Power Concavity of Low-Dimensional Unimodular Fiber Counts

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Abstract

We prove weighted power-concavity inequalities for lattice-point counts of bounded unimodular fibers. In central affine dimension at most two, the square root of the corrected count $N - 1/4$ satisfies weighted Jensen; the correction $1/4$ is best possible. If every endpoint count is at least $m \geq 1$, the power $(2m-1)/(4m-1)$ is valid. The best universal power obeys $A(m) = 1/2 - 1/(8m) + O(m^{-3/2})$, with explicit finite-threshold bounds. The obstruction uses a fixed six-edge network. We also prove that $4/13$ is the optimal concavity power of the classical continuous piecewise polynomial extension of the type-A_3 Kostant partition function. For actual integral demands, cube-root concavity already fails on this four-vertex graph, including a midpoint triple with three-dimensional fibers. An explicit lattice-only upper bound complements the $4/13$ lower bound. The planar argument combines unimodular elimination, Pick's theorem and classical Brunn–Minkowski concavity; the A_3 argument uses the classical chamber formulas and a Hessian comparison. These results give restricted global Jensen inequalities.

1 Statements and relation to earlier work

Throughout, a full-row-rank integer matrix $M \in \mathbb{Z}^{p\times n}$ is called *unimodular* if every nonzero maximal column minor is ± 1. For an integral right-hand side a put

$$P(a) = \{x \in \mathbb{R}^n_{\geq 0} : Mx = a\}, \qquad N(a) = |P(a) \cap \mathbb{Z}^n|.$$

Every fiber used below is nonempty and bounded. A finite family of integral demands a_i and weights $\lambda_i > 0$ is *admissible* when $\sum_i \lambda_i = 1$ and $b = \sum_i \lambda_i a_i$ is integral. We write $N_i = N(a_i)$.

Leake and Mohammadi Yekta prove log-concavity along rational lines for unimodular fiber counts and ask whether the full discrete Jensen inequality

$$N(b) \geq \prod_i N_i^{\lambda_i} \tag{1.1}$$

holds [1, Theorem 1.4, Question 1.7]. Intermediate pair averages need not be integral, so their line theorem does not automatically establish (1.1). Our first result concerns actual central fiber dimension, rather than the number of equations or columns. Approximate global inequalities for contingency tables and integer flows were established earlier by Barvinok [8]; their correction factors distinguish them from (1.1). The complete-DAG version of the finite weighted question was also stated explicitly by Leake and Morales [9, Section 7.4].

Theorem 1.1 (Low-dimensional power concavity)**.** *Let M be unimodular and let (a_i, λ_i) be admissible. If* $\dim \operatorname{aff} P(b) \leq 2$*, then*

$$\sqrt{N(b) - \tfrac{1}{4}} \geq \sum_i \lambda_i \sqrt{N_i - \tfrac{1}{4}}. \tag{1.2a}$$

No smaller real correction can replace $1/4$ *uniformly. If also* $N_i \geq m$ *for an integer* $m \geq 1$*, then*

$$N(b)^{\alpha_m} \geq \sum_i \lambda_i N_i^{\alpha_m}, \qquad \alpha_m = \frac{2m-1}{4m-1} = \frac{1}{2} - \frac{1}{8m-2}. \tag{1.2b}$$

In particular, (1.1) holds.

For $m \in \mathbb{N}_{\geq 1}$ let $A(m)$ be the supremum of the positive exponents for which the power inequality in (1.2b) holds for *every* matrix and admissible family in Theorem 1.1, with endpoint counts at least m. Neither the matrix nor the number of endpoints is fixed.

