Prescribed-gradient minimizers and coefficient stability for homogeneous real-stable polynomials

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Abstract

For each fixed degree n, we prove that among homogeneous real-stable polynomials in n variables with nonnegative coefficients, value one at the all-ones vector, and a prescribed gradient sufficiently close to the all-ones vector, the power of the corresponding linear form uniquely minimizes the squarefree coefficient. This answers Question 1.8 of Gurvits–Leake, arXiv:2609.21083v2. We strengthen the minimum statement to a uniform lower bound for the coefficient deficit by the coefficient ℓ^1 distance to that power. The distance order is optimal. We also classify strict local minimality at every positive gradient and give a cubic example separating local and global minimization. For ternary cubics we obtain the explicit closed gradient radius $1/8$ and coefficient-distance constant $1/3584$, including singular dispersion. The proof uses centered real-root dispersion, an exact coefficient identity and the classical balanced uniqueness theorem. Outside this cubic case the global neighborhood and constants remain qualitative. We also give a total-variation interpretation for stable count distributions and an illustrative nondeterminantal quartic family.

1 Introduction and main result

A nonzero polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is *real stable* if $p(z_1, \dots, z_n) \neq 0$ whenever $\operatorname{Im} z_i > 0$ for every i. Throughout, the polynomials under consideration have nonnegative coefficients and are homogeneous of degree n in n variables. Write $\mathbf{1} = (1, \dots, 1)^T$ and

$$\mathcal{A}(p) = [x_1 \cdots x_n]p, \qquad \|p\|_{\mathrm{coef},1} = \sum_{|\alpha|=n} |[x^\alpha]p|.$$

For $t \in \mathbb{R}^n_{>0}$ with $\sum_i t_i = n$, set

$$L_t(x) = \frac{t^T x}{n}, \qquad F(t) = \mathcal{A}(L_t^n) = \frac{n!}{n^n} \prod_{i=1}^n t_i.$$

The balanced coefficient inequality and its equality classification are due to Gurvits [1]; we use the following formulation, also recalled in Theorem 1.3 of Gurvits–Leake [2].

Theorem 1.1 (Balanced uniqueness)**.** *If p is homogeneous of degree n, real stable, has nonnegative coefficients, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbf{1}$, then $\mathcal{A}(p) \geq n!/n^n$, with equality if and only if $p = L_{\mathbf{1}}^n$.*

Theorem 1.1 is a prior result, not a contribution of this paper. The upper-half-plane convention above and the right-half-plane convention in [1] are equivalent for homogeneous polynomials by multiplication of every variable by i. To relate the formulation directly to the coefficient–capacity inequality and its equality case in [1, Theorem 5.7], write $p(x) = \sum_\alpha c_\alpha x^\alpha$. Here $\sum_\alpha c_\alpha = 1$ and $\sum_\alpha c_\alpha \alpha = \mathbf{1}$. Weighted AM–GM gives $p(x) \geq \prod_i x_i$ for $x_i > 0$, so $\operatorname{Cap}(p) = \inf_{x>0} p(x)/\prod_i x_i = 1$. The equality classification in that theorem forces a power of a positive linear form; normalization and the balanced gradient fix the form to $L_{\mathbf{1}}$.

Theorem 1.2. *For each fixed integer $n \geq 1$, there exist constants $\varepsilon_n, c_n > 0$ such that the following holds. Suppose p is homogeneous of degree n in n variables, real stable, has nonnegative real coefficients, $p(\mathbf{1}) = 1$, and $t = \nabla p(\mathbf{1})$ satisfies $\|t - \mathbf{1}\|_1 < \varepsilon_n$. Then*

$$\mathcal{A}(p) - F(t) \;\geq\; c_n \left\| p - L_t^n \right\|_{\mathrm{coef},1} . \tag{1}$$

In particular, L_t^n is the unique minimizer of $\mathcal{A}$ among polynomials with this normalization and prescribed gradient.

Euler's identity gives $\sum_i t_i = n$; the neighborhood will be chosen so that every $t_i > 0$. The theorem includes all feasible polynomials with such a gradient, not only polynomials in a chosen coefficient neighborhood of L_t^n. The constants may depend on n. Their general-degree global values are qualitative; the following theorem evaluates them in degree three, including the closed endpoint.

