spaces $V_n = H_0^1(D_n)$ are increasing closed subspaces of $V = H_0^1(\Omega)$ for the energy inner product

$$a(f,g) = \int_\Omega (\nabla f \cdot \overline{\nabla g} + f\overline{g}).$$

Their union is dense in V. Indeed, a compact subset supporting a function in $C_c^\infty(\Omega)$ is contained in one D_N, by a finite subcover from the increasing open exhaustion. The variational equations show that $R_n f$ is the energy-orthogonal projection of Rf onto V_n. These projections converge in energy norm, hence in L^2. □

7 Complete reducing blocks and the spectral set

Proof of Theorem 2.1. The geometry and local spectral induction have been constructed above. For each fixed j, (14) makes $P_{j,n}$ norm-Cauchy. Let its limit, starting from physical prefix $j+1$, be P_j. It is an orthogonal projection, and

$$\|P_j - Q_{j+1}\| \le 2^{-(j+1)}, \qquad \|P_j - P_{j,n}\| \le 2^{-n} \quad (n \ge j+1). \tag{16}$$

Since $2^{-(j+1)} < 1$, comparison with Q_{j+1} by injective restrictions shows that P_j has exactly its finite rank. Nestedness also passes to the norm limit.

For $n \ge j+1$, $R_n P_{j,n} = P_{j,n} R_n$. Lemma 6.1, norm convergence of the projections and $\|R_n\| \le 1$ give

$$RP_j = P_j R. \tag{17}$$

For example, for each $f \in \mathcal{H}$,

$$\|R_n P_{j,n} f - RP_j f\| \le \|P_{j,n} - P_j\|\|f\| + \|(R_n - R)P_j f\| \longrightarrow 0,$$

and the reverse product is treated similarly. The P_j are therefore reducing projections for the physical resolvent.

For every fixed $b_{k,m}$, $k \ge 2$, and every $n \ge k+m$, (13) and (16) give

$$\|(I - P_{n-1})b_{k,m}\| \le 2^{1-n} \longrightarrow 0. \tag{18}$$

The union of the spans of these targets is dense in $\mathcal{H}$: each is a basis of its entire $L^2(D_k)$, and the increasing union for $k \ge 2$ is dense in $L^2(\Omega)$. Passage volume is included in every finite prefix where it appears. Equation (18), with the contraction bound on $I - P_j$, proves $P_j \to I$ strongly.

Put $P_0 = 0$ and $E_j = P_j - P_{j-1}$. These are mutually orthogonal finite-rank projections commuting with R, with strong sum I. Diagonalize the self-adjoint R on each nonzero finite-dimensional range. Its eigenvalues there are strictly positive because the physical resolvent is injective. If $Rf = \mu f$ with $\mu > 0$, then $f = \mu^{-1} Rf \in \operatorname{dom} H_\Omega$ and

$$H_\Omega f = (\mu^{-1} - 1) f.$$

The finite-block eigenbases give a complete orthonormal eigenbasis for H_Ω. Both continuous spectral subspaces vanish. Coincident limiting eigenvalues in different blocks cause no difficulty; reducing finite ranges suffice for the argument.

To determine the spectral set, fix $\lambda \ge 0$ and choose $\xi \in \mathbb{R}^d$ with $|\xi|^2 = \lambda$. Let $\chi \in C_c^\infty(B_1(0))$ have L^2 norm one. Choose a subsequence n_r with $a_{n_r} \to \infty$, and define

$$u_r(x) = a_{n_r}^{-d/2} \chi\left(\frac{x - c_{n_r} e_1}{a_{n_r}}\right) e^{i\xi\cdot x}.$$

These unit vectors lie in $C_c^\infty(\Omega)$ and have disjoint supports in the balls. Thus $u_r \rightharpoonup 0$, while

$$\|(H_\Omega - \lambda)u_r\|_2 \le \frac{2|\xi|}{a_{n_r}}\|\nabla\chi\|_2 + \frac{1}{a_{n_r}^2}\|\Delta\chi\|_2 \longrightarrow 0.$$

Weyl's criterion gives $[0,\infty) \subset \sigma_{\mathrm{ess}}(H_\Omega)$. Nonnegativity gives (3).

If $H_\Omega f = 0$, the form identity gives $\nabla f = 0$. Its zero extension belongs to $H^1(\mathbb{R}^d)$ and has zero weak gradient everywhere. It is a square-integrable constant, hence zero. All eigenvalues are positive. The complete diagonal representation shows that the closure of the countable eigenvalue set is $[0,\infty)$.

Finally $P_{j,n}$ is supported in D_n. Equation (16) therefore gives

$$\|\mathbf{1}_{\Omega\setminus D_n} P_j\| \le \|P_j - P_{j,n}\| \le 2^{-n} \qquad (n \ge j+1).$$

Every eigenvector in the jth block lies in $\operatorname{ran} E_j \subset \operatorname{ran} P_j$. Applying the last bound to a unit vector and squaring proves the final assertion. □

8 Interpretation of the construction

The finite shrinking-handle comparison, Riesz continuation and the whole-cluster method are established ingredients. The construction above combines them with an explicit smooth profile and a dense approximation family for all finite passage volume. The two limiting arguments play different roles: norm convergence at a fixed attachment preserves each historical finite cluster, while strong form exhaustion identifies the ordinary Dirichlet operator on the completed domain.

Both radii and lengths in Theorem 2.1 are given before the induction. Positive passage volume permits strict eigenvalue movement as an aperture varies. Avoiding countably many resonant values then meets the next-room condition together with the finite continuation constraints. Joining the first two rooms before protecting modes handles their possible initial resonances. This gives an existential geometric realization with local C^∞ boundary and explicit reducing-block tail estimates. The tail is measured by the full smooth prefix D_n, including the connecting passages. Its interpretation in physical distance follows from the specified centers and radii.

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