# Power Concavity of Low-Dimensional Unimodular Fiber Counts

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**Abstract**

We prove a global power-Jensen inequality for bounded unimodular fibers whose central affine dimension is at most three. For every integrable integer-valued demand $A$ with feasible fibers almost surely and integral mean $b$, if $P(b)$ is bounded and has affine dimension at most three, then $\mathbb{E}N(A)^{1/6} \le N(b)^{1/6}$. In particular logarithmic Jensen holds, including countably supported distributions. The proof constructs a monotone concave extension of the sixth root of every type-$A_3$ alcoved lattice count, with all twelve directed bounds varying. It combines the classical polytrope chamber formulas and Lorentzian volume theory with a degree-six homogenization, wall gluing and shortest-path closure. A common integral rank-three reduction transfers the result to general unimodular fibers. In dimension two we retain the stronger sharp corrected square-root inequality, the optimal leading large-count exponent deficit, and a fixed network obstruction. For the complete four-vertex DAG, 4/13 is the optimal power of its classical continuous extension, while cube-root concavity fails even on integral demands. A product-preserving directed rounding theorem for opposite edges and a full-dimensional triangle obstruction distinguish local rounding methods from global Jensen.

## 1 Statements and relation to earlier work

Throughout, a full-row-rank integer matrix $M \in \mathbb{Z}^{p\times n}$ is called *unimodular* if every nonzero maximal column minor is $\pm 1$. For an integral right-hand side $a$ put

$$P(a) = \{x \in \mathbb{R}^n_{\ge 0} : Mx = a\}, \qquad N(a) = |P(a) \cap \mathbb{Z}^n|.$$

Unless explicitly stated otherwise, fibers are nonempty and bounded. A finite family of integral demands $a_i$ and weights $\lambda_i > 0$ is *admissible* when $\sum_i \lambda_i = 1$ and $b = \sum_i \lambda_i a_i$ is integral. We write $N_i = N(a_i)$.

Leake and Mohammadi Yekta prove log-concavity along rational lines for unimodular fiber counts and ask whether the full discrete Jensen inequality

$$N(b) \ge \prod_i N_i^{\lambda_i} \tag{1.1}$$

holds [1, Theorem 1.4, Question 1.7]. Intermediate pair averages need not be integral, so their line theorem does not automatically establish (1.1). Our first result concerns actual central fiber dimension, rather than the number of equations or columns. Approximate global inequalities for contingency tables and integer flows were established earlier by Barvinok [8]; their correction factors distinguish them from (1.1). The complete-DAG version of the finite weighted question was also stated explicitly by Leake and Morales [9, Section 7.4].

Our main result establishes the global inequality in central affine dimension at most three; it also allows countable support.

**Theorem 1.1** (Global Jensen in central dimension at most three)**.** *Let $M$ be unimodular, let $A$ be $\mathbb{Z}^p$-valued with $\mathbb{E}\|A\| < \infty$, and suppose $P(A)$ is nonempty almost surely. Assume $b = \mathbb{E}A$ is integral and $P(b)$ is nonempty and bounded, with* $\dim \operatorname{aff} P(b) \le 3$*. Then all endpoint fibers are bounded and*

$$\mathbb{E}N(A)^{1/6} \le N(b)^{1/6}, \qquad \mathbb{E}\log N(A) \le \log N(b).$$

*Both expectations are finite. No restriction on the number of columns or equations is imposed.*

The finite-support case answers the corresponding dimension-at-most-three part of [1, Question 1.7]. Section 6 proves the continuous count theorem, and Section 7 proves the common-lattice transfer and the countable-support extension. Classical Lorentzian theory [16] and the six polytrope formulas [17] are inputs, not new principles or new enumeration formulas. Our additional planar statements are stronger than Theorem 1.1 in their smaller domain.

**Theorem 1.2** (Low-dimensional power concavity)**.** *Let $M$ be unimodular and let $(a_i, \lambda_i)$ be admissible. If* $\dim \operatorname{aff} P(b) \le 2$*, then*

$$\sqrt{N(b) - \tfrac{1}{4}} \ge \sum_i \lambda_i \sqrt{N_i - \tfrac{1}{4}}. \tag{1.2a}$$

*No smaller real correction can replace $1/4$ uniformly. If also $N_i \ge m$ for an integer $m \ge 1$, then*

$$N(b)^{\alpha_m} \ge \sum_i \lambda_i N_i^{\alpha_m}, \qquad \alpha_m = \frac{2m-1}{4m-1} = \frac{1}{2} - \frac{1}{8m-2}. \tag{1.2b}$$

*In particular, (1.1) holds.*

For $m \in \mathbb{N}_{\ge 1}$ let $A(m)$ be the supremum of the positive exponents for which the power inequality in (1.2b) holds for *every* matrix and admissible family in Theorem 1.2, with endpoint counts at least $m$. Neither the matrix nor the number of endpoints is fixed.

**Theorem 1.3** (Sharp large-count asymptotic)**.** *Put $n(r) = 3r^2 + 3r + 1$ and $R_m = \min\{r \in \mathbb{Z}_{\ge 0} : n(r) \ge m\}$. Then*

$$\frac{1}{2} - \frac{1}{8m-2} \le A(m) \le \frac{1}{2} - \frac{1}{8n(R_m+2)-2}.$$

*In particular,*

$$A(m) = \frac{1}{2} - \frac{1}{8m} + O(m^{-3/2}), \qquad \lim_{m\to\infty} m\left(\frac{1}{2} - A(m)\right) = \frac{1}{8}. \tag{1.3}$$

*Square-root concavity fails for every endpoint-count threshold $m$. We do not determine $A(m)$ exactly at any finite threshold.*

For a DAG we use net outflow as the incidence convention. After deleting one incidence row per connected component, its incidence matrix is unimodular. Acyclicity makes every nonempty nonnegative flow fiber bounded. For the complete DAG on $0, 1, 2, 3$, let

$$(s, t, u) = (b_0, b_0 + b_1, b_0 + b_1 + b_2).$$

Its feasible integral demands correspond exactly to $(s, t, u) \in \mathbb{Z}^3_{\ge 0}$. The inverse map is $b = (s, t - s, u - t, -u)$.

**Theorem 1.4** (The complete four-vertex DAG)**.** *The integer-flow count for the complete four-vertex DAG has a positive, continuous piecewise polynomial extension $F$ to $\mathbb{R}^3_{\geq 0}$ such that $F^{4/13}$ is concave. No power $\alpha > 4/13$ makes this extension $F^\alpha$ concave. Thus, for every admissible family of feasible demands,*

$$N(b)^{4/13} \geq \sum_i \lambda_i N_i^{4/13}, \tag{1.4}$$

*and consequently (1.1) holds.*

The optimality assertion concerns this continuous extension, not the best power for inequalities restricted to integral demands. Proposition 4.2 below gives a separate obstruction on actual integral demands: even the cube-root power fails there.

Theorem 1.4 includes three-dimensional interacting fibers: when $s, t, u > 0$, all six edges can carry positive flow. Its chamber enumeration is classical, with an early seven-case treatment in Tarski's 1958 report [10, Section VII]. For the formulas used here see De Loera–Sturmfels [3, Section 5] and the earlier computations cited there. We use those polynomials, deriving them again for completeness, and prove a power-concavity property of their continuous extension. Root-direction inequalities and normalized Lorentzian series for restricted Kostant functions appear in Khare–Matherne–St. Dizier [2]; they are different from concavity of counts in demand space.

The geometry behind Theorem 1.2 is also classical. Alcoved triangulations and volume–lattice-count identities are developed by Lam–Postnikov [4, 5]. The relevant planar count formula follows from Pick and mixed area. Its continuous concavity belongs to the Brunn–Minkowski/generalized-Wills framework [6]. General lattice-point Brunn–Minkowski inequalities [7] require an enlargement that is absent from (1.1). We claim no new continuous Wills inequality or new chamber-count formula. Our results concern their exact-count applications, quantitative exponents and the global $A_3$ power inequality.

A different deformation of type-$A_3$ alcoved polytopes is treated in Section 5: the intervals on two opposite edges may vary independently, while the other four intervals remain fixed. The proof establishes a directed-midpoint product inequality for their exact multiplicities and transfers it to rectangle counts. The directed operation is classical [11], as is the four-functions summation theorem [12, 13]; the proof verifies the required interval-envelope compatibility explicitly. For integer interval multiplicities, Speyer's L-convex marginal theorem [14] supplies an alternative argument given below. A triangle-supported example then separates the count-level conclusion from local balanced-rounding methods. All six intervals are covered by the global theorem below, although the local directed-rounding mechanism does not extend to them.