Theorem 1.3 (Explicit cubic stability)**.** *Let p be a homogeneous real-stable cubic in three variables with nonnegative coefficients and $p(\mathbf{1}) = 1$. Set*

$$t = \nabla p(\mathbf{1}), \qquad L = L_t, \qquad B = tt^T/3 - \nabla^2 p(\mathbf{1})/2, \qquad d = \operatorname{Tr} B.$$

If $\|t - \mathbf{1}\|_1 \leq 1/8$, then

$$\mathcal{A}(p) - \frac{2}{9} t_1 t_2 t_3 \;\geq\; \frac{d}{256} \;\geq\; \frac{1}{3584} \left\| p - L^3 \right\|_{\mathrm{coef},1} . \tag{2}$$

The statement includes singular B. In particular, the power is the unique minimizer throughout this closed gradient ball.

Its proof in Section 5 combines an intrinsic cubic discriminant estimate with a vertex-coefficient lower bound. These estimates overlap over the entire feasible dispersion range; no unevaluated outer compactness gap is needed in this case.

Gurvits–Klein–Leake [3, Section 2.1] prove a nearbalanced permanent minimization result and explicitly leave the real-stable extension unproved. Gurvits–Leake [2, Question 1.8] ask for the prescribed-gradient conclusion of Theorem 1.2. Their balanced uniqueness result for the larger Lorentzian class does not supply the real-root estimates used below. Productization [4] gives important capacity bounds and gradient-preserving products matching an evaluation at a fixed positive point. That product may depend on the evaluation point; the result does not directly identify the squarefree-coefficient minimizer considered here.

The earlier mixed-discriminant result [8] treats the matrix subclass and answers Question 1.7 of [2]. Indeed, $p(x) = \det(\sum_i x_i X_i)$, for positive semidefinite Hermitian matrices with $\sum_i X_i = I$, has $p(\mathbf{1}) = 1$ and $\partial_i p(\mathbf{1}) = \operatorname{Tr} X_i$. The present argument does not require a determinant representation. Such a representation cannot be assumed for general stable polynomials [6, 7].

Our main step is to use the variance of centered real roots as a single size parameter controlling all higher centered coefficients. Hyperbolic eigenvalue methods are classical; see, for example, Bauschke–Güler–Lewis–Sendov [5]. Here they yield a positive leading payment for the squarefree coefficient under a perturbed gradient. Compactness and Theorem 1.1 then extend the local estimate over the entire feasible class with nearbalanced gradient.

2 Centered roots and the dispersion matrix

For the rest of the proof let $n \geq 2$, let p satisfy the normalization and stability hypotheses, and put $t = \nabla p(\mathbf{1})$. For real x, define $y = x - L_t(x)\mathbf{1}$. Then $t^T y = 0$.

Lemma 2.1. *There are homogeneous polynomials q_k of degree k, for $2 \le k \le n$, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^{n} L_t(x)^{n-k} q_k(x). \tag{3}$$

There is a real symmetric positive semidefinite matrix B with $B\mathbf{1} = 0$ such that

$$q_2(x) = -x^T B x, \qquad B = \frac{n-1}{2n} t t^T - \frac{1}{2}\nabla^2 p(\mathbf{1}). \tag{4}$$

Writing $d = \operatorname{Tr} B$, we have

$$|q_k(x)| \le \binom{n}{k} (2x^T B x)^{k/2} \le \binom{n}{k} (2d\|x\|_2^2)^{k/2}. \tag{5}$$

Moreover, $d = 0$ if and only if $p = L_t^n$.

Proof. Expand $p(z\mathbf{1} + y)$ in z. Homogeneity and $p(\mathbf{1}) = 1$ make it monic of degree n. The coefficient of z^{n-1} is $t^T y = 0$, so

$$p(z\mathbf{1} + y) = z^n + \sum_{k=2}^{n} q_k(x) z^{n-k}.$$

The map $x \mapsto y$ is unchanged by adding $a\mathbf{1}$ to x, since $L_t(\mathbf{1}) = 1$. This proves the stated invariance. Substitute $z = L_t(x)$ to obtain (3).

For every real y, all roots of $z \mapsto p(z\mathbf{1} + y)$ are real. An upper-half-plane root would contradict stability, and a lower-half-plane nonreal root would have an upper-half-plane conjugate. Let these roots be $\lambda_1, \dots, \lambda_n$. Their sum is zero. Hence

$$q_2(x) = \sum_{i<j} \lambda_i \lambda_j = -\frac{1}{2} \sum_i \lambda_i^2 \le 0.$$

Thus $q_2 = -x^T B x$ for a unique symmetric $B \succeq 0$; invariance gives $B\mathbf{1} = 0$. To check its formula, put $H = \nabla^2 p(\mathbf{1})$. Euler's identities give $H\mathbf{1} = (n-1)t$ and $\mathbf{1}^T H \mathbf{1} = n(n-1)$. The coefficient of z^{n-2} is $\frac{1}{2} y^T H y = \frac{1}{2} x^T H x - \frac{n-1}{2n}(t^T x)^2$, which is (4).