Theorem 1.2 (Sharp large-count asymptotic)**.** *Put* $n(r) = 3r^2 + 3r + 1$ *and* $R_m = \min\{r \in \mathbb{Z}_{\geq 0} : n(r) \geq m\}$*. Then*

$$\frac{1}{2} - \frac{1}{8m-2} \leq A(m) \leq \frac{1}{2} - \frac{1}{8n(R_m+2)-2}.$$

In particular,

$$A(m) = \frac{1}{2} - \frac{1}{8m} + O(m^{-3/2}), \qquad \lim_{m\to\infty} m\left(\frac{1}{2} - A(m)\right) = \frac{1}{8}. \tag{1.3}$$

Square-root concavity fails for every endpoint-count threshold m*. We do not determine* $A(m)$ *exactly at any finite threshold.*

For a DAG we use net outflow as the incidence convention. After deleting one incidence row per connected component, its incidence matrix is unimodular. Acyclicity makes every nonempty nonnegative flow fiber bounded. For the complete DAG on $0, 1, 2, 3$, let

$$(s, t, u) = (b_0, b_0 + b_1, b_0 + b_1 + b_2).$$

Its feasible integral demands correspond exactly to $(s, t, u) \in \mathbb{Z}^3_{\geq 0}$. The inverse map is $b = (s, t - s, u - t, -u)$.

Theorem 1.3 (The complete four-vertex DAG)**.** *The integer-flow count for the complete four-vertex DAG has a positive, continuous piecewise polynomial extension* F *to* $\mathbb{R}^3_{\geq 0}$ *such that* $F^{4/13}$ *is concave. No power* $\alpha > 4/13$ *makes this extension* F^α *concave. Thus, for every admissible family of feasible demands,*

$$N(b)^{4/13} \geq \sum_i \lambda_i N_i^{4/13}, \tag{1.4}$$

and consequently (1.1) holds.

The optimality assertion concerns this continuous extension, not the best power for inequalities restricted to integral demands. Proposition 4.2 below gives a separate obstruction on actual integral demands: even the cube-root power fails there.

Theorem 1.3 includes three-dimensional interacting fibers: when $s, t, u > 0$, all six edges can carry positive flow. Its chamber enumeration is classical, with an early seven-case treatment in Tarski's 1958 report [10, Section VII]. For the formulas used here see De Loera–Sturmfels [3, Section 5] and the earlier computations cited there. We use those polynomials, deriving them again for completeness, and prove a power-concavity property of their continuous extension. Root-direction inequalities and normalized Lorentzian series for restricted Kostant functions appear in Khare–Matherne–St. Dizier [2]; they are different from concavity of counts in demand space.

The geometry behind Theorem 1.1 is also classical. Alcoved triangulations and volume–lattice-count identities are developed by Lam–Postnikov [4, 5]. The relevant planar count formula follows

from Pick and mixed area. Its continuous concavity belongs to the Brunn–Minkowski/generalized-Wills framework [6]. General lattice-point Brunn–Minkowski inequalities [7] require an enlargement that is absent from (1.1). We claim no new continuous Wills inequality or new chamber-count formula. Our results concern their exact-count applications, quantitative exponents and the global A_3 power inequality.

2 Common lattice coordinates and the planar count identity

Lemma 2.1 (Active-coordinate reduction)**.** *Under the hypotheses of Theorem 1.1, all endpoint fibers lie in the coordinate face active at $P(b)$. In common integral coordinates their free dimension is $r = \dim \operatorname{aff} P(b)$. If $r = 2$, every full-dimensional fiber polygon has primitive outward normals among*

$$\pm(1,0), \quad \pm(0,1), \quad \pm(1,1), \tag{2.1}$$

after one common coordinate sign change. Coordinate maps commute with weighted averaging.

Proof. Let E_* be the coordinates positive at some point of $P(b)$. Minkowski averaging gives $\sum_i \lambda_i P(a_i) \subset P(b)$. Fixing all endpoint points but one shows that every endpoint coordinate outside E_* is zero. Averaging finitely many witnesses gives a central point positive on all of E_*. Thus its affine dimension is the kernel dimension of the retained column submatrix M_*.