## 2 Common lattice coordinates and the planar count identity

**Lemma 2.1** (Active-coordinate reduction)**.** *Under the hypotheses of Theorem 1.2, all endpoint fibers lie in the coordinate face active at $P(b)$. In common integral coordinates their free dimension is $r = \dim \operatorname{aff} P(b)$. If $r = 2$, every full-dimensional fiber polygon has primitive outward normals among*

$$\pm(1,0), \quad \pm(0,1), \quad \pm(1,1), \tag{2.1}$$

*after one common coordinate sign change. Coordinate maps commute with weighted averaging.*

*Proof.* Let $E_*$ be the coordinates positive at some point of $P(b)$. Minkowski averaging gives $\sum_i \lambda_i P(a_i) \subset P(b)$. Fixing all endpoint points but one shows that every endpoint coordinate outside $E_*$ is zero. Averaging finitely many witnesses gives a central point positive on all of $E_*$. Thus its affine dimension is the kernel dimension of the retained column submatrix $M_*$.

Let $s = \operatorname{rank} M_*$. Choose $s$ independent active columns $B_0$ and complete them to a column basis $B = (B_0, B_1)$ of $M$. Since $\det B = \pm 1$, $B^{-1}$ is integral. The active columns of $B^{-1}M$ have the form $\binom{D}{0}$, where $D$ contains $I_s$. Every $s$-column minor of $D$ is, up to sign, a maximal minor of $M$ completed by the fixed $B_1$ columns and divided by $\det B$. It is therefore 0 or $\pm 1$. Write $D = [I_s, C]$. Completing any square minor of $C$ by unused identity columns proves that $C$ is totally unimodular.

With free coordinates $t \in \mathbb{R}^r$, the active solutions are

$$x_{\text{free}} = t, \qquad x_{B_0} = (B^{-1}a)_{1:s} - Ct. \tag{2.2}$$

This is a lattice bijection and its offset is integer linear in $a$. For $r = 2$, every row of $C$ has entries $0, \pm 1$. Both diagonal row directions $(1, 1)$ and $(1, -1)$ cannot occur, since their determinant has magnitude 2. Include the free nonnegativity inequalities and change one coordinate sign if needed to obtain (2.1). All fibers are integral: choosing an independent basis at a vertex and completing it as above gives integral basic coordinates. Finally, (2.2) and linearity give $\sum_i \lambda_i Q(a_i) \subset Q(b)$ in the common free coordinates. □

Set

$$K = \{(x, y) : |x| \le \tfrac{1}{2},\ |y| \le \tfrac{1}{2},\ |x + y| \le \tfrac{1}{2}\}.$$

It has area $3/4$ and support value $1/2$ at each primitive normal in (2.1).

**Lemma 2.2** (Exact planar representation)**.** *If $Q$ is an integral polygon with normals (2.1), or a degenerate fiber segment or point in this normal system, then*

$$|Q \cap \mathbb{Z}^2| = \operatorname{area}(Q + K) + \frac{1}{4}. \tag{2.3}$$

*Proof.* For a full-dimensional $Q$, mixed area gives

$$\operatorname{area}(Q + K) = \operatorname{area}(Q) + \frac{B(Q)}{2} + \frac{3}{4},$$

where $B(Q)$ is the sum of lattice edge lengths. Indeed, an edge with primitive normal $v$ contributes its Euclidean length times $h_K(v)/\|v\|$, which is half its lattice length. Pick's formula proves (2.3). For a point, (2.3) reads $1 = 3/4 + 1/4$. A degenerate fiber segment is parallel to a primitive integer vector perpendicular to one of the normals (2.1). The transverse width of $K$ times that primitive vector's Euclidean length is 1. A segment of lattice length $\ell$ therefore adds area $\ell$ to $K$, and both sides of (2.3) equal $\ell + 1$. □

For clarity, the continuous functional in (2.3) has a generalized-Wills representation. If $R$ is uniform on $[0, 2]$, then

$$\mathbb{E} \operatorname{area}(Q + RK) = \operatorname{area}(Q) + 2V(Q, K) + 1 = \operatorname{area}(Q + K) + \tfrac{1}{4}. \tag{2.4}$$

Equivalently it is the integral of the radial tent $(1 - \operatorname{dist}_K(x, Q)/2)_+$. Classical Borell–Brascamp–Lieb yields its cube-root concavity; see the generalized-Wills setting in [6]. The following calculation retains the larger domain available when counts are large.

## 3 Count-dependent powers and the optimal leading deficit

*Proof of Theorem 1.2.* If the central affine dimension is zero, all endpoint fibers are points. If it is one, their direction spaces lie in the same rational line: vary one endpoint in the Minkowski inclusion. Choose its primitive integer direction. Endpoint and central fibers are integral segments of normalized lengths $\ell_i, \ell$, allowing length zero. Then $\ell \geq \sum_i \lambda_i \ell_i$, so $N(b) \geq \sum_i \lambda_i N_i$. Apply the increasing concave function $x \mapsto \sqrt{x - 1/4}$ on $[1, \infty)$ to prove (1.2a). The claimed power inequality follows as well.

For dimension two use Lemmas 2.1–2.2. Put $c = 1/4$ and $s_i = \sqrt{\operatorname{area}(Q(a_i) + K)} = \sqrt{N_i - c}$. Brunn–Minkowski, common-coordinate inclusion and monotonicity imply

$$\sqrt{N(b) - c} = \sqrt{\operatorname{area}(Q(b) + K)} \geq \sum_i \lambda_i s_i. \tag{3.1}$$

This proves (1.2a). Every $s_i \geq \sqrt{m - c}$. For $g_\alpha(s) = (s^2 + c)^\alpha$,

$$g''_\alpha(s) = 2\alpha(s^2 + c)^{\alpha-2}\big[c + (2\alpha - 1)s^2\big].$$

At $\alpha = \alpha_m$, $1 - 2\alpha = c/(m - c)$. Thus $g_{\alpha_m}$ is increasing and concave on $[\sqrt{m - c}, \infty)$. Apply it to (3.1), then use scalar Jensen. Weighted AM–GM proves (1.1). □

**Lemma 3.1** (A fixed network obstruction)**.** *Orient $K_{2,3}$ from its two sources to its three sinks. For net outflow*

$$a(r) = (3r, 3r, -2r, -2r, -2r), \qquad r \in \mathbb{Z}_{\geq 0},$$

*the count is $n(r) = 3r^2 + 3r + 1$. For $r > 0$ the fiber dimension is two.*

*Proof.* If $t_j$ is the flow from the first source to sink $j$, the other flow into that sink is $2r - t_j$. Hence $0 \leq t_j \leq 2r$ and $\sum_j t_j = 3r$. Inclusion–exclusion gives $\binom{3r+2}{2} - 3\binom{r+1}{2} = 3r^2 + 3r + 1$ for $r \geq 1$; two upper-bound violations are impossible. At $r = 0$ the count is 1. For $r > 0$, $(t_1, t_2, t_3) = (r, r, r)$ is a relative interior point of the two-dimensional slice. □

*Proof of Theorem 1.3.* Theorem 1.2 gives $A(m) \geq \alpha_m$. For $f_\alpha(r) = n(r)^\alpha$, the sign of $f''_\alpha(r)$ is the sign of

$$\alpha - \tau(r), \qquad \tau(r) = \frac{1}{2} - \frac{1}{6(2r + 1)^2} = \frac{1}{2} - \frac{1}{8n(r) - 2}. \tag{3.2}$$

At $\alpha = 1/2$ this is positive everywhere. Taking $R = R_m$ gives $2n(R+1)^{1/2} < n(R)^{1/2} + n(R+2)^{1/2}$, proving the square-root failure at arbitrarily large counts.