Each $|\lambda_i| \le (2x^T B x)^{1/2}$. The elementary-symmetric root formulas give (5); positivity of B implies $x^T B x \le d\|x\|_2^2$. If $d = 0$, then $B = 0$, all the roots vanish for every real x, and all q_k vanish. Equation (3) gives $p = L_t^n$. The converse follows directly. □

3 A uniform local coefficient estimate

Lemma 3.1. *Suppose $t_i > 0$, put $v_i = 1/t_i$ and $D = \operatorname{diag}(v)$. Then*

$$\mathcal{A}(L_t^{n-2} q_2) = \frac{(n-2)! \prod_i t_i}{n^{n-2}} \left(\operatorname{Tr}(D^2 B) - v^T B v \right). \tag{6}$$

If $\|t - \mathbf{1}\|_\infty \le 1/(8\sqrt{n})$, then

$$\mathcal{A}(L_t^{n-2} q_2) \ge a_n d, \qquad a_n = \frac{(n-2)!(7/8)^n}{2n^{n-2}} > 0. \tag{7}$$

Proof. In $-x^T B x$, a diagonal term cannot contribute to a squarefree monomial. The ordered pair $i \ne j$ contributes

$$-B_{ij} \frac{(n-2)!}{n^{n-2}} \prod_{\ell \ne i,j} t_\ell.$$

Summing these terms proves (6). Since $B\mathbf{1} = 0$,

$$v^T Bv = (v-\mathbf{1})^T B(v-\mathbf{1}) \le d\|v-\mathbf{1}\|_2^2, \qquad \mathrm{Tr}(D^2 B) \ge (\min_i v_i^2)d.$$

Under the stated neighborhood condition, $7/8 \le t_i \le 9/8$, so $\min_i v_i^2 \ge 64/81$. Also

$$|v_i - 1| \le \frac{1/(8\sqrt{n})}{1 - 1/(8\sqrt{n})} \le \frac{1}{7\sqrt{n}}, \qquad \|v-\mathbf{1}\|_2^2 \le \frac{1}{49}.$$

The difference $64/81 - 1/49$ exceeds $1/2$, and $\prod_i t_i \ge (7/8)^n$. This proves (7). □

Lemma 3.2. *There is $\eta_n \in (0,1]$, depending only on n, such that*

$$\mathcal{A}(p) - F(t) \ge \frac{a_n}{2} d$$

whenever $\|t-\mathbf{1}\|_\infty \le 1/(8\sqrt{n})$ and $d \le \eta_n$.

Proof. For every homogeneous polynomial R of degree n,

$$\mathcal{A}(R) = \sum_{S \subseteq [n]} (-1)^{n-|S|} R(\mathbf{1}_S). \tag{8}$$

Indeed, a monomial survives the alternating sum exactly when all variables occur; degree n then makes it squarefree. Here $\mathbf{1}_S$ is the indicator vector of S.

Nonnegativity of t and $\sum_i t_i = n$ imply $0 \le L_t(\mathbf{1}_S) \le 1$, while $\mathbf{1}_S^T B \mathbf{1}_S \le nd$. Thus, for $k \ge 3$, Lemma 2.1 and (8) give

$$|\mathcal{A}(L_t^{n-k} q_k)| \le 2^n \binom{n}{k} (2nd)^{k/2}.$$

For $0 \le d \le 1$, the sum is at most $C_n d^{3/2}$, where

$$C_n = 2^n \sum_{k=3}^{n} \binom{n}{k} (2n)^{k/2}.$$

For $n = 2$, the sum is empty and $C_2 = 0$. Choose $\eta_2 = 1$, and for $n \ge 3$ take $\eta_n = \min\{1, (a_n/(2C_n))^2\}$. Combining (3) and Lemma 3.1 proves the assertion. □

4 Compactness and the global deficit

Let $\mathcal{K}_n$ be the set of homogeneous degree-n real-stable polynomials with nonnegative coefficients and $p(\mathbf{1}) = 1$.

