Discrete Jensen inequalities for unimodular fiber counts of dimension at most three

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Luminary Oddities

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Abstract

Leake and Mohammadi Yekta asked whether the lattice-point counts $N(a)$ of the fibers $\{y \in \mathbb{Z}^n_{\geq 0} : My = a\}$ of a unimodular matrix M satisfy the discrete Jensen inequality $N(b) \geq \prod_a N(a)^{\mu(a)}$ for every probability distribution μ with integral mean b. We prove this inequality whenever the fiber over the mean has dimension at most three, for every integrable law with countable support. In particular it holds for every unimodular matrix of corank at most three. The proof gives the stronger corrected fourth-root inequality $\mathbb{E}(N(A) - 1/12)^{1/4} \leq (N(b) - 1/12)^{1/4}$, which also yields Jensen for the power $2/11$. Its core is a continuous, monotone extension of type-A_3 alcoved lattice counts in all twelve directed bounds. The extension is built from the Todd-operator form of the chamber polynomials. A quartic Lorentzian certificate, shortest-path closure and a matroid reduction of rank-three totally unimodular systems to the A_3 root system complete the argument. In dimension two the sharp correction is $1/4$, and the optimal power deficit at count threshold m_0 is $1/(8m_0)$ to leading order. On integral demands in dimension three the admissible power is at most $\alpha_K \approx 0.30337$, the root of $2 \cdot 15^\alpha = 1 + 65^\alpha$; this bound is witnessed by the dilations $0, K, 2K$ of the polytope with all twelve bounds equal to one, which give equality at α_K. For star-bound (L-convex) families, logarithmic Jensen holds in every dimension by Speyer's L-convex marginal theorem. Finally, dimension four reduces to type-A_4 alcoved counts and 3×3 cell-bounded contingency tables.

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1 Statement and relation to earlier work

1.1 Setting

Following [1], an integer matrix $M \in \mathbb{Z}^{m \times n}$ is *unimodular* if every nonsingular $\operatorname{rank} M \times \operatorname{rank} M$ submatrix has determinant ± 1. For $a \in \mathbb{Z}^m$ put

$$P(a) = \{x \in \mathbb{R}^n_{\geq 0} : Mx = a\}, \qquad N(a) = |P(a) \cap \mathbb{Z}^n|.$$

An empty fiber has $N(a) = 0$. A nonempty fiber has an integral vertex, and if it is unbounded its recession cone $\{x \geq 0 : Mx = 0\}$ contains a nonzero integral vector, so $N(a) = \infty$.

The definition is rank-based, while several arguments below use a full row rank. The two agree after deleting dependent rows. Choose $\operatorname{rank} M$ linearly independent rows, forming M'. A demand with nonempty fiber lies in the column space of M, so its fiber is cut out by M' and the corresponding

coordinates a'. The nonsingular maximal minors of M' are nonsingular rank $M \times \operatorname{rank} M$ minors of M, hence equal to ± 1. Finally, $a \mapsto a'$ is linear and preserves means. Thus we may and do assume, whenever convenient, that M has full row rank and that every nonzero maximal minor is ± 1.

1.2 The question of Leake and Mohammadi Yekta

Leake and Mohammadi Yekta proved that unimodular fiber counts are log-concave on lines: if $b = (b^- + b^+)/2$ and all fibers involved are finite, then $N(b)^2 \geq N(b^-)N(b^+)$ [1, Theorem 1.4]. This gives the Ferroni–Higashitani conjecture on Ehrhart evaluations of unimodular polytopes [1, Corollary 1.5], [16]. They then asked the global question [1, Question 1.7]: for a unimodular $M \in \mathbb{Z}^{m\times n}$ and a probability distribution μ on $\mathbb{Z}^m$ "(perhaps with finite support)" with mean $b \in \mathbb{Z}^m$, is it the case that

$$\#\{y \in \mathbb{Z}^n_{\geq 0} : My = b\} \;\geq\; \prod_{a\in\mathbb{Z}^m} \#\{y \in \mathbb{Z}^n_{\geq 0} : My = a\}^{\mu(a)} \;? \tag{1.1}$$

As they note, convexity on lines does not by itself give global convexity over $\mathbb{Z}^m$: intermediate pair averages need not be integral. Their question generalizes the 0–1 case of Barvinok's question [3, (2.3)], which is posed for every nonnegative weight matrix W. For weighted contingency tables Barvinok asked whether $T(R, C; W) \geq \prod_k T(R_k, C_k; W)^{\gamma_k}$ holds when $(R, C) = \sum_k \gamma_k(R_k, C_k)$ are positive integer margins. He singled out the unweighted case $W = \mathbf{1}$, and his Theorem 2.1 proves such an inequality up to an explicit correction factor. For a 0–1 matrix W, $T(R, C; W)$ counts integer flows in a bipartite graph, a totally unimodular fiber count. The bound (1.1) also connects to capacity lower bounds for unimodular counts [5, 4, 6, 1, 2].

We write μ also as the law of a random demand A, and read the right side of (1.1) as $\exp \mathbb{E} \log N(A)$. If μ charges a demand with empty fiber, the right side is 0; we use the convention $0 \cdot \infty = 0$ for this product, so the right side is 0 even when another charged fiber is infinite.

1.3 Main results

Theorem 1.1 (Logarithmic Jensen in central dimension at most three)**.** *Let M be unimodular and let A be a $\mathbb{Z}^m$-valued random vector with $\mathbb{E}\|A\| < \infty$ and $P(A) \neq \varnothing$ almost surely. Assume $b = \mathbb{E}A \in \mathbb{Z}^m$ and $\dim P(b) \leq 3$. Then*

$$N(b) \;\geq\; \prod_a N(a)^{\mu(a)}.$$

More precisely, $P(b)$ is nonempty. If $P(b)$ is bounded, then every fiber $P(A)$ is bounded, $\mathbb{E} \log N(A)$ is finite, and $\mathbb{E} \log N(A) \leq \log N(b)$. If $P(b)$ is unbounded, then $N(b) = \infty$.

The dimension of a nonempty fiber is at most the corank $n - \operatorname{rank} M$. Hence:

Corollary 1.2 (Corank at most three)**.** *Question 1.7 of [1] has an affirmative answer for every unimodular matrix of corank at most three. Inequality* (1.1) *holds for every probability distribution on $\mathbb{Z}^m$ with finite first moment and integral mean, including distributions with countable support.*

Corollary 1.3 (Barvinok's inequality in dimension at most three)**.** *Let W be an $m \times n$ 0–1 matrix, and let $(R, C) = \sum_k \gamma_k(R_k, C_k)$ be as in [3, Theorem 2.1] (finitely many k). Suppose the transportation polytope at the margins (R, C), restricted to the support of W, has dimension at most three. Then $T(R, C; W) \geq \prod_k T(R_k, C_k; W)^{\gamma_k}$. In particular this holds for $W = \mathbf{1}$ when $(m-1)(n-1) \leq 3$, that is, when $\min(m, n) = 1$, or $\min(m, n) = 2$ and $\max(m, n) \leq 4$.*

For $W = \mathbf{1}$ and positive margins the transportation polytope contains the positive point $(r_i c_j/N)$, so its dimension is $(m-1)(n-1)$. Corollary 1.3 therefore covers $1 \times n$ tables and $2 \times n$ tables with $n \le 4$. The 3×3 case has dimension four, and Section 9 reduces it to cell-bounded 3×3 tables.

Theorem 1.1 is the logarithmic case of the following stronger statement.

Theorem 1.4 (Corrected fourth-root Jensen)**.** *Under the hypotheses of Theorem 1.1, assume in addition that $P(b)$ is bounded. Then all fibers $P(A)$ are bounded and*

$$\mathbb{E}\big(N(A) - \tfrac{1}{12}\big)^{1/4} \le \big(N(b) - \tfrac{1}{12}\big)^{1/4}. \tag{1.2}$$

If $N(A) \ge m_0$ almost surely for an integer $m_0 \ge 1$, then

$$\mathbb{E}N(A)^{\beta_{m_0}} \le N(b)^{\beta_{m_0}}, \qquad \beta_{m_0} = \frac{3m_0 - 1}{12m_0 - 1}.$$

In particular $\beta_1 = 2/11$. All displayed expectations are finite. The number of columns and equations is arbitrary.

The correction $1/12$ is the least one for which the continuous extension used in the proof has a concave fourth root, and $2/11$ is that extension's optimal power (Theorem 4.1). In dimension at most two the inequality sharpens further.

Theorem 1.5 (Planar power concavity)**.** *Let M be unimodular and let $a_i \in \mathbb{Z}^m$ have nonempty fibers, with weights $\lambda_i > 0$, $\sum_i \lambda_i = 1$, and integral mean $b = \sum_i \lambda_i a_i$. Write $N_i = N(a_i)$. If $P(b)$ is bounded and $\dim P(b) \le 2$, then*

$$\sqrt{N(b) - \tfrac{1}{4}} \ge \sum_i \lambda_i \sqrt{N_i - \tfrac{1}{4}}, \tag{1.3}$$

and no smaller real correction than $1/4$ is valid uniformly. If also $N_i \ge m_0$ for an integer $m_0 \ge 1$, then

$$N(b)^{\alpha_{m_0}} \ge \sum_i \lambda_i N_i^{\alpha_{m_0}}, \qquad \alpha_{m_0} = \frac{2m_0 - 1}{4m_0 - 1} = \frac{1}{2} - \frac{1}{8m_0 - 2}. \tag{1.4}$$

For $m_0 \in \mathbb{N}_{\ge 1}$ let $\mathsf{A}(m_0)$ be the supremum of the exponents for which (1.4) holds for every matrix and family in Theorem 1.5 with endpoint counts at least m_0.