Let $s = \operatorname{rank} M_*$. Choose s independent active columns B_0 and complete them to a column basis $B = (B_0, B_1)$ of M. Since $\det B = \pm 1$, B^{-1} is integral. The active columns of $B^{-1}M$ have the form $\binom{D}{0}$, where D contains I_s. Every s-column minor of D is, up to sign, a maximal minor of M completed by the fixed B_1 columns and divided by $\det B$. It is therefore 0 or ± 1. Write $D = [I_s, C]$. Completing any square minor of C by unused identity columns proves that C is totally unimodular.

With free coordinates $t \in \mathbb{R}^r$, the active solutions are

$$x_{\text{free}} = t, \qquad x_{B_0} = (B^{-1}a)_{1:s} - Ct. \tag{2.2}$$

This is a lattice bijection and its offset is integer linear in a. For $r = 2$, every row of C has entries $0, \pm 1$. Both diagonal row directions $(1,1)$ and $(1,-1)$ cannot occur, since their determinant has magnitude 2. Include the free nonnegativity inequalities and change one coordinate sign if needed to obtain (2.1). All fibers are integral: choosing an independent basis at a vertex and completing it as above gives integral basic coordinates. Finally, (2.2) and linearity give $\sum_i \lambda_i Q(a_i) \subset Q(b)$ in the common free coordinates. □

Set

$$K = \{(x,y) : |x| \le \tfrac{1}{2},\ |y| \le \tfrac{1}{2},\ |x+y| \le \tfrac{1}{2}\}.$$

It has area $3/4$ and support value $1/2$ at each primitive normal in (2.1).

Lemma 2.2 (Exact planar representation)**.** *If Q is an integral polygon with normals (2.1), or a degenerate fiber segment or point in this normal system, then*

$$|Q \cap \mathbb{Z}^2| = \operatorname{area}(Q + K) + \frac{1}{4}. \tag{2.3}$$

Proof. For a full-dimensional Q, mixed area gives

$$\operatorname{area}(Q + K) = \operatorname{area}(Q) + \frac{B(Q)}{2} + \frac{3}{4},$$

where $B(Q)$ is the sum of lattice edge lengths. Indeed, an edge with primitive normal v contributes its Euclidean length times $h_K(v)/\|v\|$, which is half its lattice length. Pick's formula proves (2.3). For a point, (2.3) reads $1 = 3/4 + 1/4$. A degenerate fiber segment is parallel to a primitive integer vector perpendicular to one of the normals (2.1). The transverse width of K times that primitive vector's Euclidean length is 1. A segment of lattice length ℓ therefore adds area ℓ to K, and both sides of (2.3) equal $\ell + 1$. □

For clarity, the continuous functional in (2.3) has a generalized-Wills representation. If R is uniform on $[0, 2]$, then

$$\mathbb{E}\ \mathrm{area}(Q + RK) = \mathrm{area}(Q) + 2V(Q, K) + 1 = \mathrm{area}(Q + K) + \tfrac{1}{4}. \tag{2.4}$$

Equivalently it is the integral of the radial tent $(1 - \mathrm{dist}_K(x, Q)/2)_+$. Classical Borell–Brascamp–Lieb yields its cube-root concavity; see the generalized-Wills setting in [6]. The following calculation retains the larger domain available when counts are large.

3 Count-dependent powers and the optimal leading deficit

Proof of Theorem 1.1. If the central affine dimension is zero, all endpoint fibers are points. If it is one, their direction spaces lie in the same rational line: vary one endpoint in the Minkowski inclusion. Choose its primitive integer direction. Endpoint and central fibers are integral segments of normalized lengths ℓ_i, ℓ, allowing length zero. Then $\ell \geq \sum_i \lambda_i \ell_i$, so $N(b) \geq \sum_i \lambda_i N_i$. Apply the increasing concave function $x \mapsto \sqrt{x - 1/4}$ on $[1, \infty)$ to prove (1.2a). The claimed power inequality follows as well.

