Positive relative spectral measures for Dirichlet apertures: Stieltjes structure, thresholds and shrinking limits

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Abstract

Opening a window in a Dirichlet wall lowers every eigenvalue of a bounded guide, so the heat trace increases at each time. We prove a much stronger positivity statement for the limiting *relative* heat trace of two semi-infinite Dirichlet cylinders joined across an arbitrary open aperture in their common end plane: it is the Laplace transform of a positive energy measure, i.e. it is completely monotone in time. The cross-sections may be nonnested and of any dimension, and bounded repulsive contact interactions on the aperture are allowed. The measure has an explicit analytic density between transverse thresholds, atoms of half the matching-channel multiplicity, and inverse-square-root threshold singularities whose coefficients give the leading chemical-potential singularities of the relative population of an ideal Bose gas. Positivity is sharp: a finite capped arm and an attractive contact both produce signed measures. The analytic engine is a matrix theorem: the log-determinant derivative of a positive matrix response $M_0 + \sum_j (z + a_j)^{s_j} M_j$ with positive semidefinite, noncommuting coefficients is a Stieltjes transform whenever the exponents are ordered like the thresholds, and this ordering is exactly the condition that works for all weights. We compute its total mass by a rank integral and realize every such response by coupled weighted Bessel channels. Finally, when the aperture shrinks to a point, the relative measure divided by ϵ^{d+1} converges in weighted total variation to a sum of inverse-square-root threshold densities weighted by half the fractional torsional rigidity of the aperture shape.

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Overview

The paper has three parts. Sections 1–5 prove complete monotonicity of the aperture heat gain (Theorem 1.1), describe the measure at and between thresholds (Theorem 4.1), derive the Bose-gas consequences (Proposition 5.1), and show by two explicit models that positivity can fail without the geometric hypotheses. Sections 7–9 isolate the underlying matrix mechanism and extend it to mixed fractional exponents (Theorem 7.1), matrix-valued channel distributions (Theorem 8.1) and an operator realization by weighted Bessel channels (Theorem 9.1); the common-exponent case is the sector lemma used in the first part. The final sections treat a shrinking aperture (Theorem 10.1) and its simultaneous threshold limits (Proposition 15.1).

1 The aperture and its relative heat trace

Opening a Dirichlet screen in a waveguide decreases the eigenvalues in a finite capped guide. It follows that the heat trace increases at each positive time. We ask for a stronger sign property of the limiting difference when the caps recede: is the gain a positive mixture of exponential decays in time? Equivalently, by Bernstein's theorem, is the relative heat trace completely monotone?

The planar nested-strip theorem in [7, Sections 5–7] answers this question when one complete transverse interval is the aperture and the other interval contains it. Its proof uses a finite square-root phase formula and a trace-class first relative resolvent. Here we allow arbitrary open common apertures and higher transverse dimension. In higher dimension, the first relative resolvent has singular values of order $n^{-2/d}$, where d is the transverse dimension. We preserve the finite-dimensional Stieltjes sign through a higher-resolvent limit and use the cut geometry to identify the same measure in the actual heat trace.

Relative trace formulas in terms of a Dirichlet-to-Neumann operator and its logarithm are established tools [1, Theorem 4.1]. Positivity of the accumulated shift is the sign supplied by a resolvent order in that framework. The present problem concerns its energy derivative as a measure. For each pure half-cylinder the boundary operator is a transverse square root, as in standard product gluing [5, Proposition 2.4, Example 3.2 and Section 5]. The sum of two compressed roots retains a sign on the cut of the complex plane. The resulting density formula involves only finitely many open channels at each fixed energy, even when the first relative resolvent is outside the trace class. Modal threshold expansions for straight guides with compact volume perturbations have a related finite-dimensional reduction [6, Proposition 3.2]. Our expansion determines the energy density and threshold atoms for the aperture itself.

Let $d \geq 1$ and let $U_-, U_+ \subset \mathbb{R}^d$ be bounded nonempty Lipschitz open sets. Fix a nonempty open set $K \subset U_- \cap U_+$. With s the longitudinal coordinate, set

$$\Omega = ((-\infty, 0) \times U_-) \cup (\{0\} \times K) \cup ((0, \infty) \times U_+).$$

This is an open set. Write $A_j = -\Delta^D_{U_j}$ for the transverse operators, for $j = -, +$, and let

$$a = \min\{\lambda_1(A_-), \lambda_1(A_+)\} > 0.$$

We use the Dirichlet Laplacian convention in which the free heat kernel on $\mathbb{R}^q$ is

$$p_t^{(q)}(x, y) = (4\pi t)^{-q/2} e^{-|x-y|^2/(4t)}.$$

The connected operator A_ρ is associated with the form

$$q_\rho[f] = \int_\Omega |\nabla f|^2 + \int_K \rho(y)|\gamma f(y)|^2 \, dy, \qquad f \in H^1_0(\Omega), \qquad 0 \leq \rho \in L^\infty(K),$$

where γf is the common aperture trace. We call $\rho = 0$ the transparent aperture. The cut operator B is the direct sum of the two half-cylinder Dirichlet Laplacians, with a Dirichlet condition at their entire end planes. Both operators act in $L^2(\Omega)$ by ignoring the aperture in the volume measure. For $L > 0$, denote the corresponding operators with Dirichlet caps at $s = \pm L$ by $A_{\rho,L}$ and B_L, and set

$$D_{\rho,L}(t) = \operatorname{Tr}(e^{-tA_{\rho,L}} - e^{-tB_L}), \qquad t > 0.$$

Theorem 1.1. *The limit $D_\rho(t) = \lim_{L\to\infty} D_{\rho,L}(t)$ exists for every $t > 0$. There is a unique positive Radon measure ν_ρ supported on $[a, \infty)$ such that*

$$D_\rho(t) = \int_a^\infty e^{-tu} \, d\nu_\rho(u). \tag{1}$$

The function D_ρ is completely monotone. For every integer $m > d/2$ and $x > 0$, the relative resolvent power belongs to the trace class and satisfies

$$\operatorname{Tr}((A_\rho + x)^{-m} - (B + x)^{-m}) = \int_a^\infty (x + u)^{-m}\, d\nu_\rho(u). \tag{2}$$

For each fixed $x > 0$, the nonzero singular values of the first relative resolvent satisfy

$$s_n((A_\rho + x)^{-1} - (B + x)^{-1}) \asymp n^{-2/d}.$$

In particular, the first relative resolvent belongs to $\mathcal{S}_p$ exactly for $p > d/2$.

The accumulated shift in this orientation is $\Xi(u) = \nu_\rho([a, u])$. By Tonelli's theorem the trace in (2) is also equal to

$$m \int_a^\infty \Xi(u)(x + u)^{-m-1}\, du.$$

The positive measure is the distributional derivative of this accumulated shift.

For any enclosing open box $Q \supset U_- \cup U_+$, the heat gain satisfies

$$0 \le D_\rho(t) \le \frac{1}{2} \operatorname{Tr}(e^{-t(-\Delta_Q^D)}). \tag{3}$$

For a transparent aperture with $|\partial K| = 0$, the same measure gives the area laws

$$D_0(t) \sim \frac{|K|}{2(4\pi t)^{d/2}} \quad (t \downarrow 0), \qquad \nu_0([0, R]) \sim \frac{|K| R^{d/2}}{2(4\pi)^{d/2}\Gamma(1 + d/2)} \quad (R \to \infty). \tag{4}$$

Section 4 describes the density and all its possible atoms. Section 5 uses the lowest-threshold part of that description for ideal Bose occupation. This is a relative occupation at fixed guide geometry. The heat trace comparison used in the weak-twist opening limit of [8, Sections 4–5] concerns a different limit, with compact resolvent for each positive twisting parameter.

2 The aperture form and a Stieltjes sign

Let $E_j : L^2(K) \to L^2(U_j)$ be zero extension. The aperture form space $\mathcal{V}$ is the closure of $C_c^\infty(K)$ in the norm from the form

$$s_K[g] = \sum_{j=-,+} \|A_j^{1/4} E_j g\|^2. \tag{5}$$

This is a densely defined closed form on $L^2(K)$. Indeed $s_K[g] \ge \sqrt{a}\, \|g\|^2$, and closability follows from closedness of the transverse powers. Denote its positive self-adjoint operator by S. The aperture embedding is compact because either isometric injection E_j maps a form-bounded sequence into the compact form embedding for $A_j^{1/2}$. Min–max with that same injection, followed by comparison of U_j with an enclosing box, gives

$$\lambda_n(S) \ge c n^{1/d}, \qquad \lambda_n(A_j) \ge c' n^{2/d}. \tag{6}$$

For $x \ge 0$, define the forms and bounded normalized operator

$$M(x) = \sum_j E_j^*(A_j + x)^{1/2} E_j, \qquad N_\rho(x) = S^{-1/2}(M(x) + \rho) S^{-1/2}. \tag{7}$$

The operator $N_\rho(x)$ is bounded on $L^2(K)$ and satisfies $N_\rho(x) \geq I$. Boundedness follows from $(\lambda + x)^{1/2} \leq (1 + x/a)^{1/2}\sqrt{\lambda}$ on the transverse spectra and the boundedness of ρ.

Let $K_x g$ denote the pair of half-cylinder Poisson extensions with aperture data g. In arm j it is

$$(K_x g)_j(r) = e^{-r(A_j+x)^{1/2}} E_j g, \qquad r = |s|.$$

Separation in the transverse eigenbasis proves that this extension minimizes the x-shifted energy in each arm and has total energy $M(x)[g]$. For a smooth compactly supported function in Ω, its aperture trace belongs to $C_c^\infty(K)$. The minimum-energy inequality at $x = 0$ extends the common trace map to $H_0^1(\Omega)$, continuously with values in $\mathcal{V}$. The trace is zero precisely on the direct sum of the two cut form domains. In fact, a zero common trace and the zero lateral traces give zero trace on the boundary of each Lipschitz half-cylinder. Conversely, extending the two cut form functions by zero gives a zero-trace element of $H_0^1(\Omega)$.

Every $g \in \mathcal{V}$ has the stated Poisson extension in $H_0^1(\Omega)$. For $g \in C_c^\infty(K)$ take the smooth bridge $\chi(s)g(y)$, where χ is compactly supported and equals one at zero. On either arm the difference between the Poisson extension and the bridge has zero trace, so its zero extension belongs to $H_0^1(\Omega)$. Adding the bridge and then passing to the form closure gives the assertion. In particular the trace space used here covers arbitrary open K without any trace assumption on its edge.

Put $L_x = K_x S^{-1/2}$. In the x-shifted energy form the Poisson extensions are orthogonal to the cut domain. The added contact term vanishes against that domain, which has zero trace. The difference between the connected and cut resolvent solutions for data f is therefore $K_x g$, where

$$(M(x) + \rho)[g, v] = \langle f, K_x v \rangle \qquad (v \in \mathcal{V}).$$

The resulting factorization and the longitudinal integral of the Poisson semigroups give

$$(A_\rho + x)^{-1} - (B + x)^{-1} = L_x N_\rho(x)^{-1} L_x^*, \qquad L_x^* L_x = N_\rho'(x). \tag{8}$$

For the second equality, $K_x^* K_x = \frac{1}{2}\sum_j E_j^* (A_j + x)^{-1/2} E_j$.

Lemma 2.1. *Let $0 < \alpha \leq 1$ and $a_0 > 0$. Let M be a finite positive semidefinite $n \times n$ matrix measure on $[a_0, \infty)$ with positive definite total mass and finite α-moment. For a positive semidefinite constant F_0, put*

$$F(z) = F_0 + \int_{a_0}^\infty (z + v)^\alpha \, d\mathsf{M}(v), \qquad f(z) = \operatorname{tr}(F(z)^{-1} F'(z)),$$

using principal powers. Then f is the Stieltjes transform of a positive measure on $[a_0, \infty)$ of mass $n\alpha$.

Proof. For $\operatorname{Im} z > 0$ let $\theta = \arg(z + a_0) \in (0, \pi)$. The numerical range of $F(z)$ lies in the sector $[0, \alpha\theta]$ in argument. Rotating by $-\alpha\theta/2$ gives strictly positive real part, so $F(z)$ is invertible. If $w = F^{-1}v$, then $v^* F^{-1} v = \overline{w^* F w}$. Thus the numerical range of F^{-1} lies in the opposite sector $[-\alpha\theta, 0]$. Taking the scalar trace of F^{-1} times a positive semidefinite matrix preserves this sector. Use the positive matrix density of M with respect to its trace measure. In the integrand for f, the additional factor $\alpha(z + v)^{\alpha-1}$ shifts the argument by an amount in $[-(1 - \alpha)\theta, 0]$. The resulting sector is contained in $[-\theta, 0]$. In particular $\operatorname{Im} f(z) \leq 0$.