Lemma 4.1. *The set $\mathcal{K}_n$ is compact in coefficient space. The functions $t(p)$, $B(p)$, $d(p)$, and $\mathcal{A}(p)$ are continuous on it, and*

$$0 \le d(p) \le M_n := \frac{n(n-1)}{2}. \tag{9}$$

Proof. The coefficients lie in a fixed finite-dimensional probability simplex. It remains to show stability is closed within that simplex. Let $p_j \to p$ coefficientwise, with $p_j \in \mathcal{K}_n$. Fix real a and strictly positive real b. The polynomial $z \mapsto p_j(a+zb)$ has only real roots by the stability and conjugation argument. Its leading coefficient satisfies

$$p_j(b) \ge (\min_i b_i)^n > 0.$$

Therefore its coefficient limit still has degree n, and closure of the real-rooted univariate polynomials shows that $z \mapsto p(a + zb)$ has only real roots. If $p(a + ib) = 0$, this polynomial would have the nonreal root i. As a, b were arbitrary, p is stable. The other defining conditions are closed, proving compactness.

Continuity follows from coefficient differentiation and (4). Nonnegative coefficients give $H_{ii} \geq 0$, where $H = \nabla^2 p(\mathbf{1})$. Also $t_i \geq 0$ and $\sum_i t_i = n$, so

$$d = \frac{n-1}{2n}\|t\|_2^2 - \frac{1}{2}\operatorname{Tr} H \leq \frac{n-1}{2n}n^2 = M_n.$$

□

Proposition 4.2. *There exist $\varepsilon_n, b_n > 0$ such that*

$$\mathcal{A}(p) - F(t) \geq b_n d(p) \tag{10}$$

for every $p \in \mathcal{K}_n$ with $\|t - \mathbf{1}\|_1 < \varepsilon_n$.

Proof. Take ε_n no larger than $1/(8\sqrt{n})$. Lemma 3.2 handles $d \leq \eta_n$. Consider the compact set

$$E = \{p \in \mathcal{K}_n : \ t(p) = \mathbf{1}, \ d(p) \geq \eta_n\}.$$

If $E \neq \varnothing$, Theorem 1.1 and Lemma 2.1 give

$$m_n := \min_{p \in E}\big(\mathcal{A}(p) - F(\mathbf{1})\big) > 0.$$

For a sufficiently small gradient neighborhood, every $p \in \mathcal{K}_n$ with $d(p) \geq \eta_n$ has $\mathcal{A}(p) - F(t) \geq m_n/2$. Otherwise a sequence with $t \to \mathbf{1}$ would have a convergent subsequence with limit in E and gap at most $m_n/2$, a contradiction. By (9), take $b_n = \min\{a_n/2, m_n/(2M_n)\}$.

If $E = \varnothing$, compactness instead implies that in a sufficiently small gradient neighborhood there is no $p \in \mathcal{K}_n$ with $d(p) \geq \eta_n$: a contrary sequence would converge to a member of E. Then $b_n = a_n/2$ suffices. □

Lemma 4.3. *There is $K_n > 0$, depending only on n, such that*

$$\left\|p - L_t^n\right\|_{\mathrm{coef},1} \leq K_n d(p) \qquad (p \in \mathcal{K}_n).$$

Proof. For real $\|x\|_2 \leq 1$, nonnegativity and $\sum_i t_i = n$ give $|L_t(x)| \leq \|t\|_2/n \leq 1$. Using the exact quadratic term in (3) yields

$$|p(x) - L_t(x)^n| \leq d + \sum_{k=3}^{n} \binom{n}{k}(2d)^{k/2} \leq K_n' d,$$

because $0 \leq d \leq M_n$. For example, one can take $K_n' = 1 + \sum_{k=3}^{n} \binom{n}{k} 2^{k/2} M_n^{k/2-1}$.

On the finite-dimensional space of real homogeneous degree-n polynomials, coefficient ℓ^1 norm is bounded by a constant times the supremum norm on the Euclidean unit ball. For completeness, on the compact coefficient-ℓ^1 unit sphere the latter norm is continuous and strictly positive: a nonzero polynomial cannot vanish on the entire ball. It therefore has a positive minimum. This proves the required norm comparison, and hence the lemma. □

Proof of Theorem 1.2. For $n \geq 2$, combine Proposition 4.2 and Lemma 4.3, taking $c_n = b_n/K_n$. The power L_t^n is stable, has nonnegative coefficients, value one at $\mathbf{1}$, and gradient t. Equality in the minimum forces $\left\|p - L_t^n\right\|_{\mathrm{coef},1} = 0$. For $n = 1$, normalization forces $p = x_1$, so the theorem holds with any positive constants. □