The function $\tau$ is increasing on $[0, \infty)$. For every $\alpha > \tau(R + 2)$, equation (3.2) is positive throughout $[R, R + 2]$. Thus the integral endpoints $R, R + 2$, with integral midpoint $R + 1$, violate power Jensen and have counts at least $m$. No such exponent is admissible, giving $A(m) \leq \tau(R+2)$. Minimality gives $R = \sqrt{m/3} + O(1)$ and $n(R + 2) = m + O(\sqrt{m})$. Expanding the two explicit bounds proves (1.3). □

*Sharpness of the correction in Theorem 1.2.* For the same network, $n(r) - c = 3(r+1/2)^2 + 1/4 - c$. If $c < 1/4$, the function $\sqrt{n(r) - c}$ is strictly convex on $[0, \infty)$. Every adjacent integral triple therefore violates the proposed corrected square-root inequality. For $c = 1/4$ this function is affine, so these families also give equality in (1.2a). □

## 4 A three-dimensional case: the type-$A_3$ count

Use chord flows $x_{02} = a$, $x_{03} = v$, $x_{13} = c$. The other flows are $s - a - v$, $t - a - v - c$, $u - v - c$. Thus the count is

$$|\{(a, v, c) \in \mathbb{Z}^3_{\geq 0} : a + v \leq s,\ v + c \leq u,\ a + v + c \leq t\}|. \tag{4.1}$$

Set

$$H(s, u) = \frac{(s+1)(s+2)(3u - s + 3)}{6}, \qquad B(d) = \frac{d(d+1)(d+2)}{6}.$$

The classical chamber formulas [3, Section 5], in these coordinates, define the extension $F$ for $0 \leq s \leq u$ by

$$F(s, t, u) = \begin{cases} (t+1)(t+2)(t+3)/6, & 0 \leq t \leq s, \\ (s+1)(s+2)(3t - 2s + 3)/6, & s \leq t \leq u, \\ H(s, u) - B(s + u - t), & u \leq t \leq s + u, \\ H(s, u), & s + u \leq t. \end{cases} \tag{4.2}$$

For $s > u$ interchange $s, u$.

Here is a direct count check. In the first chamber, (4.1) is a simplex. In the second, sum $(w + 1)(t - w + 1)$ over $0 \leq w \leq s$. In the last two chambers, fix $v$ and begin with the rectangle $0 \leq a \leq s - v$, $0 \leq c \leq u - v$. Summing its sizes over $v = 0, \ldots, s$ gives $H(s, u)$. If $d = s + u - t \geq 0$, the points excluded at its upper corner have total count $\sum_{v=0}^{d-1} \binom{d-v+1}{2} = \binom{d+2}{3} = B(d)$. This proves (4.2) on integer points. Its values agree on all walls. In the middle chamber $0 \leq d \leq s \leq u$, and $H(s, u) \geq H(d, d) > B(d)$, so $F$ is positive everywhere.

**Lemma 4.1** (A Hessian comparison)**.** *In the middle chamber of (4.2), $F^{4/13}$ is concave.*

*Proof.* Use affine coordinates $(s, L, d)$, where $L = 3u - s + 3$, and write $H = f(s)L$ with $f(s) = (s+1)(s+2)/6$. Put $\beta = 9/13$ and $C = \beta \nabla F \nabla F^\top - F \operatorname{Hess} F$. Concavity of $F^{4/13}$ is equivalent to $C \succeq 0$. Eliminating its positive $d, d$ entry gives the Schur complement

$$F\big[khh^\top - \operatorname{Hess} H\big], \qquad k = \frac{\beta B''}{\beta (B')^2 + F B''}, \quad h = \nabla H, \tag{4.3}$$

where the remaining derivatives are in $(s, L)$. The $L, L$ entry of the bracket is $kf^2 > 0$, and its determinant is

$$kLf(2(f')^2 - ff'') - (f')^2.$$

Consequently $C \succeq 0$ is equivalent to $\beta(HJ - Q) \geq H - B$, where

$$J = 2 - \frac{ff''}{(f')^2} = \frac{3}{2} + \frac{1}{2(2s+3)^2}, \qquad Q = \frac{(B')^2}{B''}.$$

For $w = 2s + 3$ this condition is $E \geq 0$, with

$$E = \frac{H(w^2 + 9)}{2w^2} + 13B - 9Q. \tag{4.4}$$

The first term increases with $u$. At $u = s$ it equals $(w^3 + 8w - 9/w)/48$, which increases with $s \geq 0$. Since $0 \leq d \leq s \leq u$, decreasing first $u$ to $s$ and then $s$ to $d$ therefore gives

$$E(s, u, d) \geq E(d, d, d) = \frac{d(2d^4 + 17d^3 + 46d^2 + 58d + 30)}{12(d+1)(2d+3)} \geq 0.$$

This proves the required positive semidefiniteness. □

*Proof of Theorem 1.4.* The other three chamber polynomials in (4.2) are products of three positive affine functions; their cube roots, and hence their 4/13 powers, are concave. Lemma 4.1 handles the remaining chamber.

We verify the global gluing, since chamberwise concavity alone would not suffice. Gradients below are ordered as $(s, t, u)$. For increasing $t$ with $s < u$, the gradient jumps of $F$ are

| wall | normal into the next chamber | $\nabla F_{\text{next}} - \nabla F_{\text{previous}}$ |
|---|---|---|
| $t = s$ | $(-1, 1, 0)$ | $-\frac{3s+5}{6}(-1, 1, 0)$ |
| $t = u$ | $(0, 1, -1)$ | $-\frac{3s+4}{6}(0, 1, -1)$ |
| $t = s + u$ | $(-1, 1, -1)$ | $-\frac{1}{3}(-1, 1, -1)$. |

(4.5)

On the symmetry wall $s = u$, crossing into $s > u$, the jump is $-\frac{1}{6}(1, 0, -1)$ in the last two chambers and zero in the first. The second chamber meets this wall only at $s = t = u$. All these identities follow by differentiating (4.2).

Because wall values coincide, the jumps for $F^{4/13}$ have the same signs, multiplied by the common positive factor $\frac{4}{13}F^{-9/13}$. Thus the derivative along a line jumps downward at every generic wall crossing. Its restriction is concave on each chamber interval and is continuous, so it is concave on the whole line segment. Approximate nongeneric segments by generic ones in the positive orthant and pass to the boundary by continuity. This proves global concavity. Prefix sums are linear in the demand, so the resulting Jensen inequality is exactly (1.4).

To prove optimality for this extension, use affine coordinates $(s, u, d)$ in its middle-chamber polynomial. At $(s, u, d) = (0, 0, 0)$, direct differentiation gives

$$\det\big[(1-\alpha)\nabla F \nabla F^{\top} - F \operatorname{Hess} F\big] = \frac{4 - 13\alpha}{4}. \tag{4.6}$$

For every $\alpha > 4/13$ the determinant is negative. By continuity it remains negative at some interior points $0 < d < s < u$ arbitrarily near the origin. The controlling matrix is then not positive semidefinite, so $F^{\alpha}$ cannot be concave. This is an obstruction on real demands. □

### 4.1 An obstruction on integral demands

The continuous optimality argument near the origin does not itself produce an integral-demand counterexample. The count polynomials are inhomogeneous, so scaling rational demands to clear denominators changes the inequality. The following exact examples supply a lattice obstruction.

**Proposition 4.2** (Failure of cube-root concavity)**.** *For the complete forward four-vertex DAG, cube-root Jensen fails at the integral midpoint triples*

$$(0,0,0),\ (2,3,2),\ (4,6,4) \quad \textit{and} \quad (2,3,2),\ (4,6,4),\ (6,9,6)$$

*in prefix-demand coordinates. All fibers in the second triple are three-dimensional.*

*Let $A_{K_4}$ be the supremum of the positive powers satisfying Jensen for every admissible family of integral demands on this fixed graph. Then*

$$\frac{4}{13} \le A_{K_4} \le \alpha_* < \frac{1}{3}, \qquad 2\,13^{\alpha_*} = 1 + 51^{\alpha_*},$$

*where the displayed equation has a unique positive solution.*

*Proof.* Substitution in (4.2), or direct enumeration in (4.1), gives

$$N(0,0,0) = 1, \quad N(2,3,2) = 13, \quad N(4,6,4) = 51, \quad N(6,9,6) = 130.$$

The middle demand in each triple is the integral midpoint of its two endpoints. Integer cubes give

$$23514^3 > 13 \cdot 10000^3, \qquad 37084^3 < 51 \cdot 10000^3,$$

and hence

$$2\sqrt[3]{13} < \frac{47028}{10000} < \frac{47084}{10000} < 1 + \sqrt[3]{51}.$$

For the second triple use

$$\begin{aligned} 2351334^3 &< 13 \cdot 1000000^3, \\ 5065797^3 &< 130 \cdot 1000000^3, \\ 3708430^3 &> 51 \cdot 1000000^3. \end{aligned}$$

They imply

$$2\sqrt[3]{51} < \frac{7416860}{1000000} < \frac{7417131}{1000000} < \sqrt[3]{13} + \sqrt[3]{130}.$$

On the positive ray $(s,t,u) = (2r,3r,2r)$, take the three chord flows $a = v = c = r/2$ in (4.1). Every chord and each remaining edge has positive flow for $r > 0$. Thus each fiber in the second triple has full affine dimension three.

