Prescribed-gradient minimizers and effective stability for homogeneous real-stable polynomials

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Luminary Oddities

Abstract

For every fixed degree, we determine the unique minimizer of the squarefree coefficient among normalized homogeneous real-stable polynomials with nonnegative coefficients and a prescribed gradient sufficiently close to the balanced gradient. The minimizer is the power of the corresponding linear form. We give explicit positive rational neighborhood and stability constants in every degree: the coefficient deficit controls both centered-root dispersion and coefficient distance to the minimizer, including singular dispersion and boundary coefficients. The distance order is optimal. A radial extension to a degree-drop boundary and a positive gradient correction make the global estimate constructive, extending classical balanced minimization to perturbed gradients. We also classify strict local minimality at every positive gradient and exhibit a cubic separation between local and global minimization. In degree three, complementary discriminant and vertex-coefficient bounds give a stronger explicit neighborhood and stability constant. Applications give a Frobenius-distance reserve for Hermitian mixed discriminants and a total-variation reserve for stable count distributions. In the larger Lorentzian class, explicit fixed-gradient competitors show failure of local minimization when a gradient coordinate reaches $n/(n-1)$; consequently no positive global gradient radius can be uniform in degree. A separate cubic family gives a smaller upper bound in degree three.

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1 Introduction and main results

Coefficient bounds for stable polynomials unify permanent and mixed-discriminant inequalities and give quantitative information about dependent count distributions. Classical results identify the unique minimizer at a balanced gradient. The question here is whether that minimizer persists throughout an entire class with a prescribed, slightly unbalanced gradient, and how the coefficient excess controls distance from it. We prove this persistence with an effective stability estimate in every fixed degree.

A nonzero real polynomial is *real stable* if it has no zero when every variable has strictly positive imaginary part. Throughout, p is homogeneous of degree n in n variables, has nonnegative coefficients, is real stable, and satisfies $p(\mathbf{1}) = 1$. Write

$$t = \nabla p(\mathbf{1}), \quad L_t(x) = \frac{t^T x}{n}, \quad \mathcal{A}(p) = [x_1 \cdots x_n]p, \quad F(t) = \frac{n!}{n^n} \prod_i t_i.$$

Euler's identity gives $\sum_i t_i = n$. For a polynomial q, put $\|q\|_{\mathrm{coef},1} = \sum_\alpha |[x^\alpha]q|$. For degree one the normalized polynomial is simply $p(x_1) = x_1$, so minimization is trivial. The positive rational constants $\beta_n, \varepsilon_n, c_n$ used below are specified explicitly in Section 2.

Theorem 1.1. *Let p satisfy the standing hypotheses, with $n \geq 2$, and set*

$$B = \frac{n-1}{2n} tt^T - \frac{1}{2}\nabla^2 p(\mathbf{1}), \qquad d = \operatorname{Tr} B.$$

Then $B \succeq 0$. If $\|t - \mathbf{1}\|_1 \leq \varepsilon_n$, then

$$\mathcal{A}(p) - F(t) \;\geq\; \frac{\beta_n}{8} d \;\geq\; c_n \left\| p - L_t^n \right\|_{\mathrm{coef},1}. \tag{1}$$

In particular, L_t^n is the unique minimizer of $\mathcal{A}$ on the entire feasible class with that prescribed gradient. The closed endpoint and singular B are included.

The estimate is global over all feasible polynomials with the prescribed nearbalanced gradient, not merely a coefficient neighborhood of the power. Since $d = 0$ exactly at that power, it proves uniqueness as well as stability. The constants depend on degree and are sufficient rather than optimal. The linear coefficient-distance order is optimal even at balance; see Proposition 5.1.

The following low-degree result gives a substantially larger explicit gradient neighborhood than the general construction.

Theorem 1.2 (Explicit cubic stability)**.** *Let p be a homogeneous real-stable cubic in three variables with nonnegative coefficients and $p(\mathbf{1}) = 1$. Set*

$$t = \nabla p(\mathbf{1}), \qquad L = L_t, \qquad B = tt^T/3 - \nabla^2 p(\mathbf{1})/2, \qquad d = \operatorname{Tr} B.$$

If $\|t - \mathbf{1}\|_1 \leq 1/8$, then

$$\mathcal{A}(p) - \frac{2}{9} t_1 t_2 t_3 \;\geq\; \frac{d}{256} \;\geq\; \frac{1}{3584} \left\| p - L^3 \right\|_{\mathrm{coef},1}. \tag{2}$$

The statement includes singular B. In particular, the power is the unique minimizer throughout this closed gradient ball.

Its full analytic proof is in Appendix A. No determinant representation or nonsingularity of the dispersion matrix is assumed.

1.1 Prior work and the contribution

Gurvits's coefficient–capacity theorem and its equality case establish the balanced minimum [3]. Gurvits–Klein–Leake obtain a nearbalanced permanent minimization result [9]. Gurvits–Leake extend balanced uniqueness to Lorentzian polynomials and ask, in Question 1.8, for prescribed-gradient minimization over the full real-stable class [4]. Theorem 1.1 answers that question and supplies an effective coefficient-distance reserve. Productization bounds [6] preserve relevant evaluations but do not by themselves identify this prescribed-gradient coefficient minimizer. Lavi's quantitative permanent bounds [7] give a stronger balanced estimate in a product-of-linear-forms subclass; that representation is not assumed here.

The constructive proof gives explicit constants without an unevaluated compactness minimum. The exact local criterion, cubic local/global separation and sharp distance order accompany the effective global theorem; all required arguments are given below.

The real-stable hypothesis also has a quantitative boundary. Section 7 constructs positive-coefficient Lorentzian competitors at exactly fixed gradients, giving the necessary local condition $\max_i t_i < n/(n-1)$ and excluding a degree-uniform positive global radius in that larger class. These are not counterexamples to the stable theorem.

1.2 Proof architecture

Centered real roots produce a positive semidefinite dispersion matrix that controls every higher centered coefficient. At balance, a radial extension reaches a degree-drop boundary, where the classical refined capacity theorem gives a strict gap. A derivative-based Bernstein expansion transfers that gap back to the original polynomial. An explicit positive substitution then corrects the gradient while preserving all centered factors. This yields the effective global theorem without an unevaluated compactness minimum. Section 6 identifies the exact local region; the cubic proof and illustrative obstructions are included after the main argument.