For dimension two use Lemmas 2.1–2.2. Put $c = 1/4$ and $s_i = \sqrt{\mathrm{area}(Q(a_i) + K)} = \sqrt{N_i - c}$. Brunn–Minkowski, common-coordinate inclusion and monotonicity imply

$$\sqrt{N(b) - c} = \sqrt{\mathrm{area}(Q(b) + K)} \geq \sum_i \lambda_i s_i. \tag{3.1}$$

This proves (1.2a). Every $s_i \geq \sqrt{m - c}$. For $g_\alpha(s) = (s^2 + c)^\alpha$,

$$g''_\alpha(s) = 2\alpha(s^2 + c)^{\alpha-2}\big[c + (2\alpha - 1)s^2\big].$$

At $\alpha = \alpha_m$, $1 - 2\alpha = c/(m - c)$. Thus g_{α_m} is increasing and concave on $[\sqrt{m - c}, \infty)$. Apply it to (3.1), then use scalar Jensen. Weighted AM–GM proves (1.1). □

Lemma 3.1 (A fixed network obstruction)**.** *Orient $K_{2,3}$ from its two sources to its three sinks. For net outflow*

$$a(r) = (3r, 3r, -2r, -2r, -2r), \qquad r \in \mathbb{Z}_{\geq 0},$$

the count is $n(r) = 3r^2 + 3r + 1$. For $r > 0$ the fiber dimension is two.

Proof. If t_j is the flow from the first source to sink j, the other flow into that sink is $2r - t_j$. Hence $0 \leq t_j \leq 2r$ and $\sum_j t_j = 3r$. Inclusion–exclusion gives $\binom{3r+2}{2} - 3\binom{r+1}{2} = 3r^2 + 3r + 1$ for $r \geq 1$; two upper-bound violations are impossible. At $r = 0$ the count is 1. For $r > 0$, $(t_1, t_2, t_3) = (r, r, r)$ is a relative interior point of the two-dimensional slice. □

Proof of Theorem 1.2. Theorem 1.1 gives $A(m) \geq \alpha_m$. For $f_\alpha(r) = n(r)^\alpha$, the sign of $f''_\alpha(r)$ is the sign of

$$\alpha - \tau(r), \qquad \tau(r) = \frac{1}{2} - \frac{1}{6(2r + 1)^2} = \frac{1}{2} - \frac{1}{8n(r) - 2}. \tag{3.2}$$

At $\alpha = 1/2$ this is positive everywhere. Taking $R = R_m$ gives $2n(R+1)^{1/2} < n(R)^{1/2} + n(R+2)^{1/2}$, proving the square-root failure at arbitrarily large counts.

The function τ is increasing on $[0, \infty)$. For every $\alpha > \tau(R+2)$, equation (3.2) is positive throughout $[R, R+2]$. Thus the integral endpoints $R, R+2$, with integral midpoint $R+1$, violate power Jensen and have counts at least m. No such exponent is admissible, giving $A(m) \le \tau(R+2)$. Minimality gives $R = \sqrt{m/3} + O(1)$ and $n(R+2) = m + O(\sqrt{m})$. Expanding the two explicit bounds proves (1.3). □

Sharpness of the correction in Theorem 1.1. For the same network, $n(r) - c = 3(r+1/2)^2 + 1/4 - c$. If $c < 1/4$, the function $\sqrt{n(r) - c}$ is strictly convex on $[0, \infty)$. Every adjacent integral triple therefore violates the proposed corrected square-root inequality. For $c = 1/4$ this function is affine, so these families also give equality in (1.2a). □

4 A three-dimensional case: the type-A_3 count

Use chord flows $x_{02} = a$, $x_{03} = v$, $x_{13} = c$. The other flows are $s - a - v$, $t - a - v - c$, $u - v - c$. Thus the count is

$$|\{(a, v, c) \in \mathbb{Z}^3_{\ge 0} : a + v \le s,\ v + c \le u,\ a + v + c \le t\}|. \tag{4.1}$$

