Prescribed-gradient minimizers and coefficient stability for homogeneous real-stable polynomials

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October 22, 2026

Abstract

For each fixed degree n, we prove that among homogeneous real-stable polynomials in n variables with nonnegative coefficients, value one at the all-ones vector, and a prescribed gradient sufficiently close to the all-ones vector, the power of the corresponding linear form uniquely minimizes the squarefree coefficient. This answers Question 1.8 of Gurvits–Leake, arXiv:2609.21083v2. We strengthen the minimum statement to a uniform lower bound for the coefficient deficit by the coefficient ℓ^1 distance to that power. The distance order is optimal. The proof uses centered real-root dispersion, an exact coefficient identity and the classical balanced uniqueness theorem. The neighborhood and constants are dimension-dependent and qualitative. We also give a total-variation interpretation for stable count distributions and an illustrative nondeterminantal quartic family.

1 Introduction and main result

A nonzero polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is *real stable* if $p(z_1, \dots, z_n) \neq 0$ whenever $\operatorname{Im} z_i > 0$ for every i. Throughout, the polynomials under consideration have nonnegative coefficients and are homogeneous of degree n in n variables. Write $\mathbf{1} = (1, \dots, 1)^T$ and

$$\mathcal{A}(p) = [x_1 \cdots x_n]p, \qquad \|p\|_{\mathrm{coef},1} = \sum_{|\alpha|=n} |[x^\alpha]p|.$$

For $t \in \mathbb{R}^n_{>0}$ with $\sum_i t_i = n$, set

$$L_t(x) = \frac{t^T x}{n}, \qquad F(t) = \mathcal{A}(L_t^n) = \frac{n!}{n^n} \prod_{i=1}^n t_i.$$

The balanced coefficient inequality and its equality classification are due to Gurvits [1]; we use the following formulation, also recalled in Theorem 1.3 of Gurvits–Leake [2].

Theorem 1.1 (Balanced uniqueness)**.** *If p is homogeneous of degree n, real stable, has nonnegative coefficients, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbf{1}$, then $\mathcal{A}(p) \geq n!/n^n$, with equality if and only if $p = L_{\mathbf{1}}^n$.*

Theorem 1.1 is a prior result, not a contribution of this paper. The upper-half-plane convention above and the right-half-plane convention in [1] are equivalent for homogeneous polynomials by multiplication of every variable by i. To relate the formulation directly to the coefficient–capacity inequality and its equality case in [1, Theorem 5.7], write $p(x) = \sum_\alpha c_\alpha x^\alpha$. Here $\sum_\alpha c_\alpha = 1$ and $\sum_\alpha c_\alpha \alpha = \mathbf{1}$. Weighted AM–GM gives $p(x) \geq \prod_i x_i$ for $x_i > 0$, so $\operatorname{Cap}(p) = \inf_{x>0} p(x)/\prod_i x_i = 1$. The equality classification in that theorem forces a power of a positive linear form; normalization and the balanced gradient fix the form to $L_{\mathbf{1}}$.

Theorem 1.2. *For each fixed integer $n \geq 1$, there exist constants $\varepsilon_n, c_n > 0$ such that the following holds. Suppose p is homogeneous of degree n in n variables, real stable, has nonnegative real coefficients, $p(\mathbf{1}) = 1$, and $t = \nabla p(\mathbf{1})$ satisfies $\|t - \mathbf{1}\|_1 < \varepsilon_n$. Then*

$$\mathcal{A}(p) - F(t) \;\geq\; c_n \, \|p - L_t^n\|_{\mathrm{coef},1} \, . \tag{1}$$

In particular, L_t^n is the unique minimizer of $\mathcal{A}$ among polynomials with this normalization and prescribed gradient.

Euler's identity gives $\sum_i t_i = n$; the neighborhood will be chosen so that every $t_i > 0$. The theorem includes all feasible polynomials with such a gradient, not only polynomials in a chosen coefficient neighborhood of L_t^n. The constants may depend on n. No useful explicit global value of ε_n or c_n is computed here.

Gurvits–Klein–Leake [3, Section 2.1] prove a nearbalanced permanent minimization result and explicitly leave the real-stable extension unproved. Gurvits–Leake [2, Question 1.8] ask for the prescribed-gradient conclusion of Theorem 1.2. Their balanced uniqueness result for the larger Lorentzian class does not supply the real-root estimates used below. Productization [4] gives important capacity bounds and gradient-preserving products matching an evaluation at a fixed positive point. That product may depend on the evaluation point; the result does not directly identify the squarefree-coefficient minimizer considered here.