2 Explicit constants and centered-root control

Fix $n \geq 2$. All constants in this display are positive rational numbers:

$$v_n = \frac{n!}{n^n}, \qquad M_n = \frac{n(n-1)}{2}, \qquad h_n = \frac{(n-2)!}{n^{n-2}}, \tag{3}$$

$$\kappa_2 = 1, \qquad \kappa_n = \frac{(n-1)!(n-2)^{n-2}}{(n-1)^{2n-3}} \quad (n \geq 3), \qquad g_n = \kappa_n - v_n, \tag{4}$$

$$\beta_n = \min\left\{ \frac{g_n}{2M_n},\, h_n \left(\frac{g_n}{4n}\right)^{n-2} \right\}, \tag{5}$$

$$K_{n,k} = \binom{n}{k} \frac{2^k n^{2k-2} k^k}{k!}, \qquad W_n = \sum_{k=2}^{n} K_{n,k}, \tag{6}$$

$$\varepsilon_n = \min\left\{ \frac{1}{2}, \frac{\beta_n}{8nW_n} \right\}, \qquad c_n = \frac{\beta_n}{8W_n}. \tag{7}$$

As usual, a zeroth power equals one, including at an endpoint.

Lemma 2.1. *There are homogeneous polynomials q_k of degree k, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^{n} L_t(x)^{n-k} q_k(x), \qquad q_2(x) = -x^T B x. \tag{8}$$

The matrix B in Theorem 1.1 is PSD, $B\mathbf{1} = 0$, and

$$|q_k(x)| \leq \binom{n}{k} (2x^T B x)^{k/2} \leq \binom{n}{k} (2d\|x\|_2^2)^{k/2}. \tag{9}$$

Moreover, $d = 0$ exactly when $p = L_t^n$, and $0 \leq d \leq M_n$.

Proof. Put $y = x - L_t(x)\mathbf{1}$. The polynomial $p(z\mathbf{1} + y)$ is monic of degree n, with real roots and zero z^{n-1} coefficient. Reality of the roots follows from stability and complex conjugation. Denote its roots by λ_j. Their sum is zero, and its quadratic coefficient is $-\frac{1}{2}\sum_j \lambda_j^2 \leq 0$. Expanding in z and substituting $z = L_t(x)$ gives (8), translation invariance, and a PSD quadratic form $q_2 = -x^T B x$. Differentiating twice at $\mathbf{1}$ gives the stated formula for B. Translation invariance gives $B\mathbf{1} = 0$. Each root has absolute value at most $(2x^T B x)^{1/2}$; bounding every elementary symmetric term gives (9). If $d = 0$, PSD forces $B = 0$ and all higher q_k vanish. The converse is immediate. Finally, every diagonal Hessian entry is nonnegative and $\sum_i t_i = n$, so $d \leq (n-1)\sum_i t_i^2/(2n) \leq M_n$. □

Lemma 2.2. *For every polynomial in the standing class,*

$$\|q_k\|_{\mathrm{coef},1} \leq K_{n,k} d, \qquad \|p - L_t^n\|_{\mathrm{coef},1} \leq W_n d. \tag{10}$$

Proof. Let T_k be the symmetric k-linear polar form of q_k. The real polarization identity expresses $T_k(e_{i_1}, \dots, e_{i_k})$ as a sum of 2^k evaluations at signed sums of coordinate vectors, divided by $2^k k!$. Their Euclidean norms are at most k, so

$$|T_k(e_{i_1}, \dots, e_{i_k})| \le \binom{n}{k} (2d)^{k/2} \frac{k^k}{k!}.$$

Summing the n^k ordered tensor coefficients bounds the polynomial coefficient norm. Since $d \le n^2$ and $k \ge 2$, $d^{k/2} \le d n^{k-2}$; also $2^{k/2} \le 2^k$. These give exactly the first estimate with (6). The coefficients of L_t are nonnegative and sum to one, so $\|L_t\|_{\mathrm{coef},1} = 1$. Summing the intrinsic expansion proves the second estimate. At $d = 0$ use Lemma 2.1 directly. □

3 Effective global stability

3.1 Balanced derivatives and Bernstein payment

Call a normalized polynomial *balanced* when its gradient at $\mathbf{1}$ equals $\mathbf{1}$. We use Gurvits's coefficient–capacity theorem and its variable-degree refinement [3, Theorem 2.4 and Corollary 2.5]. If p is balanced, coefficient-weighted AM–GM gives $p(x) \ge \prod_i x_i$ for $x_i > 0$. Thus $\mathrm{Cap}(p) = 1$ and $\mathcal{A}(p) \ge v_n$.

Lemma 3.1. *If q is balanced and satisfies $q(e_i) = 0$ for some coordinate vertex, then $\mathcal{A}(q) \ge \kappa_n = v_n + g_n$, with $g_n > 0$.*

Proof. Nonnegative coefficients and $q(e_i) = 0$ imply degree at most $n-1$ in that variable. Eliminate it first in the degree-refined capacity induction. Its first factor is $((n-2)/(n-1))^{n-2}$; the remaining factors multiply to $(n-1)!/(n-1)^{n-1}$. Their product is (4). For $n = 2$ the first factor is one, giving $\kappa_2 = 1$. The function $G(m) = ((m-1)/m)^{m-1}$ strictly decreases for integers $m \ge 2$, with $G(1) = 1$. Replacing the degree-n first factor by $G(n-1) > G(n)$ therefore gives $g_n > 0$. The capacity theorem includes zero coefficients and degree drops. □

For a balanced q, define

$$E_k(q) = \sum_{|S|=k} \partial_S q(\mathbf{1}), \qquad E_0(q) = 1, \qquad T_s x = (1-s) L_{\mathbf{1}}(x) \mathbf{1} + sx.$$

Lemma 3.2. *For $0 \le k \le n$,*

$$E_k(q) \ge \binom{n}{k} \frac{n!}{(n-k)! n^k}. \tag{11}$$

For $0 \le s \le 1$ one has the exact identity

$$\mathcal{A}(q \circ T_s) = \sum_{k=0}^{n} s^k (1-s)^{n-k} \frac{(n-k)!}{n^{n-k}} E_k(q), \tag{12}$$

and consequently

$$\mathcal{A}(q \circ T_s) - v_n \ge h_n s^2 (1-s)^{n-2} d(q). \tag{13}$$

Proof. Write $D_{\mathbf{1}} = \sum_i \partial_i$ and set $r_k = (k!/n!) D_{\mathbf{1}}^{n-k} q$. Directional differentiation preserves stability; Euler's identities give $r_k(\mathbf{1}) = 1$ and $\nabla r_k(\mathbf{1}) = (k/n)\mathbf{1}$. Therefore $r_k L_{\mathbf{1}}^{n-k}$ is a balanced stable polynomial of degree n. For every squarefree S of size k, homogeneity gives

$$\partial_S r_k(0) = \frac{k!(n-k)!}{n!} \partial_S q(\mathbf{1}).$$

Its product with $L_{\mathbf{1}}^{n-k}$ has squarefree coefficient $\frac{(n-k)!}{n^{n-k}}E_k(q)/\binom{n}{k}$. Applying the balanced coefficient bound proves (11), including $k = 0, 1$. This is a general stable-polynomial derivative consequence, with no factorization into linear forms assumed.