Theorem 1.4 gives $A_{K_4} \geq 4/13$. For the upper bound set $x = 13^\alpha$ and $q = \log(51)/\log(13)$. Since $1 < q < 2$, the function $x^q - 2x + 1$ is strictly convex, vanishes at $x = 1$, has negative derivative there, and tends to positive infinity. It has exactly one further zero $x_* > 1$, defining the unique positive $\alpha_*$. For every $\alpha > \alpha_*$, the first midpoint triple violates power Jensen. Its strict violation at $\alpha = 1/3$ proves $\alpha_* < 1/3$. □

These examples do not refute logarithmic Jensen: $13^2 > 51$ and $51^2 > 13 \cdot 130$. They do not determine $A_{K_4}$. The graph and the count polynomials are classical; the obstruction follows from those exact counts and the displayed elementary inequalities.

## 5 Opposite-edge interval deformations

Let $z_0 = 0$ and give each of the six differences $z_i - z_j$ an integer interval. These define bounded type-$A_3$ alcoved lattice polytopes. We allow the intervals on the opposite edges 01 and 23 to vary, retaining four arbitrary fixed cross-edge intervals:

$$A \leq z_3 \leq B, \quad C \leq z_1 - z_3 \leq D, \quad E \leq z_2 \leq F, \quad G \leq z_1 - z_2 \leq H. \tag{1}$$

For $\xi = (a,b,c,d) \in \mathbb{Z}^4$, let $\mathcal{N}(\xi)$ count the integer triples satisfying (1) and

$$a \leq z_1 \leq b, \qquad c \leq z_2 - z_3 \leq d. \tag{2}$$

An empty system, including an interval with reversed endpoints, has count zero. The normal matrix is totally unimodular: its rows are signed rows of a transposed incidence matrix. This is also a unimodular-fiber model of the form used in Section 1. Indeed, the fixed intervals bound $z_3, z_2, z_1$ uniformly; after a fixed integer translation making these coordinates nonnegative, adjoining one

nonnegative slack per inequality gives a totally unimodular matrix containing an identity submatrix and a lattice bijection preserving counts and parameter means.

For integers $s, t$ define the directed midpoint

$$\mu(s,t) = \begin{cases} \lfloor (s+t)/2 \rfloor, & s \le t, \\ \lceil (s+t)/2 \rceil, & s \ge t. \end{cases} \tag{3}$$

The two formulas agree at equality. Apply $\mu$ coordinatewise; then $\mu(x,y) + \mu(y,x) = x + y$. This is the directed rounding operation of [11].

**Theorem 5.1** (Full opposite-edge rounding)**.** *For every choice of the fixed integer intervals in* (1) *and every* $x, y \in \mathbb{Z}^4$*,*

$$\mathcal{N}(\mu(x,y))\mathcal{N}(\mu(y,x)) \ge \mathcal{N}(x)\mathcal{N}(y). \tag{4}$$

*Consequently, if* $x_i \in \mathbb{Z}^4$*,* $\mathcal{N}(x_i) > 0$*,* $\lambda_i > 0$*,* $\sum_i \lambda_i = 1$ *and* $b = \sum_i \lambda_i x_i \in \mathbb{Z}^4$*, then*

$$\mathcal{N}(b) \ge \prod_i \mathcal{N}(x_i)^{\lambda_i}. \tag{5}$$

*The finite family and the positive real weights are arbitrary.*

The proof needs a multiplicity inequality, rather than only closure of the feasible lattice set under directed midpoints. Put

$$S = \{(0,0), (1,0), (0,-1), (1,-1)\}, \qquad \mathcal{R} = \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2), 0\}.$$

**Lemma 5.2** (Two-dimensional balancing)**.** *Let* $U$ *be a minimum of affine functions with gradients in* $S$ *and* $L$ *a maximum of affine functions with gradients in* $S$*. Set* $h = 1 + U - L$ *and* $w = \max(0, h)$*. For* $x, y \in \mathbb{Z}^2$*, writing* $u = \mu(x,y)$ *and* $v = \mu(y,x)$*, one has*

$$h(u) + h(v) \ge h(x) + h(y), \qquad \min\{h(u), h(v)\} \ge \min\{h(x), h(y)\},$$

*and hence* $w(u)w(v) \ge w(x)w(y)$*.*

*Proof.* Write $\delta = x - y$ and let $\varepsilon_i$ equal $\operatorname{sgn}(\delta_i)$ when $\delta_i$ is odd, and zero otherwise. Then $u = (x + y + \varepsilon)/2$ and $v = (x + y - \varepsilon)/2$. For every $r \in \mathcal{R}$,

$$|r \cdot \varepsilon| \le |r \cdot \delta|.$$

For coordinate normals this is immediate. For a diagonal normal, if the two signed coordinates of $\delta$ sum to zero, their parity residues cancel. If their sum has absolute value one, exactly one coordinate is odd, and the residue sum has absolute value one. Otherwise its absolute value is at least two, whereas the residue sum has absolute value at most two. Therefore $u, v$ lie in every slab

$$\min(r \cdot x, r \cdot y) \le r \cdot z \le \max(r \cdot x, r \cdot y), \qquad r \in \mathcal{R}, \tag{6}$$

containing $x, y$.

Write $U = \min_i \ell_i$, and choose active pieces $i, j$ at $u, v$. The difference of their gradients lies in $S - S \subset \mathcal{R}$. Since $u + v = x + y$, (6) gives

$$U(u) + U(v) = \ell_i(u) + \ell_j(v) \ge \min\{\ell_i(x) + \ell_j(y), \ell_i(y) + \ell_j(x)\} \ge U(x) + U(y).$$

The same argument applies to $-L$, whose gradient set is $-S$ and whose pairwise gradient differences again lie in $\mathcal{R}$. Adding proves the assertion about sums. Moreover, $h$ is the minimum of the affine

functions $1+\ell_i-m_j$, where $L=\max_j m_j$. Each has gradient in $\mathcal{R}$. By (6), its value at $u$ or $v$ is at least its smaller value at $x,y$, which is at least $\min\{h(x),h(y)\}$. Taking the minimum over $i,j$ proves minimum retention.

If either input weight is zero the product assertion is immediate. Otherwise write $m=\min\{h(x),h(y)\}>0$, $M=\max\{h(x),h(y)\}$, $c=h(u)$ and $d=h(v)$. Then $c,d\geq m$ and $c+d\geq m+M$, so

$$cd-mM=m(c+d-m-M)+(c-m)(d-m)\geq 0.$$

In particular both output heights are positive, proving the assertion for the truncated weights as well. □

**An alternative source of the weight inequality.** For integer endpoints the weight inequality also follows from Speyer's L-convex marginal theorem [14, Theorem 5.1]. For fixed integer intervals $J_i$, the function

$$\Phi(v_0,v_1,v_2,v_3)=\left|\mathbb{Z}\cap\bigcap_{i=0}^{3}(J_i+v_i)\right|$$

is L-log-concave: the set of $(v,t)$ with $t-v_i\in J_i$ is L-convex, and its fibers are finite. In particular, $\Phi(v')\Phi(v'')\geq\Phi(v)\Phi(\widetilde{v})$ whenever $v'+v''=v+\widetilde{v}$ and every coordinate difference of $v'$ lies between the corresponding differences of $v,\widetilde{v}$ [14, Definition 1.5]. For the weight below take

$$(J_0,J_1,J_2,J_3)=([A,B],[-D,-C],[E,F],[-H,-G]),\quad V(p,q)=(0,p,-q,p-q).$$

Then $w=\Phi\circ V$. Every difference of the four components of $V$ has a normal in $\mathcal{R}$, so (6) verifies the required coordinate-difference condition for $V(\mu(x,y))$. Thus the integer weight inequality is an application of existing L-convex theory; the preceding direct proof records its sum and minimum balancing mechanism.