Set

$$H(s, u) = \frac{(s+1)(s+2)(3u - s + 3)}{6}, \qquad B(d) = \frac{d(d+1)(d+2)}{6}.$$

The classical chamber formulas [3, Section 5], in these coordinates, define the extension F for $0 \le s \le u$ by

$$F(s, t, u) = \begin{cases} (t+1)(t+2)(t+3)/6, & 0 \le t \le s, \\ (s+1)(s+2)(3t - 2s + 3)/6, & s \le t \le u, \\ H(s, u) - B(s + u - t), & u \le t \le s + u, \\ H(s, u), & s + u \le t. \end{cases} \tag{4.2}$$

For $s > u$ interchange s, u.

Here is a direct count check. In the first chamber, (4.1) is a simplex. In the second, sum $(w+1)(t - w + 1)$ over $0 \le w \le s$. In the last two chambers, fix v and begin with the rectangle $0 \le a \le s - v$, $0 \le c \le u - v$. Summing its sizes over $v = 0, \dots, s$ gives $H(s, u)$. If $d = s + u - t \ge 0$, the points excluded at its upper corner have total count $\sum_{v=0}^{d-1} \binom{d-v+1}{2} = \binom{d+2}{3} = B(d)$. This proves (4.2) on integer points. Its values agree on all walls. In the middle chamber $0 \le d \le s \le u$, and $H(s, u) \ge H(d, d) > B(d)$, so F is positive everywhere.

Lemma 4.1 (A Hessian comparison)**.** *In the middle chamber of (4.2), $F^{4/13}$ is concave.*

Proof. Use affine coordinates (s, L, d), where $L = 3u - s + 3$, and write $H = f(s)L$ with $f(s) = (s+1)(s+2)/6$. Put $\beta = 9/13$ and $C = \beta \nabla F \nabla F^\top - F \operatorname{Hess} F$. Concavity of $F^{4/13}$ is equivalent to $C \succeq 0$. Eliminating its positive d, d entry gives the Schur complement

$$F\big[khh^\top - \operatorname{Hess} H\big], \qquad k = \frac{\beta B''}{\beta (B')^2 + F B''}, \quad h = \nabla H, \tag{4.3}$$

where the remaining derivatives are in (s, L). The L, L entry of the bracket is $kf^2 > 0$, and its determinant is

$$kLf\big(2(f')^2 - ff''\big) - (f')^2.$$

Consequently $C \succeq 0$ is equivalent to $\beta(HJ - Q) \geq H - B$, where

$$J = 2 - \frac{ff''}{(f')^2} = \frac{3}{2} + \frac{1}{2(2s+3)^2}, \qquad Q = \frac{(B')^2}{B''}.$$

For $w = 2s + 3$ this condition is $E \geq 0$, with

$$E = \frac{H(w^2+9)}{2w^2} + 13B - 9Q. \tag{4.4}$$

The first term increases with u. At $u = s$ it equals $(w^3 + 8w - 9/w)/48$, which increases with $s \geq 0$. Since $0 \leq d \leq s \leq u$, decreasing first u to s and then s to d therefore gives

$$E(s,u,d) \geq E(d,d,d) = \frac{d(2d^4 + 17d^3 + 46d^2 + 58d + 30)}{12(d+1)(2d+3)} \geq 0.$$

This proves the required positive semidefiniteness. □

Proof of Theorem 1.3. The other three chamber polynomials in (4.2) are products of three positive affine functions; their cube roots, and hence their 4/13 powers, are concave. Lemma 4.1 handles the remaining chamber.

We verify the global gluing, since chamberwise concavity alone would not suffice. Gradients below are ordered as (s,t,u). For increasing t with $s < u$, the gradient jumps of F are

wall	normal into the next chamber	$\nabla F_{\text{next}} - \nabla F_{\text{previous}}$
$t = s$	$(-1,1,0)$	$-\frac{3s+5}{6}(-1,1,0)$
$t = u$	$(0,1,-1)$	$-\frac{3s+4}{6}(0,1,-1)$
$t = s + u$	$(-1,1,-1)$	$-\frac{1}{3}(-1,1,-1)$.