Taylor expansion at $(1-s)L_{\mathbf{1}}(x)\mathbf{1}$ and squarefree extraction give (12). By (11) each displayed coefficient is at least $\binom{n}{k}v_n$. Subtract the binomial expansion of v_n; all resulting Bernstein deficit coefficients are nonnegative. The total Hessian sum is $n(n-1)$, whereas its trace is $(n-1)-2d(q)$. Hence

$$E_2(q) = \frac{(n-1)^2}{2} + d(q).$$

The baseline $k = 2$ value is $(n-1)^2/2$. Retaining this single deficit proves (13). For $n = 2$ its zeroth power is one; all endpoints are included by direct evaluation or continuity. □

3.2 A radial degree-drop boundary and the balanced reserve

We use the following classical hyperbolicity-cone facts [5, 3]. A homogeneous real-stable polynomial with nonnegative coefficients is hyperbolic in $\mathbf{1}$. Its cone C containing $\mathbf{1}$ is open and convex, it is hyperbolic in every direction in C, and its closed cone consists of the vectors with nonnegative hyperbolic eigenvalues. Positive combinations of directions in the closed cone, with an interior summand, are interior. Pullback along closed-cone directions whose sum is interior is hyperbolic in every strictly positive direction, hence is real stable and has nonnegative coefficients. Equivalently, the coefficient signs follow from the standard mixed-directional-derivative positivity on that cone. These facts do not require strictly hyperbolic roots or a pointed cone.

Lemma 3.3. *If p is balanced and $d(p) > 0$, there is a finite $D \geq 1$ such that $q = p \circ T_D$ is balanced, stable, normalized, has nonnegative coefficients, and satisfies $q(e_i) = 0$ for some i. Moreover,*

$$p = q \circ T_{1/D}, \qquad d(q) = D^2 d(p). \tag{14}$$

Proof. Let a_{ij} be the roots in z of $p(z\mathbf{1} - e_i)$. They are nonnegative because $e_i \in \overline{C}$, and balance gives $\sum_j a_{ij} = 1$. Let $a = \min_{i,j} a_{ij}$. If $a = 1/n$, every coordinate's eigenvalues equal $1/n$. Its centered quadratic variance is zero, so every $B_{ii} = 0$ and PSD forces $d = 0$, a contradiction. Thus $0 \leq a < 1/n$. Set $D = (1-na)^{-1}$.

The eigenvalues of $T_D e_i$ are $(1-D)/n + Da_{ij} \geq 0$, with at least one zero. Thus every $T_D e_i$ lies in $\overline{C}$ and their sum is $\mathbf{1} \in C$. For $x > 0$,

$$T_D x = (\min_i x_i)\mathbf{1} + \sum_i (x_i - \min_j x_j) T_D e_i \in C.$$

The cone facts just stated prove the stability and coefficient signs of q. Also $q(\mathbf{1}) = 1$ and $T_D^T \mathbf{1} = \mathbf{1}$, so q is balanced. The zero eigenvalue gives $q(e_i) = 0$ at some vertex. The identities $T_D T_s = T_{Ds}$ and the scaling by D on centered directions give (14). If $a = 0$, then $D = 1$ and p already has a degree-drop vertex. Repeated roots cause no difficulty. The case $D > 1$ is justified by the cone argument, not by entrywise positivity: T_D has negative off-diagonal entries. □

Proposition 3.4. *Every balanced polynomial in the standing class satisfies*

$$\mathcal{A}(p) - v_n \geq \beta_n d(p). \tag{15}$$

Proof. The case $d = 0$ follows from Lemma 2.1. Otherwise take q, D from Lemma 3.3, put $s = 1/D$ and $\theta = 1 - s$. The positive contraction T_s has row forms of coefficient norm one, whose distances

from the corresponding coordinate forms are at most 2θ. Since the nonnegative coefficients of q sum to one, telescoping each degree-n monomial product gives

$$\|q - q \circ T_s\|_{\mathrm{coef},1} \le 2n\theta. \tag{16}$$

If $\theta \le g_n/(4n)$, the boundary gap gives

$$\mathcal{A}(p) - v_n \ge g_n - 2n\theta \ge g_n/2 \ge \frac{g_n}{2M_n} d(p).$$

If $\theta > g_n/(4n)$, apply (13) to q and use $s^2 d(q) = d(p)$:

$$\mathcal{A}(p) - v_n \ge h_n \theta^{n-2} d(p) \ge h_n \left(\frac{g_n}{4n}\right)^{n-2} d(p).$$

The minimum of the two constants is β_n. This includes $n = 2$ and equality at the dividing threshold. No compactness minimum or norm estimate for the negative-entry extension T_D is used. □

3.3 Positive gradient correction

Proof of Theorem 1.1. PSD of B is Lemma 2.1. Put $e = \|t - \mathbf{1}\|_1$. When $e = 0$, Proposition 3.4 is stronger than the required estimate. Suppose $0 < e \le \varepsilon_n \le 1/2$, and define

$$z_i = \frac{1 - (1-e)t_i}{ne}, \qquad Sx = (1-e)x + e(z^T x)\mathbf{1}. \tag{17}$$

The vector z is a probability vector. Indeed its entries sum to one. For $t_i \le 1$ nonnegativity is immediate; for $t_i > 1$, $t_i - 1 \le e$ and $et_i - (t_i - 1) \ge e - (t_i - 1) \ge 0$. The matrix S is nonnegative, has positive diagonal, and satisfies $S\mathbf{1} = \mathbf{1}$ and $S^T t = \mathbf{1}$. Thus $r = p \circ S$ is normalized, balanced, stable and has nonnegative coefficients. Stability follows directly because S maps strictly positive imaginary parts to strictly positive imaginary parts.