Theorem 1.6 (Sharp large-count asymptotic)**.** *Put $n(r) = 3r^2 + 3r + 1$ and $R_{m_0} = \min\{r \in \mathbb{Z}_{\ge 0} : n(r) \ge m_0\}$. Then*

$$\frac{1}{2} - \frac{1}{8m_0 - 2} \le \mathsf{A}(m_0) \le \frac{1}{2} - \frac{1}{8n(R_{m_0} + 2) - 2}, \qquad \mathsf{A}(m_0) = \frac{1}{2} - \frac{1}{8m_0} + O(m_0^{-3/2}).$$

Square-root concavity fails at every count threshold m_0.

On integral demands the admissible powers in dimension three are bounded by an explicit integral ray.

Proposition 1.7 (Integral obstruction in dimension three)**.** *Let A_3 be the supremum of the exponents $\alpha > 0$ such that $\mathbb{E}N(A)^\alpha \le N(b)^\alpha$ for every unimodular matrix and every finitely supported family as in Theorem 1.4 (so $\dim P(b) \le 3$). Then*

$$\frac{2}{11} \le \mathsf{A}_3 \le \alpha_K, \qquad 2 \cdot 15^{\alpha_K} = 1 + 65^{\alpha_K},$$

where α_K is the unique positive root and $3/10 < \alpha_K < 4/13$; numerically $\alpha_K = 0.3033717301\ldots$. The upper bound is witnessed by a corank-three family with counts $1, 15, 65$, which gives equality at $\alpha = \alpha_K$.

Logarithmic Jensen extends to all dimensions for families whose moving parameters form an L-convex system. The basic case is type-A_d alcoved counts in which only the bounds at one anchor vertex vary.

Theorem 1.8 (Star-bound families in every dimension)**.** *Let $d \geq 1$ and fix $h_{ij} \in \mathbb{Z} \cup \{+\infty\}$ for distinct $i, j \in [d]$. For $\ell, u \in \mathbb{Z}^d$ put*

$$N(\ell, u) = \#\{z \in \mathbb{Z}^d : \ \ell \leq z \leq u, \ z_i - z_j \leq h_{ij} \ (i \neq j)\}.$$

If (ℓ_k, u_k) satisfy $N(\ell_k, u_k) > 0$, with weights $\lambda_k > 0$, $\sum_k \lambda_k = 1$, and $\sum_k \lambda_k(\ell_k, u_k) \in \mathbb{Z}^{2d}$, then $N(\sum_k \lambda_k(\ell_k, u_k)) \geq \prod_k N(\ell_k, u_k)^{\lambda_k}$.

These counts are fibers of a totally unimodular system of corank d whose demand is integer affine in (ℓ, u) (Remark 8.2). The general L-convex statement is Theorem 8.1.

1.4 Organization and method

Sections 2–4 construct a continuous, coordinatewise monotone extension $\widehat{N}$ of the type-A_3 alcoved counts in all twelve directed bounds, such that $(\widehat{N} - 1/12)^{1/4}$ is concave (Theorem 4.1). Section 2 records the chamber polynomials in their Todd-operator form. Section 3 proves chamberwise concavity by a Lorentzian certificate, and Section 4 glues the chambers and adds shortest-path closure. Section 5 transfers the result to every unimodular system, using a matroid lemma that brings every rank-three totally unimodular normal system to the A_3 root system. Sections 6–9 contain the planar refinements, the integral obstructions, the L-convex families and the reduction of dimension four. Appendix A describes the computer certificates. Appendix B treats an opposite-edge rounding theorem.

Table 1: Results and their domains.

Result	Domain and conclusion
Theorems 1.1, 1.4	Any unimodular system, feasible fibers almost surely, integrable demand with integral mean, central fiber of dimension at most three. Logarithmic, corrected fourth-root and 2/11-power Jensen, countable support.
Theorems 1.5, 1.6	Central dimension at most two. Sharp correction 1/4; optimal leading power deficit $1/(8m_0)$.
Proposition 1.7	The optimal integral power A_3 in dimension three lies in $[2/11, \alpha_K]$, $\alpha_K \approx 0.30337$.
Theorem 7.2, Proposition 7.3	Complete four-vertex DAG: continuous power 4/13 (optimal for its classical extension); integral power at most $\alpha_* \approx 0.33102$.
Theorems 1.8, 8.1	L-convex (star-bound) families in every dimension: logarithmic Jensen.
Proposition 9.1	Dimension four reduces to type-A_4 alcoved counts and 3×3 cell-bounded tables.
Theorem B.1	Opposite-edge interval deformations: directed product rounding.

2 Type-A_3 chambers and the Riemann–Roch form

Write $h = (h_{ij})_{i \neq j}$ for twelve real numbers indexed by ordered pairs from $\{1, 2, 3, 4\}$, and anchor $z_4 = 0$. Define

$$P_h = \{z \in \mathbb{R}^3 : z_i - z_j \leq h_{ij} \ (i \neq j)\}, \qquad D = \{h : P_h \neq \varnothing\}.$$

These are the type-A_3 alcoved polytopes of Lam and Postnikov [22]. They are bounded because both directions of every root occur. The set D is the closed convex polyhedron defined by nonnegative directed cycle sums. For integral $h \in D$ put $N_3(h) = |P_h \cap \mathbb{Z}^3|$; the root matrix is totally unimodular, so these counts are positive.

Remark 2.1 (Alcoved counts as unimodular fibers). For each h, put $y_i = z_i + h_{4i}$ $(i \le 3)$ and $s_{ij} = h_{ij} - z_i + z_j$ for all twelve pairs. Then $z \in P_h \cap \mathbb{Z}^3$ corresponds bijectively to the nonnegative integral solutions $(s, y) \in \mathbb{Z}^{15}_{\ge 0}$ of

$$s_{ij} + y_i - y_j = h_{ij} + h_{4i} - h_{4j}, \quad s_{i4} + y_i = h_{i4} + h_{4i}, \quad s_{4i} - y_i = 0 \qquad (i, j \le 3).$$

The matrix $[I_{12} \mid N]$ is totally unimodular, since the rows of N are rows of a transposed directed incidence matrix with one vertex grounded. It has full row rank 12 and corank 3, and the right side is integer linear in h. Thus every family of type-A_3 alcoved counts is a unimodular fiber family of corank three.

2.1 Chambers

A tight support vector a satisfies the twenty-four triangle inequalities $\tau_{ijk} = a_{ij} + a_{jk} - a_{ik} \ge 0$ for distinct i, j, k; they imply $a_{ij} + a_{ji} \ge 0$. The corresponding cone C is convex. Translations $a_{ij} \mapsto a_{ij} + u_i - u_j$ translate the polytope and are factored out whenever we refer to a pointed cone. Define the six square forms

$$\begin{aligned} g_1 &= a_{13} + a_{24} - a_{14} - a_{23}, & g_2 &= a_{31} + a_{42} - a_{32} - a_{41}, \\ g_3 &= a_{12} + a_{34} - a_{14} - a_{32}, & g_4 &= a_{21} + a_{43} - a_{23} - a_{41}, \\ g_5 &= a_{12} + a_{43} - a_{13} - a_{42}, & g_6 &= a_{21} + a_{34} - a_{24} - a_{31}, \end{aligned}$$

which obey $g_3 + g_4 = g_1 + g_2 + g_5 + g_6$. For a sign vector $\sigma \in \{\pm 1\}^6$ let C_σ be the cone defined by the twenty-four triangle inequalities and $\sigma_s g_s \ge 0$; these thirty inequalities are the *base inequalities* of σ. The three-dimensional polytrope chambers [11, 12] have six representatives. In the coordinate order $12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43$ we use the seeds

1	29	20	11	19	21	20	20	11	29	21	20	19
2	42	32	22	22	32	42	32	32	32	42	22	32
3	50	36	24	26	36	48	48	24	34	36	36	34
4	77	60	43	77	43	60	60	43	77	43	60	77
5	78	60	44	60	44	78	78	44	60	44	60	78
6	78	62	44	78	44	62	44	62	78	62	44	78

Proposition 2.2 (Chamber structure; computer-certified). *(i) At each seed, P_a is a simple polytope with twenty vertices and twelve facets, and the three edge directions at each vertex form a basis of $\mathbb{Z}^3$. Every one of its forty-eight vertex-feasibility inequalities is a sum of at most two base inequalities of its sign vector, and all thirty base inequalities occur.*

(ii) Relabelling the four vertices and reversing all directed edges carry the six seed sign vectors to 62 distinct sign vectors, in disjoint orbits of sizes $12, 8, 24, 6, 6, 6$. The two remaining sign vectors $\pm(1, 1, -1, -1, 1, 1)$ violate $g_3 + g_4 = g_1 + g_2 + g_5 + g_6$.

(iii) Consequently every strict tight vector a, meaning $\tau_{ijk}(a) > 0$ and $g_s(a) \ne 0$ for all indices, lies in the interior of the cone $C_{\sigma(a)}$ of its own sign vector, and on that interior P_a has the normal fan of the corresponding relabelled seed.

Proof. Items (i) and (ii) are finite rational computations, carried out by `verify_A3_structure.py` (Appendix A). For each seed it solves every nonsingular triple of root rows exactly and retains the feasible vertices. It forms the forty-eight feasibility rows $a_k - A_k A_I^{-1} a_I$ and checks their decomposition into base inequalities. It also applies all 48 relabellings and transposes to the six sign lists. For (iii), the interior of C_σ makes every base inequality strict, hence every seed vertex formula strictly feasible. The vertex set and normal fan are therefore those of the seed, and every strict tight vector has one of the 62 realizable sign vectors. □

This chamber coverage is the computer-assisted step of the argument; all other steps are proved in the text, and the remaining finite identities are also replayed by the same script. Boundary points of C are limits of strict points; adding a small positive constant to every bound and perturbing generically shows that the closed cones C_σ cover C.