We use the following specialization of the four-functions theorem of Halikias–Klartag–Slomka [12, Theorem 1.3]; see also [13, Theorems 1.1–1.2]. For nonnegative finitely supported functions $f,g,k,l$ on $\mathbb{Z}^2$, the pointwise hypothesis

$$f(r)g(s)\leq k(\mu(r,s))l(\mu(s,r))\quad(r,s\in\mathbb{Z}^2)$$

implies $(\sum f)(\sum g)\leq(\sum k)(\sum l)$. Indeed, the coordinatewise operation $\mu$ is translation equivariant and nondecreasing in each scalar argument, so it satisfies the theorem's Knothe monotonicity hypothesis with coordinate groups $\mathbb{Z}$. For fixed one argument, each unit increase of the other changes its value by zero or one, including the switch at equality in (3). This verifies the required monotonicity for negative integers as well as positive ones.

*Proof of Theorem 5.1.* Use the lattice bijection

$$(p,q,t)=(z_1,z_2-z_3,z_3),\qquad(z_1,z_2,z_3)=(p,t+q,t).$$

For fixed $(p,q)$ the four cross intervals give $L(p,q)\leq t\leq U(p,q)$, where

$$U(p,q)=\min\{B,p-C,F-q,p-q-G\},$$
$$L(p,q)=\max\{A,p-D,E-q,p-q-H\}.$$

Thus its multiplicity is $w(p,q) = \max\{0, 1 + U(p,q) - L(p,q)\}$. Both sets of gradients are $S$, so Lemma 5.2 applies. If $R_\xi = [a,b] \times [c,d]$, then

$$\mathcal{N}(\xi) = \sum_{r \in \mathbb{Z}^2} w(r) \mathbf{1}_{R_\xi}(r).$$

For $r \in R_x$ and $s \in R_y$, scalar monotonicity of $\mu$ places $\mu(r,s)$ in $R_{\mu(x,y)}$ and $\mu(s,r)$ in $R_{\mu(y,x)}$. The weight product inequality, multiplied by these rectangle indicators, therefore satisfies the pointwise four-functions hypothesis. Its sum conclusion is exactly (4). All four functions are finitely supported, and an empty input rectangle makes the inequality immediate.

For rational weights in (5), replicate the parameter vectors in a finite list with mean $b$. Whenever a pair differs by at least two in any coordinate, replace it by its two directed midpoints. The sum is unchanged, the product of counts does not decrease, and the nonnegative integer potential given by the sum of squared Euclidean norms strictly decreases. Hence the process terminates with coordinate spreads at most one. An integer coordinate mean then forces every entry of the list to equal $b$, proving (5). For real weights, approximate by positive rational weights in the same rational affine space defined by total weight one and mean exactly $b$. Such approximants exist by rational linear algebra and strict positivity. Pass to the limit in the finite weighted sum of logarithms. □

The common directed rule is essential to this proof; choosing one uniform floor orientation for every coordinate is a different assertion. Theorem 5.1 permits interacting cross constraints and all four moving endpoints, but does not permit the four cross intervals to vary between members of the family. Theorem 6.1 below resolves arbitrary simultaneous deformation of all six intervals by another method; the next example explains why local balanced rounding cannot supply that extension.

### 5.1 Triangle-supported deformations: a local obstruction

Fix the star box $[0,2] \times [0,2] \times [0,3]$ for $(z_1, z_2, z_3)$. For $\eta = (a,b,c,d,e,f) \in \mathbb{Z}^6$, let $\mathcal{T}(\eta)$ count its integer points subject to

$$a \le z_1 - z_2 \le b, \qquad c \le z_1 - z_3 \le d, \qquad e \le z_2 - z_3 \le f.$$

Here the six moving parameters are supported on the triangle $12, 13, 23$. For $m \in \mathbb{R}^6$ let its integer neighborhood be

$$C(m) = \{v \in \mathbb{Z}^6 : \lfloor m_j \rfloor \le v_j \le \lceil m_j \rceil \text{ for every } j\}.$$

**Proposition 5.3** (Failure of local parameter balancing)**.** *Set*

$$x = (-2, 0, -3, -1, -2, -1), \qquad y = (-2, 1, -2, 1, -2, 0), \qquad m = (x+y)/2.$$

*Then* $\mathcal{T}(x) = 9$ *and* $\mathcal{T}(y) = 19$*, whereas*

$$\max_{u \in C(m)} \mathcal{T}(u)\mathcal{T}(x+y-u) = 169 < 171.$$

*More strongly, every finite probability law* $(v_i, \lambda_i)$ *supported in* $C(m)$ *with mean* $m$ *satisfies*

$$\prod_i \mathcal{T}(v_i)^{\lambda_i} \le 13 < \sqrt{171}.$$

*Both endpoint fibers and every fiber indexed by* $C(m)$ *have affine dimension three. In particular, the negative logarithm of the count is not integrally convex in these six parameters.*

*Proof.* The eight neighboring vectors are

$$v(i,j,k) = (-2, i, j-3, 0, -2, k-1), \qquad (i,j,k) \in \{0,1\}^3,$$

and their counts are

$$\mathcal{T}(v(i,j,k)) = 9 + 4i - 2j + 6k. \tag{7}$$

For completeness, at $(i,j,k) = (0,0,0)$, sum by $z_3 = 0,1,2,3$: the respective counts are $0,1,3,5$, totaling 9. Raising $i$ adds exactly four points with $z_1 = z_2 + 1$; raising $k$ adds six with $z_2 = z_3$ and $z_1 \le z_2$; raising $j$ removes the two points with $z_1 = 0, z_3 = 3$. These three changes are disjoint and independent, proving (7). For $x$, the upper bound $z_1 - z_3 \le -1$ is automatic from $z_1 \le z_2 \le z_3 - 1$, so its count is 9. The vector $y$ has the restrictions of $v(1,1,1)$ except that $z_1 - z_3 \le 1$ replaces $z_1 - z_3 \le 0$. This adds exactly $(1,0,0)$ and $(2,1,1)$, giving $17 + 2 = 19$.

A law on these eight vectors with mean $m$ has $\mathbb{E}i = \mathbb{E}j = \mathbb{E}k = 1/2$. Equation (7) therefore gives $\mathbb{E}\mathcal{T}(v) = 13$. Weighted AM–GM proves the geometric bound. Equality is attained by equal weights on $v(0,1,1)$ and $v(1,0,0)$, each of count 13. Every complementary balanced pair has this mean, proving its product bound and the stated maximum. The four points

$$(0,0,1), \quad (0,0,2), \quad (0,1,2), \quad (1,1,2)$$

belong to both endpoint fibers and all eight neighboring fibers. The three differences from the first point have determinant $-1$, so all these fibers have dimension three.

For $f = -\log \mathcal{T}$, the local convex extension at $m$ is the infimum of $\sum_i \lambda_i f(v_i)$ over the laws just considered. It equals $-\log 13$, strictly greater than $(f(x) + f(y))/2 = -\frac{1}{2}\log 171$. Hence this local extension is not convex, which is precisely the obstruction to integral convexity; see [15, Section 3.4] for its local-extension formulation. □

The midpoint $m = (-2, 1/2, -5/2, 0, -2, -1/2)$ is not integral. Proposition 5.3 consequently does not refute the integral-mean Jensen question, nor the existence of a nonlocal concave extension of the logarithmic counts. It excludes the full balanced six-endpoint analogue of Theorem 5.1, including repairs using arbitrarily many local corners. The global theorem proves integral-mean Jensen in this case through a nonlocal continuous extension, despite this local obstruction.

## 6 All directed bounds in type $A_3$

Write $h = (h_{ij})_{i \ne j}$ for twelve real numbers, with indices in $\{1,2,3,4\}$, and anchor $z_4 = 0$. Define

$$P_h = \{z \in \mathbb{R}^3 : z_i - z_j \le h_{ij} \ (i \ne j)\}, \qquad D = \{h : P_h \ne \varnothing\}.$$

All these polytopes are bounded because both directions of every root are present. The set $D$ is the closed convex polyhedron defined by nonnegative directed cycle sums. Put $N_3(h) = |P_h \cap \mathbb{Z}^3|$ for integral $h \in D$. The root matrix is totally unimodular, so these counts are positive.