(4.5)

On the symmetry wall $s = u$, crossing into $s > u$, the jump is $-\frac{1}{6}(1,0,-1)$ in the last two chambers and zero in the first. The second chamber meets this wall only at $s = t = u$. All these identities follow by differentiating (4.2).

Because wall values coincide, the jumps for $F^{4/13}$ have the same signs, multiplied by the common positive factor $\frac{4}{13}F^{-9/13}$. Thus the derivative along a line jumps downward at every generic wall crossing. Its restriction is concave on each chamber interval and is continuous, so it is concave on the whole line segment. Approximate nongeneric segments by generic ones in the positive orthant and pass to the boundary by continuity. This proves global concavity. Prefix sums are linear in the demand, so the resulting Jensen inequality is exactly (1.4).

To prove optimality for this extension, use affine coordinates (s,u,d) in its middle-chamber polynomial. At $(s,u,d) = (0,0,0)$, direct differentiation gives

$$\det\big[(1-\alpha)\nabla F \nabla F^{\top} - F \operatorname{Hess} F\big] = \frac{4 - 13\alpha}{4}. \tag{4.6}$$

For every $\alpha > 4/13$ the determinant is negative. By continuity it remains negative at some interior points $0 < d < s < u$ arbitrarily near the origin. The controlling matrix is then not positive semidefinite, so F^{α} cannot be concave. This is an obstruction on real demands. □

4.1 An obstruction on integral demands

The continuous optimality argument near the origin does not itself produce an integral-demand counterexample. The count polynomials are inhomogeneous, so scaling rational demands to clear denominators changes the inequality. The following exact examples supply a lattice obstruction.

Proposition 4.2 (Failure of cube-root concavity)**.** *For the complete forward four-vertex DAG, cube-root Jensen fails at the integral midpoint triples*

$$(0,0,0),\ (2,3,2),\ (4,6,4) \quad \textit{and} \quad (2,3,2),\ (4,6,4),\ (6,9,6)$$

in prefix-demand coordinates. All fibers in the second triple are three-dimensional.

Let A_{K_4} be the supremum of the positive powers satisfying Jensen for every admissible family of integral demands on this fixed graph. Then

$$\frac{4}{13} \le A_{K_4} \le \alpha_* < \frac{1}{3}, \qquad 2\,13^{\alpha_*} = 1 + 51^{\alpha_*},$$

where the displayed equation has a unique positive solution.

Proof. Substitution in (4.2), or direct enumeration in (4.1), gives

$$N(0,0,0) = 1, \quad N(2,3,2) = 13, \quad N(4,6,4) = 51, \quad N(6,9,6) = 130.$$

The middle demand in each triple is the integral midpoint of its two endpoints. Integer cubes give

$$23514^3 > 13 \cdot 10000^3, \qquad 37084^3 < 51 \cdot 10000^3,$$

and hence

$$2\sqrt[3]{13} < \frac{47028}{10000} < \frac{47084}{10000} < 1 + \sqrt[3]{51}.$$

For the second triple use

$$\begin{aligned} 2351334^3 &< 13 \cdot 1000000^3, \\ 5065797^3 &< 130 \cdot 1000000^3, \\ 3708430^3 &> 51 \cdot 1000000^3. \end{aligned}$$

They imply

$$2\sqrt[3]{51} < \frac{7416860}{1000000} < \frac{7417131}{1000000} < \sqrt[3]{13} + \sqrt[3]{130}.$$

On the positive ray $(s,t,u) = (2r, 3r, 2r)$, take the three chord flows $a = v = c = r/2$ in (4.1). Every chord and each remaining edge has positive flow for $r > 0$. Thus each fiber in the second triple has full affine dimension three.