The function is positive on the real interval $x > -a_0$, where both $F(x)$ and $F'(x)$ are positive definite. Analyticity and the upper-half-plane sign imply that f is nonincreasing on that interval by taking vertical difference quotients. The shifted Stieltjes characterization [4, Theorem 2.11

and (2.13)] now gives the measure, including a possible atom at a_0, and a nonnegative constant. Dominated convergence gives

$$x^{-\alpha}F(x) \longrightarrow \mathsf{M}([a_0,\infty)), \qquad x^{1-\alpha}F'(x) \longrightarrow \alpha\mathsf{M}([a_0,\infty)).$$

Hence $xf(x) \to n\alpha$. The constant is zero, and monotone convergence in the Stieltjes formula gives mass $n\alpha$. □

3 From finite compressions to the physical trace

Let P_n project onto the first n eigenvectors of S, with an ordered orthonormal basis fixed for now. Let $N_{\rho,n} = P_n N_\rho P_n|_{\operatorname{ran} P_n}$, and extend its inverse by zero as J_n. Coercivity gives $\|J_n(x)\| \le 1$ and strong convergence $J_n(x) \to N_\rho(x)^{-1}$ for $x \ge 0$ by the Galerkin error estimate. The transverse spectral measures compressed through $E_j S^{-1/2} P_n$ are positive matrix measures with finite half-moment. Their summed total matrix is positive definite by injectivity. Lemma 2.1 with $\alpha = 1/2$ yields positive measures ν_n supported on $[a,\infty)$ with

$$f_n(x) = \operatorname{tr}(N_{\rho,n}(x)^{-1}N'_{\rho,n}(x)) = \operatorname{Tr}(L_x J_n(x) L_x^*) = \int_a^\infty (x+u)^{-1}\, d\nu_n(u). \tag{9}$$

The trace in the middle is finite rank. We differentiate it before passing to the limit.

We use the singular-value product inequality

$$s_{p+q-1}(XY) \le s_p(X)s_q(Y),$$

the corresponding sum inequality, and Hölder's inequality in the Schatten ideals [9, Chapter 2]. By (6), $s_\ell(S^{-1/2}) = O(\ell^{-1/(2d)})$. For $k \ge 1$ the k-th derivative of N_ρ is a fixed scalar times a sum of terms

$$S^{-1/2}E_j^*(A_j + x)^{1/2-k}E_j S^{-1/2}.$$

The transverse power has singular values $O(\ell^{-(2k-1)/d})$. Thus

$$s_\ell(N_\rho^{(k)}(x)) = O(\ell^{-2k/d}), \qquad s_\ell(L_x^{(j)}) = O(\ell^{-(2j+1)/d}) \quad (j \ge 0). \tag{10}$$

For the second bound, the j-th derivative of $e^{-r\sqrt{b}}$, for $b = \lambda + x$, is b^{-j} times a fixed polynomial in $r\sqrt{b}$ times $e^{-r\sqrt{b}}$. Its $L^2(dr)$ norm is $O(b^{-j-1/4})$. In either arm, distinct transverse modes give orthogonal output functions. Apply (6) and multiply on the right by $S^{-1/2}$. The constants in (10) are uniform for $x \ge 0$. The indicated operators tend to zero in operator norm as $x \to \infty$, so their Schatten norms tend to zero whenever the exponent in (10) gives a summable bound. The same estimates and operator-norm smoothness give continuity in those ideals on compact x-intervals. For instance, control the singular values of a difference by the sum inequality and use dominated convergence.

Fix an integer $m > d/2$. In the $(m-1)$-st derivative of $L_x J_n L_x^*$, every term contains two factors $L_x^{(j)}$, $(L_x^{(l)})^*$, compressed factors $N_\rho^{(k_i)}$ with $k_i \ge 1$, and uniformly bounded inverse factors. The positive derivative orders satisfy $j + l + \sum k_i = m - 1$. From (10), the Schatten exponents for these factors can be taken as $2m/(2j+1)$, $2m/(2l+1)$, and m/k_i, respectively. Their inverses sum to one. Compact compressions converge in the indicated ideals. Strong convergence of J_n and its adjoint, with their uniform norm bounds, gives convergence in an ideal when the inverse is multiplied by a factor in that ideal. This follows by first approximating the ideal factor by

finite rank. Consequently each product converges in trace norm to its uncompressed counterpart. Using the operator-norm derivative of (8), we obtain

$$\int_a^\infty (x+u)^{-m}\, d\nu_n(u) \longrightarrow G_m(x) = \mathrm{Tr}((A_\rho + x)^{-m} - (B+x)^{-m}). \tag{11}$$

The derivative factor on both sides is $(-1)^{m-1}/(m-1)!$. The same product bounds show that the uncompressed relative power tends to zero in trace norm as $x \to \infty$.

Choose for the moment $m > (d+1)/2$. On the one-point compactification of $[a, \infty)$, the positive measures $(1+u)^{-m} d\nu_n(u)$ have uniformly bounded mass by (11) at $x = 1$. Along a weakly convergent subsequence, the test function $((1+u)/(x+u))^m$ is continuous at infinity with value one. Therefore

$$G_m(x) = b + \int_a^\infty (x+u)^{-m}\, d\nu_\rho(u), \qquad b \geq 0.$$

Trace-norm decay at infinity forces $b = 0$. This yields a positive candidate measure with finite $(1+u)^{-m}$-weighted mass.

3.1 Cap exhaustion and heat trace identification

For the transparent aperture, the difference of diagonal heat kernels is the free diagonal kernel times the probability of a Brownian bridge that survives in the connected domain and hits the aperture. Under increasing caps this event increases. Place both sections in Q and allow the bridge to hit the entire end plane in the enclosing cylinder. Independence of the transverse and longitudinal bridges gives (3). In the longitudinal coordinate the same-point bridge at s hits zero with probability $e^{-s^2/t}$, whose integral against $(4\pi t)^{-1/2}\, ds$ is $1/2$. This also provides an integrable bound for the diagonal difference which is uniform when t lies in a compact positive interval.

The heat kernel for a bounded nonnegative contact term lies between those of the cut and the transparent guide. For clarity, take nonnegative data in the resolvent and test by the negative part to obtain positivity. To compare the cut and contact solutions, test their difference by the positive part of the cut solution minus the contact solution. This test has zero aperture trace since the cut solution has zero trace and the contact solution is nonnegative. The contact term vanishes on the test. To compare the contact solution with the transparent one, test by the positive part of the contact solution minus the transparent solution. The contact term on this test is nonnegative. The resolvent comparisons pass to positive semigroup order by the exponential product formula. They apply with caps and to domain exhaustion using zero extension. The individual positive kernels increase to their full-domain limits under exhaustion, by the order and strong semigroup convergence of closed forms on increasing domains. The diagonal is obtained from the square integral of the half-time kernel, so the same order and convergence hold there almost everywhere. In particular the contact-minus-cut diagonal difference with caps is dominated by the transparent-minus-cut diagonal difference with the same caps, and the latter by the infinite transparent difference. Dominated integration proves the cap limit and identifies it with the integral of the full diagonal difference. This proves (3) for D_ρ. Continuity in positive time follows from the same domination and continuity of the diagonal. For example, the diagonal for $t > 2t_0 > 0$ can be written as the matrix element of $e^{-(t-2t_0)A_\rho}$ on the L^2 kernel column at t_0.

Localize to the slab $|s| \leq R$. For $m > (d+1)/2$, the individual localized powers have finite traces by the free heat bound in dimension $d+1$ and the gamma integral. Subtract their trace identities, and let $R \to \infty$. The resolvent side converges by trace-class continuity under strong

localization. On the heat side the diagonal difference is nonnegative, so monotone convergence gives

$$G_m(x) = \frac{1}{(m-1)!} \int_0^\infty e^{-xt} t^{m-1} D_\rho(t)\, dt.$$

The generalized Stieltjes formula gives the same expression with $D_\rho(t)$ replaced by $\int e^{-tu}\, d\nu_\rho(u)$, by Tonelli's theorem. The weighted mass bound makes the latter function continuous for $t > 0$. Uniqueness of the scalar Laplace transform first identifies the heat traces in (1) and then identifies the measure. It also shows that the entire sequence of weighted Galerkin measures converges to this measure with the stated weight and no mass at infinity: every subsequential weak limit gives the same measure. Differentiation of (1) is justified by the weighted mass bound for all orders, proving complete monotonicity.

For the integer $m > d/2$ possibly omitted by $m > (d+1)/2$, apply the preceding identification at $m+1$. The relative m-th power is differentiable in trace norm with derivative equal to $-m$ times the relative $(m+1)$-st power. To see this, differentiate the products used for (11) in their stated Schatten ideals. For each positive derivative factor, its next derivative is continuous in the same ideal by (10), so integration in that ideal verifies the differentiation. The two L_x factors obey the same argument. As the relative m-th power tends to zero in trace norm, integrating the trace derivative from x to infinity, then applying Tonelli to the representation at $m+1$, gives (2) at m.

3.2 Sharpness of the first resolvent estimate and the area law

The upper singular-value bound in Theorem 1.1 follows from the two L_x factors in (8). For the lower bound, the positive relative resolvent and $N_\rho^{-1/2} N_\rho' N_\rho^{-1/2}$ have the same nonzero eigenvalues. The generalized min–max quotient on aperture data is

$$\frac{M'(x)[g]}{M(x)[g] + \langle \rho g, g \rangle}.$$

Choose a box $Q_0 \Subset K$ and take the first n Dirichlet modes on Q_0, extended by zero to K. They belong to $\mathcal{V}$ by $H_0^1(Q_0)$ approximation. A unit vector in this span has shifted transverse first moment at most $\lambda_n(-\Delta_{Q_0}^D) + x$ in both arms. By concavity of the square root, the denominator is at most $2\sqrt{\lambda_n + x} + \|\rho\|_\infty$. Convexity of the reciprocal square root gives the numerator bound $(\lambda_n + x)^{-1/2}$ from below. Box eigenvalue growth gives the required order $n^{-2/d}$. This includes the claimed strict Schatten threshold.

To prove (4) for $\rho = 0$, put $s = \sqrt{t}\, z$ in the diagonal bridge formula. At each fixed $y \in K$, the transverse bridge remains in a ball contained in K with probability tending to one. On that event the surviving path hits the aperture precisely when the longitudinal bridge hits zero. At $y \notin \overline{K}$, an aperture hit requires a transverse displacement bounded below by a positive distance, so its probability tends to zero. The longitudinal factor is bounded by e^{-z^2}, and y ranges over a bounded set. The null-boundary assumption and dominated convergence give the first formula in (4). The second is Karamata's Laplace Tauberian theorem for the positive cumulative measure [2, Theorem 1.7.1].

4 Density and atoms at the transverse thresholds

Let $\mathcal{T} = \sigma(A_-) \cup \sigma(A_+)$ be the set of distinct transverse thresholds. It is locally finite, and its minimum is a. Let Π_{j,u_0} denote the spectral projection of A_j at u_0, with value zero if that

threshold is absent in arm j. For $u_0 \in \mathcal{T}$, define the space of matching aperture data

$$W_{u_0} = \{g \in \mathcal{V} : E_j g = \Pi_{j,u_0} E_j g \text{ for } j = -,+, \quad \rho^{1/2} g = 0\}. \tag{12}$$

For $u \notin \mathcal{T}$, write $U(u) + iV(u) = N_\rho(-u + i0)$ for the upper boundary value. Explicitly,

$$U(u) = S^{-1/2} \left(\rho + \sum_j E_j^* (A_j - u)^{1/2} \mathbf{1}_{(u,\infty)}(A_j) E_j \right) S^{-1/2}, \tag{13}$$

$$V(u) = S^{-1/2} \left(\sum_j E_j^* (u - A_j)^{1/2} \mathbf{1}_{(-\infty,u)}(A_j) E_j \right) S^{-1/2}. \tag{14}$$

The roots act on the indicated spectral subspaces. Dots below denote derivatives with respect to u. In particular $-\dot{U} \geq 0$, $\dot{V} \geq 0$, and $V, \dot{V}$ have finite rank locally away from the thresholds.

