Pure-Point Dirichlet Spectrum in an Enlarging Cube Tower

Scott Schubr
Luminary Oddities

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Abstract

For every dimension $d \geq 2$, we construct a connected quasi-conical open set whose Dirichlet Laplacian has a complete orthonormal basis of eigenfunctions and spectrum $[0, \infty)$. The domain is a tower of adjacent enlarging cubes joined through positive windows tending to zero. Cube sizes and windows are chosen together. The construction gives an affirmative existence answer to the singular-continuous-spectrum question raised for this geometry by Krejčiřík and Lotoreichik and repeated in a recent spectral-geometry survey. At each finite attachment we continue a nested family of whole spectral clusters, including their multiplicities. A summable projection budget and a refreshed dense family of test vectors make the limiting finite-dimensional reducing subspaces complete. We also obtain a room-index L^2 tail bound and retain the possibility of placing one prescribed positive embedded eigenvalue by dilation.

Keywords: Dirichlet Laplacian; quasi-conical domain; pure-point spectral type; embedded eigenvalue; completeness; domain perturbation.

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1 Introduction

An open set in $\mathbb{R}^d$ is *quasi-conical* if it contains balls of arbitrarily large radius. Its Dirichlet Laplacian is nonnegative and has spectrum $[0, \infty)$, as follows by placing cutoff plane waves inside those balls. The spectral set alone does not determine its spectral type. Krejčiřík and Lotoreichik [2] constructed a connected quasi-conical domain with a prescribed positive embedded eigenvalue. Their domain consists of enlarging cubes connected by small windows; the windows can also be chosen to remove the absolutely continuous part. Section 5 of [2] leaves the singular-continuous part open. The same question appears as Open Problem 1 in [3].

We construct a tower in this geometric class with pure-point spectral type. The room sizes, as well as the windows, are chosen adaptively. Write $Q_a(x) = x + (-a, a)^d$ and let e_1 be the first coordinate vector.

Theorem 1.1. *For every integer $d \geq 2$, there exist half-widths $a_n > 0$, centres $c_n e_1$, and window half-widths $\delta_n > 0$ satisfying*

$$c_1 = a_1, \qquad c_{n+1} = c_n + a_n + a_{n+1}, \qquad a_{n+1} > a_n + 1, \qquad 0 < \delta_n < \min\{a_n, 2^{-n}\},$$

such that, with

$$\omega_n = Q_{a_n}(c_n e_1), \quad W_n = Q_{\delta_n}((c_n + a_n)e_1), \quad \Omega = \bigcup_{n \geq 1} \omega_n \cup \bigcup_{n \geq 1} W_n, \tag{1}$$

the ordinary Dirichlet Laplacian $H = H_\Omega$ has a complete orthonormal eigenfunction basis and

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty).$$

Both its absolutely continuous and singular-continuous spectral subspaces are zero. All its eigenvalues are positive, and their set is countable and dense in $[0, \infty)$.

Pure-point type here means completeness of the point spectral subspace. The stated conclusion concerns the spectrum as a set. The theorem gives an existence answer within the adaptive cube architecture of [2]; it does not prescribe an arbitrary room sequence in advance.

The small-window convergence and the cube geometry are inherited from [2], with general perturbation results in [1]. The step needed for completeness is to preserve a growing family of finite-dimensional spectral subspaces, rather than a fixed finite list of eigenfunctions. At each stage we add a spectral cutoff approximating more room-basis vectors with a smaller error. The continued projections have norm limits. Those limits reduce the actual infinite-domain resolvent, and their ranges exhaust $L^2(\Omega)$. Degenerate eigenvalues are retained as whole clusters throughout.

Section 2 gives the finite-stage perturbation lemmas. Section 3 constructs the tower and projections. Section 4 identifies the limiting operator and proves completeness. Section 5 gives the tail and scaling consequences. The argument is analytic.

2 Geometry and finite-stage perturbation

For an open set D, let H_D denote the self-adjoint operator associated with the form

$$h_D(u,v) = \int_D \nabla u \cdot \overline{\nabla v}, \qquad \operatorname{dom} h_D = H_0^1(D).$$

For bounded D the resolvent is compact: zero extension embeds $H_0^1(D)$ into $H_0^1(B)$ for a surrounding box B, and Rellich compactness applies. Its eigenvalues are strictly positive. We use $\mathbb{N} = \{1, 2, \dots\}$. All Hilbert spaces may be taken complex.