2.2 Count polynomials and their Todd form

Fix a strict chamber C_σ. Its polytopes are simple and unimodular with the twelve primitive facet normals $e_i - e_j$, and a_{ij} is the support number at $e_i - e_j$. By the Riemann–Roch theorem of Khovanskii and Pukhlikov [8] (see also Brion–Vergne [9]), the lattice count is the Todd operator applied to volume:

$$F(a) = \mathrm{Td}(\partial_a) V(a), \qquad \mathrm{Td}(\partial) = \prod_{i \neq j} \frac{\partial_{ij}}{1 - e^{-\partial_{ij}}} = 1 + \tfrac{1}{2} \sum \partial_{ij} + \cdots, \tag{2.1}$$

where V is the volume polynomial of the chamber. Equivalently, by Brion's theorem [10], F is the constant coefficient in t of

$$\sum_I e^{t\langle \xi, v_I \rangle} \prod_{j=1}^{3} \left(1 - e^{t\langle \xi, d_{Ij} \rangle}\right)^{-1} \tag{2.2}$$

for any integer ξ nonorthogonal to the primitive edge directions d_{Ij} at the twenty vertices v_I. The homogeneous components of F are the Todd components of degrees $0, 1, 2, 3$, and they take the form

$$F(a) = V(a) + Q(a) + L(a) + 1, \qquad L(a) = \frac{1}{3} \sum_{i \neq j} a_{ij} - \frac{1}{12} \sum_{s=1}^{6} |g_s(a)|. \tag{2.3}$$

Here V is the Euclidean volume and $Q = \frac{1}{2} D_{\mathbf{1}} V$ is the degree-one Todd term, the same quadratic in every chamber; L is linear within a chamber. If $B = \mathrm{Hess}\, Q$, then

$$(Ba)_{ij} = \frac{r_i + c_j}{2} - 2a_{ij} = \frac{1}{2} \sum_{k \neq i,j} (a_{ik} + a_{kj} - a_{ij}), \quad r_i = \sum_{j \neq i} a_{ij}, \quad c_j = \sum_{i \neq j} a_{ij}, \tag{2.4}$$

and $Q = a^T B a / 2$. With $K = P_{\mathbf{1}}$,

$$V(K) = 4, \qquad B\mathbf{1} = \mathbf{1}, \qquad Q = \tfrac{1}{2} D_{\mathbf{1}} V. \tag{2.5}$$

These are coefficient identities of (2.2) on each chamber: the cubic coefficient gives V, the quadratic coefficient gives (2.4), and the linear coefficient is $\mathbf{1}/3 - \sum_s \sigma_s g_s / 12$. They are reconstructed exactly, without importing the classical polynomials [11, 14], by the script of Appendix A. Each coordinate coefficient of L in every chamber is $1/6$, $1/3$ or $1/2$.

2.3 Minkowski additivity

The normal fan of every chamber refines the fan of K: at $\mathbf{1}$ every triangle slack equals 1 and every g_s vanishes, so all thirty base inequalities hold for every sign vector, and by Proposition 2.2(i) every chamber vertex formula remains feasible at K. Thus $\mathbf{1}$ lies in every closed chamber. On a closed chamber all vertex formulas remain feasible and their normal cones cover the dual space, so support functions add:

$$P_{a+b} = P_a + P_b \qquad \text{for } a, b \text{ in the same closed chamber,} \tag{2.6}$$

including degenerate summands. In particular $P_{a+t\mathbf{1}} = P_a + tK$. Modulo translations a closed chamber is a pointed rational cone with finitely many generators a_j. The gauge $a_{14} = a_{24} = a_{34} = 0$, obtained from the potential $u_i = -a_{i4}$, makes every other coordinate nonnegative by the triangle inequalities. A point polytope represents only the zero class, so every nonzero generator gives a nonpoint polytope and $L(a_j) > 0$.

3 The Lorentzian certificate

We use the following facts from Brändén–Huh [7]; theorem numbers refer to arXiv:1902.03719v8.

(L1) [7, Theorem 2.25 and the characterization following it]. The closure of the strictly Lorentzian polynomials is the space of Lorentzian polynomials. Consequently, a homogeneous polynomial f of degree $d \geq 2$ with nonnegative coefficients is Lorentzian if and only if its support is M-convex and every partial derivative $\partial^\alpha f$ with $|\alpha| = d - 2$ is a quadratic form with at most one positive eigenvalue.

(L2) [7, Theorem 2.30 and Proposition 2.33]. Lorentzian polynomials are strongly log-concave. Hence $f^{1/d}$ is concave on the open positive orthant, and by continuity on the closed orthant.

(L3) [7, Theorem 4.1 and Corollary 4.2]. For convex bodies $K_1, \dots, K_k$ in $\mathbb{R}^3$, the volume polynomial $w \mapsto \mathrm{vol}(\sum_i w_i K_i)$ is Lorentzian and has M-convex support. The chamber polytopes are nef divisors on the smooth projective toric threefold of the chamber fan, so [7, Theorem 4.6] gives the same conclusion.

(L4) [7, Definition 2.6 and Lemma 2.5]. The partial derivatives of a Lorentzian polynomial are Lorentzian, and a quadratic form with nonnegative coefficients is Lorentzian exactly when it has at most one positive eigenvalue.

Theorem 3.1 (Chamberwise corrected fourth root)**.** *On every closed chamber, $a \mapsto (F(a) - 1/12)^{1/4}$ is concave.*

Proof. Fix a closed chamber with generators a_j, put $a(y) = \sum_j y_j a_j$ for $y \geq 0$, and define the homogeneous quartic

$$\mathcal{H}(y,t) = tV(a(y)) + t^2 Q(a(y)) + t^3 L(a(y)) + \tfrac{11}{12} t^4,$$

so that $\mathcal{H}(y,1) = F(a(y)) - 1/12$. Let $W(y,t) = \mathrm{vol}\big(\sum_j y_j P_{a_j} + tK\big)$. By (2.6), $W(y,t) = V(a(y) + t\mathbf{1})$, a Lorentzian cubic by (L3). All coefficients of $\mathcal{H}$ are nonnegative: those of $V(a(y))$ are mixed volumes of the P_{a_j}, those of $Q(a(y))$ are $d^T B e = 3V(P_d, P_e, K) \geq 0$, and $L(a_j) > 0$.

Support. Taylor expansion and (2.5) give $W = V + 2tQ + t^2 \sum_{ij} a_{ij} + 4t^3$, since $\frac{1}{2} D^2_{\mathbf{1}} V(a) = \sum a_{ij}$ is the mixed volume $3V(P_a, K, K)$. The monomials of tW and $\mathcal{H}$ therefore agree: compare Q with

$2Q$, and note that $L(a_j) > 0$ and $\sum_{ik}(a_j)_{ik} > 0$ for nonpoint P_{a_j}. The support of tW is a translate of the M-convex support of W, so the support of $\mathcal{H}$ is M-convex.

Second derivatives. By (L1) it suffices that each of the following quadratic forms in (y,t) has at most one positive eigenvalue.

(a) $\partial_t^2\mathcal{H} = 2Q + 6tL + 11t^2$. On the chamber write $L = \ell^T a$, $\ell = \mathbf{1}/3 - R/12$ with $R = \sum_s \sigma_s g_s$. The matrix B has orthogonal eigenspaces: constants (eigenvalue 1, dimension 1); symmetric arrays $u_i + u_j$ with $\sum u_i = 0$ (eigenvalue -1, dimension 3); arrays with zero row and column sums (eigenvalue -2, dimension 5); and potentials $u_i - u_j$ (eigenvalue 0, dimension 3). Thus B has inertia $(1,8)$ on the translation quotient. Each g_s has zero row and column sums, so $Bg_s = -2g_s$. The Gram matrix of the g_s has diagonal 4, zeros within the pairs $(1,2),(3,4),(5,6)$, entries 1 between the first and second pairs, -1 between the first and third, and 1 between the second and third. With $U = \sigma_1 + \sigma_2$, $W' = \sigma_3 + \sigma_4$, $T = \sigma_5 + \sigma_6$ this gives $\|R\|^2 = 24 + 2(UW' - UT + W'T) \le 32$; the multiaffine bracket takes only the values 4 and -12 at the vertices of $[-2,2]^3$. With $u = B^{-1}\ell$ on the quotient,

$$\rho = \ell^T B^{-1}\ell = \frac{4}{3} - \frac{\|R\|^2}{288} \ge \frac{11}{9}, \qquad \partial_t^2\mathcal{H} = (a+3tu)^T B(a+3tu) + (11-9\rho)t^2.$$

The last coefficient is nonpositive and B has one positive direction, so this form has at most one positive eigenvalue, also after pullback along $y \mapsto a(y)$.

(b) $\partial_{y_d}\partial_t\mathcal{H}$ for a generator d. Differentiating $W = V + 2tQ + t^2\sum a_{ij} + 4t^3$ gives $\partial_{y_d}W = D_dV + 2tD_dQ + t^2\sum d_{ij}$, so

$$\partial_{y_d}\partial_t\mathcal{H} = D_dV + 2tD_dQ + 3t^2L(d) = \partial_{y_d}W(y,t) + t^2\Big(3L(d) - \sum_{ij} d_{ij}\Big).$$

The first term is a first partial derivative of the Lorentzian volume polynomial W. By (L4) it is a Lorentzian quadratic form in (y,t) with at most one positive eigenvalue. On the closed chamber, $3L(d) - \sum d_{ij} = -\frac{1}{4}\sum_s \sigma_s g_s(d) \le 0$, and adding a negative semidefinite rank-one term does not create a positive eigenvalue.

(c) $\partial_{y_d}\partial_{y_e}\mathcal{H} = t\,D_dD_eV(a(y)) + t^2B(d,e)$, for generators d,e. Both $D_dD_eV(a(y))$, a linear form in y, and $B(d,e)$ are nonnegative mixed volumes. The form is t times a linear form, so it has at most one positive eigenvalue.