For a threshold u_0, use the notation

$$C_{u_0} = \sum_j E_j^* \Pi_{j,u_0} E_j, \qquad C_0 = (S^{-1/2} C_{u_0} S^{-1/2})|_{W_{u_0}^\perp}.$$

The normalized regular operator $R(z)$ at this threshold is the restriction to $W_{u_0}^\perp$ of $N_\rho(z) - \sqrt{z + u_0}\, S^{-1/2} C_{u_0} S^{-1/2}$, using the upper boundary branches for the other roots. We show below that $R(-u_0)$ is well-defined and boundedly invertible.

Theorem 4.1. *The measure ν_ρ has an analytic density h_ρ on each open interval between thresholds, assigns mass $\dim W_{u_0}/2$ to the singleton $\{u_0\}$ at each threshold u_0, and has no singular continuous part. For $u \notin \mathcal{T}$, put $J = (U + iV)^{-1}$. The density is*

$$h_\rho(u) = \frac{1}{\pi} \operatorname{Tr}\left[J^* V J \, (-\dot{U}) + J^* U J \, \dot{V} \right]. \tag{15}$$

Both products under the trace are trace class and contribute nonnegative terms. With

$$c_{u_0} = \operatorname{Tr}(R(-u_0)^{-1} C_0),$$

the one-sided threshold coefficients are

$$h_\rho(u_0 - v) = \frac{-\operatorname{Im} c_{u_0}}{2\pi\sqrt{v}} + O(1), \qquad h_\rho(u_0 + v) = \frac{\operatorname{Re} c_{u_0}}{2\pi\sqrt{v}} + O(1) \qquad (v \downarrow 0). \tag{16}$$

At the lowest threshold a, the number c_a is nonnegative and the upper-side remainder is $O(\sqrt{v})$.

Proof. On each compact energy interval away from the thresholds, $U + V$ is bounded below by a positive multiple of the identity modulo a finite-rank operator. To see this, compare the high closed-mode roots in U with S using the ratio $\sqrt{(\lambda - u)/\lambda}$ for λ sufficiently large. The low transverse projections omitted from S are finite rank. The kernel of $U + V$ is zero because the two sums together include every transverse mode with a strictly positive weight. A nonnegative operator that is coercive modulo finite rank is bounded below on the orthogonal complement of its kernel. It follows that $U + V \geq cI$, locally uniformly by norm continuity. Rotation by $e^{-i\pi/4}$ gives coercivity of $U + iV$. The same bounds hold for each compression. The associated Galerkin inverses and their adjoints converge strongly, with locally uniform bounds, by the coercive Galerkin error estimate for the operator and its adjoint.

In each finite compression,

$$\operatorname{Re} J_n = J_n^* U_n J_n, \qquad \operatorname{Im} J_n = -J_n^* V_n J_n.$$

The boundary value of f_n in (9) uses the derivative $-\dot{U}_n - i\dot{V}_n$. Finite-dimensional trace cyclicity and Stieltjes inversion give (15) at this level. The expressions on the right converge to their infinite-dimensional counterparts because the compressed $V, \dot{V}$ converge in trace norm, and multiplication by the strongly convergent bounded operators preserves convergence when the other factors are trace class. The remaining factors have locally uniform operator-norm bounds. Weak convergence of the weighted measures from Section 3 and dominated convergence on compact subintervals identify the density of the physical measure. The expression is real analytic in u on each such interval by the locally norm-analytic root operators, inversion, and the finite rank of the open-mode factors. The sign follows from $-\dot{U}, \dot{V}, U, V \geq 0$.

Fix $u_0 \in \mathcal{T}$, and use a disk about $-u_0$ that meets no other threshold. The branches of all other roots, chosen from the upper half-plane, extend analytically through the center of the disk. For the infinitely many high roots, their power expansions after dividing by the root at the center converge uniformly in the transverse eigenvalue. After S-normalization this yields operator-norm analyticity of the regular part.

At the center its real and imaginary parts are nonnegative. Their sum is coercive modulo finite rank by the high-mode comparison. Its kernel consists exactly of the normalized vectors whose aperture data lie in W_{u_0}: all nonthreshold modal coefficients and $\rho^{1/2} g$ must vanish. For $g \in W_{u_0}$, the isometric injections and (5) give $Sg = 2\sqrt{u_0}\, g$ and $C_{u_0} g = 2g$. The other roots and ρ annihilate W_{u_0}. Thus normalization preserves the subspace and all the forms split over it and its orthogonal complement. On the complement the regular part is invertible at the center by the same rotation as above, and remains invertible in a smaller disk by a Neumann series.

We may choose the eigenbasis of S to include these matching subspaces, as those at distinct thresholds belong to distinct eigenspaces of S. Fix n large enough to include W_{u_0}. The contribution of this subspace to $f_n(z)$ is $\dim W_{u_0}/[2(z+u_0)]$. On the complement, factor $C_0 = TT^*$ for a map $T : \mathbb{C}^r \to W_{u_0}^{\perp}$, where $r = \operatorname{rank} C_0$. Let $J_{\mathrm{reg},n}(z)$ be the inverse of the compressed regular part on the complement, extended by zero. Coercivity at the center gives boundedness and strong convergence there. Uniform Neumann expansion about the center gives strong convergence locally uniformly on a smaller disk, for the inverse and its adjoint at the center. In detail, with $E(z) = R(z) - R(-u_0)$ the inverse on a compressed space, extended by zero, equals

$$J_{\mathrm{reg},n}(-u_0)\big(I + E(z) J_{\mathrm{reg},n}(-u_0)\big)^{-1}.$$

Shrink the disk so the norm in the Neumann series is uniformly below $1/2$. The matrices $B_n(z) = T^* J_{\mathrm{reg},n}(z) T$ then converge in finite-dimensional norm locally uniformly, and their derivatives converge by Cauchy's formula. The case $r = 0$ is read with a unit determinant.

The finite-rank determinant identity now gives

$$f_n(z) = \frac{\dim W_{u_0}}{2(z+u_0)} + f_{\mathrm{reg},n}(z) + \partial_z \log\det(I + \sqrt{z+u_0}\, B_n(z)).$$

Writing $w = \sqrt{z+u_0}$, the analytic logarithm with value zero at $w = 0$ starts with $w \operatorname{tr} B_n(-u_0)$. Its z-derivative is $\operatorname{tr} B_n(-u_0)/(2w) + b_n + O(w)$, with uniform Taylor bounds. The imaginary part of the regular trace $f_{\mathrm{reg},n}$ is uniformly bounded through the threshold by the earlier finite-rank formula applied after omitting the threshold mode. All other roots stay bounded away from zero there, so the inverse and positive-order derivative factors used in that formula have uniform bounds.

Stieltjes inversion now shows that near the threshold the absolutely continuous part of ν_n has at most a uniformly integrable $O(|u-u_0|^{-1/2})$ singularity. The exact pole from W_{u_0} gives that threshold's sole atom. On either side the remaining boundary value is analytic away from the threshold, which excludes a singular part there. The uniform integrable bound and weak convergence identify the same atom and density in the physical measure. Since $\mathcal{T}$ is locally finite, these local representations account for the entire measure. Substituting $w=\sqrt{v}$ below the threshold and $w=i\sqrt{v}$ above it gives (16). The numerical range of $R(-u_0)^{-1}$ lies in the fourth quadrant by the argument in Lemma 2.1, so the two leading coefficients are nonnegative. At $u_0=a$, the regular form is real self-adjoint on real z near the center, hence its logarithmic derivative is real there. The constant b_n is real as well because the Taylor expansion in real w is real. This improves the upper-side error to $O(\sqrt{v})$ and gives $c_a \geq 0$. □

For example, if the lowest threshold belongs to only one arm, $W_a = 0$. When that threshold is simple with normalized eigenfunction ϕ, its coefficient is

$$c_a = \langle R(-a)^{-1} v_a, v_a \rangle, \qquad v_a = S^{-1/2} E_j^* \phi.$$

It is positive exactly when the transverse mode has nonzero aperture restriction. In the identical full-aperture guide with $\rho = 0$, each common transverse mode contributes exactly a half-atom. With a constant repulsive contact term, the same homogeneous channels have a density at threshold instead, as illustrated below.

5 Relative occupation of an ideal Bose gas

Fix $\beta > 0$ and $\mu < a$. The capped operators have compact resolvent and spectra bounded below by a, by the transverse Dirichlet Poincaré inequality in each arm. Define the relative mean occupation as the limit of the grand-canonical occupations for $A_{\rho,L}$ minus those for B_L, at the same β, μ. We use units in which the one-body Hamiltonian is the operator above. By Theorem 1.1, this relative quantity is

$$N_{\mathrm{rel}}(\beta,\mu) = \sum_{\ell \geq 1} e^{\ell\beta\mu} D_\rho(\ell\beta) = \int_a^\infty \frac{d\nu_\rho(u)}{e^{\beta(u-\mu)}-1}. \tag{17}$$

Here is the cap-limit justification. Pair the sorted capped eigenvalues $\lambda_k(A_{\rho,L}) \leq \lambda_k(B_L)$ by min–max. The contact term vanishes on the cut form domain. If $t \geq 1$, integrating each paired gap against te^{-tu} gives

$$0 \leq D_{\rho,L}(t) \leq te^{-(t-1)a} D_{\rho,L}(1).$$

The time-one trace gain is uniformly bounded by the aperture enclosure. This controls the large-ℓ tail in (17) uniformly in L. For each remaining term the cap limit holds. Summing the exponential representation by Tonelli gives the measure formula.

Proposition 5.1. *At fixed $\beta > 0$, write $\epsilon = a - \mu$. As $\epsilon \downarrow 0$,*

$$N_{\mathrm{rel}}(\beta, a-\epsilon) = \frac{\dim W_a}{2\beta\epsilon} + \frac{c_a}{2\beta\sqrt{\epsilon}} + O(1). \tag{18}$$

The corresponding long-time heat law is

$$e^{at} D_\rho(t) = \frac{\dim W_a}{2} + \frac{c_a}{2\sqrt{\pi t}} + O(t^{-3/2}). \tag{19}$$

For a transparent aperture with $|\partial K| = 0$, at fixed $\mu < a$, the relative grand potential satisfies

$$\Omega_{\mathrm{rel}}(\beta,\mu) := -\frac{1}{\beta}\sum_{\ell\geq 1}\frac{e^{\ell\beta\mu}D_0(\ell\beta)}{\ell} \sim -\frac{|K|\zeta(1+d/2)}{2(4\pi)^{d/2}}\,\beta^{-1-d/2} \qquad (\beta\downarrow 0).$$

Here ζ is the Riemann zeta function, evaluated in its convergent positive real half-line.

Proof. In (17), the atom at a contributes $\dim W_a/(2\beta\epsilon) + O(1)$. On a fixed small band above a, replace the Bose factor by $1/[\beta(v+\epsilon)]$ with uniformly bounded error. Inserting $h_\rho(a+v) = c_a/(2\pi\sqrt{v}) + O(\sqrt{v})$, the density remainder is integrable against this bound uniformly in ϵ. The identity $\int_0^\infty v^{-1/2}(v+\epsilon)^{-1}\,dv = \pi/\sqrt{\epsilon}$ gives the second term of (18). Truncating the integral at the chosen band edge changes it by a bounded quantity. The rest of the physical measure contributes uniformly $O(1)$ by exponential decay of the Bose factor and the weighted-mass bound. The same threshold splitting in the Laplace integral gives (19) by integrating $v^{-1/2}$ and $O(\sqrt{v})$.