Define the first n rooms with their already opened windows by

$$D_n = \bigcup_{k=1}^{n} \omega_k \;\cup\; \bigcup_{k=1}^{n-1} W_k, \qquad H_n = H_{D_n}.$$

Each D_n is bounded and connected. Since $\delta_k < a_k < a_{k+1}$, the two off-interface halves of W_k lie inside its adjacent cubes. Windows therefore change the disjoint room union only by Lebesgue-null interface sets. After all choices have been made, the common Hilbert space is canonically

$$\mathcal{H} = L^2(\Omega) = \bigoplus_{k\ge 1} L^2(\omega_k), \qquad \mathcal{H}_n = L^2(D_n) = \bigoplus_{k=1}^{n} L^2(\omega_k). \tag{2}$$

Finite-stage operators and projections will be extended by zero on $\mathcal{H}_n^{\perp}$. In particular,

$$R_n = i_n (H_n + 1)^{-1} i_n^* \tag{3}$$

is compact, positive and self-adjoint on $\mathcal{H}$, with $\|R_n\| \le 1$. Its restriction to $\mathcal{H}_n$ is injective. Zero is excluded from all spectral contours used below. The space in (2) can be defined after the induction: each finite-stage norm comparison takes place on finitely many already chosen rooms and is unchanged by later zero extension.

Lemma 2.1 (Small-window convergence)**.** *Fix the first n rooms and windows. Attach a cube ω_{n+1} with $a_{n+1} > a_n + 1$ at the next interface. Put*

$$D_{n+1}(\delta) = D_n \cup \omega_{n+1} \cup Q_\delta((c_n + a_n)e_1).$$

On $\mathcal{H}_n \oplus L^2(\omega_{n+1})$, as $\delta \downarrow 0$ with $0 < \delta < a_n$,

$$(H_{D_{n+1}(\delta)} + 1)^{-1} \longrightarrow T_0 := (R_n|_{\mathcal{H}_n}) \oplus (H_{\omega_{n+1}} + 1)^{-1} \tag{4}$$

in operator norm, using the canonical almost-everywhere identification of these spaces.

Proof. The domains stay in one bounded ambient set. As the window closes, the only exceptional point in the decreasing-domain limit is its interface centre. A point has zero H^1 capacity in every dimension $d \geq 2$, including dimension two. The decreasing-domain Mosco criterion and bounded-ambient norm-resolvent convergence therefore apply. This is the same local geometry used in the first interconnection in [2, Section 3]; see its Propositions 2.2–2.3. The earlier, finitely many windows are fixed and are away from this point. The underlying results are also given in [1, Corollary 5.2.5 and Section 5.4]. □

Lemma 2.2 (Nonresonance by room scaling)**.** *Let Λ be a finite set of nonzero eigenvalues of R_n. One can choose $a > a_n + 1$, arbitrarily large, so that no element of Λ is an eigenvalue of the resolvent of the new cube Q_a.*

Proof. The cube Dirichlet eigenvalues are

$$\lambda_\nu(a) = \frac{\pi^2}{4a^2}|\nu|^2, \qquad \nu \in \mathbb{N}^d.$$

For a fixed $\mu \in \Lambda \subset (0, 1)$, resonance excludes only

$$a = \frac{\pi}{2}\sqrt{\frac{|\nu|^2}{\mu^{-1} - 1}}, \qquad \nu \in \mathbb{N}^d.$$

A finite union of these countable sets cannot fill any open interval. No simplicity of the new cube spectrum is required. □

Lemma 2.3 (Simultaneous cluster continuation)**.** *Let $P_1 \leq \cdots \leq P_s$ be finite-rank spectral projections of R_n, selecting whole nonzero eigenspaces. Choose the next cube by Lemma 2.2 for all eigenvalues selected by P_s. For every $\eta > 0$, a sufficiently small positive next window gives finite-rank spectral projections $P'_1 \leq \cdots \leq P'_s$ of R_{n+1} such that*

$$\operatorname{rank} P'_j = \operatorname{rank} P_j, \qquad \|P'_j - P_j\| < \eta \quad (1 \leq j \leq s).$$

The window can satisfy any additional prescribed positive upper bound.

Proof. In the disconnected operator T_0 of (4), every selected eigenvalue is isolated from the unselected spectrum and from zero. Nonresonance ensures that its full eigenspace is exactly the old selected eigenspace, extended by zero to the new room. Choose disjoint small circles around the distinct eigenvalues selected by P_s. Each P_j is a sum over a nested subcollection of those circles.