5 Explicit whole-class stability for ternary cubics

We prove Theorem 1.3 without a determinant representation or an external tensor-norm theorem. Throughout this section $\|t - \mathbf{1}\|_1 \le 1/8$ and $n = 3$. Lemma 2.1 supplies the exact expansion

$$p = L^3 - Lx^T Bx + q, \qquad B \succeq 0, \quad B\mathbf{1} = 0, \tag{11}$$

where q is a homogeneous cubic invariant under translation by $\mathbf{1}$. Each centered line has three real roots, so the discriminant of $z^3 - (x^T Bx)z + q(x)$ is nonnegative. Thus

$$|q(x)| \le \frac{2}{3\sqrt{3}}(x^T Bx)^{3/2} \qquad (x \in \mathbb{R}^3). \tag{12}$$

If $d = 0$, then $p = L^3$; assume $d > 0$ below.

5.1 Two complementary coefficient bounds

Write $b_i = B_{ii}$ and $\Delta = \mathcal{A}(p) - (2/9)t_1t_2t_3$. Translation invariance and Boolean inclusion–exclusion give $\mathcal{A}(q) = 2\sum_i q(e_i)$: indeed, $q(\mathbf{1} - e_i) = q(-e_i) = -q(e_i)$ and $q(\mathbf{1}) = q(0) = 0$. The row-sum condition on B gives $B_{jk} = (b_i - b_j - b_k)/2$ for distinct indices. Consequently

$$\Delta = d - \frac{2}{3}\sum_i t_i b_i + 2\sum_i q(e_i). \tag{13}$$

The vertex coefficients are nonnegative, hence

$$0 \le p(e_i) = t_i^3/27 - t_i b_i/3 + q(e_i).$$

Substitution in (13) cancels the weighted dispersion term exactly and yields the global outer bound

$$\Delta \ge d - \frac{2}{27}\sum_i t_i^3. \tag{14}$$

For the inner bound, let $P = I - \mathbf{1}\mathbf{1}^T/3$ and $\eta = t - \mathbf{1}$. Since $B = PBP$, $b_i \le d\|Pe_i\|^2 = 2d/3$. Equation (12) therefore implies

$$|\mathcal{A}(q)| \le \frac{4}{3\sqrt{3}}\sum_i b_i^{3/2} \le \frac{4\sqrt{2}}{9}d^{3/2}.$$

On $\mathbf{1}^\perp$, the eigenvalues of $P\,\mathrm{diag}(\eta)P$ are $\pm r$, where $r = \|\eta\|_2/\sqrt{6}$. Their sum is zero and the trace of the square is $\sum_i \eta_i^2/3$, which verifies the formula. Positivity of B now gives $\sum_i t_i b_i \le (1 + r)d$. Hence

$$\Delta \ge \left(\frac{1}{3} - \frac{2r}{3} - \frac{4\sqrt{2}}{9}\sqrt{d}\right)d. \tag{15}$$

Because $\sum_i \eta_i = 0$ and $\|\eta\|_1 \le 1/8$, the positive and negative masses are each at most $1/16$. It follows that

$$\max_i |\eta_i| \le 1/16, \qquad \sum_i \eta_i^2 \le 1/128, \qquad r \le 1/(16\sqrt{3}).$$

Using $\sqrt{3} > 5/3$, for $d \le 15/64$ equation (15) gives

$$\frac{\Delta}{d} \ge \frac{1}{3} - \frac{1}{24\sqrt{3}} - \frac{\sqrt{30}}{18} > \frac{37}{120} - \frac{\sqrt{30}}{18} > \frac{1}{256}.$$

The last comparison follows by subtracting $1/256$ and squaring the two positive sides. For the complementary range, the same gradient bounds give $|\sum_i \eta_i^3| \le 1/2048$, so

$$\sum_i t_i^3 = 3 + 3\sum_i \eta_i^2 + \sum_i \eta_i^3 \le 6193/2048.$$

When $d \ge 15/64$, equation (14) implies

$$\frac{\Delta}{d} \ge 1 - \frac{6193/27648}{15/64} = \frac{287}{6480} > \frac{1}{256}.$$

These two closed ranges cover every feasible $d > 0$.

5.2 An elementary coefficient-distance estimate

The quadratic form has coefficient norm at most $\sum_{i,j} |B_{ij}| \le 3d$, by positivity and Cauchy–Schwarz. Multiplication by L, whose coefficients are nonnegative and sum to one, does not increase this bound.

