Prescribed-gradient minimizers and coefficient stability for homogeneous real-stable polynomials

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Abstract

For each fixed degree n, we prove that among homogeneous real-stable polynomials in n variables with nonnegative coefficients, value one at the all-ones vector, and a prescribed gradient sufficiently close to the all-ones vector, the power of the corresponding linear form uniquely minimizes the squarefree coefficient. This answers Question 1.8 of Gurvits–Leake, arXiv:2609.21083v2. We strengthen the minimum statement to a uniform lower bound for the coefficient deficit by the coefficient ℓ^1 distance to that power. The distance order is optimal. We also classify strict local minimality at every positive gradient and give a cubic example separating local and global minimization. The proof uses centered real-root dispersion, an exact coefficient identity and the classical balanced uniqueness theorem. The neighborhood and constants are dimension-dependent and qualitative. We also give a total-variation interpretation for stable count distributions and an illustrative nondeterminantal quartic family.

1 Introduction and main result

A nonzero polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is *real stable* if $p(z_1, \dots, z_n) \neq 0$ whenever $\operatorname{Im} z_i > 0$ for every i. Throughout, the polynomials under consideration have nonnegative coefficients and are homogeneous of degree n in n variables. Write $\mathbf{1} = (1, \dots, 1)^T$ and

$$\mathcal{A}(p) = [x_1 \cdots x_n]p, \qquad \|p\|_{\mathrm{coef},1} = \sum_{|\alpha|=n} |[x^\alpha]p|.$$

For $t \in \mathbb{R}^n_{>0}$ with $\sum_i t_i = n$, set

$$L_t(x) = \frac{t^T x}{n}, \qquad F(t) = \mathcal{A}(L_t^n) = \frac{n!}{n^n} \prod_{i=1}^n t_i.$$

The balanced coefficient inequality and its equality classification are due to Gurvits [1]; we use the following formulation, also recalled in Theorem 1.3 of Gurvits–Leake [2].

Theorem 1.1 (Balanced uniqueness)**.** *If p is homogeneous of degree n, real stable, has nonnegative coefficients, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbf{1}$, then $\mathcal{A}(p) \geq n!/n^n$, with equality if and only if $p = L_{\mathbf{1}}^n$.*

Theorem 1.1 is a prior result, not a contribution of this paper. The upper-half-plane convention above and the right-half-plane convention in [1] are equivalent for homogeneous polynomials by multiplication of every variable by i. To relate the formulation directly to the coefficient–capacity inequality and its equality case in [1, Theorem 5.7], write $p(x) = \sum_\alpha c_\alpha x^\alpha$. Here $\sum_\alpha c_\alpha = 1$ and $\sum_\alpha c_\alpha \alpha = \mathbf{1}$. Weighted AM–GM gives $p(x) \geq \prod_i x_i$ for $x_i > 0$, so $\operatorname{Cap}(p) = \inf_{x>0} p(x)/\prod_i x_i = 1$. The equality classification in that theorem forces a power of a positive linear form; normalization and the balanced gradient fix the form to $L_{\mathbf{1}}$.

Theorem 1.2. *For each fixed integer $n \geq 1$, there exist constants $\varepsilon_n, c_n > 0$ such that the following holds. Suppose p is homogeneous of degree n in n variables, real stable, has nonnegative real coefficients, $p(\mathbf{1}) = 1$, and $t = \nabla p(\mathbf{1})$ satisfies $\|t - \mathbf{1}\|_1 < \varepsilon_n$. Then*

$$\mathcal{A}(p) - F(t) \;\geq\; c_n \left\| p - L_t^n \right\|_{\mathrm{coef},1} . \tag{1}$$

In particular, L_t^n is the unique minimizer of $\mathcal{A}$ among polynomials with this normalization and prescribed gradient.

Euler's identity gives $\sum_i t_i = n$; the neighborhood will be chosen so that every $t_i > 0$. The theorem includes all feasible polynomials with such a gradient, not only polynomials in a chosen coefficient neighborhood of L_t^n. The constants may depend on n. No useful explicit global value of ε_n or c_n is computed here.