Lemma 2.1 and the resolvent identity give uniform operator-norm convergence of $(z - T_\delta)^{-1}$ to $(z - T_0)^{-1}$ on every circle. Thus the corresponding Riesz projections

$$\frac{1}{2\pi i}\int_\Gamma (z - T_\delta)^{-1}\, dz$$

converge in norm. They are orthogonal projections because the operators are self-adjoint. For small enough δ their ranks are unchanged: orthogonal projections at norm distance below one have equal finite ranks, by injectivity of the restrictions in both directions. Summing the same nested circle subcollections preserves nestedness. There are only finitely many constraints, so all norm bounds and any positive aperture upper bound can be met together. A degenerate eigenvalue may split within its circle; the full cluster is retained. □

3 The projection-preserving induction

Choose the first room. Once room ω_k is chosen, fix an orthonormal Dirichlet eigenbasis $\{b_{k,m} : m \geq 1\}$ of $L^2(\omega_k)$, extended by zero. These vectors serve only as approximation targets. All pairs (k, m) eventually form an orthonormal basis of $\mathcal{H}$.

At stage n maintain nested finite-rank spectral projections

$$P_{1,n} \le \cdots \le P_{n,n}$$

of R_n, each selecting whole positive eigenspaces. Older projections have already been continued to this stage. Choose the new one, $Q_n = P_{n,n}$, as

$$Q_n = \mathbf{1}_{[\tau_n,\infty)}(R_n), \qquad 0 < \tau_n \notin \sigma(R_n), \tag{5}$$

with τ_n small enough that Q_n has nonzero rank, contains $P_{n-1,n}$ when $n > 1$, and satisfies

$$\|(I - Q_n)b_{k,m}\| \le 2^{-n} \quad \text{whenever } k, m \ge 1, \quad k + m \le n. \tag{6}$$

This is possible because the list is finite, lies in $\mathcal{H}_n$, and positive spectral cutoffs of the compact resolvent tend strongly to the identity on $\mathcal{H}_n$. Taking the threshold outside the spectrum includes whole multiplicities. For $n = 1$ the test list is empty.

Choose $a_{n+1} > a_n + 1$ by Lemma 2.2 for the finite spectrum selected by Q_n, and put $c_{n+1} = c_n + a_n + a_{n+1}$. Apply Lemma 2.3 to all n tracked projections, choosing the positive aperture also to satisfy $\delta_n < \min\{a_n, 2^{-n}\}$. Require

$$\|P_{j,n+1} - P_{j,n}\| \le 2^{-(n+1)} \qquad (1 \le j \le n). \tag{7}$$

Then choose Q_{n+1} by (5)–(6) at the next stage. Every choice uses only finite data already fixed.

The widths increase without bound and the centres escape to infinity. There is no finite accumulation of rooms or windows. The increasing union $\Omega = \bigcup_n D_n$ is open, connected, and contains each cube. It is therefore quasi-conical.

4 The infinite operator and completeness

Lemma 4.1 (Physical resolvent limit)**.** *On $\mathcal{H} = L^2(\Omega)$, the zero-extended operators in* (3) *satisfy*

$$R_n \longrightarrow R := (H_\Omega + 1)^{-1} \quad \textit{strongly}.$$

Proof. Extend all Sobolev functions by zero. The spaces $V_n = H_0^1(D_n)$ are increasing closed subspaces of $V = H_0^1(\Omega)$ for the energy inner product

$$a(u, v) = \int_\Omega \big(\nabla u \cdot \overline{\nabla v} + u\bar{v}\big).$$

Their union is dense in V. Indeed, the compact support of any function in $C_c^\infty(\Omega)$ is covered by finitely many of the increasing open sets D_n and hence lies in one D_N. For $f \in \mathcal{H}$, the variational equations show that $R_n f$ is the energy-orthogonal projection of Rf onto V_n. Density gives convergence in the energy norm and thus in L^2. This proves the asserted limit without a uniform spectral gap or operator-norm convergence on the infinite tower. □

Proof of Theorem 1.1. The preceding induction constructs the required geometry. For fixed j, (7) makes $P_{j,n}$ norm-Cauchy as $n \to \infty$. Let its limit be P_j. It is an orthogonal projection and has the same finite rank as Q_j: eventually its distance from a fixed tail projection is below one. Nestedness passes to the norm limit. Moreover,

$$\|P_j - Q_j\| \le \sum_{n=j}^{\infty} 2^{-(n+1)} = 2^{-j}. \tag{8}$$

For $n \ge j$ we have $R_n P_{j,n} = P_{j,n} R_n$. Lemma 4.1, norm convergence of the projections and $\sup_n \|R_n\| \le 1$ allow both products to pass strongly to the limit. For example, for $f \in \mathcal{H}$,

$$\|R_n P_{j,n} f - R P_j f\| \le \|P_{j,n} - P_j\|\,\|f\| + \|(R_n - R)P_j f\| \longrightarrow 0.$$

The reverse product is treated in the same way. Therefore

$$RP_j = P_j R. \tag{9}$$

These limits are reducing projections; no final isolated spectral cluster is required.