**Theorem 6.1** (A monotone sixth-root extension)**.** *There is a continuous, nonnegative, coordinatewise nondecreasing concave function $G : D \to \mathbb{R}$ such that $G(h) = N_3(h)^{1/6}$ at every integral $h \in D$, including lower-dimensional polytopes. Consequently all twelve raw bounds satisfy finite weighted power Jensen at integral means.*

Here coordinatewise monotonicity compares any two feasible bound vectors in the coordinate order. We prove the continuous statement, which is needed when intermediate means are nonintegral.

### 6.1 Chambers and their count polynomials

A tight support vector $a$ satisfies the twenty-four triangle inequalities

$$\tau_{ijk} = a_{ij} + a_{jk} - a_{ik} \geq 0 \qquad (i, j, k \text{ distinct}).$$

They imply $a_{ij} + a_{ji} \geq 0$. The corresponding cone $C$ is convex. Translations $a_{ij} \mapsto a_{ij} + u_i - u_j$ translate the polytope and are factored out whenever we refer to a pointed cone. Define the six forms

$$\begin{aligned} g_1 &= a_{13} + a_{24} - a_{14} - a_{23}, \quad g_2 = a_{31} + a_{42} - a_{32} - a_{41}, \\ g_3 &= a_{12} + a_{34} - a_{14} - a_{32}, \quad g_4 = a_{21} + a_{43} - a_{23} - a_{41}, \\ g_5 &= a_{12} + a_{43} - a_{13} - a_{42}, \quad g_6 = a_{21} + a_{34} - a_{24} - a_{31}. \end{aligned}$$

They obey $g_3 + g_4 = g_1 + g_2 + g_5 + g_6$. The classical three-dimensional polytrope chambers [17] have six representatives. To specify fully the finite data used here, the following seeds use the order $12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43$:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 29 | 20 | 11 | 19 | 21 | 20 | 20 | 11 | 29 | 21 | 20 | 19 |
| 2 | 42 | 32 | 22 | 22 | 32 | 42 | 32 | 32 | 32 | 42 | 22 | 32 |
| 3 | 50 | 36 | 24 | 26 | 36 | 48 | 48 | 24 | 34 | 36 | 36 | 34 |
| 4 | 77 | 60 | 43 | 77 | 43 | 60 | 60 | 43 | 77 | 43 | 60 | 77 |
| 5 | 78 | 60 | 44 | 60 | 44 | 78 | 78 | 44 | 60 | 44 | 60 | 78 |
| 6 | 78 | 62 | 44 | 78 | 44 | 62 | 44 | 62 | 78 | 62 | 44 | 78 |

Here and below the finite verifications can be reproduced by `verify_A3_structure.py`; the following description also specifies an exact reconstruction without any polynomial interpolation. Let $A$ be the twelve-row root matrix. For each seed enumerate all three-row subsets $I$ with $\det A_I \neq 0$, form $v_I = A_I^{-1} a_I$, and retain it if $Av_I \leq a$. There are twenty distinct simple vertices. The three columns of $-A_I^{-1}$ are the primitive edge directions at $v_I$; all determinants have magnitude one. Form every nonzero feasibility row $a_k - A_k A_I^{-1} a_I$ and divide by its positive rational content. The resulting forty-eight inequalities are exactly the twenty-four triangles, the six signs of $g_s$ at that seed, and eighteen sums of two of these thirty base inequalities. Conversely every base inequality occurs. These instructions are finite rational arithmetic on twelve rows; they certify equality of the cones, not only validity at a seed.

Relabelling the four nodes and transposing all directed edges gives six disjoint sign-pattern orbits of sizes $12, 8, 24, 6, 6, 6$, respectively. Their sum is 62. The remaining two of the 64 sign patterns are impossible by the displayed circuit relation. These assertions can also be checked simply by applying the 48 relabellings/transposes to the six lists of signs; hence all feasible strict sign patterns are covered. A strict triangle metric has no active directed two-path, since it would force a triangle equality. Its generic active three-edge tree is therefore bipartite; a nonsimple active graph would contain a bipartite square, forcing a $g_s$ to vanish. This proves coverage of the generic domain; adding a small positive constant to every bound, perturbing generically, and taking limits proves coverage by the six cone closures and their symmetries on all of $C$.

For completeness, the polynomial reconstruction is as follows. Choose an integer objective $\xi$ nonorthogonal to all of the finitely many edge directions $d_{Ij}$, and use the constant coefficient in $t$ of

$$\sum_I e^{t\langle \xi, v_I \rangle} \prod_{j=1}^{3} (1 - e^{t\langle \xi, d_{Ij} \rangle})^{-1}. \tag{8}$$

The sum is over the twenty vertices. Expanding $(1 - e^x)^{-1} = -x^{-1} + 1/2 - x/12 + O(x^3)$ and the exponential gives a cubic polynomial in $a$, independent of $\xi$. Brion's theorem [18] identifies it with

the lattice count on integral strict chambers. The same polynomials were computed in [17]; we use their classical formulas. Their homogeneous decomposition has the particularly useful form

$$F(a) = V(a) + Q(a) + L(a) + 1, \qquad L(a) = \frac{1}{3}\sum_{i\neq j} a_{ij} - \frac{1}{12}\sum_{s=1}^{6} |g_s(a)|. \tag{9}$$

Here $V$ is the actual Euclidean volume, $Q$ is the same quadratic in every chamber, and $L$ is linear within a chamber. If $B = \operatorname{Hess} Q$, then

$$(Ba)_{ij} = \frac{r_i + c_j}{2} - 2a_{ij} = \frac{1}{2}\sum_{k\neq i,j} (a_{ik} + a_{kj} - a_{ij}), \quad r_i = \sum_{j\neq i} a_{ij}, \quad c_j = \sum_{i\neq j} a_{ij}. \tag{10}$$

In particular $Q = a^T Ba/2$. If $K = P_{\mathbf{1}}$, then

$$V(K) = 4, \qquad B\mathbf{1} = \mathbf{1}, \qquad Q = \tfrac{1}{2} D_{\mathbf{1}} V. \tag{11}$$

These identities follow coefficientwise from (8): the cubic coefficient gives $V$, the quadratic coefficient gives (10), and the linear coefficient is $\mathbf{1}/3 - \sum_s \operatorname{sign}(g_s) g_s/12$. Thus they are a finite identity on each explicitly specified cone. The expansion recipe, seed table and base-inequality test make this calculation reproducible without importing the classical polynomials. Each coordinate coefficient of $L$ in every representative chamber is $1/6$, $1/3$ or $1/2$; relabellings and transpose preserve this fact.

The normal fan of each generic chamber refines the fan of $K$. Indeed, at $\mathbf{1}$ any active bipartite tree gives potentials with two levels differing by one, feasible for every root inequality. Thus each chamber vertex formula remains feasible at $K$. On a closed chamber all vertex formulas remain feasible and their normal cones cover the dual space. Support functions therefore add:

$$P_{a+b} = P_a + P_b \quad \text{for } a, b \text{ in the same closed chamber.} \tag{12}$$

This includes degenerate summands. Modulo translations that chamber is a pointed rational cone with finitely many rational generators $a_j$. The gauge $a_{14} = a_{24} = a_{34} = 0$ makes every other coordinate nonnegative: it is obtained by the potential $u_i = -a_{i4}$, and the triangle inequalities give the assertion. A point polytope represents only the zero class. Consequently every nonzero generator gives a nonpoint polytope and $L(a_j) > 0$.

### 6.2 A degree-six Lorentzian certificate

Fix a closed chamber and its Minkowski generators, write $a(y) = \sum_j y_j a_j$ for $y_j \geq 0$, and define

$$H(y,t) = t^3 V(a(y)) + t^4 Q(a(y)) + t^5 L(a(y)) + t^6.$$

We use the classical characterization of Lorentzian polynomials [16, Theorem 2.25]: a homogeneous nonnegative-coefficient polynomial with M-convex support is Lorentzian if all its derivatives of order two less than its degree are zero or quadratics with at most one positive Hessian eigenvalue. M-convexity is the symmetric exchange property of the exponent support. Volume polynomials of Minkowski sums have that support property [16, Theorem 4.1 and Corollary 4.2].