Gurvits–Klein–Leake [3, Section 2.1] prove a nearbalanced permanent minimization result and explicitly leave the real-stable extension unproved. Gurvits–Leake [2, Question 1.8] ask for the prescribed-gradient conclusion of Theorem 1.2. Their balanced uniqueness result for the larger Lorentzian class does not supply the real-root estimates used below. Productization [4] gives important capacity bounds and gradient-preserving products matching an evaluation at a fixed positive point. That product may depend on the evaluation point; the result does not directly identify the squarefree-coefficient minimizer considered here.

The earlier mixed-discriminant result [8] treats the matrix subclass and answers Question 1.7 of [2]. Indeed, $p(x) = \det(\sum_i x_i X_i)$, for positive semidefinite Hermitian matrices with $\sum_i X_i = I$, has $p(\mathbf{1}) = 1$ and $\partial_i p(\mathbf{1}) = \operatorname{Tr} X_i$. The present argument does not require a determinant representation. Such a representation cannot be assumed for general stable polynomials [6, 7].

Our main step is to use the variance of centered real roots as a single size parameter controlling all higher centered coefficients. Hyperbolic eigenvalue methods are classical; see, for example, Bauschke–Güler–Lewis–Sendov [5]. Here they yield a positive leading payment for the squarefree coefficient under a perturbed gradient. Compactness and Theorem 1.1 then extend the local estimate over the entire feasible class with nearbalanced gradient.

2 Centered roots and the dispersion matrix

For the rest of the proof let $n \geq 2$, let p satisfy the normalization and stability hypotheses, and put $t = \nabla p(\mathbf{1})$. For real x, define $y = x - L_t(x)\mathbf{1}$. Then $t^T y = 0$.

Lemma 2.1. *There are homogeneous polynomials q_k of degree k, for $2 \leq k \leq n$, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^{n} L_t(x)^{n-k} q_k(x). \tag{2}$$

There is a real symmetric positive semidefinite matrix B with $B\mathbf{1} = 0$ such that

$$q_2(x) = -x^T B x, \qquad B = \frac{n-1}{2n} t t^T - \frac{1}{2}\nabla^2 p(\mathbf{1}). \tag{3}$$

Writing $d = \operatorname{Tr} B$, we have

$$|q_k(x)| \leq \binom{n}{k} (2x^T B x)^{k/2} \leq \binom{n}{k} (2d\|x\|_2^2)^{k/2}. \tag{4}$$

Moreover, $d = 0$ if and only if $p = L_t^n$.

Proof. Expand $p(z\mathbf{1}+y)$ in z. Homogeneity and $p(\mathbf{1})=1$ make it monic of degree n. The coefficient of z^{n-1} is $t^Ty=0$, so

$$p(z\mathbf{1}+y)=z^n+\sum_{k=2}^{n}q_k(x)z^{n-k}.$$

The map $x\mapsto y$ is unchanged by adding $a\mathbf{1}$ to x, since $L_t(\mathbf{1})=1$. This proves the stated invariance. Substitute $z=L_t(x)$ to obtain (2).

For every real y, all roots of $z\mapsto p(z\mathbf{1}+y)$ are real. An upper-half-plane root would contradict stability, and a lower-half-plane nonreal root would have an upper-half-plane conjugate. Let these roots be $\lambda_1,\dots,\lambda_n$. Their sum is zero. Hence

$$q_2(x)=\sum_{i<j}\lambda_i\lambda_j=-\frac{1}{2}\sum_i\lambda_i^2\le 0.$$

Thus $q_2=-x^TBx$ for a unique symmetric $B\succeq 0$; invariance gives $B\mathbf{1}=0$. To check its formula, put $H=\nabla^2p(\mathbf{1})$. Euler's identities give $H\mathbf{1}=(n-1)t$ and $\mathbf{1}^TH\mathbf{1}=n(n-1)$. The coefficient of z^{n-2} is $\frac{1}{2}y^THy=\frac{1}{2}x^THx-\frac{n-1}{2n}(t^Tx)^2$, which is (3).