Fix $b_{k,m}$. For every $j \geq k+m$, (6) and (8) give

$$\|(I - P_j)b_{k,m}\| \leq \|(I - Q_j)b_{k,m}\| + \|Q_j - P_j\| \leq 2^{1-j} \longrightarrow 0. \tag{10}$$

The nested ranges of P_j thus have dense union in $\mathcal{H}$. Put $P_0 = 0$ and $E_j = P_j - P_{j-1}$. The E_j are mutually orthogonal finite-rank projections, commute with R, and sum strongly to the identity. Diagonalize the self-adjoint R on each finite-dimensional range. Its eigenvalues on nonzero vectors are strictly positive because $R = (H_\Omega + 1)^{-1}$ is injective. If $Ru = \mu u$, $\mu > 0$, then $u = \mu^{-1}Ru \in \operatorname{dom} H_\Omega$ and

$$H_\Omega u = (\mu^{-1} - 1)u.$$

The resulting finite-block orthonormal bases give a complete orthonormal eigenbasis of H_Ω. Its two continuous spectral subspaces vanish.

It remains to identify the spectral set and the endpoint. Fix $\lambda \geq 0$ and choose $\xi \in \mathbb{R}^d$ with $|\xi|^2 = \lambda$. Take $\chi \in C_c^\infty((-1,1)^d)$ with $\|\chi\|_2 = 1$, and set

$$u_n(x) = a_n^{-d/2}\chi\left(\frac{x - c_n e_1}{a_n}\right)e^{i\xi\cdot x}.$$

These unit vectors lie in $C_c^\infty(\omega_n)$ and have disjoint supports. Hence $u_n \rightharpoonup 0$, while

$$\|(H_\Omega - \lambda)u_n\|_2 \leq \frac{2|\xi|}{a_n}\|\nabla\chi\|_2 + \frac{1}{a_n^2}\|\Delta\chi\|_2 \longrightarrow 0.$$

Weyl's criterion yields $[0,\infty) \subset \sigma_{\mathrm{ess}}(H_\Omega)$; nonnegativity supplies the reverse inclusion. If $H_\Omega u = 0$, its form identity gives $\|\nabla u\|_2 = 0$. Its zero extension belongs to $H^1(\mathbb{R}^d)$, has zero weak gradient, and is a square-integrable constant, hence zero. Thus zero is not an eigenvalue. The eigenvalue set of the complete basis is countable and its closure is $[0,\infty)$. □

Remark 4.2. The limiting P_j need not be Borel spectral projections of H_Ω. Coincident limiting eigenvalues can occur in different blocks. Finite-dimensional reducing ranges are sufficient for the proof.

5 Tail control and a prescribed eigenvalue

Corollary 5.1. *The projections in the construction satisfy, for $n \geq j$,*

$$\|\mathbf{1}_{\Omega\setminus D_n}P_j\| \leq 2^{-n}.$$

Consequently the eigenbasis in Theorem 1.1 may be grouped into finite blocks so that every normalized eigenvector u in the jth block satisfies

$$\int_{\Omega\setminus D_n}|u(x)|^2\,dx \leq 4^{-n} \qquad (n \geq j).$$

Proof. The projection $P_{j,n}$ is supported in D_n, and

$$\|P_j - P_{j,n}\| \leq \sum_{k=n}^{\infty} 2^{-(k+1)} = 2^{-n}.$$

Multiplication by the indicator of $\Omega \setminus D_n$ therefore gives the operator bound. Apply it to a unit vector in $\operatorname{ran} E_j \subset \operatorname{ran} P_j$ and square. □

This bound measures mass by room index. Adaptive sizes may grow arbitrarily fast; the estimate gives neither a pointwise physical-distance decay rate nor an effective formula for the apertures.

Corollary 5.2. *For every $\lambda_* > 0$ and $d \geq 2$, a connected enlarging-cube tower with positive vanishing windows has the spectral-type and spectral-set conclusions of Theorem 1.1 and has λ_* as an eigenvalue.*

Proof. Choose any positive eigenvalue λ_0 of the constructed operator and dilate the entire domain by $\alpha = \sqrt{\lambda_0/\lambda_*}$. Unitary dilation multiplies Dirichlet eigenvalues by α^{-2} and preserves pure-point type, the half-line spectral set and the qualitative growing-room geometry. The room-tail mass bound is preserved under unitary normalization. The auxiliary numerical growth and window bounds rescale with α. □

The construction gives an existential, adaptively selected tower. The argument uses the finite-stage convergence of [2, 1], but proves completeness directly rather than deducing it from a dense set of eigenvalues or from a trace-class comparison.

AI assistance

AI tools assisted research, proof development, writing, and verification.

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