The cap-limit justification for the grand potential is the same as for (17), with the additional factor $1/\ell$. From (3) and positivity with lower spectral support a, we have $e^{\mu t}D_0(t) \lesssim_\mu t^{-d/2}$ for all $t > 0$. Indeed the short-time bound follows from the enclosing box and the long-time bound follows by comparing the Laplace integral at $t \geq 1$ with its value at one. The normalized summands for the grand potential are dominated by a constant times $\ell^{-1-d/2}$. Applying (4) termwise and using dominated convergence gives the claimed coefficient. □

6 Geometries and couplings with a signed measure

6.1 A finite capped arm

Take one fixed transverse section U, put a Dirichlet cap at $s = L > 0$, and compare the connected cylinder $(-\infty, L)\times U$ with the cut at zero. Write a_j for the transverse eigenvalues. Separation of variables and the reflected longitudinal heat kernels give the exact relative heat trace

$$\mathrm{Tr}(e^{-tA_U})\left[\frac{L}{2\sqrt{\pi t}} - \sum_{n\geq 1} e^{-t(\pi n/L)^2}\right], \qquad A_U = -\Delta_U^D. \tag{20}$$

For the longitudinal calculation, the integrated reflected contribution on $(-\infty, L)$ maps by translation to the reflected contribution on $(-\infty, 0)$. Their difference therefore leaves the free diagonal heat term integrated over a length L, followed by subtraction of the Dirichlet heat trace of the cut finite arm. This computation also agrees with the cap exhaustion on the infinite side by the positive diagonal bridge formula.

The Laplace transform in (20) has the locally finite signed measure

$$\sum_j \left[\frac{L}{2\pi}\mathbf{1}_{u>a_j}(u-a_j)^{-1/2}\,du - \sum_{n\geq 1}\delta_{a_j+(\pi n/L)^2}\right].$$

It has finite absolute mass after an exponential weight at every positive time and a strictly negative atom at $a_1 + (\pi/L)^2$. The positive part shown is absolutely continuous. Uniqueness of Laplace transforms rules out a positive representing measure. Thus this pure guide with one finite arm fails complete monotonicity.

6.2 An attractive contact term

Consider identical full-aperture guides, $U_- = U_+ = K = U$, and an attractive constant contact strength $-2b$ with $0 < b < \sqrt{\lambda_1(A_U)}$. In each transverse channel the odd longitudinal subspace agrees with the Dirichlet comparison and the even subspace is the half-line with Robin condition $f'(0) = -bf(0)$. The longitudinal bound state has energy $-b^2$, so the full guide remains positive.

For a transverse energy a_j, put $w = \sqrt{z + a_j}$. The relative even-channel resolvent has kernel $e^{-w(s+r)}/(w - b)$, for $s, r > 0$, by imposing the Robin and Dirichlet boundary conditions on the half-line Green functions. Its trace is

$$\frac{1}{2w(w-b)} = \frac{1}{z + a_j - b^2} - \frac{1}{2w(w+b)}. \tag{21}$$

The last term is the repulsive-channel Stieltjes transform with density

$$\frac{b}{2\pi\sqrt{u - a_j}(u - a_j + b^2)}\, \mathbf{1}_{u > a_j}.$$

This follows from the principal-root boundary value or direct substitution in the transform. For comparison, constant contact heat formulas can also be expressed using the complementary error function [3, Theorem 1].

To justify the cap limit, write p_t for the free longitudinal heat kernel. Before multiplying by e^{-ta_j}, the attractive half-line Robin-minus-Dirichlet heat kernel difference is

$$2p_t(s + r) + 2b \int_0^\infty e^{bq} p_t(s + r + q)\, dq.$$

Its Laplace transform is the stated Green function difference, and its diagonal is integrable at every positive time. The finite-cap heat kernel difference is the contribution of reflected paths hitting zero, with Robin weight and killing at the finite endpoint. With the reflecting half-line diffusion of generator ∂_s^2, the reflection regulator ℓ_t has coefficient one in the boundary term of Itô's formula. The multiplicative factor $e^{b\ell_t}$ gives the condition $f'(0) = -bf(0)$. On paths avoiding zero this weight is one and the Dirichlet and Robin path contributions agree. Thus the capped difference increases to the integrable half-line difference as the cap recedes. The transverse heat trace is finite and multiplies this separated longitudinal difference.

It follows from (21) that the relative measure has an atom at each $a_j - b^2$ and a negative continuum contribution above each a_j. The channel sum has finite total variation after any positive exponential weight. Just above the lowest threshold its continuum density is strictly negative, away from the bound-state atoms. Laplace uniqueness again precludes complete monotonicity. The choice of reference also matters for repulsive couplings. In a homogeneous channel, for $0 < b_1 < b_2$, the weaker-minus-stronger comparison has density with sign equal to that of $b_1 b_2 - (u - a_j)$. Choosing a transverse gap larger than $b_1 b_2$ gives a negative interval in the first open channel.

7 The matrix result

Let $a_1, \dots, a_m > 0$ be thresholds and $0 < s_j \le 1$ be their exponents. Let $M_0, M_1, \dots, M_m$ be positive semidefinite Hermitian $n \times n$ matrices whose sum is positive definite. With principal powers, put

$$F(z) = M_0 + \sum_{j=1}^{m} (z + a_j)^{s_j} M_j, \qquad f(z) = \operatorname{tr}(F(z)^{-1} F'(z)). \tag{22}$$

The constant term is allowed to supply directions absent from all channel matrices. The matrices need not commute. Set $a_* = \min_j a_j$ and $s_{\max} = \max_j s_j$.

Theorem 7.1. *Suppose $s_i \le s_j$ whenever $a_i < a_j$. There is a unique finite positive measure ν supported on $[a_*, \infty)$ such that*

$$f(z) = \int_{a_*}^{\infty} \frac{d\nu(u)}{z+u}. \tag{23}$$

The exact mass is

$$\nu([a_*, \infty)) = \int_0^1 \operatorname{rank}\left(\sum_{j:s_j \ge t} M_j\right) dt. \tag{24}$$

In particular it is at most $n s_{\max}$. For any two distinct thresholds with $a_1 < a_2$ and $s_1 > s_2$, there are scalar data with both channel weights positive for which (23) *fails.*

The common-exponent case already follows from the sector lemma in Section 2, which treats finite sums and matrix integrals of a fixed fractional power. In that argument the sector of the inverse has width controlled by the common exponent and the derivative contributes the complementary width. With mixed exponents, the separate sector bounds for those factors can exceed the required width. We instead pair the channel contributions inside the trace and order their boundary phases and logarithmic amplitude derivatives.

Matrix complete Bernstein and Stieltjes functions give a natural framework for inverses of positive matrix responses [10, Appendix B]. The Stieltjes sign here concerns the derivative of a scalar logarithm of the full response. The scalar Thorin representation supplies the corresponding logarithmic integral once that derivative is Stieltjes [11, Chapters 8–9]. In spectral theory a boundary matrix logarithm is also a standard way of representing an accumulated shift [12, Theorem 4.1]. Our question is whether its derivative in energy is a positive measure. The ordered pairing and the reversed example distinguish these two properties directly.

7.1 The mixed-exponent phase identity

For $\operatorname{Im} z > 0$, the numerical range of $F(z)$ lies in the sector of arguments between 0 and $s_{\max} \arg(z + a_*) < \pi$. Rotating by half the sector angle makes the real part strictly positive definite, since the sum of the positive coefficient matrices is positive definite. Thus F is invertible, its principal matrix logarithm is holomorphic, and

$$\mathcal{L}(z) = \operatorname{tr} \log F(z), \qquad \mathcal{L}'(z) = f(z), \qquad 0 \le \operatorname{Im} \mathcal{L}(z) \le n\pi s_{\max}. \tag{25}$$

The derivative follows from the scalar determinant identity. The bound follows by summing the principal eigenvalue arguments, counted with multiplicity. On the real line to the right of $-a_*$, F is positive definite.

For the boundary calculation assume first that the thresholds are distinct and $s_{\max} < 1$. On a real-energy interval free of thresholds let $J = F(-u + i0)^{-1}$, and write $F(-u + i0) = \sum_{j=0}^{m} b_j M_j$. Set $b_j = r_j e^{i\phi_j}$ and $c_j = \partial_u \log r_j$, with $r_0 = 1$, $\phi_0 = c_0 = 0$. For $j > 0$ the phase and amplitude derivatives are

$$(\phi_j, c_j) = \begin{cases} (0, -s_j/(a_j - u)), & u < a_j, \\ (\pi s_j, s_j/(u - a_j)), & u > a_j. \end{cases}$$

Define

$$T_{ij} = \tfrac{1}{2}\left[\operatorname{tr}(J^* M_i J M_j) + \operatorname{tr}(J^* M_j J M_i)\right].$$

Each trace is a squared Hilbert–Schmidt norm, so $T_{ij} = T_{ji} \geq 0$. Starting with $\operatorname{tr}(J\,\partial_u F)$, substitute $J = J^*F^*J$ to get the coefficient $\operatorname{tr}(J^*M_iJM_j)$ of $\bar{b}_i b_j c_j$. The identity $J = JF^*J^*$ gives the reversed coefficient by cyclicity. Averaging and pairing (i,j) with (j,i) yields

$$\partial_u \operatorname{Im} \mathcal{L}(-u+i0) = \sum_{i<j} T_{ij} r_i r_j (c_j - c_i) \sin(\phi_j - \phi_i). \tag{26}$$

This pairing is invariant under swapping the labels of a pair. Two closed channels have phase zero. A closed or constant channel paired with an open one has smaller phase and smaller c_j. For two open channels order the thresholds so that $a_i < a_j$. Then $\phi_i \leq \phi_j$ by hypothesis and $s_i/(u-a_i) \leq s_j/(u-a_j)$ by the same exponent order and the threshold order. Each phase difference lies in $(-\pi, \pi)$. Thus every summand in (26) is nonnegative.

7.2 Threshold jumps and the transform

At a distinct threshold a_j let K be the common kernel of all matrices other than M_j. On its orthogonal complement the nonthreshold sum at the upper boundary is invertible by sector rotation. These nonthreshold matrices annihilate K. In the decomposition $K \oplus K^\perp$, the K-block and off-diagonal blocks are therefore proportional to $(z+a_j)^{s_j}$, and the compression of M_j to K is positive definite. Schur reduction gives

$$\det F(z) = (z+a_j)^{s_j \dim K}(C + o(1)), \qquad C \neq 0,$$

uniformly in a small upper half-disk. The other roots are holomorphic there on their continued upper branches. The logarithm of the nonzero factor has matching one-sided limits. As u crosses a_j upwards, the first factor increases the unwrapped argument by $\pi s_j \dim K$. Consequently the energy phase $\Xi(u) = \operatorname{Im} \mathcal{L}(-u+i0)/\pi$ has a nondecreasing representative, vanishing for $u < a_*$.

Here is a logarithmic representation useful also without the exponent order. For any finite channel data in (22), the bounded harmonic imaginary part in (25) has almost-everywhere boundary values, and we use them to define $\Xi(u)$ as above. It vanishes below a_*. The holomorphic function

$$\int_{a_*}^{\infty} \left(\frac{1}{1+u} - \frac{1}{z+u} \right) \Xi(u)\, du$$

has the same bounded imaginary part as $\mathcal{L}(z)$, by the upper-half-plane Poisson formula for bounded boundary data and uniqueness of bounded harmonic extension. The integral converges locally uniformly by its subtraction. The difference from $\mathcal{L}$ is thus a real constant. Differentiating gives

$$f(z) = \int_{a_*}^{\infty} \frac{\Xi(u)}{(z+u)^2}\, du, \qquad 0 \leq \Xi(u) \leq n s_{\max}. \tag{27}$$

For the distinct ordered data just proved, $\nu = d\Xi$ on $\mathbb{R}$ is positive and finite. Fubini, first at $z = x > 0$, gives (23). Holomorphic continuation supplies the result throughout the cut plane. Scalar Laplace uniqueness, applied twice to the Stieltjes kernel, gives uniqueness of the measure.

To include tied thresholds, separate each tied group by small increases of its thresholds in exponent order. To include exponent one, multiply all exponents by a common factor increasing to one. The approximations preserve the ordering, keep the support floor at or above a_*, and converge locally in F, F' off the cut. Inverses converge there by the sector argument. The approximating measures are bounded in mass by $n s_{\max}$. Weak compactness on the one-point compactification gives a positive limit, with any mass at infinity discarded since the Stieltjes kernel vanishes there. This proves (23) for all finite ordered data. By comparison with (27), the energy phase equals the cumulative positive measure almost everywhere for these data.

7.3 Exact mass and tied atoms

Factor each coefficient matrix into rank-one columns, assigning exponent s_j to columns of M_j and exponent zero to columns of M_0. By Cauchy–Binet, $\det F(x)$ is a finite sum of products of powers with nonnegative coefficients, each the squared modulus of an n-column determinant. A nonzero term grows with power equal to the sum of its selected exponents. If b is the largest such basis weight, $xf(x) \to b$ because each dominant term has scaled logarithmic derivative tending to b. The Stieltjes formula then gives total mass b by monotone convergence. Greedily admitting independent columns in decreasing exponent order gives a maximum-weight basis. The number of its columns of weight at least $t > 0$ is the rank of the span of all columns of weight at least t. Integrating this count proves (24).