Mixed volumes, (11), and $L(a_j) > 0$ give nonnegative coefficients of $H$. Its support equals that of $t^3 \operatorname{vol}(\sum_j y_j P_{a_j} + tK)$: cubic terms agree; quadratic terms are positive multiples by $Q = \frac{1}{2} D_K V$; every linear coefficient in either is positive because $P_{a_j}$ is nonpoint and $K$ is full-dimensional; and both constant terms are positive. Multiplication by $t^3$ translates the exponent support and

preserves M-convexity. Thus the support requirement holds even for lower-dimensional generator polytopes.

We give every quadratic derivative check explicitly. First $B$ has orthogonal eigenspaces: constants (eigenvalue 1, dimension 1); symmetric arrays $a_{ij} = u_i + u_j$, $\sum u_i = 0$ (eigenvalue $-1$, dimension 3); arrays with all row and column sums zero (eigenvalue $-2$, dimension 5); and potentials $a_{ij} = u_i - u_j$ (eigenvalue 0, dimension 3). Formula (10) verifies this decomposition directly. Thus the translation quotient has inertia $(1, 8)$.

On the chosen chamber write $\ell = \mathbf{1}/3 - R/12$, where $R = \sum_s \sigma_s g_s$ and $\sigma_s = \operatorname{sign} g_s$. Each $g_s$ has row and column sums zero, hence $Bg_s = -2g_s$. Its squared norm is 4; inner products within the pairs $(1, 2)$, $(3, 4)$, $(5, 6)$ are zero. Between the first and second pairs they are 1, between the first and third $-1$, and between the second and third 1. If $U = \sigma_1 + \sigma_2$, $W = \sigma_3 + \sigma_4$ and $T = \sigma_5 + \sigma_6$, then

$$\|R\|^2 = 24 + 2(UW - UT + WT) \leq 32.$$

Indeed $UW - UT + WT$ is multiaffine on $[-2, 2]^3$; at its eight vertices its values are 4 or $-12$. Orthogonality to constants now gives, with inverse taken on the translation quotient,

$$\rho = \ell^T B^{-1} \ell = \frac{4}{3} - \frac{\|R\|^2}{288} \geq \frac{11}{9} > \frac{6}{5}.$$

This bound holds even for sign patterns not realized by a chamber. Put $u = B^{-1}\ell$. The fourth $t$ derivative is

$$24Q + 120tL + 360t^2 = 12(a + 5tu)^T B(a + 5tu) + (360 - 300\rho)t^2.$$

The final coefficient is negative, so this quadratic has at most one positive direction. Pullback to redundant generator coordinates cannot increase that index.

For a generator direction $d$, put $H_d = \operatorname{Hess} V(d)$, using cubic polarization. Alexandrov–Fenchel gives at most one positive direction on the Minkowski-generator span, by continuity also for boundary $d$. The identities (11) give $H_d\mathbf{1} = 2Bd$ and $\mathbf{1}^T H_d \mathbf{1} = 2\sum d_{ij}$. The derivative with one shape and three $t$ derivatives is therefore

$$\begin{aligned} 6D_dV + 24tD_dQ + 60t^2 L(d) = {} & 3(a + 2t\mathbf{1})^T H_d (a + 2t\mathbf{1}) \\ & + t^2(60L(d) - 24\sum d_{ij}). \end{aligned}$$

The last coefficient is nonpositive because $L(d) \leq \sum d_{ij}/3$ and the chosen gauge has nonnegative coordinates. No inverse of $H_d$ is required. The two-shape/two-$t$ derivative is $6tD_dD_eV + 12t^2 B(d, e)$, a product of $t$ and a nonnegative linear form. The three-shape/one-$t$ derivative is $3t^2 D_dD_eD_fV \geq 0$; four shape derivatives give zero. These five cases exhaust the order-four coordinate derivatives. Each quadratic has at most one positive Hessian eigenvalue. The Lorentzian characterization and homogeneous power-concavity [16, Proposition 2.33] prove $H^{1/6}$ concave. Set $t = 1$ to obtain $F^{1/6}$ concave on the entire closed chamber. Choose any nonnegative generator representations of two points and interpolate them; this transfers concavity to support coordinates regardless of nonuniqueness of those representations.

### 6.3 Gluing, raw bounds, and exact boundary values

Inside the strict metric domain, volume is $C^1$ even across changes of combinatorial type: its derivative in a fixed support is facet area divided by the fixed normal length, continuously tending to zero when that facet disappears. Thus $V$ has no derivative jump at a square wall; $Q$ is the same

global quadratic, and $L$ in (9) has only downward derivative jumps. Also $F \geq 1$: after the nonnegative gauge, mixed-volume positivity and $L \geq 0$ give $V, Q, L \geq 0$. The derivative jumps of $F^{1/6}$ have the same sign as those of $F$. Piecewise concavity consequently glues along generic segments; approximation covers segments in walls and the boundary of $C$. Hence $F^{1/6}$ is concave on all tight supports.

We need monotonicity separately. Volume is inclusion-monotone. Equation (10) gives $\partial_{ij} Q$ as half the sum of two nonnegative triangle slacks, so $Q$ is nondecreasing along comparable points of the convex cone $C$. Each coordinate occurs in exactly two $g_s$. For any nonnegative increment $d$,

$$\sum_s (|g_s(a+d)| - |g_s(a)|) \leq 2 \sum_{ij} d_{ij}, \qquad L(a+d) - L(a) \geq \tfrac{1}{6} \sum_{ij} d_{ij}.$$

It follows that $F$, and its sixth root, are coordinatewise monotone on $C$, including across chamber walls.

For raw $h \in D$, let $c_{ij}(h)$ be the shortest directed path length from $j$ to $i$ with the edge $j \to i$ assigned length $h_{ij}$. Feasibility excludes negative cycles, so a minimum over the finite set of simple paths suffices. This map is continuous, entrywise concave, and integral on integral inputs; $c(h) \in C$ and $P_{c(h)} = P_h$. Therefore

$$c\left(\sum_s \lambda_s h_s\right) \geq \sum_s \lambda_s c(h_s).$$

Concavity and the separately proved monotonicity on $C$ imply that $G(h) = F(c(h))^{1/6}$ is concave and monotone on $D$.

It remains essential to prove exact integer values at degenerate points; continuity of lattice counts would not suffice. Let $a \in C$ be integral and choose a closed chamber containing it and a strict integral vector $v$ in that chamber. Equation (12) gives $P_{ra+sv} = rP_a + sP_v$. The mixed Ehrhart theorem [19, Theorem 2.1] makes its lattice count a polynomial in the nonnegative integers $r, s$, including zero. For $r \geq 0, s \geq 1$ it agrees with the chamber polynomial $F(ra+sv)$ by (8). Polynomial identity then gives agreement also at $(r, s) = (1, 0)$. This covers lower-dimensional summands because $P_v$ is full-dimensional. Closure preserves the polytope and integrality, so $G(h) = N_3(h)^{1/6}$ for every integral feasible raw $h$. This proves Theorem 6.1.

# 7 Integral rank-three transfer and countable demands

We supply the common-lattice and unbounded-support details to avoid imposing a finite-support cap implicitly.

**Lemma 7.1** (Rank-three normal directions)**.** *A totally unimodular matrix with three columns containing $I_3$ has one right transformation in* $\mathrm{GL}(3, \mathbb{Z})$ *taking all its rows to zero or signed roots in* $\{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}$.