Each $|\lambda_i|\le(2x^TBx)^{1/2}$. The elementary-symmetric root formulas give (4); positivity of B implies $x^TBx\le d\|x\|_2^2$. If $d=0$, then $B=0$, all the roots vanish for every real x, and all q_k vanish. Equation (2) gives $p=L_t^n$. The converse follows directly. □

3 A uniform local coefficient estimate

Lemma 3.1. *Suppose $t_i>0$, put $v_i=1/t_i$ and $D=\operatorname{diag}(v)$. Then*

$$\mathcal{A}(L_t^{n-2}q_2)=\frac{(n-2)!\prod_i t_i}{n^{n-2}}\left(\operatorname{Tr}(D^2B)-v^TBv\right). \tag{5}$$

If $\|t-\mathbf{1}\|_\infty\le 1/(8\sqrt{n})$, then

$$\mathcal{A}(L_t^{n-2}q_2)\ge a_nd,\qquad a_n=\frac{(n-2)!(7/8)^n}{2n^{n-2}}>0. \tag{6}$$

Proof. In $-x^TBx$, a diagonal term cannot contribute to a squarefree monomial. The ordered pair $i\ne j$ contributes

$$-B_{ij}\frac{(n-2)!}{n^{n-2}}\prod_{\ell\ne i,j}t_\ell.$$

Summing these terms proves (5). Since $B\mathbf{1}=0$,

$$v^TBv=(v-\mathbf{1})^TB(v-\mathbf{1})\le d\|v-\mathbf{1}\|_2^2,\qquad \operatorname{Tr}(D^2B)\ge(\min_i v_i^2)d.$$

Under the stated neighborhood condition, $7/8\le t_i\le 9/8$, so $\min_i v_i^2\ge 64/81$. Also

$$|v_i-1|\le\frac{1/(8\sqrt{n})}{1-1/(8\sqrt{n})}\le\frac{1}{7\sqrt{n}},\qquad \|v-\mathbf{1}\|_2^2\le\frac{1}{49}.$$

The difference $64/81-1/49$ exceeds $1/2$, and $\prod_i t_i\ge(7/8)^n$. This proves (6). □

Lemma 3.2. *There is $\eta_n\in(0,1]$, depending only on n, such that*

$$\mathcal{A}(p)-F(t)\ge\frac{a_n}{2}d$$

whenever $\|t-\mathbf{1}\|_\infty\le 1/(8\sqrt{n})$ and $d\le\eta_n$.

Proof. For every homogeneous polynomial R of degree n,

$$\mathcal{A}(R) = \sum_{S \subseteq [n]} (-1)^{n-|S|} R(\mathbf{1}_S). \tag{7}$$

Indeed, a monomial survives the alternating sum exactly when all variables occur; degree n then makes it squarefree. Here $\mathbf{1}_S$ is the indicator vector of S.

Nonnegativity of t and $\sum_i t_i = n$ imply $0 \le L_t(\mathbf{1}_S) \le 1$, while $\mathbf{1}_S^T B \mathbf{1}_S \le nd$. Thus, for $k \ge 3$, Lemma 2.1 and (7) give

$$|\mathcal{A}(L_t^{n-k} q_k)| \le 2^n \binom{n}{k} (2nd)^{k/2}.$$

For $0 \le d \le 1$, the sum is at most $C_n d^{3/2}$, where

$$C_n = 2^n \sum_{k=3}^{n} \binom{n}{k} (2n)^{k/2}.$$

For $n = 2$, the sum is empty and $C_2 = 0$. Choose $\eta_2 = 1$, and for $n \ge 3$ take $\eta_n = \min\{1, (a_n/(2C_n))^2\}$. Combining (2) and Lemma 3.1 proves the assertion. □

4 Compactness and the global deficit

Let $\mathcal{K}_n$ be the set of homogeneous degree-n real-stable polynomials with nonnegative coefficients and $p(\mathbf{1}) = 1$.

Lemma 4.1. *The set $\mathcal{K}_n$ is compact in coefficient space. The functions $t(p)$, $B(p)$, $d(p)$, and $\mathcal{A}(p)$ are continuous on it, and*

$$0 \le d(p) \le M_n := \frac{n(n-1)}{2}. \tag{8}$$

Proof. The coefficients lie in a fixed finite-dimensional probability simplex. It remains to show stability is closed within that simplex. Let $p_j \to p$ coefficientwise, with $p_j \in \mathcal{K}_n$. Fix real a and strictly positive real b. The polynomial $z \mapsto p_j(a + zb)$ has only real roots by the stability and conjugation argument. Its leading coefficient satisfies

$$p_j(b) \ge (\min_i b_i)^n > 0.$$

Therefore its coefficient limit still has degree n, and closure of the real-rooted univariate polynomials shows that $z \mapsto p(a + zb)$ has only real roots. If $p(a + ib) = 0$, this polynomial would have the nonreal root i. As a, b were arbitrary, p is stable. The other defining conditions are closed, proving compactness.