These cases exhaust the second partials. Hence $\mathcal{H}$ is Lorentzian, and by (L2) $\mathcal{H}^{1/4}$ is concave on $\mathbb{R}^{k+1}_{\ge 0}$. At $t = 1$, $y \mapsto (F(a(y)) - 1/12)^{1/4}$ is concave. Choosing nonnegative generator representations of two support vectors and interpolating transfers this to support coordinates, even when the representations are not unique. □

4 Gluing and closure

Theorem 4.1 (A monotone corrected fourth-root extension)**.** *There is a continuous, coordinatewise nondecreasing function $\widehat{N} : D \to [1,\infty)$ with $\widehat{N}(h) = N_3(h)$ for every integral $h \in D$, including lower-dimensional polytopes, such that*

$$G(h) = \big(\widehat{N}(h) - \tfrac{1}{12}\big)^{1/4} \quad \textit{and} \quad \widehat{N}(h)^{2/11}$$

are concave on D. For this extension $\widehat{N}$, $2/11$ is the largest power α with $\widehat{N}^\alpha$ concave and $1/12$ is the least real correction giving a concave fourth root.

4.1 Gluing across chambers

For polytopes with fixed facet normals, Aleksandrov's variational lemma [18, Lemma 7.5.3] gives the first variation of volume: at every h with P_h full-dimensional, $h \mapsto \mathrm{vol}(P_h)$ is differentiable, and its partial derivative in h_{ij} is the area of the face with normal $e_i - e_j$ divided by $|e_i - e_j|$. Let a lie on a wall between closed chambers C_σ and $C_{\sigma'}$ in the interior of C. For every direction v in the open cone of directions pointing into C_σ, the derivative of the volume along v equals $\langle \nabla V_\sigma(a), v\rangle$, where V_σ is the chamber polynomial. It also equals the pairing of v with the gradient from the variational lemma. Two linear forms that agree on an open cone are equal, so $\nabla V_\sigma(a) = \nabla V_{\sigma'}(a)$. Thus V is C^1 on the interior of C, across changes of combinatorial type as well; a facet that disappears does so with area tending to zero. Moreover Q is one global quadratic, and L in (2.3) is concave, with only downward derivative jumps. Also $F \geq 1$: in the nonnegative gauge, mixed-volume positivity and $L \geq 0$ give $V, Q, L \geq 0$. The derivative jumps of $(F - 1/12)^{1/4}$ have the signs of those of F. Chamberwise concavity (Theorem 3.1) therefore glues along generic segments; approximation covers segments inside walls and on the boundary of C. Hence $(F - 1/12)^{1/4}$ is concave on C.

4.2 Monotonicity

Volume is inclusion-monotone. By (2.4), $\partial_{ij} Q$ is half a sum of two nonnegative triangle slacks, so Q is nondecreasing along comparable points of C. Each coordinate occurs in exactly two g_s, so for every nonnegative increment d

$$\sum_s \bigl(|g_s(a+d)| - |g_s(a)|\bigr) \leq 2 \sum_{ij} d_{ij}, \qquad L(a+d) - L(a) \geq \tfrac{1}{6} \sum_{ij} d_{ij}.$$

Hence F and its corrected fourth root are coordinatewise monotone on C, across chamber walls as well.

4.3 Shortest-path closure

For $h \in D$ let $c_{ij}(h)$ be the shortest directed path length from j to i, where the edge $j \to i$ has length h_{ij}. Feasibility excludes negative cycles, so a minimum over the finitely many simple paths suffices. The map c is continuous, entrywise concave, integral on integral inputs, and satisfies $c(h) \in C$ and $P_{c(h)} = P_h$. Therefore $c(\sum_s \lambda_s h_s) \geq \sum_s \lambda_s c(h_s)$. Define $\widehat{N}(h) = F(c(h))$ and $G(h) = (\widehat{N}(h) - 1/12)^{1/4}$. Concavity and monotonicity on C make G concave and monotone on D.

4.4 Exact values at degenerate points

Let $a \in C$ be integral and choose a closed chamber containing it and a strict integral vector v in that chamber. By (2.6), $P_{ra+sv} = rP_a + sP_v$, and the Bernstein–McMullen theorem [17, Theorem 2.3] makes its lattice count a polynomial in the nonnegative integers r, s, including zero. For $r \geq 0$, $s \geq 1$ it agrees with the chamber polynomial $F(ra + sv)$ by (2.2). Polynomial identity gives agreement at $(r, s) = (1, 0)$ as well, which covers lower-dimensional P_a because P_v is full-dimensional. Closure preserves the polytope and integrality, so $G(h) = (N_3(h) - 1/12)^{1/4}$ for every integral $h \in D$.

4.5 Count-dependent powers

For $m_0 \geq 1$ put $\beta_{m_0} = (3m_0 - 1)/(12m_0 - 1)$. The scalar map $\phi_\alpha(x) = (x^4 + c)^\alpha$, $c = 1/12$, satisfies

$$\phi''_\alpha(x) = 4\alpha x^2 (x^4 + c)^{\alpha - 2} \bigl[(4\alpha - 1)x^4 + 3c\bigr],$$

so it is increasing and concave for $x^4 \geq m_0 - c$ when $\alpha = \beta_{m_0}$. Since $G \geq (1-c)^{1/4}$, composition gives concavity and monotonicity of $\widehat{N}^{2/11}$ on D.

4.6 Optimality for this extension

The fourth seed $a_* = (77, 60, 43, 77, 43, 60, 60, 43, 77, 43, 60, 77)$ lies in a strict chamber with all six $g_s > 0$. There

$$\ell = (\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}), \quad u = (\tfrac{5}{12}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{5}{12}),$$

with $Bu = \ell$ and $\ell^T u = 11/9$. The chamber polynomial has $F = 1$, $\nabla F = \ell$ and $\operatorname{Hess} F = B$ at the origin, so

$$D_u^2 F^\alpha(0) = \alpha\rho\big[1 - (1-\alpha)\rho\big], \qquad \rho = \tfrac{11}{9},$$

which is positive for every $\alpha > 2/11$. By continuity the same holds at εa_* for small $\varepsilon > 0$, an interior point of the strict chamber, so F^α is not concave there. Similarly, for $c < 1/12$,

$$D_u^2 (F - c)^{1/4}(0) = \tfrac{1}{4}(1-c)^{-7/4}\rho\big[(1-c) - \tfrac{3}{4}\rho\big] > 0.$$

These obstructions use real interior bounds; integral demands are treated in Section 7. This completes the proof of Theorem 4.1.

5 Transfer and the rank-three matroid lemma

Lemma 5.1 (Rank-three normal systems)**.** *Let $\mathcal{A}$ be a totally unimodular matrix with three columns whose rows include e_1, e_2, e_3. There is $T \in \mathrm{GL}(3, \mathbb{Z})$ such that every row of $\mathcal{A}T$ is zero or a signed root in $\Phi = \{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}$.*

Proof. Two parallel nonzero rows of a $\{0, \pm 1\}$ matrix are equal up to sign, so after deleting zero rows and repetitions up to sign the rows form a simple matroid $\mathcal{M}$ of rank three. It is regular, being represented by a totally unimodular matrix. By Tutte's excluded-minor theorem [19], a regular matroid is graphic unless it has a minor $M^*(K_5)$ or $M^*(K_{3,3})$. Those have ranks 6 and 4, and minors do not increase rank, so $\mathcal{M}$ is graphic; this also follows from Seymour's decomposition theorem [20]. A simple graphic matroid of rank three is $M(G)$ for a simple connected graph on four vertices, that is, a restriction of $M(K_4)$.

Under this isomorphism the basis e_1, e_2, e_3 corresponds to a spanning tree with edges t_1, t_2, t_3. Orient the edges of K_4 and send the edge $i \to j$ to $\rho(i \to j) = \varepsilon_i - \varepsilon_j$ with $\varepsilon_4 = 0$. The vectors $\rho(t_k)$ form a unimodular basis, the grounded incidence matrix of a tree. Let $c(f)$ be the coordinate vector of $\rho(f)$ in this basis. The matrix of the $c(f)$ is a network matrix, hence totally unimodular. The row n_f of $\mathcal{A}$ is its own coordinate vector in the basis e_1, e_2, e_3. Both vectors have support equal to the fundamental circuit of f minus f, so the two matrices have the same support. By Camion's theorem [21], totally unimodular matrices with the same support differ by multiplying rows and columns by -1: $n_f = \epsilon_f \, c(f) \operatorname{diag}(\delta)$. Let T have rows $e_k T = \delta_k \rho(t_k)$. Then $n_f T = \epsilon_f \rho(f)$ is a signed root, and $\det T = \pm 1$. Restoring repetitions and signs proves the lemma. □

The finite version of Lemma 5.1 is also checked exhaustively: `verify_rank_three.py` tests all 2^{10} sets of additional $\{0, \pm 1\}$ directions, finds 51 totally unimodular systems (16 maximal), and exhibits an explicit T for each maximal one.

Proof of Theorem 1.4. By the reduction of Section 1 we may assume full row rank. If $n = 0$ every fiber is a point. Assume $n > 0$. Boundedness of $P(b)$ gives $\ker M \cap \mathbb{R}^n_{\geq 0} = \{0\}$, so the continuous function $\|Mx\|$ has a positive minimum η on the nonnegative unit ℓ^1 simplex, and

$$\|x\|_1 \leq \eta^{-1}\|Mx\| \qquad (x \geq 0). \tag{5.1}$$

Every feasible fiber is therefore bounded, and any selection of feasible points at the atoms is integrable.

Let J be the coordinates not identically zero on $P(b)$. If some point of an endpoint fiber were positive outside J, select it at that atom and arbitrary feasible points elsewhere. Integer-valued demand has countable support, so this selection is measurable and, by (5.1), integrable. Its expectation would lie in $P(b)$ and be positive outside J, which is impossible. Hence all fibers lie in one active face. Averaging finitely many central witnesses gives a point positive on J, and $r = |J| - \operatorname{rank} M_J = \dim P(b) \leq 3$.