*Proof.* Suppress zero rows and repetitions and normalize row signs by first nonzero positive. Besides the three units, the ten possibilities are

$$\begin{array}{lll} d_0 = (0,1,-1), & d_1 = (0,1,1), & d_2 = (1,-1,-1), \\ d_3 = (1,-1,0), & d_4 = (1,-1,1), & d_5 = (1,0,-1), \\ d_6 = (1,0,1), & d_7 = (1,1,-1), & d_8 = (1,1,0), \\ d_9 = (1,1,1). & & \end{array}$$

The following is a fully finite minor classification. A pair is compatible with $I_3$ precisely for

$$(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(1,2),(1,3),(1,5),(1,6),(1,8),(1,9),$$
$$(2,3),(2,5),(3,4),(3,5),(3,6),(4,6),(5,7),(5,8),(6,8),(6,9),(7,8),(8,9).$$

The graph has twenty triangles; exactly $(0,3,6),(0,5,8),(1,3,5),(1,6,8)$ fail a determinant test. Its only four-cliques are $(0,3,4,6),(0,5,7,8),(1,2,3,5),(1,6,8,9)$, each containing a forbidden triangle. Every pair and singleton extends to a valid triangle. Thus there are sixteen maximal systems. The table gives the three rows of a suitable transformation $T$ for each; direct multiplication checks $\det T = \pm 1$ and that all six directions become signed roots.

| extras | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| $1,2,3$ | $(-1,0,0)$ | $(0,-1,0)$ | $(0,1,-1)$ |
| $0,3,4$ | $(-1,0,0)$ | $(0,-1,0)$ | $(0,-1,1)$ |
| $1,2,5$ | $(-1,0,0)$ | $(0,1,-1)$ | $(0,-1,0)$ |
| $0,3,5$ | $(-1,0,0)$ | $(0,-1,0)$ | $(0,0,-1)$ |
| $2,3,5$ | $(-1,0,0)$ | $(0,-1,0)$ | $(-1,0,1)$ |
| $1,3,6$ | $(-1,0,0)$ | $(0,-1,0)$ | $(0,0,1)$ |
| $0,4,6$ | $(-1,0,0)$ | $(0,1,-1)$ | $(0,1,0)$ |
| $3,4,6$ | $(-1,0,0)$ | $(0,-1,0)$ | $(1,0,-1)$ |
| $0,5,7$ | $(-1,0,0)$ | $(0,-1,1)$ | $(0,-1,0)$ |
| $1,5,8$ | $(-1,0,0)$ | $(0,1,0)$ | $(0,0,-1)$ |
| $0,6,8$ | $(-1,0,0)$ | $(0,1,0)$ | $(0,0,1)$ |
| $0,7,8$ | $(-1,0,0)$ | $(0,1,0)$ | $(0,1,-1)$ |
| $5,7,8$ | $(-1,0,0)$ | $(0,1,0)$ | $(-1,0,1)$ |
| $1,6,9$ | $(-1,0,0)$ | $(0,-1,1)$ | $(0,1,0)$ |
| $1,8,9$ | $(-1,0,0)$ | $(0,1,0)$ | $(0,-1,1)$ |
| $6,8,9$ | $(-1,0,0)$ | $(0,1,0)$ | $(1,0,-1)$ |

The pair list is obtained by checking the $2\times 2$ and $3\times 3$ minors after appending the two rows to $I_3$; the triangle test adds only their three-row determinant. These are exhaustive tests because there are only three columns. Restoring signs and repetitions proves the assertion. The accompanying `verify_rank_three.py` independently tests all $2^{10} = 1024$ masks (51 valid systems). □

*Proof of Theorem 1.1.* If $n = 0$, the only feasible fiber is a point and the claim is immediate. Assume $n > 0$. Central boundedness implies $\ker M \cap \mathbb{R}^n_{\geq 0} = \{0\}$. The continuous function $\|Mx\|$ has positive minimum $\eta$ on the compact nonnegative unit $\ell^1$ simplex. Hence

$$\|x\|_1 \leq \eta^{-1}\|Mx\| \qquad (x \geq 0). \tag{13}$$

Every feasible endpoint fiber is consequently bounded, and any choice of a feasible point at each demand atom is integrable.

Let $J$ be the coordinates not identically zero on $P(b)$. If an endpoint point were positive outside $J$, select it at that atom and arbitrary feasible points at every other atom. Integer-valued demand has countable support, so this selection is measurable and integrable by (13). Its expectation would be central feasible and positive outside $J$, a contradiction. All endpoint fibers therefore lie in this one active face. Averaging finitely many central witnesses gives a point positive in every retained coordinate, and $r = |J| - \operatorname{rank} M_J \leq 3$.

Normalize $M$ by a full unimodular column basis before restricting columns: $B_0^{-1}M = [I, D_0]$ is totally unimodular, since every minor of $D_0$ completes with identity columns to a maximal minor.

Restrict to $J$, choose independent rows, and then an integral pivot basis $H_0$. The solutions have common coordinates

$$x_{\text{pivot}} = H_0^{-1}c(a) - H_0^{-1}D_1 z, \qquad x_{\text{free}} = z,$$

where $c(a)$ is integer linear in $a$. Feasibility ensures consistency of all omitted equations. Complementary minors show the kernel matrix $[-H_0^{-1}D_1; I_r]$ is TU, and this parameterization is a lattice bijection, not a finite-index substitution. If $r < 3$, pad with coordinates fixed to zero by their two opposite inequalities; the resulting three-column normal matrix is TU and contains $I_3$ up to row signs. Lemma 7.1 supplies one unimodular transformation for the entire countable family. Thus every fiber becomes

$$Q(a) = \{w \in \mathbb{R}^3 : n_j \cdot w \le c_j(a),\ j \in \mathcal{J}\},$$

with fixed signed $A_3$ roots $n_j$, integer-linear offsets $c_j$, and possible zero-normal feasibility inequalities. Counts are preserved.

Interpret a root $w_i - w_j \le c$ as the edge $j \to i$ of length $c$, retaining repeated edges. This fixed graph is strongly connected. Otherwise its condensation graph admits a nonconstant potential with $v_i - v_j \le 0$ on every edge; subtract the anchor value to produce a nonzero recession vector, contradicting boundedness of $Q(b)$. For every ordered pair define $h_{ij}(a)$ as the minimum length over its finitely many simple directed paths. Each length is integer linear in $a$, so $h$ is componentwise concave, integral on integral demands, and obeys $|h(a)| \le C\|a\|$. At feasible demands negative cycles are absent, and these path bounds give exactly $Q(a) = P_{h(a)}$. No finite cap on the countable family is assumed.

The random vector $Y = h(A)$ is integrable and lies in $D$. For every path $p$, $\mathbb{E}h_{ij}(A) \le \mathbb{E}c_p(A) = c_p(b)$; taking the finite minimum gives $\mathbb{E}Y \le h(b)$. Theorem 6.1 and countable Jensen now give

$$N(b)^{1/6} = G(h(b)) \ge G(\mathbb{E}Y) \ge \mathbb{E}G(Y) = \mathbb{E}N(A)^{1/6}.$$

To justify the middle inequality without assuming integrability of $G(Y)$ or an interior mean, enumerate atoms $y_s$ with probabilities $p_s$ and group the tail after $k$ at its conditional mean $y_{>k}$. The domain $D$ is a closed convex polyhedron, so this conditional mean belongs to $D$. Finite concavity and nonnegativity give

$$G(\mathbb{E}Y) \ge \sum_{s\le k} p_s G(y_s) + \Big(\sum_{s>k} p_s\Big) G(y_{>k}) \ge \sum_{s\le k} p_s G(y_s).$$

An empty tail is omitted. Monotone convergence proves both the inequality and finiteness of the expectation. Every count is a positive integer by unimodularity. Thus $0 \le \log N(A) \le 6N(A)^{1/6}$ is integrable, and scalar logarithmic Jensen yields $\mathbb{E}\log N(A) \le 6\log \mathbb{E}N(A)^{1/6} \le \log N(b)$. □

# 8 Block products and scope

**Corollary 8.1.** *Let a finite DAG be assembled at articulation vertices from blocks that each have cycle rank at most three. Then its integer-flow counts satisfy (1.1) for every admissible family.*

*Proof.* For a leaf block attached at $v$, its induced demand is the original demand away from $v$ and minus the sum of those demands at $v$. The remaining demand at $v$ is the original demand minus this induced block demand. These maps are integer linear. Restricting flows to the two edge sets is a Cartesian-product bijection, so counts multiply. Iterate over blocks, apply Theorem 1.1 to each, and multiply the logarithmic inequalities. Disconnected components factor as well. □

For a fixed number $k$ of blocks, the geometric mean on $\mathbb{R}^k_{\geq 0}$ is increasing and concave. Applying it to the block sixth-root inequalities gives power $1/(6k)$ for the product count. Theorem 1.3 concerns the universal low-central-dimension class, not a fixed graph and not the best finite-$m$ exponent. The optimality in Theorem 1.4 concerns its specified continuous extension. Proposition 4.2 bounds that lattice-only power between $4/13$ and $\alpha_* < 1/3$ for the complete four-vertex DAG. The opposite-edge result in Theorem 5.1 addresses another restricted family and does not change this lattice-power bound. Theorem 1.1 resolves (1.1) in central affine dimension at most three. The structural checkers verify exhaustive finite identities used in the proof; they are not sample-based evidence for the all-parameter theorem. The remaining concavity, closure and integrability arguments are analytic.

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