Continuity follows from coefficient differentiation and (3). Nonnegative coefficients give $H_{ii} \ge 0$, where $H = \nabla^2 p(\mathbf{1})$. Also $t_i \ge 0$ and $\sum_i t_i = n$, so

$$d = \frac{n-1}{2n} \|t\|_2^2 - \frac{1}{2} \operatorname{Tr} H \le \frac{n-1}{2n} n^2 = M_n.$$

□

Proposition 4.2. *There exist $\varepsilon_n, b_n > 0$ such that*

$$\mathcal{A}(p) - F(t) \ge b_n d(p) \tag{9}$$

for every $p \in \mathcal{K}_n$ with $\|t - \mathbf{1}\|_1 < \varepsilon_n$.

Proof. Take ε_n no larger than $1/(8\sqrt{n})$. Lemma 3.2 handles $d \le \eta_n$. Consider the compact set

$$E = \{p \in \mathcal{K}_n : \ t(p) = \mathbf{1}, \ d(p) \ge \eta_n\}.$$

If $E \neq \varnothing$, Theorem 1.1 and Lemma 2.1 give

$$m_n := \min_{p \in E} \big(\mathcal{A}(p) - F(\mathbf{1})\big) > 0.$$

For a sufficiently small gradient neighborhood, every $p \in \mathcal{K}_n$ with $d(p) \ge \eta_n$ has $\mathcal{A}(p) - F(t) \ge m_n/2$. Otherwise a sequence with $t \to \mathbf{1}$ would have a convergent subsequence with limit in E and gap at most $m_n/2$, a contradiction. By (8), take $b_n = \min\{a_n/2, m_n/(2M_n)\}$.

If $E = \varnothing$, compactness instead implies that in a sufficiently small gradient neighborhood there is no $p \in \mathcal{K}_n$ with $d(p) \ge \eta_n$: a contrary sequence would converge to a member of E. Then $b_n = a_n/2$ suffices. □

Lemma 4.3. *There is $K_n > 0$, depending only on n, such that*

$$\left\|p - L_t^n\right\|_{\mathrm{coef},1} \le K_n d(p) \qquad (p \in \mathcal{K}_n).$$

Proof. For real $\|x\|_2 \le 1$, nonnegativity and $\sum_i t_i = n$ give $|L_t(x)| \le \|t\|_2/n \le 1$. Using the exact quadratic term in (2) yields

$$|p(x) - L_t(x)^n| \le d + \sum_{k=3}^{n} \binom{n}{k} (2d)^{k/2} \le K_n' d,$$

because $0 \le d \le M_n$. For example, one can take $K_n' = 1 + \sum_{k=3}^{n} \binom{n}{k} 2^{k/2} M_n^{k/2-1}$.

On the finite-dimensional space of real homogeneous degree-n polynomials, coefficient ℓ^1 norm is bounded by a constant times the supremum norm on the Euclidean unit ball. For completeness, on the compact coefficient-ℓ^1 unit sphere the latter norm is continuous and strictly positive: a nonzero polynomial cannot vanish on the entire ball. It therefore has a positive minimum. This proves the required norm comparison, and hence the lemma. □

Proof of Theorem 1.2. For $n \ge 2$, combine Proposition 4.2 and Lemma 4.3, taking $c_n = b_n/K_n$. The power L_t^n is stable, has nonnegative coefficients, value one at $\mathbf{1}$, and gradient t. Equality in the minimum forces $\left\|p - L_t^n\right\|_{\mathrm{coef},1} = 0$. For $n = 1$, normalization forces $p = x_1$, so the theorem holds with any positive constants. □

5 Optimal distance order and a probability formulation

Proposition 5.1. *For every $n \ge 2$, no positive uniform lower bound by $\left\|p - L_t^n\right\|_{\mathrm{coef},1}^{\alpha}$ with $\alpha < 1$ can replace the right side of* (1)*, even when $t = \mathbf{1}$.*