For the cubic term, set $a_i = q(e_i)$, let c_{ij} denote the coefficient of $x_i^2 x_j$ in q, and let c be its squarefree coefficient. Translation invariance gives

$$c = 2\sum_i a_i, \qquad c_{ij} + c_{ji} = -\sum_i a_i.$$

Evaluation at $e_i - e_j$ then yields, for distinct i, j, k,

$$c_{ij} = -a_j - \tfrac{1}{2}a_k - \tfrac{1}{2}q(e_i - e_j).$$

Summing the absolute values of these ten coefficients proves

$$\|q\|_{\mathrm{coef},1} \le 6\sum_i |a_i| + \sum_{i<j} |q(e_i - e_j)|.$$

Equation (12), $b_i \le 2d/3$, and $(e_i - e_j)^T B(e_i - e_j) \le 2d$ imply

$$\|q\|_{\mathrm{coef},1} \le \left(\frac{8\sqrt{2}}{3} + 4\sqrt{\frac{2}{3}}\right) d^{3/2} < 8d^{3/2}. \tag{16}$$

For example, the final strict bound follows from $\sqrt{2} < 3/2$ and $\sqrt{3} > 5/3$. Nonnegative coefficients also imply $\partial_i^2 p(\mathbf{1}) \ge 0$, whence

$$d \le \frac{\|t\|_2^2}{3} \le \frac{385}{384} < \left(\frac{129}{128}\right)^2.$$

Combining (11) and (16),

$$\left\|p - L^3\right\|_{\mathrm{coef},1} \le 3d + 8d^{3/2} < \left(3 + 8\frac{129}{128}\right) d < 14d.$$

Together with $\Delta \ge d/256$, this proves Theorem 1.3. No invertibility of B was used. The exact coefficient identities and scalar comparisons are reproduced by the optional standard-library checker `verify_cubic_constants.py`; the universal stable-domain argument is the analytic proof above.

6 Optimal distance order and a probability formulation

Proposition 6.1. *For every $n \geq 2$, no positive uniform lower bound by $\left\|p - L_t^n\right\|_{\mathrm{coef},1}^{\alpha}$ with $\alpha < 1$ can replace the right side of* (1)*, even when $t = \mathbf{1}$.*

Proof. Put $L = L_{\mathbf{1}}$, $a = x_1 - x_2$, and, for $0 < s < 1/n$, let

$$p_s = L^{n-2}(L + sa)(L - sa) = L^n - s^2 L^{n-2} a^2.$$

Every linear factor has nonnegative coefficients and positive coefficient sum, so the product is stable. Moreover, $p_s(\mathbf{1}) = 1$ and $\nabla p_s(\mathbf{1}) = \mathbf{1}$. Only the term $-2x_1x_2$ in a^2 can contribute to the squarefree coefficient, giving

$$\mathcal{A}(p_s) - \mathcal{A}(L^n) = \frac{2(n-2)!}{n^{n-2}} s^2, \qquad \|p_s - L^n\|_{\mathrm{coef},1} = s^2 \left\|L^{n-2} a^2\right\|_{\mathrm{coef},1} > 0.$$

The ratio of the gap to this distance raised to $\alpha < 1$ tends to zero. This proves the assertion. □

This establishes optimal order, not the best constant in Theorem 1.2.

Corollary 6.2. *Let N take values in $\{\alpha \in \mathbb{Z}_{\geq 0}^n : \sum_i \alpha_i = n\}$, and suppose its probability generating polynomial $p(x) = \mathbb{E}[x^N]$ is real stable. Put $t = \mathbb{E}N$, and let M_t have the multinomial count law for n independent draws with probabilities t_i/n. Under the gradient condition of Theorem 1.2,*

$$\mathbb{P}(N = \mathbf{1}) - \frac{n!}{n^n} \prod_i t_i \geq 2c_n d_{\mathrm{TV}}(\mathcal{L}(N), \mathcal{L}(M_t)).$$

Proof. The coefficients of p are the probabilities of N, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbb{E}N = t$. The generating polynomial of M_t is L_t^n. Coefficient ℓ^1 distance is twice total variation, where $d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_\alpha |\mu(\alpha) - \nu(\alpha)|$. The result is precisely (1). □

The unique minimizing object here is a count law, not an underlying labeled sampling mechanism. Differentiation gives a further interpretation of the dispersion matrix:

$$2B = \mathrm{diag}(t) - \frac{tt^T}{n} - \mathrm{Cov}(N) = \mathrm{Cov}(M_t) - \mathrm{Cov}(N). \tag{17}$$

Indeed, $\nabla^2 p(\mathbf{1}) = \mathrm{Cov}(N) + tt^T - \mathrm{diag}(t)$; substitution into (4) proves (17). The coefficient comparison supplies the stated inequality.