Normalize M by a unimodular column basis B_0: $B_0^{-1}M = [I, D_0]$ is totally unimodular, since every minor of D_0 completes with identity columns to a maximal minor. Restrict to J, choose independent rows, and an integral pivot basis H_0. The solutions have common coordinates

$$x_{\text{pivot}} = H_0^{-1}c(a) - H_0^{-1}D_1 z, \qquad x_{\text{free}} = z,$$

with $c(a)$ integer linear in a; feasibility ensures consistency of the omitted equations. Complementary minors show that $[-H_0^{-1}D_1; I_r]$ is totally unimodular, and the parameterization is a lattice bijection. If $r < 3$, pad with coordinates fixed to zero by two opposite inequalities. The resulting three-column normal matrix is totally unimodular and contains $-I_3$; negating its columns gives a matrix containing I_3, as in Lemma 5.1. Lemma 5.1 gives one unimodular transformation for the entire family, after which every fiber is

$$Q(a) = \{w \in \mathbb{R}^3 : n_j \cdot w \leq c_j(a),\ j \in \mathcal{J}\},$$

with fixed signed roots n_j, integer-linear offsets c_j, possibly zero-normal feasibility rows, and preserved counts.

Read a root inequality $w_i - w_j \leq c$ as an edge $j \to i$ of length c, keeping repeated edges. This graph is strongly connected: otherwise its condensation admits a nonconstant potential with $v_i - v_j \leq 0$ on every edge, and subtracting the anchor value gives a nonzero recession vector of $Q(b)$. For each ordered pair let $h_{ij}(a)$ be the minimum length over the finitely many simple directed paths. Each length is integer linear in a, so h is componentwise concave, integral on integral demands, and $|h(a)| \leq C\|a\|$. At feasible demands there are no negative cycles and $Q(a) = P_{h(a)}$.

The random vector $Y = h(A)$ is integrable and lies in D. For every path p, $\mathbb{E}h_{ij}(A) \leq \mathbb{E}c_p(A) = c_p(b)$, so $\mathbb{E}Y \leq h(b)$. Theorem 4.1 and countable Jensen give

$$(N(b) - \tfrac{1}{12})^{1/4} = G(h(b)) \geq G(\mathbb{E}Y) \geq \mathbb{E}G(Y) = \mathbb{E}(N(A) - \tfrac{1}{12})^{1/4}.$$

For the middle inequality, enumerate the atoms y_s with probabilities p_s and group the tail after k at its conditional mean $y_{>k} \in D$ (the domain D is closed and convex). Finite concavity and $G \geq 0$ give

$$G(\mathbb{E}Y) \geq \sum_{s \leq k} p_s G(y_s) + \Big(\sum_{s>k} p_s\Big) G(y_{>k}) \geq \sum_{s \leq k} p_s G(y_s),$$

and monotone convergence proves the inequality and the finiteness of the expectation. Every count is a positive integer. If $N(A) \geq m_0$ almost surely, then $G(Y) \geq (m_0 - 1/12)^{1/4}$, and (1.2) gives $N(b) \geq m_0$. The function $\phi_{\beta_{m_0}}$ of Section 4 is increasing and concave on this interval, and its tangent at the left endpoint dominates it, so $\phi_{\beta_{m_0}}(G(Y))$ is integrable; scalar Jensen gives the β_{m_0}-power inequality. □

Proof of Theorem 1.1. For each feasible atom choose a vertex of $P(a)$. Vertices are basic solutions $x_B = B^{-1}a$ for finitely many bases B, so this selection is integrable, and its mean lies in $P(b)$. Thus $P(b) \neq \varnothing$. If $P(b)$ is unbounded, then $N(b) = \infty$ (Section 1). If $P(b)$ is bounded, take $m_0 = 1$ in Theorem 1.4: $0 \le \log N(A) \le \frac{11}{2} N(A)^{2/11}$ gives integrability, and $\mathbb{E} \log N(A) \le \frac{11}{2} \log \mathbb{E} N(A)^{2/11} \le \log N(b)$. □

Proof of Corollaries 1.2 and 1.3. If μ charges a demand with empty fiber, the right side of (1.1) is zero, with the convention $0 \cdot \infty = 0$. Otherwise Theorem 1.1 applies, since every nonempty fiber has dimension at most the corank. For Corollary 1.3, the bipartite incidence matrix of the support of W is totally unimodular, the margins are linear in the demand, and $T(R, C; W)$ is its fiber count. □

Corollary 5.2 (Block products)**.** *Let a finite DAG be assembled at articulation vertices from blocks of cycle rank at most three. Its integer-flow counts satisfy* (1.1) *for every finitely supported family with integral mean.*

Proof. For a leaf block attached at v, its induced demand is the original demand away from v and minus the sum of those demands at v; the remaining demand at v is the original one minus this induced demand. These maps are integer linear, and restricting flows to the two edge sets is a product bijection, so counts multiply. Iterate, apply Theorem 1.1 to each block, and add the logarithmic inequalities. □

6 Planar sharpness

Lemma 6.1 (Active-coordinate reduction)**.** *Under the hypotheses of Theorem 1.5, all fibers lie in the coordinate face active at $P(b)$. In common integral coordinates their free dimension is $r = \dim P(b)$. If $r = 2$, every full-dimensional fiber polygon has primitive outward normals among*

$$\pm(1,0), \quad \pm(0,1), \quad \pm(1,1) \tag{6.1}$$

after one common coordinate sign change, and coordinate maps commute with weighted averaging.

Proof. Let E_* be the coordinates positive at some point of $P(b)$. Minkowski averaging gives $\sum_i \lambda_i P(a_i) \subset P(b)$; fixing all endpoint points but one shows that every endpoint coordinate outside E_* vanishes. Averaging witnesses gives a central point positive on E_*, so the affine dimension is the kernel dimension of the retained columns M_*. Let $s = \operatorname{rank} M_*$, choose s independent active columns B_0, and complete them to a column basis $B = (B_0, B_1)$ of M. Since $\det B = \pm 1$, the active columns of $B^{-1}M$ have the form $\binom{D}{0}$ with $D = [I_s, C]$, and every square minor of C is a maximal minor of M up to sign; so C is totally unimodular. With free coordinates $t \in \mathbb{R}^r$, the active solutions are $x_{\text{free}} = t$, $x_{B_0} = (B^{-1}a)_{1:s} - Ct$, a lattice bijection with integer-linear offset. For $r = 2$ the rows of C lie in $\{0, \pm 1\}^2$, and $(1,1)$ and $(1,-1)$ cannot both occur because their determinant is ± 2. Including the free nonnegativity inequalities and changing one sign gives (6.1). All fibers are integral, and linearity gives $\sum_i \lambda_i Q(a_i) \subset Q(b)$ in the common coordinates. □

Set $K_2 = \{(x, y) : |x| \le \frac{1}{2},\ |y| \le \frac{1}{2},\ |x + y| \le \frac{1}{2}\}$, of area $3/4$ and support value $1/2$ at each normal in (6.1).

Lemma 6.2 (Exact planar representation)**.** *If Q is an integral polygon with normals* (6.1)*, or a degenerate fiber segment or point in this normal system, then $|Q \cap \mathbb{Z}^2| = \operatorname{area}(Q + K_2) + \frac{1}{4}$.*

Proof. For full-dimensional Q, mixed area gives $\operatorname{area}(Q+K_2) = \operatorname{area}(Q) + \frac{1}{2}B(Q) + \frac{3}{4}$, where $B(Q)$ is the lattice perimeter: an edge with primitive normal v contributes its length times $h_{K_2}(v)/\|v\|$, half its lattice length. Pick's formula concludes. For a point both sides equal 1. A degenerate segment of lattice length ℓ is perpendicular to one of the normals; the transverse width of K_2 times the primitive length is 1, so both sides equal $\ell + 1$. □

The functional $Q \mapsto \operatorname{area}(Q + K_2) + \frac{1}{4}$ equals $\mathbb{E}\operatorname{area}(Q + RK_2)$ for R uniform on $[0, 2]$, a generalized Wills functional [23]; lattice Brunn–Minkowski inequalities in the sense of [24] include an enlargement absent here.

Proof of Theorem 1.5. In dimension zero all fibers are points. In dimension one the fibers are integral segments in a common rational line, of normalized lengths ℓ_i and $\ell \geq \sum_i \lambda_i \ell_i$, so $N(b) \geq \sum_i \lambda_i N_i$; apply the increasing concave map $x \mapsto \sqrt{x - 1/4}$ on $[1, \infty)$. In dimension two put $c = 1/4$ and $s_i = \sqrt{\operatorname{area}(Q(a_i) + K_2)} = \sqrt{N_i - c}$. Brunn–Minkowski, the inclusion of Lemma 6.1 and monotonicity give

$$\sqrt{N(b) - c} = \sqrt{\operatorname{area}(Q(b) + K_2)} \geq \sum_i \lambda_i s_i, \tag{6.2}$$

which is (1.3). For $g_\alpha(s) = (s^2 + c)^\alpha$, $g''_\alpha(s) = 2\alpha(s^2 + c)^{\alpha-2}[c + (2\alpha - 1)s^2]$, and at $\alpha = \alpha_{m_0}$ one has $1 - 2\alpha = c/(m_0 - c)$. Thus $g_{\alpha_{m_0}}$ is increasing and concave on $[\sqrt{m_0 - c}, \infty)$; apply it to (6.2) and use scalar Jensen. □

Lemma 6.3 (A fixed network family)**.** *Orient $K_{2,3}$ from its two sources to its three sinks. For net outflow $a(r) = (3r, 3r, -2r, -2r, -2r)$, $r \in \mathbb{Z}_{\geq 0}$, the count is $n(r) = 3r^2 + 3r + 1$, and for $r > 0$ the fiber has dimension two.*

Proof. If t_j is the flow from the first source to sink j, the other inflow is $2r - t_j$, so $0 \leq t_j \leq 2r$ and $\sum_j t_j = 3r$. Inclusion–exclusion gives $\binom{3r+2}{2} - 3\binom{r+1}{2} = 3r^2 + 3r + 1$; two violations are impossible. For $r > 0$, (r, r, r) is a relative interior point of the two-dimensional slice. □