Proof. Put $L = L_{\mathbf{1}}$, $a = x_1 - x_2$, and, for $0 < s < 1/n$, let

$$p_s = L^{n-2}(L + sa)(L - sa) = L^n - s^2 L^{n-2} a^2.$$

Every linear factor has nonnegative coefficients and positive coefficient sum, so the product is stable. Moreover, $p_s(\mathbf{1}) = 1$ and $\nabla p_s(\mathbf{1}) = \mathbf{1}$. Only the term $-2x_1x_2$ in a^2 can contribute to the squarefree coefficient, giving

$$\mathcal{A}(p_s) - \mathcal{A}(L^n) = \frac{2(n-2)!}{n^{n-2}} s^2, \qquad \|p_s - L^n\|_{\mathrm{coef},1} = s^2 \left\|L^{n-2} a^2\right\|_{\mathrm{coef},1} > 0.$$

The ratio of the gap to this distance raised to $\alpha < 1$ tends to zero. This proves the assertion. □

This establishes optimal order, not the best constant in Theorem 1.2.

Corollary 5.2. *Let N take values in $\{\alpha \in \mathbb{Z}^n_{\geq 0} : \sum_i \alpha_i = n\}$, and suppose its probability generating polynomial $p(x) = \mathbb{E}[x^N]$ is real stable. Put $t = \mathbb{E}N$, and let M_t have the multinomial count law for n independent draws with probabilities t_i/n. Under the gradient condition of Theorem 1.2,*

$$\mathbb{P}(N = \mathbf{1}) - \frac{n!}{n^n} \prod_i t_i \geq 2c_n d_{\mathrm{TV}}(\mathcal{L}(N), \mathcal{L}(M_t)).$$

Proof. The coefficients of p are the probabilities of N, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbb{E}N = t$. The generating polynomial of M_t is L^n_t. Coefficient ℓ^1 distance is twice total variation, where $d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2}\sum_\alpha |\mu(\alpha) - \nu(\alpha)|$. The result is precisely (1). □

The unique minimizing object here is a count law, not an underlying labeled sampling mechanism. Differentiation gives a further interpretation of the dispersion matrix:

$$2B = \mathrm{diag}(t) - \frac{tt^T}{n} - \mathrm{Cov}(N) = \mathrm{Cov}(M_t) - \mathrm{Cov}(N). \tag{10}$$

Indeed, $\nabla^2 p(\mathbf{1}) = \mathrm{Cov}(N) + tt^T - \mathrm{diag}(t)$; substitution into (3) proves (10). The coefficient comparison supplies the stated inequality.

6 Exact local geometry and a global obstruction

The nearbalanced global conclusion does not identify the full gradient region in which the power is a local minimizer. We now classify strict local minimality in the entire stable-polynomial class. The corresponding matrix Hessian criterion already appears in [8]; the point here is that centered-root control suffices without a matrix representation.

Proposition 6.1. *Fix $n \geq 2$ and $t_i > 0$ with $\sum_i t_i = n$, and set*

$$\gamma(t) = \frac{n^2}{\|t\|_2^2} - (n-1).$$

In the coefficient topology on normalized nonnegative homogeneous real-stable polynomials with gradient t, the power L^n_t is a strict local minimizer of $\mathcal{A}$ if and only if $\gamma(t) > 0$. If $\gamma(t) < 0$, it is not a local minimizer. If $\gamma(t) = 0$, a nonconstant stable product curve through L^n_t has the same squarefree coefficient throughout.

Proof. Let $T = \mathrm{diag}(t)$ and $S = T^{-1}BT^{-1}$. Then $S \geq 0$ and $St = 0$. Define

$$r = \mathbf{1} - \frac{nt}{\|t\|_2^2}, \qquad \|r\|_2^2 = 1 - \gamma(t).$$

The exact coefficient identity of Lemma 3.1 becomes

$$A_2 := \mathcal{A}(L_t^{n-2} q_2) = F(t)\frac{n}{n-1}(\mathrm{Tr}\, S - r^T S r).$$