The correction preserves the intrinsic factors exactly:

$$L_t(Sx) = L_{\mathbf{1}}(x), \quad q_k(Sx) = (1-e)^k q_k(x), \quad d(r) = (1-e)^2 d(p). \tag{18}$$

The middle identity uses translation invariance and homogeneity, not a loss estimate or inverse-dispersion bound. The balanced theorem therefore gives

$$\mathcal{A}(r) - v_n \ge \beta_n (1-e)^2 d \ge \beta_n d/4.$$

Now $\|L_t\|_{\mathrm{coef},1} = \|L_{\mathbf{1}}\|_{\mathrm{coef},1} = 1$ and $\|L_t - L_{\mathbf{1}}\|_{\mathrm{coef},1} = e/n$. Telescoping powers and $1-(1-e)^k \le ke$ yield

$$\left\| L_t^{n-k} - (1-e)^k L_{\mathbf{1}}^{n-k} \right\|_{\mathrm{coef},1} \le \frac{n-k}{n} e + ke \le ne.$$

Subtract the two intrinsic expansions after removing their respective linear-form powers. Lemma 2.2 bounds the resulting coefficient norm by $neW_n d$. Since the squarefree coefficient is bounded in absolute value by that norm,

$$\mathcal{A}(p) - F(t) \ge \beta_n d/4 - neW_n d \ge \beta_n d/8.$$

The final inequality uses (7). Applying $\|p - L_t^n\|_{\mathrm{coef},1} \le W_n d$ proves (1). If the deficit is zero, $d = 0$ and $p = L_t^n$, proving uniqueness. □

4 Mixed-discriminant and probability consequences

Corollary 4.1. *Let $X_1, \dots, X_n$ be complex Hermitian PSD matrices of order n, with $\sum_i X_i = I$. Put $t_i = \operatorname{Tr} X_i$ and use the normalized mixed discriminant*

$$\mathrm{MD}(X_1, \dots, X_n) = \frac{1}{n!}[x_1 \cdots x_n] \det\left(\sum_i x_i X_i\right).$$

If $\|t - \mathbf{1}\|_1 \le \varepsilon_n$, then

$$\mathrm{MD}(X_1, \dots, X_n) - \frac{\prod_i t_i}{n^n} \ge \frac{\beta_n}{16n!} \sum_i \left\| X_i - \frac{t_i}{n} I \right\|_F^2. \tag{19}$$

Individual X_i may be singular and have complex entries.

Proof. The polynomial $p(x) = \det(\sum_i x_i X_i)$ has nonnegative coefficients and $p(\mathbf{1}) = 1$. If all imaginary parts of x_i are positive, the pencil has positive-definite imaginary part, so is invertible; hence p is real stable. Differentiation at I gives $\partial_i p(\mathbf{1}) = t_i$ and $\partial_{ii} p(\mathbf{1}) = t_i^2 - \operatorname{Tr}(X_i^2)$. Therefore

$$d = \frac{1}{2} \sum_i \left(\operatorname{Tr}(X_i^2) - \frac{t_i^2}{n} \right) = \frac{1}{2} \sum_i \left\| X_i - \frac{t_i}{n} I \right\|_F^2.$$

Divide the dispersion reserve of Theorem 1.1 by $n!$. □

The coefficient probability distribution of p is a count law on $\{\alpha \in \mathbb{Z}^n_{\ge 0} : |\alpha| = n\}$ with mean t. The coefficients of L_t^n give the multinomial law with probabilities t_i/n. Their coefficient distance is twice total variation. Consequently (1) also reads

$$\Pr_p\{\alpha = \mathbf{1}\} - F(t) \ge 2c_n d_{\mathrm{TV}}(p, \mathrm{Mult}(n, t/n)).$$

This gives a direct probabilistic interpretation of the coefficient bound.

5 Optimal distance order

Proposition 5.1. *For every $n \ge 2$, no positive uniform lower bound by $\|p - L_t^n\|_{\mathrm{coef},1}^{\alpha}$ with $\alpha < 1$ can replace the right side of (1), even when $t = \mathbf{1}$.*

Proof. Put $L = L_{\mathbf{1}}$, $a = x_1 - x_2$, and, for $0 < s < 1/n$, let

$$p_s = L^{n-2}(L + sa)(L - sa) = L^n - s^2 L^{n-2} a^2.$$

Every linear factor has nonnegative coefficients and positive coefficient sum, so the product is stable. Moreover, $p_s(\mathbf{1}) = 1$ and $\nabla p_s(\mathbf{1}) = \mathbf{1}$. Only the term $-2x_1x_2$ in a^2 can contribute to the squarefree coefficient, giving

$$\mathcal{A}(p_s) - \mathcal{A}(L^n) = \frac{2(n-2)!}{n^{n-2}} s^2, \qquad \|p_s - L^n\|_{\mathrm{coef},1} = s^2 \left\| L^{n-2} a^2 \right\|_{\mathrm{coef},1} > 0.$$

The ratio of the gap to this distance raised to $\alpha < 1$ tends to zero. This proves the assertion. □

This establishes optimal order, not the best constant in Theorem 1.1.

6 Exact local geometry and global obstructions

The nearbalanced global conclusion does not identify the full gradient region in which the power is a local minimizer. We now classify strict local minimality in the entire stable-polynomial class. Centered-root control gives the criterion without a matrix representation.

6.1 The exact quadratic payment and the remainder

Lemma 6.1. *For positive t_i, put $v_i = 1/t_i$ and $D = \operatorname{diag}(v)$. Then*

$$\mathcal{A}(L_t^{n-2} q_2) = \frac{(n-2)! \prod_i t_i}{n^{n-2}} \left(\operatorname{Tr}(D^2 B) - v^T B v \right). \tag{20}$$

Proof. A diagonal term of $-x^T B x$ cannot contribute to a squarefree monomial. The ordered pair $i \neq j$ contributes

$$-B_{ij} \frac{(n-2)!}{n^{n-2}} \prod_{\ell \neq i,j} t_\ell.$$

Summing these terms gives the identity. □

Lemma 6.2. *For $n \geq 2$, set*

$$C_n = 2^n \sum_{k=3}^{n} \binom{n}{k} (2n)^{k/2},$$

where an empty sum is zero. If $t_i \geq 0$, $\sum_i t_i = n$ and $0 \leq d \leq 1$, then

$$|\mathcal{A}(p) - F(t) - \mathcal{A}(L_t^{n-2} q_2)| \leq C_n d^{3/2}.$$

Proof. For every homogeneous degree-n polynomial R,

$$\mathcal{A}(R) = \sum_{S \subseteq [n]} (-1)^{n-|S|} R(\mathbf{1}_S).$$

A monomial survives exactly when every variable occurs, and degree n then makes it squarefree. Also $0 \leq L_t(\mathbf{1}_S) \leq 1$ and $\mathbf{1}_S^T B \mathbf{1}_S \leq nd$. Lemma 2.1 bounds the contribution of each $k \geq 3$ by $2^n \binom{n}{k} (2nd)^{k/2}$. Sum and use $d^{k/2} \leq d^{3/2}$. □

6.2 The local criterion

Proposition 6.3. *Fix $n \geq 2$ and $t_i > 0$ with $\sum_i t_i = n$, and set*

$$\gamma(t) = \frac{n^2}{\|t\|_2^2} - (n-1).$$

In the coefficient topology on normalized nonnegative homogeneous real-stable polynomials with gradient t, the power L_t^n is a strict local minimizer of $\mathcal{A}$ if and only if $\gamma(t) > 0$. If $\gamma(t) < 0$, it is not a local minimizer. If $\gamma(t) = 0$, a nonconstant stable product curve through L_t^n has the same squarefree coefficient throughout.