7 Exact local geometry and a global obstruction

The nearbalanced global conclusion does not identify the full gradient region in which the power is a local minimizer. We now classify strict local minimality in the entire stable-polynomial class. The corresponding matrix Hessian criterion already appears in [8]; the point here is that centered-root control suffices without a matrix representation.

Proposition 7.1. *Fix $n \geq 2$ and $t_i > 0$ with $\sum_i t_i = n$, and set*

$$\gamma(t) = \frac{n^2}{\|t\|_2^2} - (n-1).$$

In the coefficient topology on normalized nonnegative homogeneous real-stable polynomials with gradient t, the power L_t^n is a strict local minimizer of $\mathcal{A}$ if and only if $\gamma(t) > 0$. If $\gamma(t) < 0$, it is not a local minimizer. If $\gamma(t) = 0$, a nonconstant stable product curve through L_t^n has the same squarefree coefficient throughout.

Proof. Let $T = \operatorname{diag}(t)$ and $S = T^{-1}BT^{-1}$. Then $S \succeq 0$ and $St = 0$. Define

$$r = \mathbf{1} - \frac{nt}{\|t\|_2^2}, \qquad \|r\|_2^2 = 1 - \gamma(t).$$

The exact coefficient identity of Lemma 3.1 becomes

$$A_2 := \mathcal{A}(L_t^{n-2} q_2) = F(t)\frac{n}{n-1}(\operatorname{Tr} S - r^T S r).$$

In particular, the bracket is the negative sum of all ordered off-diagonal entries $B_{ij}/(t_i t_j)$; its diagonal cancels. Since $r^T S r \le \|r\|_2^2 \operatorname{Tr} S$,

$$A_2 \ge F(t)\frac{n}{n-1}\gamma(t)\operatorname{Tr} S \ge \alpha(t) d, \qquad \alpha(t) = \frac{F(t) n \gamma(t)}{(n-1)(\max_i t_i)^2},$$

when $\gamma(t) > 0$. The cube estimate in the proof of Lemma 3.2 required only $t_i \ge 0$ and $\sum_i t_i = n$. It therefore gives, for $d \le 1$,

$$|\mathcal{A}(p) - F(t) - A_2| \le C_n d^{3/2}.$$

For $n \ge 3$, put $\eta(t) = \min\{1, (\alpha(t)/(2C_n))^2\}$. For $n = 2$, put $\eta(t) = 1$, since the remainder vanishes. If $d \le \eta(t)$, then

$$\mathcal{A}(p) - F(t) \ge \tfrac{1}{2}\alpha(t) d.$$

The rigidity $d = 0 \iff p = L_t^n$ and continuity of d establish strict local minimality. Lemma 4.3 also gives local coefficient-distance coercivity with constant $\alpha(t)/(2K_n)$. This is an explicit local dispersion radius, not an explicit global gradient neighborhood.

For the converse, if $\gamma(t) \le 0$ then $r \ne 0$. Set $b = Tr$, so $b^T \mathbf{1} = t^T r = 0$. For sufficiently small $s > 0$, all coefficients of $L_t \pm s b^T x$ are positive. Hence

$$p_s = L_t^{n-2}(L_t + s b^T x)(L_t - s b^T x) = L_t^n - s^2 L_t^{n-2}(b^T x)^2$$

is feasible and nonconstant. Here $B = s^2 b b^T$, $S = s^2 r r^T$, and there are no higher centered terms. Consequently,

$$\mathcal{A}(p_s) - F(t) = F(t)\frac{n}{n-1} s^2 \gamma(t) \|r\|_2^2.$$

It is negative when $\gamma(t) < 0$ and zero when $\gamma(t) = 0$. □

This criterion is local. Failure of global rank-one minimization away from balance is already known for permanents; see [2, Section 6] and [3]. The following small example shows that even strict local positivity does not provide a global theorem.