In particular, the bracket is the negative sum of all ordered off-diagonal entries $B_{ij}/(t_i t_j)$; its diagonal cancels. Since $r^T S r \leq \|r\|_2^2 \,\mathrm{Tr}\, S$,

$$A_2 \geq F(t)\frac{n}{n-1}\gamma(t)\,\mathrm{Tr}\, S \geq \alpha(t) d, \qquad \alpha(t) = \frac{F(t) n \gamma(t)}{(n-1)(\max_i t_i)^2},$$

when $\gamma(t) > 0$. The cube estimate in the proof of Lemma 3.2 required only $t_i \geq 0$ and $\sum_i t_i = n$. It therefore gives, for $d \leq 1$,

$$|\mathcal{A}(p) - F(t) - A_2| \leq C_n d^{3/2}.$$

For $n \geq 3$, put $\eta(t) = \min\{1, (\alpha(t)/(2C_n))^2\}$. For $n = 2$, put $\eta(t) = 1$, since the remainder vanishes. If $d \leq \eta(t)$, then

$$\mathcal{A}(p) - F(t) \geq \tfrac{1}{2}\alpha(t)d.$$

The rigidity $d = 0 \iff p = L_t^n$ and continuity of d establish strict local minimality. Lemma 4.3 also gives local coefficient-distance coercivity with constant $\alpha(t)/(2K_n)$. This is an explicit local dispersion radius, not an explicit global gradient neighborhood.

For the converse, if $\gamma(t) \leq 0$ then $r \neq 0$. Set $b = Tr$, so $b^T\mathbf{1} = t^T r = 0$. For sufficiently small $s > 0$, all coefficients of $L_t \pm sb^T x$ are positive. Hence

$$p_s = L_t^{n-2}(L_t + sb^T x)(L_t - sb^T x) = L_t^n - s^2 L_t^{n-2}(b^T x)^2$$

is feasible and nonconstant. Here $B = s^2 bb^T$, $S = s^2 rr^T$, and there are no higher centered terms. Consequently,

$$\mathcal{A}(p_s) - F(t) = F(t)\frac{n}{n-1}s^2\gamma(t)\|r\|_2^2.$$

It is negative when $\gamma(t) < 0$ and zero when $\gamma(t) = 0$. □

This criterion is local. Failure of global rank-one minimization away from balance is already known for permanents; see [2, Section 6] and [3]. The following small example shows that even strict local positivity does not provide a global theorem.

Proposition 6.2. *In degree three there are strictly positive stable product polynomials with $\gamma(t) > 0$ but $\mathcal{A}(p) < F(t)$. Moreover, any universal open gradient neighborhood $\|t - \mathbf{1}\|_1 < \varepsilon$ on which the power is always the unique global minimizer must have $\varepsilon \leq 1$.*

Proof. Let $0 < a < 1/2$, $t = (1 + 2a, 1 - a, 1 - a)$, $L = L_t$ and $b^T x = -2x_1 + x_2 + x_3$. Set

$$p_{a,e} = (L - eb^T x)^2(L + 2eb^T x).$$

All three factors have coefficient sum one, and their gradients sum to t. Their coefficients are strictly positive if $0 < e < \min\{(1 - a)/3, (1 + 2a)/12\}$. Direct expansion gives

$$p_{a,e} = L^3 - 3e^2 L(b^T x)^2 + 2e^3(b^T x)^3, \qquad \mathcal{A}(p_{a,e}) - F(t) = 6e^2(1 - 2a - 4e).$$

For every $1/4 < a < 1/2$, the interval

$$(1 - 2a)/4 < e < (1 + 2a)/12$$

is nonempty and satisfies the positivity constraints, while the displayed gap is negative. Yet $\|t\|_2^2 = 3 + 6a^2 < 9/2$, so $\gamma(t) > 0$.

For example, $a = 1/3$, $e = 1/8$ gives categorical factor rows

$$(29/36, 7/72, 7/72), \quad (29/36, 7/72, 7/72), \quad (1/18, 17/36, 17/36).$$

Here $\gamma = 5/11$, $A_2 = 1/32$, the cubic contribution is $-3/64$, and the total gap is $-1/64$. At $a = 1/4$, $e = 1/8$, allowing one zero factor coefficient, the rows are $(3/4, 1/8, 1/8)$ twice and $(0, 1/2, 1/2)$. This nonpower has the same coefficient as L_t^3 and $\|t - \mathbf{1}\|_1 = 1$. Thus every open uniqueness radius greater than one contains a counterexample. Strictly positive failures also approach this gradient distance from above. □

7 Quartic examples and scope

For $n = 4$, the expansion is

$$p = L_t^4 - L_t^2 x^T B x + L_t q_3 + q_4.$$

At $t = \mathbf{1}$, the leading squarefree contribution is $d/8$. The remaining contributions are $O(d^{3/2})$ and $O(d^2)$; the constants depend only on degree. Thus the argument controls actual stable quartics without choosing a smooth parameterization of their cone.