Proposition 7.2. *Assume the finite data in Theorem 7.1 satisfy $s_{\max} < 1$. The measure has a real-analytic density on each open band between thresholds. At a threshold a, let K be the common kernel of M_0 and the channel matrices with $a_j \neq a$. The mass $\nu(\{a\})$ is the smallest sum of exponents of a basis of K chosen from the rank-one columns of the compressions $P_K M_j|_K$ with $a_j = a$. It is zero for $K = \{0\}$. If the tied channels have a single exponent s, the mass is $s \dim K$.*

Proof. The boundary values in (25) are analytic on each open band by sector invertibility and analytic continuation of each root there. The accumulated phase is the cumulative measure, so its derivative gives the density on these bands.

Put $z = -a + iy$, $0 < y \leq 1$, and set $H = \sum_{j=0}^{m} |b_j(z)| M_j$, where $b_0 = 1$, $b_j = (z + a_j)^{s_j}$. The matrices $P_j = H^{-1/2}|b_j|M_j H^{-1/2}$ are positive and sum to I. The matrix $H^{-1/2} F H^{-1/2}$ has norm at most one by Cauchy–Schwarz applied to the forms of P_j in a matrix element. After rotation by half the sector width, its real part is at least $\cos(\pi s_{\max}/2) I$, so all its singular values are at least that constant. It follows that $|\det F|$ is between $\cos(\pi s_{\max}/2)^n \det H$ and $\det H$.

The nontied radii in H stay bounded above and below, and the tied radii are y^{s_j}. Cauchy–Binet shows that $\log \det H / \log y$ tends to the smallest basis exponent after assigning weight zero to the nontied columns. A minimum-weight basis first admits a basis of the nontied span $K^{\perp}$, then admits compressed tied columns by increasing exponent until they span K. Hence this limit is the weight specified in the proposition.

Integrating (23) shows $\log |\det F(z)| = \text{constant} + \int \log(|z+u|/(1+u))\, d\nu(u)$. For $z = -a+iy$, the ratio $|z+u|/(1+u)$ is bounded above by a constant and below by $C_a y$, uniformly for $u \geq a_*$. Dividing by $\log y$ and applying dominated convergence identifies its limit as exactly $\nu(\{a\})$. Since there are only finitely many thresholds, the band densities and these masses account for the whole measure. □

The exponent restriction in Proposition 7.2 allows the uniform boundary sector and is specific to the atom formula. At exponent one new atom locations can occur. For example, $F(z) = b + z + a$ for $b > 0$ gives a unit atom at $a + b$.

7.4 Sharpness

Let $a_1 < a_2$ and $0 < s_2 < s_1 \leq 1$, and take $F(z) = (z + a_1)^{s_1} + (z + a_2)^{s_2}$. For $v = u - a_2 \downarrow 0$, factor out $e^{i\pi s_1}(a_2 - a_1 + v)^{s_1}$ in the upper boundary value. Expanding the logarithm of the remaining factor gives the leading phase derivative

$$\frac{s_2 \sin(\pi(s_2 - s_1))}{(a_2 - a_1)^{s_1}}\, v^{s_2 - 1} < 0.$$

The error is $o(v^{s_2-1})$ by differentiating the finite scalar expression. Since $f(-u+i0) = -\partial_u \mathcal{L}(-u+i0)$, the imaginary part of f is positive on a short open interval. A positive Stieltjes transform has nonpositive imaginary part in the upper half-plane. This proves the sharpness assertion, which is a guarantee across all choices of weights.

8 Matrix-valued distributions of channels

Theorem 8.1. *Let M be a finite positive semidefinite $n \times n$ matrix measure on $[a_*, \infty)$, $a_* > 0$, and $s : [a_*, \infty) \to (0, 1]$ a measurable nondecreasing function. Assume*

$$M_0 + M([a_*, \infty)) > 0, \qquad \int_{a_*}^{\infty} (1+a)^{s(a)} \operatorname{tr} dM(a) < \infty,$$

with $M_0 \geq 0$. Put $F(z) = M_0 + \int (z+a)^{s(a)} dM(a)$. Then $\operatorname{tr}(F^{-1}F')$ has a finite positive Stieltjes measure ν supported on $[a_*, \infty)$, with mass

$$\nu([a_*, \infty)) = \int_0^1 \operatorname{rank} M(\{a : s(a) \geq t\})\, dt. \tag{28}$$

Proof. Truncate at a finite upper threshold so the combined total remains positive definite, which holds for all sufficiently large cutoffs. On each bounded joint rectangle for $(a, s(a))$, use a finite partition and select tags from the graph in the bins with positive trace mass. These tags have the same threshold ordering, and their coefficient matrices are the masses of the bins. On bounded joint rectangles the integrands for F, F' converge by uniform continuity on compact upper-half-plane z sets. On the removed tail their norms are bounded there by constants times $(1+a)^{s(a)}$, which is integrable. The inverses converge by the numerical-sector argument. Uniform mass bounds from Theorem 7.1 and compactness for the Stieltjes transforms then give the positive finite measure.

For the mass identity take $x \geq 1$, and quantize $s(a)$ down and up in steps of $1/N$. The scalar bounds $x^{s(a)} \leq (x+a)^{s(a)} \leq x^{s(a)}(1+a)^{s(a)}$ give lower and upper finite matrix power sums, in the Loewner order. On each quantization bin the two coefficient matrices have the same range: positive weighting by $(1+a)^{s(a)}$ preserves the kernel of the integrated positive matrix. Add M_0 at exponent zero in both sums. Monotonicity of the determinant of positive matrices sandwiches $\log\det F(x)/\log x$ between the limits of their largest basis exponents as $x \to \infty$. By the finite formula these exponents are the integrals of the lower and upper quantized rank profiles. They bracket the right hand side of (28), and differ by at most n/N, since each column exponent changes by at most $1/N$. Let $N \to \infty$. Finally, the Stieltjes formula gives $xf(x) \to \nu([a_*, \infty))$ and its integral in x gives the same limit for $\log\det F(x)/\log x$. □

Corollary 8.2. *For either ordered model, $\det F(0)/\det F(x)$, $x \geq 0$, is the Laplace transform of a scalar generalized gamma convolution with Thorin measure ν.*

Proof. Integration of the Stieltjes formula gives

$$\frac{\det F(0)}{\det F(x)} = \exp\left[-\int_{a_*}^{\infty} \log(1 + x/u)\, d\nu(u)\right].$$

The logarithm inside the integral equals $\int_0^\infty (1 - e^{-xt}) e^{-ut}\, dt/t$. Thus the Lévy measure has density $t^{-1} \int e^{-ut}\, d\nu(u)$, with finite first moment near zero and finite mass at infinity because $a_* > 0$ and ν is finite. The Thorin form gives the stated class [11, Chapters 8–9]. For zero ν the law is concentrated at zero. □

9 Realization by weighted Bessel channels

We realize every finite matrix data set from (22), first interpreting the channels with $s_j < 1$ as weighted rays. The conormal powers follow from the weighted extension of Caffarelli–Silvestre and Stinga–Torrea [13, 14]. Scalar Bessel resolvent differences and their spectral shifts are explicit in [15, Section 5]. Demirel [16, Theorem 1.1] derives relative traces for a Kirchhoff star with regular rays and integrable potentials. Here the rays can have distinct Bessel exponents and energy floors, and share a general matrix coupling. The exponent-one term is realized by a finite vertex state, as specified below.

Factor $M_j = B_j^* B_j$ with $B_j : \mathcal{G} \to \mathcal{E}_j$ between finite-dimensional Hilbert spaces and $\mathcal{G} = \mathbb{C}^n$. For j with $0 < s_j < 1$, let

$$w_j(r) = r^{1-2s_j}, \qquad \mathcal{H}_j = L^2((0,\infty), w_j(r)\,dr; \mathcal{E}_j), \qquad d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}, \qquad L_j = d_{s_j}^{-1/2} B_j.$$

The branch energy is $\int_0^\infty (|p'|^2 + a_j|p|^2) w_j\,dr$ on the weighted first Sobolev domain. Its trace at zero exists and is continuous in the form norm. Indeed for $0 < t < r$,

$$|p(r) - p(t)|^2 \le \int_t^r |p'(v)|^2 w_j(v)\,dv\,\frac{r^{2s_j} - t^{2s_j}}{2s_j}.$$

Taking $t \downarrow 0$ and averaging r in a fixed interval away from zero gives trace continuity. The zero-trace form is the Friedrichs closure of its smooth interior core. In fact the cutoff derivative on $[\delta, 2\delta]$ contributes at most $C\int_0^{2\delta} |p'|^2 w_j\,dr$, by the last inequality with zero trace and $r^{2s_j} w_j(r) = r$. Then cut off at infinity and mollify away from zero. A smooth compactly supported function constant with the trace near zero handles general traces. Multiplying by $r^{1/2-s_j}$ into the unweighted L^2 space gives the differential expression

$$-\partial_r^2 + (s_j^2 - \tfrac{1}{4}) r^{-2} + a_j.$$

When $s_j = 1$, a raw $L^2(r^{-1}dr)$ ray only admits zero as a finite trace. The Liouville potential at that endpoint is in the limit-point regime [17, Theorem 1.1]. We use a finite vertex state for each linear contribution. Let $Cg = (B_j g)_{s_j=1}$ and let $\mathcal{X} = \operatorname{ran} C$ with the norm inherited from the direct sum of those $\mathcal{E}_j$. If there are no exponent-one channels, $\mathcal{X} = \{0\}$. The Hilbert space for the coupled system is

$$\mathcal{H} = \left(\bigoplus_{s_j<1} \mathcal{H}_j\right) \oplus \mathcal{X}.$$

The allowed data have $p_j(0) = L_j g$ and vertex coordinate Cg for a common $g \in \mathcal{G}$. Let $N = \ker((L_j)_{s_j<1}, C)$. For a fixed allowed trace and vertex coordinate, minimize $g^* M_0 g$ over their preimages. This gives a unique minimizing g, since M_0 is positive definite on N, and it is characterized by $P_N M_0 g = 0$. The minimizing selection is linear and bounded on the finite-dimensional space of allowed data.

Add $g^* M_0 g$ with this selection and the vertex potential $\sum_{s_j=1} a_j \|B_j g\|^2$ to the ray energies. This is a closed nonnegative form: the joint trace restriction is closed in the weighted energy norm, and the finite terms are bounded there and nonnegative. Its domain is dense in $\mathcal{H}$. For prescribed Cg, approximate any ray L^2 data by compactly supported interior tests and attach cutoffs of radius δ with traces $L_j g$. Their squared $L^2(w_j\,dr)$ norms are $O(\delta^{2-2s_j})$. The form defines a self-adjoint $H \ge a_*$.

Let B be the direct sum of the ray Dirichlet operators with trace zero, on the ray Hilbert space alone. Let ι be its isometric embedding into $\mathcal{H}$. Thus the cut removes the vertex mass states when they occur. A finite vertex Hilbert space is the usual mechanism for linear spectral-parameter terms in a boundary condition [18]. Here its removal determines the precise relative trace. For a purely continuous model $\iota = I$. The form condition on the operator domain then gives $\sum_j L_j^* \ell_j + M_0 g = 0$, where $\ell_j = -\lim_{r\downarrow 0} w_j p_j'(r)$ is the outward conormal flux. This follows by integration by parts against arbitrary allowed traces. The flux has a finite endpoint limit for operator-domain vectors, since its derivative is integrable near the endpoint from the $L^2(w_j\,dr)$ equation and the local integrability of w_j.