The earlier mixed-discriminant result [8] treats the matrix subclass and answers Question 1.7 of [2]. Indeed, $p(x) = \det(\sum_i x_i X_i)$, for positive semidefinite Hermitian matrices with $\sum_i X_i = I$, has $p(\mathbf{1}) = 1$ and $\partial_i p(\mathbf{1}) = \operatorname{Tr} X_i$. The present argument does not require a determinant representation. Such a representation cannot be assumed for general stable polynomials [6, 7].

Our main step is to use the variance of centered real roots as a single size parameter controlling all higher centered coefficients. Hyperbolic eigenvalue methods are classical; see, for example, Bauschke–Güler–Lewis–Sendov [5]. Here they yield a positive leading payment for the squarefree coefficient under a perturbed gradient. Compactness and Theorem 1.1 then extend the local estimate over the entire feasible class with nearbalanced gradient.

2 Centered roots and the dispersion matrix

For the rest of the proof let $n \geq 2$, let p satisfy the normalization and stability hypotheses, and put $t = \nabla p(\mathbf{1})$. For real x, define $y = x - L_t(x)\mathbf{1}$. Then $t^T y = 0$.

Lemma 2.1. *There are homogeneous polynomials q_k of degree k, for $2 \leq k \leq n$, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^{n} L_t(x)^{n-k} q_k(x). \tag{2}$$

There is a real symmetric positive semidefinite matrix B with $B\mathbf{1} = 0$ such that

$$q_2(x) = -x^T B x, \qquad B = \frac{n-1}{2n} t t^T - \frac{1}{2} \nabla^2 p(\mathbf{1}). \tag{3}$$

Writing $d = \operatorname{Tr} B$, we have

$$|q_k(x)| \leq \binom{n}{k} (2x^T B x)^{k/2} \leq \binom{n}{k} (2d\|x\|_2^2)^{k/2}. \tag{4}$$

Moreover, $d = 0$ if and only if $p = L_t^n$.

Proof. Expand $p(z\mathbf{1}+y)$ in z. Homogeneity and $p(\mathbf{1})=1$ make it monic of degree n. The coefficient of z^{n-1} is $t^Ty=0$, so

$$p(z\mathbf{1}+y)=z^n+\sum_{k=2}^{n}q_k(x)z^{n-k}.$$

The map $x\mapsto y$ is unchanged by adding $a\mathbf{1}$ to x, since $L_t(\mathbf{1})=1$. This proves the stated invariance. Substitute $z=L_t(x)$ to obtain (2).

For every real y, all roots of $z\mapsto p(z\mathbf{1}+y)$ are real. An upper-half-plane root would contradict stability, and a lower-half-plane nonreal root would have an upper-half-plane conjugate. Let these roots be $\lambda_1,\dots,\lambda_n$. Their sum is zero. Hence

$$q_2(x)=\sum_{i<j}\lambda_i\lambda_j=-\frac{1}{2}\sum_i\lambda_i^2\le 0.$$

Thus $q_2=-x^TBx$ for a unique symmetric $B\succeq 0$; invariance gives $B\mathbf{1}=0$. To check its formula, put $H=\nabla^2p(\mathbf{1})$. Euler's identities give $H\mathbf{1}=(n-1)t$ and $\mathbf{1}^TH\mathbf{1}=n(n-1)$. The coefficient of z^{n-2} is $\frac{1}{2}y^THy=\frac{1}{2}x^THx-\frac{n-1}{2n}(t^Tx)^2$, which is (3).

Each $|\lambda_i|\le(2x^TBx)^{1/2}$. The elementary-symmetric root formulas give (4); positivity of B implies $x^TBx\le d\|x\|_2^2$. If $d=0$, then $B=0$, all the roots vanish for every real x, and all q_k vanish. Equation (2) gives $p=L_t^n$. The converse follows directly. □

3 A uniform local coefficient estimate

Lemma 3.1. *Suppose $t_i>0$, put $v_i=1/t_i$ and $D=\operatorname{diag}(v)$. Then*

$$\mathcal{A}(L_t^{n-2}q_2)=\frac{(n-2)!\prod_i t_i}{n^{n-2}}\left(\operatorname{Tr}(D^2B)-v^TBv\right). \tag{5}$$

If $\|t-\mathbf{1}\|_\infty\le 1/(8\sqrt{n})$, then

$$\mathcal{A}(L_t^{n-2}q_2)\ge a_nd,\qquad a_n=\frac{(n-2)!(7/8)^n}{2n^{n-2}}>0. \tag{6}$$