For an example outside the determinant class, consider the specialized Vámos polynomial from [7, Section 3]:

$$h(x) = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4)e_3(x), \qquad h(\mathbf{1}) = 65,$$

where $e_3 = \sum_{i<j<k} x_i x_j x_k$. Its real stability (inherited from the stable Vámos matroid polynomial by specialization) and the obstruction to a definite determinantal representation of any positive integer power are prior results; see [6, 7]. Let J be the all-ones matrix and define

$$T_\delta = (1-\delta)J/4 + \delta I, \qquad p_\delta(x) = h(T_\delta x)/65, \qquad 0 < \delta \le 1.$$

Every row of T_δ is nonnegative with sum one, so it maps vectors with positive imaginary parts to such vectors; hence p_δ is stable. The transformation is invertible and fixes $\mathbf{1}$. A definite determinantal representation of a positive power would pull back to one for a power of h, so the obstruction persists. Its gradient is

$$t_\delta = (1 + \delta/65, 1 + \delta/65, 1 - \delta/65, 1 - \delta/65).$$

Direct expansion gives

$$\mathcal{A}(p_\delta) = \frac{3}{32} + \frac{19}{208}\delta^2 + \frac{3}{52}\delta^3 + \frac{7}{2080}\delta^4,$$
$$\mathcal{A}(p_\delta) - F(t_\delta) = \frac{3089}{33800}\delta^2 + \frac{3}{52}\delta^3 + \frac{480593}{142805000}\delta^4 > 0.$$

These are illustrative identities.

Stability cannot simply be omitted. Put $L = (x_1 + x_2 + x_3 + x_4)/4$. For $0 < r < 3/32$, the polynomial

$$q_r = \frac{L^4 - r x_1 x_2 x_3 x_4}{1 - r}$$

has nonnegative coefficients, value one at $\mathbf{1}$, and gradient $\mathbf{1}$, but $\mathcal{A}(q_r) = (3/32 - r)/(1 - r) < 3/32$. Under $x = (z + 1, z - 1, z, z)$ it becomes $z^4 + rz^2/(1 - r)$, which has a nonreal upper-half-plane root. All four corresponding coordinates have positive imaginary part; thus q_r is not stable.

The proof does not extend the perturbed-gradient theorem to every Lorentzian polynomial: real-rooted line restrictions are essential to Lemma 2.1. Effective radii and dimension dependence remain natural questions.

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References

[1] L. Gurvits, Van der Waerden/Schrijver–Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all, *Electronic Journal of Combinatorics* **15** (2008), R66. doi:10.37236/790. arXiv:0711.3496.

[2] L. Gurvits and J. Leake, Unique minimizers for permanents, mixed discriminants, and log-concave polynomials, arXiv:2609.21083v2 (15 October 2026).

[3] L. Gurvits, N. Klein and J. Leake, From trees to polynomials and back again: new capacity bounds with applications to TSP, *51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)*, LIPIcs **297** (2024), 79:1–79:20. doi:10.4230/LIPIcs.ICALP.2024.79. arXiv:2311.09072.

[4] L. Gurvits and J. Leake, Capacity lower bounds via productization, *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing (STOC 2021)*, 847–858. arXiv:2007.08390.

[5] H. H. Bauschke, O. Güler, A. S. Lewis and H. S. Sendov, Hyperbolic polynomials and convex analysis, *Canadian Journal of Mathematics* **53** (2001), 470–488. doi:10.4153/CJM-2001-020-6.

[6] P. Brändén, Obstructions to determinantal representability, *Advances in Mathematics* **226** (2011), 1202–1212. doi:10.1016/j.aim.2010.08.003.

[7] M. Kummer, A note on the hyperbolicity cone of the specialized Vámos polynomial, arXiv:1306.4483v2 (18 August 2015).

[8] S. Schubr, Trace-perturbed minimizers of the mixed discriminant, Luminary Research Archive (2026). Public research record.