Proof of Theorem 1.6. Theorem 1.5 gives $\mathsf{A}(m_0) \geq \alpha_{m_0}$. For $f_\alpha(r) = n(r)^\alpha$ the sign of f''_α is that of $\alpha - \tau(r)$, with $\tau(r) = \frac{1}{2} - \frac{1}{6(2r+1)^2} = \frac{1}{2} - \frac{1}{8n(r)-2}$, an increasing function. At $\alpha = 1/2$ this is positive everywhere, so with $R = R_{m_0}$, $2n(R+1)^{1/2} < n(R)^{1/2} + n(R+2)^{1/2}$. For $\alpha > \tau(R+2)$, f_α is strictly convex on $[R, R+2]$, so the endpoints $R, R+2$ and integral midpoint $R+1$ violate power Jensen with counts at least m_0; hence $\mathsf{A}(m_0) \leq \tau(R+2)$. Minimality gives $R = \sqrt{m_0/3} + O(1)$ and $n(R+2) = m_0 + O(\sqrt{m_0})$, and the expansion follows. □

Sharpness of the correction 1/4. For the same network, $n(r) - c = 3(r + \frac{1}{2})^2 + \frac{1}{4} - c$. If $c < 1/4$, then $\sqrt{n(r) - c}$ is strictly convex on $[0, \infty)$, and every adjacent integral triple violates the corrected inequality. For $c = 1/4$ the function is affine, and these families give equality in (1.3). □

7 Obstructions: the K-ray and the four-vertex DAG

7.1 The K-ray

Proof of Proposition 1.7. Theorem 1.4 with $m_0 = 1$ gives $\mathsf{A}_3 \geq 2/11$. Take the type-A_3 bounds $h = r\mathbf{1}$, so $P_h = rK$. On the chamber closure containing $\mathbf{1}$, (2.3) and (2.5) give $V(\mathbf{1}) = 4$, $Q(\mathbf{1}) = \frac{1}{2}\mathbf{1}^T B\mathbf{1} = 6$ and $L(\mathbf{1}) = 4$, since every $g_s(\mathbf{1}) = 0$. Exactness at integral points (Section 4) yields

$$N_3(r\mathbf{1}) = 4r^3 + 6r^2 + 4r + 1 = (r+1)^4 - r^4, \qquad N_3(0) = 1,\ N_3(\mathbf{1}) = 15,\ N_3(2 \cdot \mathbf{1}) = 65.$$

By Remark 2.1 these are fibers of a fixed unimodular matrix of corank three over the demands 0, d, $2d$ with d integral, so the two-point law on $\{0, 2d\}$ has integral mean d. Its central fiber K has dimension three. Power Jensen for this family requires $2 \cdot 15^\alpha \geq 1 + 65^\alpha$. Put $x = 15^\alpha$ and $q = \log 65 / \log 15 \in (1, 2)$. The function $x^q - 2x + 1$ is strictly convex, vanishes at $x = 1$, has negative derivative there and tends to $+\infty$, so it has one further zero $x_* > 1$. This defines the unique positive root α_K, and power Jensen fails for every $\alpha > \alpha_K$. The integer inequalities

$$15^3 \cdot 1000^{10} > 2253^{10}, \quad 65^3 \cdot 1000^{10} < 3499^{10}, \quad 15^4 \cdot 1000^{13} < 2301^{13}, \quad 65^4 \cdot 1000^{13} > 3612^{13}$$

give $2 \cdot 15^{3/10} > 4.506 > 1 + 65^{3/10}$ and $2 \cdot 15^{4/13} < 4.602 < 4.612 < 1 + 65^{4/13}$, hence $3/10 < \alpha_K < 4/13$. □

On this ray the corrected fourth root holds with margin $2(15 - \frac{1}{12})^{1/4} - (\frac{11}{12})^{1/4} - (65 - \frac{1}{12})^{1/4} > 0.11$, and logarithmic Jensen holds because $15^2 > 65$.

7.2 The complete four-vertex DAG

For a DAG we use net outflow as the incidence convention. After one incidence row per component is deleted, the incidence matrix is unimodular, and acyclicity makes every nonempty flow fiber bounded. For the complete DAG on $0, 1, 2, 3$ put $(s, t, u) = (b_0, b_0 + b_1, b_0 + b_1 + b_2)$. The feasible integral demands correspond exactly to $(s, t, u) \in \mathbb{Z}^3_{\geq 0}$, with inverse $b = (s, t - s, u - t, -u)$. With chord flows $x_{02} = a$, $x_{03} = v$, $x_{13} = c$ the count is

$$|\{(a, v, c) \in \mathbb{Z}^3_{\geq 0} : a + v \leq s,\ v + c \leq u,\ a + v + c \leq t\}|. \tag{7.1}$$

Put $H(s, u) = \frac{1}{6}(s + 1)(s + 2)(3u - s + 3)$ and $B(d) = \frac{1}{6}d(d + 1)(d + 2)$. The classical chamber formulas [14, Section 5], going back to Tarski [13, Section VII], define for $0 \leq s \leq u$

$$F(s, t, u) = \begin{cases} (t + 1)(t + 2)(t + 3)/6, & 0 \leq t \leq s, \\ (s + 1)(s + 2)(3t - 2s + 3)/6, & s \leq t \leq u, \\ H(s, u) - B(s + u - t), & u \leq t \leq s + u, \\ H(s, u), & s + u \leq t, \end{cases} \tag{7.2}$$

and for $s > u$ we interchange s, u. In the first chamber (7.1) is a simplex; in the second, sum $(w + 1)(t - w + 1)$ over $0 \leq w \leq s$. In the last two, summing the rectangles $0 \leq a \leq s - v$, $0 \leq c \leq u - v$ gives $H(s, u)$, and for $d = s + u - t \geq 0$ the excluded corner points number $\sum_{v<d} \binom{d-v+1}{2} = B(d)$. The values agree on all walls, and $F > 0$. Log-concavity of restricted Kostant partition functions along root directions is proved in [15], and capacity lower bounds for Kostant partition functions are given in [6].

Lemma 7.1 (A Hessian comparison)**.** *In the third chamber of* (7.2)*, $F^{4/13}$ is concave.*

Proof. Use affine coordinates (s, L, d) with $L = 3u - s + 3$, and write $H = f(s)L$ with $f(s) = (s + 1)(s+2)/6$. Put $\beta = 9/13$ and $C = \beta \nabla F \nabla F^\top - F \operatorname{Hess} F$; concavity of $F^{4/13}$ is $C \succeq 0$. Eliminating the positive (d, d) entry gives the Schur complement $F[khh^\top - \operatorname{Hess} H]$ with $k = \beta B''/(\beta(B')^2 + FB'')$ and $h = \nabla H$ in (s, L). Its (L, L) entry is $kf^2 > 0$, and its determinant is $kLf(2(f')^2 - ff'') - (f')^2$. Hence $C \succeq 0$ is equivalent to $\beta(HJ - \mathcal{Q}) \geq H - B$ with $J = 2 - ff''/(f')^2 = \frac{3}{2} + \frac{1}{2(2s+3)^2}$ and $\mathcal{Q} = (B')^2/B''$. For $w = 2s + 3$ this is $E \geq 0$, where $E = \frac{H(w^2+9)}{2w^2} + 13B - 9\mathcal{Q}$. The first term increases with u; at $u = s$ it equals $(w^3 + 8w - 9/w)/48$, which increases with s. Since $0 \leq d \leq s \leq u$, lowering u to s and then s to d gives

$$E(s, u, d) \geq E(d, d, d) = \frac{d(2d^4 + 17d^3 + 46d^2 + 58d + 30)}{12(d + 1)(2d + 3)} \geq 0. \qquad \square$$

Theorem 7.2 (The complete four-vertex DAG)**.** *The extension F of* (7.2) *is positive and continuous on $\mathbb{R}^3_{\geq 0}$, and $F^{4/13}$ is concave. No power $\alpha > 4/13$ makes F^α concave. For every finitely supported family of feasible demands with integral mean, $N(b)^{4/13} \geq \sum_i \lambda_i N_i^{4/13}$.*

Proof. The other three chamber polynomials are products of three positive affine functions, so their cube roots and hence their $4/13$ powers are concave; Lemma 7.1 handles the third chamber. For increasing t with $s < u$, with gradients ordered (s, t, u), the gradient jumps of F at the walls $t = s$, $t = u$, $t = s + u$ are $-\frac{3s+5}{6}(-1, 1, 0)$, $-\frac{3s+4}{6}(0, 1, -1)$ and $-\frac{1}{3}(-1, 1, -1)$, each a negative multiple of the normal into the next chamber. On the symmetry wall $s = u$ the jump is $-\frac{1}{6}(1, 0, -1)$ in the last two chambers and zero in the first; the second chamber meets it only at $s = t = u$. Since wall values agree, the jumps of $F^{4/13}$ have the same signs, so the restriction to generic lines is concave; approximation and continuity give global concavity. Prefix sums are linear in the demand. For optimality, in the coordinates (s, u, d) of the third chamber at the origin, $\det[(1-\alpha)\nabla F \nabla F^\top - F \operatorname{Hess} F] = (4 - 13\alpha)/4$, which is negative for $\alpha > 4/13$ and stays negative at interior points nearby. □

Proposition 7.3 (Integral obstruction for the four-vertex DAG)**.** *Cube-root Jensen fails on the integral midpoint triples $(0,0,0), (2,3,2), (4,6,4)$ and $(2,3,2), (4,6,4), (6,9,6)$ in prefix coordinates, with counts $1, 13, 51, 130$; all fibers in the second triple are three-dimensional. If A_{K_4} is the supremum of the powers satisfying Jensen for all finitely supported integral-mean families on this graph, then $4/13 \leq A_{K_4} \leq \alpha_* < 1/3$, where $2 \cdot 13^{\alpha_*} = 1 + 51^{\alpha_*}$ and $\alpha_* = 0.33101\ldots$.*

Proof. Direct enumeration in (7.1) gives the counts. The integer inequalities $23514^3 > 13 \cdot 10^{12}$, $37084^3 < 51 \cdot 10^{12}$ give $2\sqrt[3]{13} < 4.7028 < 4.7084 < 1 + \sqrt[3]{51}$, and $2351334^3 < 13 \cdot 10^{18}$, $5065797^3 < 130 \cdot 10^{18}$, $3708430^3 > 51 \cdot 10^{18}$ give $2\sqrt[3]{51} < 7.41686 < 7.417131 < \sqrt[3]{13} + \sqrt[3]{130}$. On the ray $(2r, 3r, 2r)$ the chord flows $a = v = c = r/2$ make every edge positive for $r > 0$. Theorem 7.2 gives the lower bound, and the convexity argument of Proposition 1.7 with $q = \log 51 / \log 13$ gives the upper one. □

These triples satisfy logarithmic Jensen, since $13^2 > 51$ and $51^2 > 13 \cdot 130$. The four-vertex DAG power $4/13$ exceeds α_K, so it is special to that graph: Proposition 1.7 shows that it does not extend to all three-dimensional fibers.