Proof. In $-x^TBx$, a diagonal term cannot contribute to a squarefree monomial. The ordered pair $i\ne j$ contributes

$$-B_{ij}\frac{(n-2)!}{n^{n-2}}\prod_{\ell\ne i,j}t_\ell.$$

Summing these terms proves (5). Since $B\mathbf{1}=0$,

$$v^TBv=(v-\mathbf{1})^TB(v-\mathbf{1})\le d\|v-\mathbf{1}\|_2^2,\qquad \operatorname{Tr}(D^2B)\ge(\min_i v_i^2)d.$$

Under the stated neighborhood condition, $7/8\le t_i\le 9/8$, so $\min_i v_i^2\ge 64/81$. Also

$$|v_i-1|\le\frac{1/(8\sqrt{n})}{1-1/(8\sqrt{n})}\le\frac{1}{7\sqrt{n}},\qquad \|v-\mathbf{1}\|_2^2\le\frac{1}{49}.$$

The difference $64/81-1/49$ exceeds $1/2$, and $\prod_i t_i\ge(7/8)^n$. This proves (6). □

Lemma 3.2. *There is $\eta_n\in(0,1]$, depending only on n, such that*

$$\mathcal{A}(p)-F(t)\ge\frac{a_n}{2}d$$

whenever $\|t-\mathbf{1}\|_\infty\le 1/(8\sqrt{n})$ and $d\le\eta_n$.

Proof. For every homogeneous polynomial R of degree n,

$$\mathcal{A}(R) = \sum_{S \subseteq [n]} (-1)^{n-|S|} R(\mathbf{1}_S). \tag{7}$$

Indeed, a monomial survives the alternating sum exactly when all variables occur; degree n then makes it squarefree. Here $\mathbf{1}_S$ is the indicator vector of S.

Nonnegativity of t and $\sum_i t_i = n$ imply $0 \le L_t(\mathbf{1}_S) \le 1$, while $\mathbf{1}_S^T B \mathbf{1}_S \le nd$. Thus, for $k \ge 3$, Lemma 2.1 and (7) give

$$|\mathcal{A}(L_t^{n-k} q_k)| \le 2^n \binom{n}{k} (2nd)^{k/2}.$$

For $0 \le d \le 1$, the sum is at most $C_n d^{3/2}$, where

$$C_n = 2^n \sum_{k=3}^{n} \binom{n}{k} (2n)^{k/2}.$$

For $n = 2$, the sum is empty and $C_2 = 0$. Choose $\eta_2 = 1$, and for $n \ge 3$ take $\eta_n = \min\{1, (a_n/(2C_n))^2\}$. Combining (2) and Lemma 3.1 proves the assertion. □

4 Compactness and the global deficit

Let $\mathcal{K}_n$ be the set of homogeneous degree-n real-stable polynomials with nonnegative coefficients and $p(\mathbf{1}) = 1$.

Lemma 4.1. *The set $\mathcal{K}_n$ is compact in coefficient space. The functions $t(p)$, $B(p)$, $d(p)$, and $\mathcal{A}(p)$ are continuous on it, and*

$$0 \le d(p) \le M_n := \frac{n(n-1)}{2}. \tag{8}$$

Proof. The coefficients lie in a fixed finite-dimensional probability simplex. It remains to show stability is closed within that simplex. Let $p_j \to p$ coefficientwise, with $p_j \in \mathcal{K}_n$. Fix real a and strictly positive real b. The polynomial $z \mapsto p_j(a + zb)$ has only real roots by the stability and conjugation argument. Its leading coefficient satisfies

$$p_j(b) \ge (\min_i b_i)^n > 0.$$

Therefore its coefficient limit still has degree n, and closure of the real-rooted univariate polynomials shows that $z \mapsto p(a + zb)$ has only real roots. If $p(a + ib) = 0$, this polynomial would have the nonreal root i. As a, b were arbitrary, p is stable. The other defining conditions are closed, proving compactness.

Continuity follows from coefficient differentiation and (3). Nonnegative coefficients give $H_{ii} \ge 0$, where $H = \nabla^2 p(\mathbf{1})$. Also $t_i \ge 0$ and $\sum_i t_i = n$, so

$$d = \frac{n-1}{2n} \|t\|_2^2 - \frac{1}{2} \operatorname{Tr} H \le \frac{n-1}{2n} n^2 = M_n.$$

□

Proposition 4.2. *There exist $\varepsilon_n, b_n > 0$ such that*

$$\mathcal{A}(p) - F(t) \ge b_n d(p) \tag{9}$$

for every $p \in \mathcal{K}_n$ with $\|t - \mathbf{1}\|_1 < \varepsilon_n$.