9.1 The channel normalization and relative resolvent

For $b > 0$ the decaying unit-trace profile for a ray with exponent $s \in (0,1)$ is

$$p_{s,b}(r) = \frac{2^{1-s}}{\Gamma(s)} (\sqrt{b}\, r)^s K_s(\sqrt{b}\, r),$$

where K_s is the modified Bessel function of the second kind. The differential equation is $-w^{-1}(wp')' + bp = 0$, $w = r^{1-2s}$. The small-argument expansion gives trace one. Using $\partial_\tau(\tau^s K_s(\tau)) = -\tau^s K_{1-s}(\tau)$ and $K_{1-s}(\tau) \sim 2^{-s}\Gamma(1-s)\tau^{s-1}$ at zero gives the outward flux $d_s b^s$. The weighted profile and derivative energies are finite at zero, with squared integrands of orders r^{1-2s} and r^{2s-1}, and the profile decays exponentially at infinity. Integration by parts and zero-trace density show that it minimizes the shifted energy with its trace, with total energy $d_s b^s$. Subtracting the polarized equations with parameters $b, c > 0$ gives

$$(b-c)\langle p_{s,b}, p_{s,c}\rangle_{L^2(w\,dr)} = d_s(b^s - c^s).$$

Taking $c \to b$ is justified by continuity in weighted L^2, using the endpoint estimates locally uniformly in $c > 0$. Thus $\|p_{s,b}\|^2_{L^2(w\,dr)} = d_s s b^{s-1}$.

For $x > 0$ write $K_x : \mathcal{G} \to \mathcal{H}$ for the Poisson map with ray components $p_{s_j, a_j + x} L_j g$ and vertex component Cg. The positive form shifted by x on this response is $g^* F(x) g$ when g is the minimizing coordinate. Indeed each flux gives $(x + a_j)^{s_j} B_j^* B_j$ in the normalized trace, and each vertex term gives $(x + a_j) B_j^* B_j$. One may equivalently choose traces $B_j g$ for continuous channels and put the constant $d_{s_j}^{-1}$ into their Hilbert weights and energies.

The cut and coupled shifted resolvent solutions differ by $K_x g$, since the difference is orthogonal to all cut tests in shifted energy and has the same allowed boundary data as $K_x g$. Tests on the Poisson part give $F(x) g = K_x^* f$ for data f. On the invisible coordinates N, these equations are exactly $P_N M_0 g = 0$; on their complement they are the variational equations with the imposed trace and vertex state. The norm identity for the ray profiles and the norm of Cg give

$$(H + x)^{-1} - \iota (B + x)^{-1} \iota^* = K_x F(x)^{-1} K_x^*, \qquad K_x^* K_x = F'(x). \tag{29}$$

The correction is positive of finite rank. Taking its trace by finite-rank cyclicity gives $f(x)$ exactly. The calculation is variational and allows complex matrix weights.

9.2 Actual heat traces and receding caps

Theorem 9.1. *For every finite data set in* (22)*, the relative heat difference $e^{-tH} - \iota e^{-tB} \iota^*$ is trace class for $t > 0$. Its trace $D(t)$ equals the limit of the connected-minus-cut trace with Dirichlet caps receding on the rays. For the phase in* (27)*,*

$$D(t) = t \int_{a_*}^{\infty} e^{-tu} \Xi(u)\, du. \tag{30}$$

It satisfies $0 \le D(t) \le n s_{\max} e^{-a_* t}$. *Its limit as* $t \downarrow 0$ *is the largest determinant basis weight, given by the right hand side of* (24). *For ordered data,*

$$D(t) = \int_{a_*}^{\infty} e^{-tu}\, d\nu(u),$$

and D *is completely monotone. Reversing the order can violate complete monotonicity for two scalar rays.*

Proof. Fix $\sigma > 0$, and put $T = (H+\sigma)^{-1}$, $T_0 = \iota(B+\sigma)^{-1}\iota^*$, and $V = T - T_0$. These bounded resolvents are positive with spectra in $[0, (a_* + \sigma)^{-1}]$, and V is positive trace class by (29). For any smooth compactly supported g on $\mathbb{R}$, Fourier inversion and the Duhamel formula give a trace-class difference $g(T) - g(T_0)$ and the spectral-averaging identity

$$\operatorname{Tr}(g(T) - g(T_0)) = \int g'(r)\, d\omega(r), \qquad \omega(\Delta) = \int_0^1 \operatorname{Tr}\Big[V^{1/2} E_{T_0+\tau V}(\Delta) V^{1/2}\Big]\, d\tau.$$

Here E_A is the spectral resolution of A. To see the identity, differentiate $e^{iu(T_0+\tau V)}$ in trace norm and cycle the bounded factors around V within the trace. Then integrate in τ and against the Fourier transform of g. The norm bound $\|e^{iuT} - e^{iuT_0}\|_1 \le |u| \|V\|_1$ justifies the integrals. This is the bounded spectral-averaging trace formula [9].

The choices $g_x(r) = r/[1+(x-\sigma)r]$ and $g_t(r) = e^{-t(1/r-\sigma)}$ for $r > 0$ give the relative resolvent and heat traces. Both vanish at zero and have smooth extensions of compact support from the common spectral interval, using flatness at zero for g_t. Thus the extra reference zero block in the vertex model contributes zero in either formula. The resolvent choice gives

$$f(x) = \int [1 + (x-\sigma)r]^{-2}\, d\omega(r).$$

For $x \ge \sigma$ the integral tends to $\omega(\{0\})$ as $x \to \infty$. The inequality $0 \le xF'(x) \le s_{\max} F(x)$ proves $f(x) = O(x^{-1})$, so this atom is zero. The heat choice now gives

$$D(t) = \int_{r>0} t r^{-2} e^{-t(1/r-\sigma)}\, d\omega(r) \ge 0, \qquad \int_0^{\infty} e^{-xt} D(t)\, dt = f(x).$$

The last identity uses Tonelli and the previous resolvent formula. The formula for $D(t)$ is continuous in positive time since g_t' varies continuously in sup norm on compact positive time intervals. Comparing with (27) by Laplace uniqueness proves (30) and the bound. With order, Ξ is the cumulative ν and Tonelli gives the desired positive measure.

For cap exhaustion, impose Dirichlet conditions at $r = L$ in the two forms. Their zero extensions are nested subspaces with form-dense union, by cutoffs at infinity preserving the vertex data. Variational projections in shifted energy give strong convergence of both embedded capped resolvents. For a fixed trace, the capped Poisson minimizers converge in shifted energy to the full ones: cutoffs of the full profiles give convergence of the minimum energies, and orthogonality to the zero-trace difference identifies the squared error with the excess minimum energy. This proves norm convergence of Poisson maps and aperture matrices, since the aperture dimension is finite. Formula (29) then gives trace-norm convergence of the capped relative resolvent corrections. The capped cut eigenvalues in ray j are $a_j + (j_{s_j,k}/L)^2$, where $j_{s,k}$ are the positive zeros of J_s. The solution with zero trace uses $r^s J_s$, and the zeros grow linearly in k [19, Section 10.21]. The compact capped resolvents differ by finite rank, so the relative eigenvalue counts differ by at most that rank. Both capped heat traces are finite. In the Duhamel formula for g_t applied to

the capped resolvents, the unitary factors and their adjoints converge strongly, and the middle trace-class differences converge in trace norm. Fourier dominated convergence yields the stated convergence of the capped relative heat traces.

To obtain the limit at small t for any ordering, let b denote the largest determinant basis weight. Uniformly for $z = Re^{i\theta}$ with $0 \le \theta \le \pi$ on the upper side, the Cauchy–Binet expansion gives $\det F(z) = CR^b e^{ib\theta}(1 + o(1))$, with $C > 0$, as $R \to \infty$. Indeed each factor has relative error $O(R^{-1})$ against its power of z, and all largest-weight terms have the same phase. At $\theta = 0$ the response is positive definite. The continuous logarithm on the arc therefore has boundary argument tending to πb at $\theta = \pi$, with the branch fixed by its real value on the positive axis. Hence $\Xi(u) \to b$ as $u \to \infty$. Substituting $v = tu$ in (30) and using bounded convergence gives $D(0+) = b$.

Finally choose the scalar reversed data from Section 7 with $s_1, s_2 < 1$, $B_1 = B_2 = 1$ and $M_0 = 0$. If its physical D were completely monotone, Bernstein's theorem and the heat-to-resolvent Laplace identity would make $f(x)$ a positive Stieltjes transform. Holomorphic uniqueness would then give a nonpositive imaginary part in the upper half-plane, contrary to the strict sign just above a_2. The same construction realizes the reversal with $s_1 = 1$ using a lower-threshold vertex state. For a single normalized ray with $M_0 = 0$ and $M_1 = 1$, the formulas give $D(t) = s_1 e^{-a_1 t}$, providing a direct normalization check. □

10 Statement of the continuum limit

Let U_-, U_+ be bounded nonempty Lipschitz open subsets of $\mathbb{R}^d$, $d \ge 1$, and let y_0 be interior to both. Fix a nonempty bounded open set $K_0 \subset \mathbb{R}^d$, and put $K_\epsilon = y_0 + \epsilon K_0$. We take $\epsilon > 0$ small enough that this aperture lies in $U_- \cap U_+$. The connected guide is the union

$$\Omega_\epsilon = ((-\infty, 0) \times U_-) \cup (\{0\} \times K_\epsilon) \cup ((0, \infty) \times U_+).$$

Let H_ϵ be the Dirichlet operator on this open set, with a possible nonnegative contact term $\int_{K_\epsilon} \rho_\epsilon |\gamma f|^2$ added to its form on $H_0^1(\Omega_\epsilon)$. Here γf is the common trace, and the ρ_ϵ are nonnegative with $\sup_\epsilon \|\rho_\epsilon\|_\infty < \infty$. The cut operator B is the direct sum of the two half-cylinder Dirichlet operators with a Dirichlet condition on the end planes. The relative heat trace $D_\epsilon(t)$ is the limit of the connected-minus-cut heat traces with Dirichlet caps at longitudinal coordinates $\pm L$, as $L \to \infty$. Its positive representing energy measure ν_ϵ satisfies

$$D_\epsilon(t) = \int e^{-tu}\, d\nu_\epsilon(u). \tag{31}$$

Theorem 1.1 gives this measure for each fixed aperture. Section 11 gives the relative-trace identities used here.

Write $A_j = -\Delta_{U_j}^D$, $j = -, +$, with eigenvalues $a_{j,k}$ and an orthonormal eigenbasis $\phi_{j,k}$. The bottom transverse threshold is $a = \min_j a_{j,1} > 0$. The values $\phi_{j,k}(y_0)$ use their interior smooth representatives. Sums over an eigenspace are independent of the basis chosen in that space. Let F be the positive self-adjoint operator in $L^2(K_0)$ obtained by closing the form

$$q(g) = \int_{\mathbb{R}^d} |\xi|\, |\widehat{Eg}(\xi)|^2\, d\xi, \qquad g \in C_c^\infty(K_0),$$

where E is extension by zero and the Fourier transform has the unitary convention with derivative multiplier $i\xi$. Define

$$\mathcal{T}(K_0) = \langle \mathbf{1}, (2F)^{-1}\mathbf{1} \rangle_{L^2(K_0)}. \tag{32}$$

The factor 2 reflects the two half-space extension energies.

Theorem 10.1. *For every integer $m > d$,*

$$\left\| (1+u)^{-m} \big(\epsilon^{-d-1}\nu_\epsilon - \nu_*\big) \right\|_{\mathrm{TV}} \longrightarrow 0, \tag{33}$$

where the positive measure ν_* is

$$d\nu_*(u) = \frac{\mathcal{T}(K_0)}{2\pi} \sum_{j,k} |\phi_{j,k}(y_0)|^2 \frac{\mathbf{1}_{\{u > a_{j,k}\}}}{\sqrt{u - a_{j,k}}}\, du. \tag{34}$$

In particular, locally uniformly for $t > 0$,

$$\epsilon^{-d-1} D_\epsilon(t) \longrightarrow \frac{\mathcal{T}(K_0)}{2\sqrt{\pi t}} \sum_{j=-,+} e^{-tA_j}(y_0, y_0). \tag{35}$$

Thin handles and disappearing Neumann windows give closely related variational coefficients for shifts of discrete eigenvalues. Abatangelo, Felli, and Terracini [20, Theorems 1.1–1.2] obtain the ambient-dimensional power for a thin handle under a nonzero boundary normal derivative. Abatangelo and Ognibene [21, Theorem 1.3 and Section 2.2] treat thin torsion and eigenvalue ramification. For a disappearing Neumann region, the half-space problem of Felli, Noris, and Ognibene [25, Lemma 3.1 and (3.2)–(3.5)] supplies the one-arm version of the constant-data variational energy. Here the transverse geometry stays fixed in each lead. The perturbed quantity is a continuum measure, and its leading coefficients involve the values of all transverse channels at y_0. Parallel waveguides coupled by a lateral window have small-window bound states in a different geometry [24, Theorem 2.2]. The axial aperture considered here retains the fixed transverse floor in both arms.