8 L-convex families in every dimension

We use Speyer's terminology [25]. For $x, y \in \mathbb{Z}^n$ let $\Pi(x, y)$ be the set of z with $\min(x_i - x_j, y_i - y_j) \leq z_i - z_j \leq \max(x_i - x_j, y_i - y_j)$ for all i, j. A set $\mathcal{K} \subseteq \mathbb{Z}^n$ is *L-convex* if $\Pi(x, y) \subseteq \mathcal{K}$ for all $x, y \in \mathcal{K}$ [25, Definition 1.2]. A function $g \geq 0$ is *L-log-concave* if $\log g$ is L-concave [25, Definition 1.5]. Every set defined by difference inequalities $w_p - w_q \leq c$ is L-convex: if x, y satisfy such an inequality, so does every $z \in \Pi(x, y)$, since $z_p - z_q \leq \max(x_p - x_q, y_p - y_q) \leq c$.

Theorem 8.1 (Logarithmic Jensen for L-convex marginals)**.** *Let $\mathcal{K} \subseteq \mathbb{Z}^{M+N}$ be L-convex and assume that $E(x) = \#\{y \in \mathbb{Z}^N : (x, y) \in \mathcal{K}\}$ is finite for every $x \in \mathbb{Z}^M$. Let $x_1, \ldots, x_k \in \mathbb{Z}^M$ with $E(x_i) > 0$, and let $\lambda_i > 0$ with $\sum_i \lambda_i = 1$ and $\bar{x} = \sum_i \lambda_i x_i \in \mathbb{Z}^M$. Then*

$$E(\bar{x}) \geq \prod_i E(x_i)^{\lambda_i}.$$

Proof. By Speyer's marginal theorem [25, Theorem 5.1], E is L-log-concave, and $E(x+\mathbf{1}_M) = E(x)$ by translating y by $\mathbf{1}_N$. The equivalence (1)⇔(3) of [25, Theorem 4.4], applied to $-\log E$, gives discrete midpoint log-concavity:

$$E\big(\lfloor \tfrac{x+y}{2} \rfloor\big) E\big(\lceil \tfrac{x+y}{2} \rceil\big) \geq E(x)E(y) \qquad (x, y \in \mathbb{Z}^M). \tag{8.1}$$

This is Murota's discrete midpoint concavity [26, Section 7.2], which makes $\log E$ an $\mathrm{L}^\natural$-concave function. Such functions are integrally concave and agree with their concave extension at integer points [26, 27]. We give the resulting Jensen inequality directly.

First let the weights be rational, and replicate the points x_i into a finite list with mean $\bar{x}$. While two list members x, y satisfy $\|x - y\|_\infty \geq 2$, replace them by $\lfloor (x+y)/2 \rfloor$ and $\lceil (x+y)/2 \rceil$. The sum is unchanged. By (8.1) the product of the values of E does not decrease, so it stays positive. The integer $\sum \|x\|_2^2$ strictly decreases, because in a coordinate where $|x_p - y_p| \geq 2$ the two new entries are strictly closer together. The process stops when, in every coordinate, all entries lie in $\{c_p, c_p+1\}$ for some integer c_p. Since the coordinate mean is the integer $\bar{x}_p$, every entry then equals $\bar{x}_p$. Hence the final list is constant, equal to $\bar{x}$, and $E(\bar{x})^{k'} \geq \prod E(x_i)^{k'\lambda_i}$ for the list length k'. For real weights, choose positive rational weights in the rational affine space $\{\lambda : \sum \lambda_i = 1,\ \sum \lambda_i x_i = \bar{x}\}$ converging to λ, and pass to the limit in $\sum_i \lambda_i \log E(x_i)$. □

Proof of Theorem 1.8. Let $\mathcal{K}$ be the set of $(\ell, u, z) \in \mathbb{Z}^{3d}$ with $\ell_i - z_i \leq 0$, $z_i - u_i \leq 0$ and $z_i - z_j \leq h_{ij}$. It is defined by difference inequalities, hence L-convex, and its fibers over (ℓ, u) lie in the box $[\ell, u]$, so they are finite. Theorem 8.1 applies with $M = 2d$, $N = d$. □

Remark 8.2 (Unimodular form). With $x_i = z_i - \ell_i \geq 0$, upper slacks $u_i - z_i \geq 0$ and internal slacks $h_{ij} - z_i + z_j \geq 0$, the counts $N(\ell, u)$ are fibers of a totally unimodular system $[I \mid \mathcal{N}]$, with $\mathcal{N}$ a grounded incidence matrix, of corank d. The demand $(u_i - \ell_i,\ h_{ij} - \ell_i + \ell_j)$ is integer affine in (ℓ, u). With an added anchor vertex 0 and $z_0 = 0$, as in Section 2, the moving parameters are exactly the star bounds $h_{i0} = u_i$ and $h_{0i} = -\ell_i$, while the bounds between nonanchor vertices stay fixed. Thus Theorem 1.8 gives (1.1) for these d-dimensional unimodular families in every dimension d. For $d \leq 3$, Theorem 1.4 gives the stronger corrected fourth root.

Speyer applies his Theorem 5.1 to skeps and Littlewood–Richardson coefficients. The consequence for lattice-point counts above is derived here from his marginal theorem and Murota's discrete midpoint concavity.

9 The dimension-four reduction

Theorem 1.4 rests on two features of dimension three: every normal system is the A_3 root system (Lemma 5.1), and the A_3 chambers carry a corrected Lorentzian certificate. In dimension four the first feature takes the following form. Let

$$\Omega = \{e_1,\ e_2,\ e_3,\ e_4,\ -e_1 - e_2,\ -e_3 - e_4,\ -e_1 - e_3,\ -e_2 - e_4,\ e_1 + e_2 + e_3 + e_4\} \subset \mathbb{Z}^4,$$

the nine cell functionals of a 3×3 table in the free cells $(x_{11}, x_{12}, x_{21}, x_{22})$. They represent the cographic matroid $M^*(K_{3,3})$.

Proposition 9.1 (Rank-four normal systems)**.** *Let $\mathcal{A}$ be a totally unimodular matrix with four columns whose rows include $e_1, \dots, e_4$. There is $T \in \mathrm{GL}(4, \mathbb{Z})$ such that every nonzero row of $\mathcal{A}T$ is $\pm$ an element of exactly one of the following systems, the same for all rows:*

(a) the type-A_4 roots $\{e_i\} \cup \{e_i - e_j\}$;

(b) Ω.

Consequently every unimodular family satisfying the hypotheses of Theorem 1.4, except that the bounded central fiber has dimension four, is count-preservingly a family of one of two kinds: (a) type-A_4 alcoved polytopes with twenty directed bounds, each the minimum of integer-linear functions of the demand; or (b) 3×3 cell-bounded contingency tables, with integer-linear margins and cell bounds that are maxima (lower) and minima (upper) of integer-linear functions of the demand.

Proof. As in Lemma 5.1, the simplified row matroid $\mathcal{M}$ is simple, regular and of rank four. If it is graphic, it is a restriction of $M(K_5)$, and the proof of Lemma 5.1 with a spanning tree of K_5 gives (a). Otherwise, by Tutte's theorem [19], $\mathcal{M}$ has a minor $M^*(K_5)$ or $M^*(K_{3,3})$. The first has rank six, so the minor is $M^*(K_{3,3})$, of rank four. A minor of full rank in a simple matroid is obtained by deletions alone, so $M^*(K_{3,3})$ is a restriction $\mathcal{M}|S$. Choose a basis of $\mathcal{M}$ inside S corresponding to the first four elements of Ω, and pivot $\mathcal{A}$ onto it; pivoting preserves total unimodularity. Camion's theorem [21] applied to the rows in S makes them equal to $\pm\Omega$ after a diagonal sign change of columns. Any further nonparallel row would then give a totally unimodular extension of Ω by a new $\{0, \pm 1\}$ direction. The script `verify_rank_three.py` checks all 31 candidate directions and finds none (Appendix A). Hence $\mathcal{M} = M^*(K_{3,3})$, which is (b).

For the families, follow the proof of Theorem 1.4 with $r = 4$. In case (a), shortest-path closure gives the twenty bounds. In case (b), write the nine cells as $x_{11} = w_1$, $x_{12} = w_2$, $x_{21} = w_3$, $x_{22} = w_4$, $x_{13} = r_1 - w_1 - w_2$, $x_{23} = r_2 - w_3 - w_4$, $x_{31} = c_1 - w_1 - w_3$, $x_{32} = c_2 - w_2 - w_4$ and $x_{33} = r_3 - c_1 - c_2 + w_1 + w_2 + w_3 + w_4$, for integer margins r_1, r_2, r_3, c_1, c_2. These are the tables with row sums r and column sums c, where $c_3 = r_1 + r_2 + r_3 - c_1 - c_2$. Each inequality $\pm\omega \cdot w \le c_j(a)$ with $\omega \in \Omega$ is then an upper or lower bound on one cell, with integer-linear offset. Parallel rows contribute minima or maxima of such offsets. □

Family (a) is governed by the polytrope chambers in dimension four, which Tran enumerated [12] as an explicit finite list of combinatorial types. Family (b) contains the unweighted 3×3 case of Barvinok's question [3, (2.3)], with nonnegativity as the only cell bounds. The transfer of Section 5 uses only continuity, concavity, nonnegativity, monotonicity in the bounds, and exactness at integral points. It therefore carries any extension with these properties for family (a) to all dimension-four families of type (a), and likewise for family (b) when all eighteen cell bounds are finite.