Proof. Take ε_n no larger than $1/(8\sqrt{n})$. Lemma 3.2 handles $d \le \eta_n$. Consider the compact set

$$E = \{p \in \mathcal{K}_n : \ t(p) = \mathbf{1}, \ d(p) \ge \eta_n\}.$$

If $E \neq \varnothing$, Theorem 1.1 and Lemma 2.1 give

$$m_n := \min_{p \in E} (\mathcal{A}(p) - F(\mathbf{1})) > 0.$$

For a sufficiently small gradient neighborhood, every $p \in \mathcal{K}_n$ with $d(p) \ge \eta_n$ has $\mathcal{A}(p) - F(t) \ge m_n/2$. Otherwise a sequence with $t \to \mathbf{1}$ would have a convergent subsequence with limit in E and gap at most $m_n/2$, a contradiction. By (8), take $b_n = \min\{a_n/2, m_n/(2M_n)\}$.

If $E = \varnothing$, compactness instead implies that in a sufficiently small gradient neighborhood there is no $p \in \mathcal{K}_n$ with $d(p) \ge \eta_n$: a contrary sequence would converge to a member of E. Then $b_n = a_n/2$ suffices. □

Lemma 4.3. *There is $K_n > 0$, depending only on n, such that*

$$\|p - L_t^n\|_{\mathrm{coef},1} \le K_n d(p) \qquad (p \in \mathcal{K}_n).$$

Proof. For real $\|x\|_2 \le 1$, nonnegativity and $\sum_i t_i = n$ give $|L_t(x)| \le \|t\|_2/n \le 1$. Using the exact quadratic term in (2) yields

$$|p(x) - L_t(x)^n| \le d + \sum_{k=3}^{n} \binom{n}{k} (2d)^{k/2} \le K_n' d,$$

because $0 \le d \le M_n$. For example, one can take $K_n' = 1 + \sum_{k=3}^{n} \binom{n}{k} 2^{k/2} M_n^{k/2-1}$.

On the finite-dimensional space of real homogeneous degree-n polynomials, coefficient ℓ^1 norm is bounded by a constant times the supremum norm on the Euclidean unit ball. For completeness, on the compact coefficient-ℓ^1 unit sphere the latter norm is continuous and strictly positive: a nonzero polynomial cannot vanish on the entire ball. It therefore has a positive minimum. This proves the required norm comparison, and hence the lemma. □

Proof of Theorem 1.2. For $n \ge 2$, combine Proposition 4.2 and Lemma 4.3, taking $c_n = b_n/K_n$. The power L_t^n is stable, has nonnegative coefficients, value one at $\mathbf{1}$, and gradient t. Equality in the minimum forces $\|p - L_t^n\|_{\mathrm{coef},1} = 0$. For $n = 1$, normalization forces $p = x_1$, so the theorem holds with any positive constants. □

5 Optimal distance order and a probability formulation

Proposition 5.1. *For every $n \ge 2$, no positive uniform lower bound by $\|p - L_t^n\|_{\mathrm{coef},1}^{\alpha}$ with $\alpha < 1$ can replace the right side of* (1)*, even when $t = \mathbf{1}$.*

Proof. Put $L = L_{\mathbf{1}}$, $a = x_1 - x_2$, and, for $0 < s < 1/n$, let

$$p_s = L^{n-2}(L + sa)(L - sa) = L^n - s^2 L^{n-2} a^2.$$

Every linear factor has nonnegative coefficients and positive coefficient sum, so the product is stable. Moreover, $p_s(\mathbf{1}) = 1$ and $\nabla p_s(\mathbf{1}) = \mathbf{1}$. Only the term $-2x_1x_2$ in a^2 can contribute to the squarefree coefficient, giving

$$\mathcal{A}(p_s) - \mathcal{A}(L^n) = \frac{2(n-2)!}{n^{n-2}} s^2, \qquad \|p_s - L^n\|_{\mathrm{coef},1} = s^2 \left\| L^{n-2} a^2 \right\|_{\mathrm{coef},1} > 0.$$

The ratio of the gap to this distance raised to $\alpha < 1$ tends to zero. This proves the assertion. □

This establishes optimal order, not the best constant in Theorem 1.2.