The proof uses a bounded perturbation of the free half-space aperture form after rescaling. A higher relative resolvent isolates the one-product term of its aperture logarithmic derivative. All multiple products are of smaller order by localized Schatten estimates. The open-channel energy formula then gives convergence of densities on bounded energy intervals. Uniform control of the weighted tails gives (33). Section 15 obtains simultaneous large-time and low chemical-potential laws. Section 16 records the shape dependence and the explicit ball normalization.

11 Relative trace identities

Throughout this section the aperture is fixed. To lighten notation write $K = K_\epsilon$, $\rho = \rho_\epsilon$, and $H = H_\epsilon$. Write $E_j : L^2(K) \to L^2(U_j)$ for zero extension. The aperture form domain $\mathcal{V}$ is the closure of $C_c^\infty(K)$ in the norm given by $\sum_j \|A_j^{1/4} E_j g\|^2$. Denote the operator of this form by S. It is bounded below by a positive constant. Its embedding into $L^2(K)$ is compact, since either zero extension is an isometry into the compact transverse form embedding. The min–max comparison with $A_j^{1/2}$ gives

$$\lambda_n(S) \gtrsim n^{1/d}, \qquad \lambda_n(A_j) \gtrsim n^{2/d}, \tag{36}$$

using enclosing boxes for the transverse eigenvalue bound. At a real parameter $x \geq 0$ set

$$M(x) = \sum_j E_j^* (A_j + x)^{1/2} E_j + \rho, \qquad N(x) = S^{-1/2} M(x) S^{-1/2}. \tag{37}$$

These are respectively a form operator and a bounded normalized operator, with $N(x) \geq I$.

The pair of Poisson extensions $K_x g$ on the arms has components $e^{-r(A_j+x)^{1/2}}E_j g$, r being the distance from the end plane. It minimizes the shifted energy at trace g. The common trace space of $H_0^1(\Omega_\epsilon)$ is $\mathcal{V}$. Indeed the energy bound for the minimizer extends the trace map from compact smooth functions to this space. Conversely, for smooth aperture data, subtract a compact smooth bridge with that trace from the extensions. The two differences have zero boundary trace in their respective Lipschitz arms, so their zero extensions belong to the cut form domain. Adding back the bridge and taking limits in the form norm proves the assertion. The kernel of the trace is exactly the cut domain by the zero boundary trace characterization on the two arms.

Let $L_x = K_x S^{-1/2}$. Variational orthogonality of the Poisson extensions to the cut domain, and integration in r, give

$$(H+x)^{-1} - (B+x)^{-1} = L_x N(x)^{-1} L_x^*, \qquad L_x^* L_x = N'(x). \tag{38}$$

For example, the correction to the cut solution for data f has trace g determined by $M(x)[g,v] = \langle f, K_x v\rangle$ for $v \in \mathcal{V}$. The second identity follows from $K_x^* K_x = \frac{1}{2}\sum_j E_j^*(A_j+x)^{-1/2}E_j$.

Let P_n project onto the first n eigenvectors of S, and let $J_n(x)$ be the inverse of the compression $P_n N(x) P_n$ on its range, extended by zero. Coercivity and the Galerkin error bound give $J_n \to N^{-1}$ strongly. Define $B_n(x) = S^{-1/2} J_n(x) S^{-1/2}$. Variational restriction gives

$$0 \le B_n(x) \le M(x)^{-1}, \qquad B_n(x) \longrightarrow M(x)^{-1} \text{ strongly.} \tag{39}$$

We need the scalar function $f_n(z) = \operatorname{tr}(N_n(z)^{-1} N_n'(z))$, where N_n is the finite compression of N. It is a Stieltjes transform of a positive measure ν_n supported on $[a,\infty)$. Here and below a transform of order one uses the kernel $(z+u)^{-1}$.

For this sign, the finite matrix $N_n(z)$ is a positive constant matrix plus an integral of principal roots $(z+u)^{1/2}$ against a positive matrix measure with finite half-moment and positive definite total mass. If $0 < \theta = \arg(z+a) < \pi$, its numerical range lies in the sector of arguments $[0,\theta/2]$, and rotation by $-\theta/4$ gives invertibility. The numerical range of the inverse lies in $[-\theta/2, 0]$, by writing $v = N_n w$ in $v^* N_n^{-1} v$. In the log derivative, the scalar root derivative contributes an argument in the same opposite sector. The trace against the positive matrix density therefore places $f_n(z)$ in $[-\theta, 0]$. On the real interval to the right of $-a$ the function is positive, and $x f_n(x) \to n/2$ as $x \to \infty$ by dominated convergence. The real values are nonincreasing by the upper-half-plane sign and vertical difference quotients, so the positive limit at the shifted endpoint exists. The holomorphic Stieltjes characterization [4, Theorem 2.11] gives the positive measure and excludes a constant term.

For each integer $m > (d+1)/2$, differentiating the finite trace $m-1$ times gives

$$\int (x+u)^{-m}\, d\nu_n(u) \longrightarrow \operatorname{Tr}((H+x)^{-m} - (B+x)^{-m}) = \int (x+u)^{-m}\, d\nu_\epsilon(u), \qquad x > 0. \tag{40}$$

We record the limit details since they also justify the energy formulas below. Bounds (36) and the singular-value product inequality give

$$s_\ell(N^{(k)}(x)) = O(\ell^{-2k/d}) \quad (k \ge 1), \qquad s_\ell(L_x^{(j)}) = O(\ell^{-(2j+1)/d}) \quad (j \ge 0).$$

For the first bound use the transverse power $(A_j+x)^{1/2-k}$ between two normalizers. For the second, the j-th derivative of the Poisson exponential at a transverse level of shifted energy b has longitudinal L^2 norm $O(b^{-j-1/4})$. This follows by differentiating $e^{-r\sqrt{b}}$ and scaling $r\sqrt{b}$. Multiply the resulting transverse singular-value rate by that of $S^{-1/2}$. The bounds are uniform for $x \ge 0$, and both sets of operators tend to zero in operator norm as $x \to \infty$.

By (38) the finite trace $f_n(x)$ equals $\mathrm{Tr}(L_x J_n(x) L_x^*)$. In a differentiated term of order $m-1$, the two Poisson factors of orders j, l and the positive-order normalized factors of orders k_i obey $j + l + \sum k_i = m - 1$. Their Schatten exponents can be taken as $2m/(2j+1), 2m/(2l+1), m/k_i$, whose reciprocals sum to one. The strong inverse limits can be multiplied against the Schatten factors in their respective ideals, by approximation by finite rank and uniform boundedness of the inverses and their adjoints. Schatten Hölder then gives the trace-norm limit of the differentiated resolvent, which proves the first limit in (40). The same bounds give trace-norm decay of the relative powers at infinity. The differentiation factor for the order-m transform is $(-1)^{m-1}/(m-1)!$.

The positive measures $(1+u)^{-m}\nu_n$ have bounded mass at $x = 1$; their limits on the one-point compactification have an ordinary finite-energy measure and a possible nonnegative mass at infinity. In the generalized transform the latter gives a constant, excluded by trace-norm decay as $x \to \infty$. To identify the measure on finite energies, localize to a bounded longitudinal slab. Each localized m-th power is trace class by the free $(d+1)$-dimensional heat bound and $m > (d+1)/2$. The transparent diagonal heat difference counts killed Brownian bridges that hit the aperture. It is dominated by the corresponding difference for a full-plane cut in an enclosing cylinder. Its integrated bound is $\frac{1}{2}\,\mathrm{Tr}(e^{-tA_Q})$, where Q is an enclosing transverse box. Indeed, the same-point longitudinal bridge at s hits zero with probability $e^{-s^2/t}$, and its integral against the free diagonal is $1/2$. For a nonnegative contact barrier the diagonal kernel is between those of the cut and transparent guide. For the first comparison, the positive part of the cut solution minus the contact solution has zero aperture trace and can be used in both resolvent equations. For the second, the positive part of the contact solution minus the transparent solution gives a nonnegative contact term. The exponential product formula transfers these inequalities to the kernels. Applying these comparisons also on capped domains gives the cap limit by dominated convergence: the individual kernels increase under domain exhaustion, and the capped differences are bounded by the infinite transparent difference. The full diagonal difference is continuous in positive time by semigroup kernel factorization and the same domination.

The localized gamma heat integral can now be subtracted and the slab can be removed by trace-class continuity on the resolvent side and monotone convergence of the diagonal integrals on the heat side. Thus the physical trace in (40) equals the Laplace transform of $t^{m-1}D_\epsilon(t)/(m-1)!$. The generalized Stieltjes measure obtained above has the same transform by Tonelli. Uniqueness of the scalar Laplace transform identifies D_ϵ with its Laplace integral, and a second use identifies its energy measure. This proves (31)–(40) and weak convergence of the entire weighted sequence $(1+u)^{-m}\nu_n$ for the fixed aperture.

We will also use the energy density from the same compressions. On an open transverse band, write $U + iV = M(-u + i0)$, with

$$U = \rho + \sum_j E_j^*(A_j - u)^{1/2}\mathbf{1}_{(u,\infty)}(A_j)E_j, \tag{41}$$

$$V = \sum_j E_j^*(u - A_j)^{1/2}\mathbf{1}_{(-\infty,u)}(A_j)E_j. \tag{42}$$

For a sufficiently small aperture and bounded energy, $J = (U+iV)^{-1}$ exists as a bounded inverse on $L^2(K)$; see Section 14 for the uniform estimate. In the finite compressions, the imaginary part of the logarithmic derivative equals the negative of

$$\mathrm{Tr}\left[J^*VJ(-\dot{U})\right] + \mathrm{Tr}\left[(\mathrm{Re}\,J)\dot{V}\right], \tag{43}$$

where dots are energy derivatives and J in the finite expression is the unnormalized extended Galerkin inverse. This identity follows by cyclicity and by $\mathrm{Im}\,N_n^{-1} = -N_n^{-*}(S^{-1/2}VS^{-1/2})_n N_n^{-1}$;

the other term uses the real part of the inverse. In each term the normalizers adjacent to the inverse combine into the unnormalized inverse, exactly as in (39). On compact band subintervals $V, \dot{V}$ have finite rank and $\dot{U}$ is bounded. When the inverses converge strongly with their adjoints and are locally bounded, Stieltjes inversion and the weighted measure limit in (40) identify the physical density as (43) divided by π. At a threshold, the finite-rank term from that energy is a square root. If the regular inverse remains bounded on the two boundary branches, the local logarithmic derivative has an integrable inverse-square-root singularity, with no atom. We use the direct uniform estimates, including this endpoint, in Section 14.

12 Rescaling the aperture form

Let $W_\epsilon : L^2(K_0) \to L^2(K_\epsilon)$ be unitary, with

$$(W_\epsilon g)(y_0 + \epsilon y) = \epsilon^{-d/2} g(y).$$

For any fixed $z \in \mathbb{R}$ with $A_j + z \geq 0$ for both arms, define $\widetilde{M}_\epsilon(z) = \epsilon W_\epsilon^* M_\epsilon(z) W_\epsilon$. Under this dilation the two terms in the normalized physical scale are the compressions of

$$\left(-\Delta^D_{(U_j - y_0)/\epsilon} + \epsilon^2 z\right)^{1/2}.$$

The contact term is bounded by $C\epsilon$. Replacing the shifted root by the unshifted root costs at most $\epsilon\sqrt{|z|}$ in operator norm, by the scalar square-root bound on their spectra.

The exterior fractional form q is strictly coercive. For a zero-extended function supported in K_0, the Fourier mass below frequency $\delta_0 > 0$ is at most $C_d |K_0| \delta_0^d \|g\|^2$ by the L^1 bound on its Fourier transform. Choosing δ_0 sufficiently small gives $q(g) \geq \delta_0 \|g\|^2/2$. The unshifted root on either scaled cross-section is bounded below by q on the aperture core: its form is the minimum harmonic extension energy in that half-cylinder, and zero extension gives a competitor in the full half-space.