A Computer certificates

The package contains four standard-library Python scripts. Each prints JSON, writes no files, and keeps its checks active under `-O`.

- `verify_A3_structure.py` certifies Proposition 2.2. For each of the six seeds it reconstructs the twenty simple vertices and their unimodular edge bases, and the forty-eight cone rows as sums of at most two of the thirty base inequalities. It verifies the 62-pattern orbit coverage and the coefficient identities (2.3)–(2.5) from Brion's formula (2.2) with two objectives. It also checks the spectral decomposition of B, the bound $\rho \ge 11/9$ on all 64 sign vectors, and 1092 order-two derivative identities of the quartic $\mathcal{H}$. It confirms the optimal constants $2/11$ and $1/12$.

- `verify_rank_three.py` replays Lemma 5.1 exhaustively (1024 masks, 51 valid systems, 16 maximal systems with explicit T). It also checks the step of Proposition 9.1 that Ω is totally unimodular and admits none of the 31 new directions.

- `verify_counterexample.py` enumerates the K-ray counts $(r+1)^4 - r^4$ for $0 \le r \le 8$. It recounts $1, 15, 65$ as fibers of the corank-three system of Remark 2.1, and certifies $3/10 < \alpha_K < 4/13$ by integer powers. It also checks the four-vertex DAG witnesses of Proposition 7.3 and the ray polynomial $(5r^3 + 11r^2 + 8r + 2)/2$ for $0 \le r \le 30$.
- `verify_rounding.py` checks finite instances of Lemma B.2 and Theorem B.1 and the counts in Proposition B.3.

The all-size analytic statements (Lorentzian, gluing, closure, transfer and countable support) are proved in the text.

B Opposite-edge rounding

Let $z_0 = 0$ and give each difference $z_i - z_j$ of type A_3 an integer interval. Fix four cross-edge intervals

$$A \le z_3 \le B, \quad C \le z_1 - z_3 \le D, \quad E \le z_2 \le F, \quad G \le z_1 - z_2 \le H, \tag{B.1}$$

and for $\xi = (a, b, c, d) \in \mathbb{Z}^4$ let $\mathcal{N}(\xi)$ count the integer triples satisfying (B.1), $a \le z_1 \le b$ and $c \le z_2 - z_3 \le d$; an empty system has count zero. For integers s, t define the directed midpoint $\mu(s,t) = \lfloor (s+t)/2 \rfloor$ if $s \le t$ and $\lceil (s+t)/2 \rceil$ if $s \ge t$, applied coordinatewise [28]; then $\mu(x,y) + \mu(y,x) = x + y$.

Theorem B.1 (Opposite-edge rounding)**.** *For all fixed intervals in* (B.1) *and all $x, y \in \mathbb{Z}^4$,*

$$\mathcal{N}(\mu(x,y))\,\mathcal{N}(\mu(y,x)) \ge \mathcal{N}(x)\,\mathcal{N}(y).$$

Consequently

$$\mathcal{N}(b) \ge \prod_i \mathcal{N}(x_i)^{\lambda_i}$$

whenever $\mathcal{N}(x_i) > 0$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$ and $b = \sum_i \lambda_i x_i \in \mathbb{Z}^4$.

Lemma B.2 (Interval weights)**.** *Put $U(p,q) = \min\{B, p-C, F-q, p-q-G\}$, $L(p,q) = \max\{A, p-D, E-q, p-q-H\}$ and $w = \max(0, 1+U-L)$. Then $w(\mu(x,y))w(\mu(y,x)) \ge w(x)w(y)$ for $x, y \in \mathbb{Z}^2$.*

Proof. Write $\delta = x - y$ and let $\varepsilon_i = \operatorname{sgn} \delta_i$ when δ_i is odd and 0 otherwise, so $\mu(x,y) = (x+y+\varepsilon)/2$. For every r in $\mathcal{R} = \{0, \pm e_1, \pm e_2, \pm(e_1+e_2), \pm(e_1-e_2)\}$ one has $|r \cdot \varepsilon| \le |r \cdot \delta|$: parity residues cancel when the signed sum of two coordinates of δ is zero, have absolute value one when it is ± 1, and are at most two otherwise. Hence $\mu(x,y)$ and $\mu(y,x)$ lie in every slab $\min(r \cdot x, r \cdot y) \le r \cdot z \le \max(r \cdot x, r \cdot y)$. Now $w = \Phi \circ V$ with $V(p,q) = (0, p, -q, p-q)$ and $\Phi(v) = |\mathbb{Z} \cap \bigcap_{i=0}^{3} (J_i + v_i)|$ for $(J_0, \dots, J_3) = ([A,B], [-D,-C], [E,F], [-H,-G])$. The set of (v,t) with $t - v_i \in J_i$ is L-convex with finite fibers, so Φ is L-log-concave by [25, Theorem 5.1]. Every difference of components of V has a normal in $\mathcal{R}$, so the slab property places $V(\mu(x,y))$ in $\Pi(V(x), V(y))$, and [25, Definition 1.5] gives the claim. □

Proof of Theorem B.1. With $(p,q,t) = (z_1, z_2 - z_3, z_3)$ the cross intervals give $L(p,q) \le t \le U(p,q)$, so $\mathcal{N}(\xi) = \sum_r w(r) \mathbf{1}_{R_\xi}(r)$ with $R_\xi = [a,b] \times [c,d]$. Scalar monotonicity of μ places $\mu(r,s)$ in $R_{\mu(x,y)}$ for $r \in R_x$, $s \in R_y$. Lemma B.2 thus verifies the pointwise hypothesis of the discrete four-functions theorem of Halikias–Klartag–Slomka [29, Theorem 1.3] (see also [30]). The coordinatewise operation μ is translation equivariant and nondecreasing in each argument, changing by 0 or 1 per unit step, which is its Knothe monotonicity hypothesis. The conclusion is the product inequality.

For rational weights, replicate the vectors and replace any pair differing by at least two in a coordinate by its two directed midpoints. The sum is preserved, the product does not decrease, and $\sum \|x\|_2^2$ strictly decreases. At termination all spreads are at most one, and the integral mean forces a constant list. Real weights follow by rational approximation in the affine space of weights with mean b. □

The four cross intervals are fixed here; Theorem 4.1 allows all twelve bounds to move. The next example shows that the global continuous extension is needed for that.

Proposition B.3 (A local rounding obstruction)**.** *In the box* $[0,2]\times[0,2]\times[0,3]$ *let* $\mathcal{T}(\eta)$*, for* $\eta = (a,b,c,d,e,f)\in\mathbb{Z}^6$*, count the integer points with* $a\le z_1-z_2\le b$*,* $c\le z_1-z_3\le d$*,* $e\le z_2-z_3\le f$*. For* $x=(-2,0,-3,-1,-2,-1)$ *and* $y=(-2,1,-2,1,-2,0)$*,* $\mathcal{T}(x)=9$ *and* $\mathcal{T}(y)=19$*, whereas every probability law on the integer neighbors of* $m=(x+y)/2$ *with mean* m *has geometric mean of* $\mathcal{T}$ *at most* $13<\sqrt{171}$*. All these fibers are three-dimensional.*

Proof. The neighbors are $v(i,j,k)=(-2,i,j-3,0,-2,k-1)$, $(i,j,k)\in\{0,1\}^3$, with $\mathcal{T}(v(i,j,k))=9+4i-2j+6k$: at $(0,0,0)$ the counts by $z_3=0,1,2,3$ are $0,1,3,5$; raising i adds four points with $z_1=z_2+1$, raising k adds six with $z_2=z_3$ and $z_1\le z_2$, and raising j removes the two points with $z_1=0$, $z_3=3$. For x the bound $z_1-z_3\le -1$ is implied, giving 9; y relaxes $v(1,1,1)$ by $z_1-z_3\le 1$, adding $(1,0,0)$ and $(2,1,1)$, giving 19. A law with mean m has $\mathbb{E}i=\mathbb{E}j=\mathbb{E}k=\frac{1}{2}$, so $\mathbb{E}\mathcal{T}=13$, and AM–GM bounds the geometric mean. The points $(0,0,1),(0,0,2),(0,1,2),(1,1,2)$ lie in all these fibers and span a unimodular tetrahedron. □

Thus $-\log\mathcal{T}$ is not integrally convex in these six parameters [27, Section 3.4], while Theorem 1.4 still gives Jensen there through the nonlocal extension of Theorem 4.1.

Use of AI tools. OpenAI Codex assisted with proof exploration, drafting and exact verification code for the prior versions. Anthropic Claude (Claude Code) assisted with the referee-style review, the literature cross-check, the restructuring of this version, the K-ray, L-convex and dimension-four sections, and the reruns of the verification scripts. The author directed and approved the work and takes full responsibility for the content.

Prior versions. This article revises the manuscript *Power Concavity of Low-Dimensional Unimodular Fiber Counts*, posted in the Luminary Research Archive in five editions, from 22 October to 2 November 2026 [31]. The present version leads with logarithmic Jensen in the form of [1, Question 1.7] and adds Proposition 1.7, Section 8 and Section 9. The same Lorentzian criterion appears in the companion study of lattice counts on smooth toric Fano threefolds [32]; the A_3 chambers are smooth toric threefolds.

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