Theorem 1.3 gives $A_{K_4} \ge 4/13$. For the upper bound set $x = 13^\alpha$ and $q = \log(51)/\log(13)$. Since $1 < q < 2$, the function $x^q - 2x + 1$ is strictly convex, vanishes at $x = 1$, has negative derivative there, and tends to positive infinity. It has exactly one further zero $x_* > 1$, defining the unique positive α_*. For every $\alpha > \alpha_*$, the first midpoint triple violates power Jensen. Its strict violation at $\alpha = 1/3$ proves $\alpha_* < 1/3$. □

These examples do not refute logarithmic Jensen: $13^2 > 51$ and $51^2 > 13 \cdot 130$. They do not determine A_{K_4}. The graph and the count polynomials are classical; the obstruction follows from those exact counts and the displayed elementary inequalities.

5 Block products and scope

Corollary 5.1. *Let a finite DAG be assembled at articulation vertices from blocks that each either have cycle rank at most two or are complete four-vertex DAGs. Then its integer-flow counts satisfy (1.1) for every admissible family.*

Proof. For a leaf block attached at v, its induced demand is the original demand away from v and minus the sum of those demands at v. The remaining demand at v is the original demand minus this induced block demand. These maps are integer linear. Restricting flows to the two edge sets is a Cartesian-product bijection, so counts multiply. Iterate over blocks, apply Theorem 1.1 or 1.3 to each, and multiply the logarithmic inequalities. Disconnected components factor as well. □

Theorem 1.2 concerns the universal low-central-dimension class, not a fixed graph and not the best finite-m exponent. The optimality in Theorem 1.3 concerns its specified continuous extension. Proposition 4.2 bounds that lattice-only power between $4/13$ and $\alpha_* < 1/3$ for the complete four-vertex DAG.

The proofs are analytic; finite diagnostic searches and symbolic cross-checks are not premises of any theorem here.

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References

[1] J. Leake and M. Mohammadi Yekta, *Log-concavity and Approximate Counting for Totally Unimodular Polytopes*, arXiv:2609.39917v1 (2026). arxiv.org/abs/2609.39917.

[2] A. Khare, J. P. Matherne and A. St. Dizier, *Log-concavity of characters of parabolic Verma modules, and of restricted Kostant partition functions*, arXiv:2504.01623v2 (2025). arxiv.org/abs/2504.01623.

[3] J. A. De Loera and B. Sturmfels, *Algebraic unimodular counting*, Math. Program. **96** (2003), 183–203. arXiv:math/0104286.

[4] T. Lam and A. Postnikov, *Alcoved polytopes I*, Discrete Comput. Geom. **38** (2007), 453–478. arXiv:math/0501246.

[5] T. Lam and A. Postnikov, *Alcoved polytopes II*, arXiv:1202.4015 (2012). arxiv.org/abs/1202.4015.

[6] D. Alonso-Gutiérrez, M. A. Hernández Cifre and J. Yepes Nicolás, *Further inequalities for the (generalized) Wills functional*, Commun. Contemp. Math. **23** (2021), 2050011. arxiv.org/abs/1912.07993.

[7] D. Iglesias, J. Yepes Nicolás and A. Zvavitch, *Brunn–Minkowski type inequalities for the lattice point enumerator*. Author manuscript (2019).

[8] A. Barvinok, *Brunn–Minkowski inequalities for contingency tables and integer flows*, Adv. Math. **211** (2007), 105–122. doi:10.1016/j.aim.2006.07.012.

[9] J. Leake and A. H. Morales, *Capacity bounds on integral flows and the Kostant partition function*, arXiv:2406.07838v3 (2024). arxiv.org/abs/2406.07838.

[10] J. Tarski, *The partition function for certain simple Lie algebras*, Technical Report No. 7, AFOSR TN58-657 (AD162188), Department of Mathematics, University of California, Berkeley (August 1958). HathiTrust scan.