Proof. Let $T = \operatorname{diag}(t)$ and $S = T^{-1} B T^{-1}$. Then $S \succeq 0$ and $St = 0$. Define

$$r = \mathbf{1} - \frac{nt}{\|t\|_2^2}, \qquad \|r\|_2^2 = 1 - \gamma(t).$$

The exact coefficient identity of Lemma 6.1 becomes

$$A_2 := \mathcal{A}(L_t^{n-2} q_2) = F(t)\frac{n}{n-1}(\operatorname{Tr} S - r^T S r).$$

In particular, the bracket is the negative sum of all ordered off-diagonal entries $B_{ij}/(t_i t_j)$; its diagonal cancels. Since $r^T S r \le \|r\|_2^2 \operatorname{Tr} S$,

$$A_2 \ge F(t)\frac{n}{n-1}\gamma(t)\operatorname{Tr} S \ge \alpha(t) d, \qquad \alpha(t) = \frac{F(t) n \gamma(t)}{(n-1)(\max_i t_i)^2},$$

when $\gamma(t) > 0$. The cube estimate in Lemma 6.2 required only $t_i \ge 0$ and $\sum_i t_i = n$. It therefore gives, for $d \le 1$,

$$|\mathcal{A}(p) - F(t) - A_2| \le C_n d^{3/2}.$$

For $n \ge 3$, put $\eta(t) = \min\{1, (\alpha(t)/(2C_n))^2\}$. For $n = 2$, put $\eta(t) = 1$, since the remainder vanishes. If $d \le \eta(t)$, then

$$\mathcal{A}(p) - F(t) \ge \tfrac{1}{2}\alpha(t) d.$$

The rigidity $d = 0 \iff p = L_t^n$ and continuity of d establish strict local minimality. Lemma 2.2 also gives local coefficient-distance coercivity with constant $\alpha(t)/(2W_n)$. This is an explicit local dispersion radius, not an explicit global gradient neighborhood.

For the converse, if $\gamma(t) \le 0$ then $r \ne 0$. Set $b = Tr$, so $b^T \mathbf{1} = t^T r = 0$. For sufficiently small $s > 0$, all coefficients of $L_t \pm s b^T x$ are positive. Hence

$$p_s = L_t^{n-2}(L_t + s b^T x)(L_t - s b^T x) = L_t^n - s^2 L_t^{n-2}(b^T x)^2$$

is feasible and nonconstant. Here $B = s^2 b b^T$, $S = s^2 r r^T$, and there are no higher centered terms. Consequently,

$$\mathcal{A}(p_s) - F(t) = F(t)\frac{n}{n-1} s^2 \gamma(t) \|r\|_2^2.$$

It is negative when $\gamma(t) < 0$ and zero when $\gamma(t) = 0$. □

This criterion is local. Failure of global rank-one minimization away from balance is already known for permanents; see [4, Section 6] and [9]. The following small example shows that even strict local positivity does not provide a global theorem.

Proposition 6.4. *In degree three there are strictly positive stable product polynomials with $\gamma(t) > 0$ but $\mathcal{A}(p) < F(t)$. Moreover, any universal open gradient neighborhood $\|t - \mathbf{1}\|_1 < \varepsilon$ on which the power is always the unique global minimizer must have $\varepsilon \le 1$.*

Proof. Let $0 < a < 1/2$, $t = (1 + 2a, 1 - a, 1 - a)$, $L = L_t$ and $b^T x = -2x_1 + x_2 + x_3$. Set

$$p_{a,e} = (L - e b^T x)^2 (L + 2e b^T x).$$

All three factors have coefficient sum one, and their gradients sum to t. Their coefficients are strictly positive if $0 < e < \min\{(1 - a)/3, (1 + 2a)/12\}$. Direct expansion gives

$$p_{a,e} = L^3 - 3e^2 L (b^T x)^2 + 2e^3 (b^T x)^3, \qquad \mathcal{A}(p_{a,e}) - F(t) = 6e^2(1 - 2a - 4e).$$

For every $1/4 < a < 1/2$, the interval

$$(1 - 2a)/4 < e < (1 + 2a)/12$$

is nonempty and satisfies the positivity constraints, while the displayed gap is negative. Yet $\|t\|_2^2 = 3 + 6a^2 < 9/2$, so $\gamma(t) > 0$.

For example, $a = 1/3, e = 1/8$ gives categorical factor rows

$$(29/36, 7/72, 7/72), \quad (29/36, 7/72, 7/72), \quad (1/18, 17/36, 17/36).$$

Here $\gamma = 5/11$, $A_2 = 1/32$, the cubic contribution is $-3/64$, and the total gap is $-1/64$. At $a = 1/4, e = 1/8$, allowing one zero factor coefficient, the rows are $(3/4, 1/8, 1/8)$ twice and $(0, 1/2, 1/2)$. This nonpower has the same coefficient as L_t^3 and $\|t - \mathbf{1}\|_1 = 1$. Thus every open uniqueness radius greater than one contains a counterexample. Strictly positive failures also approach this gradient distance from above. □

7 Limits of extension to the Lorentzian class

Real stability cannot be replaced indiscriminately by Lorentzianity in local geometry or in degree-uniform gradient assertions. We give two explicit obstructions. Neither is a counterexample to the real-stable results above.

We use the closed Lorentzian class with nonnegative coefficients of Brändén and Huh [1]. For degree $d \geq 2$, its characterization requires M-convex support and that every coordinate derivative of order $d-2$ have a Hessian with at most one positive eigenvalue. Here M-convex means the exchange condition: for support vectors α, β and $\alpha_i > \beta_i$, some j with $\alpha_j < \beta_j$ has both $\alpha - e_i + e_j$ and $\beta - e_j + e_i$ in the support. In degree one, the class consists of nonnegative linear forms. Singular quadratic derivatives and boundary coefficients are included. Nonnegative linear substitutions preserve this closed class [1, Theorem 2.10].