Proposition 7.2. *In degree three there are strictly positive stable product polynomials with $\gamma(t) > 0$ but $\mathcal{A}(p) < F(t)$. Moreover, any universal open gradient neighborhood $\|t - \mathbf{1}\|_1 < \varepsilon$ on which the power is always the unique global minimizer must have $\varepsilon \le 1$.*

Proof. Let $0 < a < 1/2$, $t = (1 + 2a, 1 - a, 1 - a)$, $L = L_t$ and $b^T x = -2x_1 + x_2 + x_3$. Set

$$p_{a,e} = (L - e b^T x)^2 (L + 2e b^T x).$$

All three factors have coefficient sum one, and their gradients sum to t. Their coefficients are strictly positive if $0 < e < \min\{(1 - a)/3, (1 + 2a)/12\}$. Direct expansion gives

$$p_{a,e} = L^3 - 3e^2 L (b^T x)^2 + 2e^3 (b^T x)^3, \qquad \mathcal{A}(p_{a,e}) - F(t) = 6e^2(1 - 2a - 4e).$$

For every $1/4 < a < 1/2$, the interval

$$(1-2a)/4 < e < (1+2a)/12$$

is nonempty and satisfies the positivity constraints, while the displayed gap is negative. Yet $\|t\|_2^2 = 3+6a^2 < 9/2$, so $\gamma(t) > 0$.

For example, $a = 1/3$, $e = 1/8$ gives categorical factor rows

$$(29/36, 7/72, 7/72), \quad (29/36, 7/72, 7/72), \quad (1/18, 17/36, 17/36).$$

Here $\gamma = 5/11$, $A_2 = 1/32$, the cubic contribution is $-3/64$, and the total gap is $-1/64$. At $a = 1/4$, $e = 1/8$, allowing one zero factor coefficient, the rows are $(3/4, 1/8, 1/8)$ twice and $(0, 1/2, 1/2)$. This nonpower has the same coefficient as L_t^3 and $\|t - \mathbf{1}\|_1 = 1$. Thus every open uniqueness radius greater than one contains a counterexample. Strictly positive failures also approach this gradient distance from above. □

8 Quartic examples and scope

For $n = 4$, the expansion is

$$p = L_t^4 - L_t^2 x^T B x + L_t q_3 + q_4.$$

At $t = \mathbf{1}$, the leading squarefree contribution is $d/8$. The remaining contributions are $O(d^{3/2})$ and $O(d^2)$; the constants depend only on degree. Thus the argument controls actual stable quartics without choosing a smooth parameterization of their cone.

For an example outside the determinant class, consider the specialized Vámos polynomial from [7, Section 3]:

$$h(x) = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4) e_3(x), \qquad h(\mathbf{1}) = 65,$$

where $e_3 = \sum_{i<j<k} x_i x_j x_k$. Its real stability (inherited from the stable Vámos matroid polynomial by specialization) and the obstruction to a definite determinantal representation of any positive integer power are prior results; see [6, 7]. Let J be the all-ones matrix and define

$$T_\delta = (1-\delta)J/4 + \delta I, \qquad p_\delta(x) = h(T_\delta x)/65, \qquad 0 < \delta \le 1.$$

Every row of T_δ is nonnegative with sum one, so it maps vectors with positive imaginary parts to such vectors; hence p_δ is stable. The transformation is invertible and fixes $\mathbf{1}$. A definite determinantal representation of a positive power would pull back to one for a power of h, so the obstruction persists. Its gradient is

$$t_\delta = (1 + \delta/65, 1 + \delta/65, 1 - \delta/65, 1 - \delta/65).$$

Direct expansion gives

$$\mathcal{A}(p_\delta) = \frac{3}{32} + \frac{19}{208}\delta^2 + \frac{3}{52}\delta^3 + \frac{7}{2080}\delta^4,$$
$$\mathcal{A}(p_\delta) - F(t_\delta) = \frac{3089}{33800}\delta^2 + \frac{3}{52}\delta^3 + \frac{480593}{142805000}\delta^4 > 0.$$

These are illustrative identities.

Stability cannot simply be omitted. Put $L = (x_1 + x_2 + x_3 + x_4)/4$. For $0 < r < 3/32$, the polynomial

$$q_r = \frac{L^4 - r x_1 x_2 x_3 x_4}{1 - r}$$

has nonnegative coefficients, value one at $\mathbf{1}$, and gradient $\mathbf{1}$, but $\mathcal{A}(q_r) = (3/32 - r)/(1 - r) < 3/32$. Under $x = (z + 1, z - 1, z, z)$ it becomes $z^4 + rz^2/(1 - r)$, which has a nonreal upper-half-plane root. All four corresponding coordinates have positive imaginary part; thus q_r is not stable.

The proof does not extend the perturbed-gradient theorem to every Lorentzian polynomial: real-rooted line restrictions are essential to Lemma 2.1. Effective global radii outside degree three and their dimension dependence remain natural questions. The cubic constants of Theorem 1.3 do not identify the largest valid radius.

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