Corollary 5.2. *Let N take values in $\{\alpha \in \mathbb{Z}^n_{\geq 0} : \sum_i \alpha_i = n\}$, and suppose its probability generating polynomial $p(x) = \mathbb{E}[x^N]$ is real stable. Put $t = \mathbb{E}N$, and let M_t have the multinomial count law for n independent draws with probabilities t_i/n. Under the gradient condition of Theorem 1.2,*

$$\mathbb{P}(N = \mathbf{1}) - \frac{n!}{n^n} \prod_i t_i \geq 2c_n d_{\mathrm{TV}}(\mathcal{L}(N), \mathcal{L}(M_t)).$$

Proof. The coefficients of p are the probabilities of N, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbb{E}N = t$. The generating polynomial of M_t is L_t^n. Coefficient ℓ^1 distance is twice total variation, where $d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2}\sum_\alpha |\mu(\alpha) - \nu(\alpha)|$. The result is precisely (1). □

The unique minimizing object here is a count law, not an underlying labeled sampling mechanism. Differentiation gives a further interpretation of the dispersion matrix:

$$2B = \mathrm{diag}(t) - \frac{tt^T}{n} - \mathrm{Cov}(N) = \mathrm{Cov}(M_t) - \mathrm{Cov}(N). \tag{10}$$

Indeed, $\nabla^2 p(\mathbf{1}) = \mathrm{Cov}(N) + tt^T - \mathrm{diag}(t)$; substitution into (3) proves (10). The coefficient comparison supplies the stated inequality.

6 Quartic examples and scope

For $n = 4$, the expansion is

$$p = L_t^4 - L_t^2 x^T B x + L_t q_3 + q_4.$$

At $t = \mathbf{1}$, the leading squarefree contribution is $d/8$. The remaining contributions are $O(d^{3/2})$ and $O(d^2)$; the constants depend only on degree. Thus the argument controls actual stable quartics without choosing a smooth parameterization of their cone.

For an example outside the determinant class, consider the specialized Vámos polynomial from [7, Section 3]:

$$h(x) = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4)e_3(x), \qquad h(\mathbf{1}) = 65,$$

where $e_3 = \sum_{i<j<k} x_i x_j x_k$. Its real stability (inherited from the stable Vámos matroid polynomial by specialization) and the obstruction to a definite determinantal representation of any positive integer power are prior results; see [6, 7]. Let J be the all-ones matrix and define

$$T_\delta = (1 - \delta)J/4 + \delta I, \qquad p_\delta(x) = h(T_\delta x)/65, \qquad 0 < \delta \leq 1.$$

Every row of T_δ is nonnegative with sum one, so it maps vectors with positive imaginary parts to such vectors; hence p_δ is stable. The transformation is invertible and fixes $\mathbf{1}$. A definite determinantal representation of a positive power would pull back to one for a power of h, so the obstruction persists. Its gradient is

$$t_\delta = (1 + \delta/65, 1 + \delta/65, 1 - \delta/65, 1 - \delta/65).$$

Direct expansion gives

$$\mathcal{A}(p_\delta) = \frac{3}{32} + \frac{19}{208}\delta^2 + \frac{3}{52}\delta^3 + \frac{7}{2080}\delta^4,$$
$$\mathcal{A}(p_\delta) - F(t_\delta) = \frac{3089}{33800}\delta^2 + \frac{3}{52}\delta^3 + \frac{480593}{142805000}\delta^4 > 0.$$

These are illustrative identities.

Stability cannot simply be omitted. Put $L = (x_1 + x_2 + x_3 + x_4)/4$. For $0 < r < 3/32$, the polynomial

$$q_r = \frac{L^4 - r x_1 x_2 x_3 x_4}{1 - r}$$

has nonnegative coefficients, value one at $\mathbf{1}$, and gradient $\mathbf{1}$, but $\mathcal{A}(q_r) = (3/32 - r)/(1 - r) < 3/32$. Under $x = (z + 1, z - 1, z, z)$ it becomes $z^4 + rz^2/(1 - r)$, which has a nonreal upper-half-plane root. All four corresponding coordinates have positive imaginary part; thus q_r is not stable.

The proof does not extend the perturbed-gradient theorem to every Lorentzian polynomial: real-rooted line restrictions are essential to Lemma 2.1. Effective radii and dimension dependence remain natural questions.

Acknowledgment of assistance

AI tools assisted with exploration, drafting, exact calculations, literature discovery and internal proof checking.

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