For completeness, the binary test needed here is elementary from that characterization. If $f(X, Y) = \sum_{k=0}^{d} \binom{d}{k} a_k X^k Y^{d-k}$, with $a_k \geq 0$ and interval support, its support is M-convex. The derivative $\partial_X^j \partial_Y^{d-2-j} f$ is

$$\frac{d!}{2}(a_j Y^2 + 2a_{j+1} XY + a_{j+2} X^2).$$

Its Hessian is $d!$ times the symmetric matrix with diagonal a_{j+2}, a_j and off-diagonal a_{j+1}. Its trace is nonnegative, so it has at most one positive eigenvalue exactly when $a_{j+1}^2 \geq a_j a_{j+2}$. Thus interval-supported normalized log-concave binary coefficients give precisely the binary Lorentzian criterion used below. Truncated binomial-reference log-affine families and their tilts have classical localization ancestry [2]; we use them for the prescribed-gradient coefficient objective rather than claim a new family or localization principle.

Theorem 7.1 (A fixed-gradient local obstruction)**.** *Let $n \geq 3$, $t_i > 0$, $\sum_i t_i = n$. If $\max_i t_i \geq n/(n-1)$, then L_t^n is not a local minimizer of $\mathcal{A}$ in the coefficient topology on normalized Lorentzian polynomials with gradient exactly t. There are competitors with every coefficient strictly positive, converging to L_t^n, whose squarefree coefficients are strictly smaller than $F(t)$.*

Proof. Relabel a coordinate attaining the threshold as $t_1 = \tau$. Positivity of the other coordinates gives $\tau < n$. Define

$$Y(x) = \frac{\sum_{j=2}^{n} t_j x_j}{n - \tau}, \qquad q_r = \frac{\tau + (n - \tau)r}{n},$$
$$\ell_r = q_r x_1 + (1 - q_r)Y, \qquad p_r = \frac{\ell_r^n - r x_1^n}{1 - r}, \qquad 0 < r < r_0 := \frac{1}{2}(\tau/n)^n.$$

Here $r_0 < 1/2$, $0 < q_r < 1$ and $q_r^n - r > (\tau/n)^n/2 > 0$. Thus every coefficient of p_r is positive, including its only depressed coefficient, that of x_1^n. To check Lorentzianity, first consider the binary polynomial $[(q_rX + (1-q_r)Y)^n - rX^n]/(1-r)$. Its binomially normalized coefficients are

$$a_k = \frac{q_r^k(1-q_r)^{n-k}}{1-r} \ (k < n), \qquad a_n = \frac{q_r^n - r}{1-r} > 0.$$

All normalized log-concavity inequalities except the last are equalities; the last has margin

$$a_{n-1}^2 - a_{n-2}a_n = \frac{rq_r^{n-2}(1-q_r)^2}{(1-r)^2} > 0.$$

The binary test and nonnegative substitution prove that p_r is Lorentzian. Positive coefficients do not assert that it is an interior point of the Lorentzian cone.

Since $\ell_r(\mathbf{1}) = 1$, one has $p_r(\mathbf{1}) = 1$, and differentiation gives

$$\partial_1 p_r(\mathbf{1}) = \frac{nq_r - nr}{1-r} = \tau, \qquad \partial_j p_r(\mathbf{1}) = \frac{n(1-q_r)t_j}{(n-\tau)(1-r)} = t_j \quad (j \geq 2).$$

Thus the gradient is fixed exactly. Only the term with x_1 exponent one contributes to the squarefree coefficient; the subtracted pure monomial contributes zero. Because $1 - q_r = (1 - \tau/n)(1-r)$, extraction yields

$$\frac{\mathcal{A}(p_r)}{F(t)} = (1+ar)(1-r)^{n-2}, \qquad a = \frac{n-\tau}{\tau} \leq n-2. \tag{21}$$

Put $m = n-2$. For $R(r) = (1+ar)(1-r)^m$,

$$\frac{R'(r)}{R(r)} = \frac{a - m - a(m+1)r}{(1+ar)(1-r)} < 0 \qquad (r > 0).$$

This includes the endpoint $a = m$. Since $R(0) = 1$, all these competitors have smaller coefficient. Their coefficients converge to those of L_t^n as $r \downarrow 0$, proving the local assertion. □

Corollary 7.2 (No degree-uniform Lorentzian radius)**.** *Let ρ_n be the supremum of radii ρ for which every positive t of sum n and $\|t - \mathbf{1}\|_1 < \rho$ has L_t^n as a global minimizer on the normalized Lorentzian class. Then*

$$\rho_n \leq \frac{2}{n-1} \qquad (n \geq 3).$$

In particular no positive radius independent of degree can guarantee this minimization. Requiring uniqueness or a positive reserve cannot increase the allowable radius.

Proof. Use the positive gradient

$$t_1 = \frac{n}{n-1}, \qquad t_j = \frac{n(n-2)}{(n-1)^2} \quad (j \geq 2).$$

Its ℓ^1 distance from $\mathbf{1}$ is $2/(n-1)$, and Theorem 7.1 gives failure at that gradient. Every open radius larger than this distance contains it; a closed radius equal to the distance already fails. Letting n grow proves the degree-uniform negative conclusion. □

This is a distinction between feasible classes. At the displayed endpoint gradient,

$$\|t - \mathbf{1}\|_2^2 = \frac{n}{(n-1)^3}, \qquad \gamma(t) = \frac{(n-1)(n-2)}{n^2 - 3n + 3} > 0.$$

Proposition 6.3 therefore proves strict local minimality on the real-stable class at this same gradient. The Lorentzian competitors are not real stable: after setting $X = 1$ in the binary polynomial, its roots are

$$Y = \frac{r^{1/n}\zeta - q_r}{1 - q_r}, \qquad \zeta^n = 1.$$

For $n \geq 3$ some are nonreal. A stable polynomial's nonzero real univariate specialization is real-rooted. If p_r were stable, identifying $x_2 = \cdots = x_n = Y$ would first give a stable binary polynomial, a contradiction. At the threshold, the ratio in (21) is $1 - (n-2)(n-1)r^2/2 + O_n(r^3)$; the exact derivative proof, rather than this expansion, establishes the endpoint case. For example $n = 3, r = 1/32$ gives $t = (3/2, 3/4, 3/4)$ and gap $-3/16384$.

Proposition 7.3 (A smaller cubic uniqueness-radius bound)**.** *Let $z_* > 0$ be the positive root of $2z^3 - 2z - 1 = 0$, and put*

$$R_* = \frac{2(3z_*^2 - 1)}{1 + 3z_* + 3z_*^2}.$$

Any universal open gradient radius on which the normalized Lorentzian ternary-cubic class has L_t^3 as its unique global minimizer is at most $R_* < 166/223 < 1$. This is an upper bound, not an optimal-radius or lower-neighborhood assertion.

Proof. For $z > 0$ set $Z = 1 + 3z + 3z^2$ and

$$f_z(X, Y) = \frac{Y^3 + 3zXY^2 + 3z^2X^2Y}{Z}, \qquad p_z(x) = f_z\left(x_1, \frac{x_2 + x_3}{2}\right).$$

The normalized binary coefficients $(1, z, z^2, 0)/Z$ have interval support and are log-concave. Hence p_z is Lorentzian and normalized. Its count mean and squarefree coefficient are

$$\mu = \frac{3z + 6z^2}{Z}, \quad t = \left(\mu, \frac{3 - \mu}{2}, \frac{3 - \mu}{2}\right), \quad \mathcal{A}(p_z) = \frac{3z}{2Z}, \quad F(t) = \frac{\mu(3 - \mu)^2}{18}.$$

Since $\mu = 3z(1 + 2z)/Z$ and $3 - \mu = 3(1 + z)^2/Z$,

$$\mathcal{A}(p_z) - F(t) = -\frac{3z^3(2z^3 - 2z - 1)}{2Z^3}.$$

Indeed $(1+2z)(1+z)^4 - Z^2 = z^2(2z^3 - 2z - 1)$. Descartes' rule and existence give one positive root z_*, and the values at 1 and $6/5$ give $1 < z_* < 6/5$. At z_* the polynomial is a nonpower equality competitor: its pure x_1^3 coefficient is zero, whereas $t_1 > 0$. For $z > z_*$ its coefficient is strictly smaller.

For $z > 1/\sqrt{3}$ the gradient distance is

$$R(z) = \|t - \mathbf{1}\|_1 = \frac{2(3z^2 - 1)}{Z}.$$

It increases strictly, since $d\mu/d(\log z)$ is the positive variance of the nonconstant exponent law with probabilities proportional to $(1, 3z, 3z^2)$. Thus $R_* < R(6/5) = 166/223 < 1$, and every open uniqueness radius larger than R_* contains the equality competitor. An equal-coefficient nonpower either is another minimizer or the power is not minimizing at all; either possibility excludes unique minimality. For a rational strict example $z = 6/5$ gives

$$t = (306/223, 363/446, 363/446), \qquad \mathcal{A}(p_z) = 45/223,$$
$$F(t) = 2240073/11089567, \qquad \mathcal{A}(p_z) - F(t) = -2268/11089567.$$

These are Lorentzian rather than stable counterexamples. Explicitly the binary quadratic factor vanishes at $X/Y = (-3+i\sqrt{3})/(6z)$. Choose $Y = 1 + i/(2\sqrt{3})$ and $X = Y(-3+i\sqrt{3})/(6z)$. Both imaginary parts are positive, with $\operatorname{Im} X = \sqrt{3}/(12z)$; putting $x_2 = x_3 = Y$ supplies a nonstable witness. Strict failures with all coefficients positive also follow by composing with $(1-s)\mathbf{1}\mathbf{1}^T/3 + sI$, for $s < 1$ sufficiently close to one. Its entries are positive, substitution preserves Lorentzianity and normalization, and the strict gap persists by continuity. □

These obstructions do not determine the true local region below $\max_i t_i = n/(n-1)$, an optimal radius in any degree, or a matching positive degree-order bound. They also do not establish a degree-uniform negative statement for the real-stable class. Their role is to identify precise limits on extending the stable hypotheses and quantitative scope.

A Explicit stability for ternary cubics

We prove Theorem 1.2 without a determinant representation or an external tensor-norm theorem. Throughout this section $\|t - \mathbf{1}\|_1 \le 1/8$ and $n = 3$. Lemma 2.1 supplies the exact expansion

$$p = L^3 - Lx^T Bx + q, \qquad B \succeq 0, \quad B\mathbf{1} = 0, \tag{22}$$

where q is a homogeneous cubic invariant under translation by $\mathbf{1}$. Each centered line has three real roots, so the discriminant of $z^3 - (x^T Bx)z + q(x)$ is nonnegative. Thus

$$|q(x)| \le \frac{2}{3\sqrt{3}}(x^T Bx)^{3/2} \qquad (x \in \mathbb{R}^3). \tag{23}$$

If $d = 0$, then $p = L^3$; assume $d > 0$ below.

A.1 Two complementary coefficient bounds

Write $b_i = B_{ii}$ and $\Delta = \mathcal{A}(p) - (2/9)t_1t_2t_3$. Translation invariance and Boolean inclusion–exclusion give $\mathcal{A}(q) = 2\sum_i q(e_i)$: indeed, $q(\mathbf{1} - e_i) = q(-e_i) = -q(e_i)$ and $q(\mathbf{1}) = q(0) = 0$. The row-sum condition on B gives $B_{jk} = (b_i - b_j - b_k)/2$ for distinct indices. Consequently

$$\Delta = d - \frac{2}{3}\sum_i t_i b_i + 2\sum_i q(e_i). \tag{24}$$

The vertex coefficients are nonnegative, hence

$$0 \le p(e_i) = t_i^3/27 - t_i b_i/3 + q(e_i).$$

Substitution in (24) cancels the weighted dispersion term exactly and yields the global outer bound

$$\Delta \ge d - \frac{2}{27}\sum_i t_i^3. \tag{25}$$

For the inner bound, let $P = I - \mathbf{1}\mathbf{1}^T/3$ and $\eta = t - \mathbf{1}$. Since $B = PBP$, $b_i \le d\|Pe_i\|^2 = 2d/3$. Equation (23) therefore implies

$$|\mathcal{A}(q)| \le \frac{4}{3\sqrt{3}}\sum_i b_i^{3/2} \le \frac{4\sqrt{2}}{9}d^{3/2}.$$

On $\mathbf{1}^\perp$, the eigenvalues of $P\,\mathrm{diag}(\eta)P$ are $\pm r$, where $r = \|\eta\|_2/\sqrt{6}$. Their sum is zero and the trace of the square is $\sum_i \eta_i^2/3$, which verifies the formula. Positivity of B now gives $\sum_i t_i b_i \le (1+r)d$. Hence

$$\Delta \ge \left(\frac{1}{3} - \frac{2r}{3} - \frac{4\sqrt{2}}{9}\sqrt{d}\right) d. \tag{26}$$

Because $\sum_i \eta_i = 0$ and $\|\eta\|_1 \le 1/8$, the positive and negative masses are each at most $1/16$. It follows that

$$\max_i |\eta_i| \le 1/16, \qquad \sum_i \eta_i^2 \le 1/128, \qquad r \le 1/(16\sqrt{3}).$$

Using $\sqrt{3} > 5/3$, for $d \le 15/64$ equation (26) gives

$$\frac{\Delta}{d} \ge \frac{1}{3} - \frac{1}{24\sqrt{3}} - \frac{\sqrt{30}}{18} > \frac{37}{120} - \frac{\sqrt{30}}{18} > \frac{1}{256}.$$

The last comparison follows by subtracting $1/256$ and squaring the two positive sides. For the complementary range, the same gradient bounds give $|\sum_i \eta_i^3| \le 1/2048$, so

$$\sum_i t_i^3 = 3 + 3\sum_i \eta_i^2 + \sum_i \eta_i^3 \le 6193/2048.$$

When $d \ge 15/64$, equation (25) implies

$$\frac{\Delta}{d} \ge 1 - \frac{6193/27648}{15/64} = \frac{287}{6480} > \frac{1}{256}.$$

These two closed ranges cover every feasible $d > 0$.

A.2 An elementary coefficient-distance estimate

The quadratic form has coefficient norm at most $\sum_{i,j} |B_{ij}| \le 3d$, by positivity and Cauchy–Schwarz. Multiplication by L, whose coefficients are nonnegative and sum to one, does not increase this bound.

For the cubic term, set $a_i = q(e_i)$, let c_{ij} denote the coefficient of $x_i^2 x_j$ in q, and let c be its squarefree coefficient. Translation invariance gives

$$c = 2\sum_i a_i, \qquad c_{ij} + c_{ji} = -\sum_i a_i.$$

Evaluation at $e_i - e_j$ then yields, for distinct i, j, k,

$$c_{ij} = -a_j - \tfrac{1}{2}a_k - \tfrac{1}{2}q(e_i - e_j).$$

Summing the absolute values of these ten coefficients proves

$$\|q\|_{\mathrm{coef},1} \le 6\sum_i |a_i| + \sum_{i<j} |q(e_i - e_j)|.$$

Equation (23), $b_i \le 2d/3$, and $(e_i - e_j)^T B(e_i - e_j) \le 2d$ imply

$$\|q\|_{\mathrm{coef},1} \le \left(\frac{8\sqrt{2}}{3} + 4\sqrt{\frac{2}{3}}\right) d^{3/2} < 8d^{3/2}. \tag{27}$$

For example, the final strict bound follows from $\sqrt{2} < 3/2$ and $\sqrt{3} > 5/3$. Nonnegative coefficients also imply $\partial_i^2 p(\mathbf{1}) \geq 0$, whence

$$d \leq \frac{\|t\|_2^2}{3} \leq \frac{385}{384} < \left(\frac{129}{128}\right)^2.$$

Combining (22) and (27),

$$\left\|p - L^3\right\|_{\mathrm{coef},1} \leq 3d + 8d^{3/2} < \left(3 + 8\frac{129}{128}\right) d < 14d.$$

Together with $\Delta \geq d/256$, this proves Theorem 1.2. No invertibility of B was used. The displayed coefficient identities and rational comparisons complete the analytic proof; no numerical stability test is a premise.

B Illustrative quartics and the role of stability

For $n = 4$, the expansion is

$$p = L_t^4 - L_t^2 x^T B x + L_t q_3 + q_4.$$

At $t = \mathbf{1}$, the leading squarefree contribution is $d/8$. The remaining contributions are $O(d^{3/2})$ and $O(d^2)$; the constants depend only on degree. Thus the argument controls actual stable quartics without choosing a smooth parameterization of their cone.

For an example outside the determinant class, consider the specialized Vámos polynomial from [8, Section 3]:

$$h(x) = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4)e_3(x), \qquad h(\mathbf{1}) = 65,$$

where $e_3 = \sum_{i<j<k} x_i x_j x_k$. Its real stability (inherited from the stable Vámos matroid polynomial by specialization) and the obstruction to a definite determinantal representation of any positive integer power are prior results; see [10, 8]. Let J be the all-ones matrix and define

$$T_\delta = (1-\delta)J/4 + \delta I, \qquad p_\delta(x) = h(T_\delta x)/65, \qquad 0 < \delta \leq 1.$$

Every row of T_δ is nonnegative with sum one, so it maps vectors with positive imaginary parts to such vectors; hence p_δ is stable. The transformation is invertible and fixes $\mathbf{1}$. A definite determinantal representation of a positive power would pull back to one for a power of h, so the obstruction persists. Its gradient is

$$t_\delta = (1 + \delta/65, 1 + \delta/65, 1 - \delta/65, 1 - \delta/65).$$

Direct expansion gives

$$\begin{aligned} \mathcal{A}(p_\delta) &= \frac{3}{32} + \frac{19}{208}\delta^2 + \frac{3}{52}\delta^3 + \frac{7}{2080}\delta^4, \\ \mathcal{A}(p_\delta) - F(t_\delta) &= \frac{3089}{33800}\delta^2 + \frac{3}{52}\delta^3 + \frac{480593}{142805000}\delta^4 > 0. \end{aligned}$$

These are illustrative identities.

Stability cannot simply be omitted. Put $L = (x_1 + x_2 + x_3 + x_4)/4$. For $0 < r < 3/32$, the polynomial

$$q_r = \frac{L^4 - r x_1 x_2 x_3 x_4}{1 - r}$$

has nonnegative coefficients, value one at $\mathbf{1}$, and gradient $\mathbf{1}$, but $\mathcal{A}(q_r) = (3/32 - r)/(1 - r) < 3/32$. Under $x = (z+1, z-1, z, z)$ it becomes $z^4 + rz^2/(1-r)$, which has a nonreal upper-half-plane root. All four corresponding coordinates have positive imaginary part; thus q_r is not stable.

C An intrinsic-coefficient obstruction and exact checks

Nonnegative Bernstein *deficits* in Lemma 3.2 do not mean every intrinsic q_k has nonnegative squarefree coefficient. For an exact quartic example, let J_4 have entries $1/4$ and put

$$E = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix}, \qquad A = J_4 + E/20, \qquad p_A(x) = \prod_{i=1}^{4} \sum_j A_{ij} x_j.$$

The matrix A is positive and doubly stochastic, so p_A satisfies the balanced hypotheses. Its intrinsic degree-four term has squarefree coefficient $\mathrm{per}(E)/20^4 = -1/8000$, since $\mathrm{per}(E) = -20$. Thus intrinsic quartic positivity cannot replace the derivative/boundary argument.

All finite identities used in the proof are displayed and derived in this article. Illustrative product and Vámos calculations corroborate the normalizations; they are not a finite proof of the universal stable class or a stability-testing algorithm.

Real-rooted line restrictions are essential; the theorem applies to the stated real-stable class. The constants are sufficient and coarse. Replacing the coefficient budget W_n or the boundary-gap threshold by a sharper dispersion-sensitive payment is a natural remaining quantitative problem.

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Data availability

All mathematical arguments supporting the results are contained in this article.

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