For the upper comparison, let Pg be the full half-space harmonic extension of a smooth aperture datum. The half-space energy equals $q(g)$ by Fourier transform. A cutoff χ_R, equal to one in the ball of radius R and supported in the ball of radius $2R$ in $\mathbb{R}^{d+1}$, preserves the aperture trace for R large. Harmonic energy orthogonality gives the exact identity

$$\|\nabla(\chi_R P g)\|_2^2 - \|\nabla P g\|_2^2 = \|P g \, \nabla \chi_R\|_2^2. \tag{44}$$

Indeed $(\chi_R^2 - 1) Pg$ has zero boundary trace. It can be used as an energy test by first replacing 1 with χ_L^2 and letting $L \to \infty$. On the outer annuli the tails of ∇Pg vanish in L^2, and the cutoff terms vanish by the following Poisson estimate. At distance at least R from the origin, with R larger than twice the radius containing K_0, the Poisson kernel gives $|Pg(s,y)| \leq C\|g\|_1 |(s,y)|^{-d}$. Taking $|\nabla \chi_R| \leq C/R$ proves that the right hand side of (44) is at most $CR^{-d-1}\|g\|_2^2$. We may take $R \asymp \epsilon^{-1}$ with a fixed small proportionality constant, so the truncated extension is admissible on each scaled half-cylinder.

We have proved two-sided bounds on the smooth aperture core,

$$|\widetilde{M}_\epsilon(z)[g] - 2q(g)| \leq C_z \epsilon \|g\|_2^2.$$

For all sufficiently small ϵ, the shifted aperture form and $2q$ have the same completion by coercivity and equivalence of their form norms. Polarization represents their difference as a bounded operator with norm $O_z(\epsilon)$. The inverse perturbation identity therefore gives

$$\|\widetilde{M}_\epsilon(z)^{-1} - (2F)^{-1}\| = O_z(\epsilon), \qquad \|M_\epsilon(z)^{-1}\| = O_z(\epsilon). \tag{45}$$

All constants here are uniform over the specified bounded contact barriers. At $z = -a$ the zero transverse root on any lowest channel is included as zero, so this comparison also applies to the regular aperture operator at the first threshold.

13 Higher-resolvent limit and weighted control

Fix $x > 0$ and an integer $m > d$. We use M_ϵ for the operator (37) with the shrinking aperture. The derivatives of this operator can be estimated by the size of the aperture before applying Schatten Hölder. For a transverse arm, the compressed heat operator $T_{\epsilon,j}(t) = E_{\epsilon,j}^* e^{-tA_j} E_{\epsilon,j}$ is positive, and

$$\|T_{\epsilon,j}(t)\| \le \min(1, C\epsilon^d t^{-d/2}), \qquad \operatorname{Tr} T_{\epsilon,j}(t) \le C\epsilon^d t^{-d/2}. \tag{46}$$

The first inequality uses heat contraction and the kernel bound with Schur's test. The second follows by integrating the diagonal on K_ϵ. For $1 \le p < \infty$, positivity gives $\|T\|_{\mathcal{S}_p} \le \|T\|^{1-1/p} (\operatorname{Tr} T)^{1/p}$.

For $k \ge 1$ the k-th derivative of $M_\epsilon(x)$ is a scalar multiple of the sum of compressions of $(A_j + x)^{1/2-k}$. Put $b = k - \frac{1}{2}$. Integrate the bound from (46) against $t^{b-1} e^{-xt}/\Gamma(b)$ in the gamma representation. On $t < \epsilon^2$ the bound is $C\epsilon^{d/p} t^{b-1-d/(2p)}$, with integral $O(\epsilon^{2b})$ when $2bp > d$. Above ϵ^2 use $C\epsilon^d t^{b-1-d/2} e^{-xt}$. This proves

$$\|M_\epsilon^{(k)}(x)\|_{\mathcal{S}_p} \le C \begin{cases} \epsilon^{2k-1}, & 2k-1 < d, \\ \epsilon^d \log(e/\epsilon), & 2k-1 = d, \\ \epsilon^d, & 2k-1 > d, \end{cases} \quad p(2k-1) > d. \tag{47}$$

In the $(m-1)$-st derivative of the finite logarithmic derivative from Section 11, the terms are constant multiples of traces of products with r inverse factors, alternating with derivatives $M_\epsilon^{(k_i)}$, where $k_i \ge 1$ and $\sum_i k_i = m$. The inverse factors are $B_{\epsilon,n}(x)$ from (39): in the finite trace, each form normalizer adjacent to an inverse becomes part of this unnormalized inverse by cyclicity. In particular $\|B_{\epsilon,n}\| \le C_x \epsilon$ uniformly in n, by (45). For each term choose $p_i \ge 1$ so that

$$\sum_i \frac{1}{p_i} = 1, \qquad \frac{1}{p_i} < \frac{2k_i - 1}{d}.$$

For $r \ge 2$ the sum of the positive capped upper bounds $\min(1, (2k_i - 1)/d)$ exceeds one: if none is capped this follows from $(2m - r)/d > 1$. For $r = 1$ use $p_1 = 1$. Thus Schatten Hölder gives convergence of the traces at fixed ϵ with M_ϵ^{-1} in place of the Galerkin inverse, by (39) and strong multiplication against the Schatten factors. Equation (40) identifies the resulting physical relative trace after multiplication by $(-1)^{m-1}/(m-1)!$.

When $r \ge 2$ and all $2k_i - 1 < d$, the term is $O(\epsilon^{2m})$ by (47). If some $2k_i - 1 \ge d$, that derivative together with its inverse gives $O(\epsilon^{d+1} \log(e/\epsilon))$ or better. The other factors contribute additional positive powers of ϵ, up to logarithms. Thus every $r \ge 2$ term is $o(\epsilon^{d+1})$. Only the single product of M_ϵ^{-1} and $M_\epsilon^{(m)}$, whose derivative coefficient is one, enters the limit.

Set $b = m - \frac{1}{2} > d/2$. The operator $(A_j + x)^{-b}$ has a jointly continuous kernel near (y_0, y_0). This follows from the gamma integral and the free heat bound, since $b > d/2$ and $x > 0$. Denote the kernel by $G_{j,x}$. After conjugation by W_ϵ and division by ϵ^d, the compressed operator has kernel $G_{j,x}(y_0 + \epsilon y, y_0 + \epsilon y')$ on $K_0 \times K_0$. It converges in operator norm to $G_{j,x}(y_0, y_0) |\mathbf{1}\rangle\langle\mathbf{1}|$. Its trace converges to that of the same operator by integration of the diagonal. Since all these operators are positive, convergence is in trace norm. To see this, decompose with respect to the

span of **1** and its orthogonal complement. The trace of the complementary positive block tends to zero, and the off-diagonal trace norm is at most the geometric mean of the two block traces by factoring the positive operator through its square root.

Use the inverse norm convergence (45) in this trace-norm limit. The rescaled relative resolvent trace satisfies

$$\epsilon^{-d-1}\operatorname{Tr}((H_\epsilon + x)^{-m} - (B + x)^{-m}) \longrightarrow \frac{\Gamma(m - \frac{1}{2})}{2\sqrt{\pi}\,\Gamma(m)}\mathcal{T}(K_0)\sum_j G_{j,x}(y_0, y_0). \tag{48}$$

The prefactor is $(-1)^{m-1}/(m-1)!$ times the m-th falling factorial of $1/2$. The spectral expansion, justified by the positive gamma heat integral on the diagonal, gives

$$G_{j,x}(y_0, y_0) = \sum_k |\phi_{j,k}(y_0)|^2 (x + a_{j,k})^{1/2-m}.$$

The beta integral of $(x + a_{j,k} + v)^{-m} v^{-1/2}/(2\pi)$ supplies exactly the prefactor in (48). Thus the right hand side is the order-m generalized Stieltjes transform of ν_* in (34).

At $x = 1$, the convergence gives a uniform total mass bound on $(1+u)^{-m}\epsilon^{-d-1}\nu_\epsilon$. Take weak limits on the one-point compactification. Every such limit has the transform in (48), possibly with a constant from the mass at infinity. The transform on the right tends to zero as $x \to \infty$, by the gamma integral of the transverse diagonal heat bound, which forces the mass at infinity to vanish. Uniqueness on finite energies follows by the two Laplace inversions used for (40). Hence the full sequence converges weakly after weighting, and its weighted tails are uniformly tight as $\epsilon \to 0$.

14 Density convergence on bounded energy intervals

Fix a finite upper energy bound $E > a$ and a positive real resolvent point x. On each open band below E, let $J_\epsilon(u) = M_\epsilon(-u+i0)^{-1}$ in the unnormalized coordinates of (41)–(43). The difference of the two aperture forms at $-u + i0$ and at x is a bounded operator $\Delta_\epsilon(u)$ with a uniform norm bound. Indeed, the difference of the corresponding square-root symbols is uniformly bounded on the transverse spectra and on the bounded energy interval. The contact terms cancel in this difference. The bounded perturbation identity and (45) give

$$J_\epsilon(u) = M_\epsilon(x)^{-1} + O_E(\epsilon^2), \qquad \|J_\epsilon(u)\| = O_E(\epsilon). \tag{49}$$

They also apply at the one-sided values at each threshold. For the finite Galerkin inverse replace $M_\epsilon(x)^{-1}$ by $B_{\epsilon,n}(x)$; the formula is

$$B_{\epsilon,n}(x)\big(I + \Delta_\epsilon(u)B_{\epsilon,n}(x)\big)^{-1}.$$

For small ϵ the series is uniformly convergent. The strong convergence in (39) gives convergence of these boundary inverses and their adjoints, via the series and its adjoint. This justifies the limiting density (43) on open bands.

For completeness, at a threshold $u_0 \le E$, the regular aperture form obtained by omitting the root at u_0 still differs from $M_\epsilon(x)$ by a uniformly bounded operator. On the appropriate analytic branches it has a uniformly bounded inverse, in (49) and in the finite compressions. The determinant lemma in the compressions separates the root from the regular log derivative. Its finite-rank factor has the form $\det(I + \sqrt{z + u_0}\,C_n(z))$ with C_n holomorphic and bounded

in a small disk. The uniform bound follows by composing the inverse perturbation series with the finite list of threshold mode restrictions. Thus its derivative contributes at most the locally integrable $O(|u-u_0|^{-1/2})$ singularity, with no simple pole, and the regular imaginary trace stays bounded by the finite-rank open-channel formula without that threshold. Stieltjes inversion gives the density across arbitrarily small punctured neighborhoods and no atom at u_0. The bounds are uniform in the Galerkin cut, so the weighted weak limit at fixed ϵ does not introduce an atom. There are finitely many thresholds on the prescribed interval. This proves that the physical measures there have exactly the densities from the open bands.

Write $f_{\epsilon,j,k} = \phi_{j,k}|_{K_\epsilon}$. For each index with $a_{j,k} \le E$, $\|f_{\epsilon,j,k}\|^2 = O_E(\epsilon^d)$ by interior regularity. On the bands, V and $\dot V$ are the finite sums of rank-one terms on these restrictions, with coefficients $\sqrt{u-a_{j,k}}$ and $1/(2\sqrt{u-a_{j,k}})$, respectively. Hence $\|V\|_1 = O_E(\epsilon^d)$, and $\|\dot V\|_1$ is bounded by $C_E\epsilon^d$ times a finite sum of upper-side inverse square roots at thresholds. The closed derivative $-\dot U$ has operator norm bounded by a constant plus a finite sum of lower-side inverse square roots, because the high transverse spectrum stays away from the energy interval. Denote by $g_E(u)$ one plus these integrable sums.

In (43), the first trace is bounded by $C_E\epsilon^{d+2}g_E(u)$ by (49). For the second, replacing $\operatorname{Re} J_\epsilon$ by $M_\epsilon(x)^{-1}$ costs another $O_E(\epsilon^{d+2}g_E(u))$. Each term of the remaining open derivative is paired with the quadratic inverse form on $f_{\epsilon,j,k}$. By (45) and interior C^1 regularity,

$$\epsilon^{-d-1}\langle f_{\epsilon,j,k}, M_\epsilon(x)^{-1} f_{\epsilon,j,k}\rangle \longrightarrow |\phi_{j,k}(y_0)|^2\mathcal{T}(K_0).$$

The trace formula now gives L^1 convergence of the rescaled densities on $[a,E]$, by domination by $C_E g_E$. This proves local total variation convergence. Combining it with weighted tightness from Section 13 proves (33).

For t in a compact positive interval, $(1+u)^m e^{-tu}$ is uniformly bounded on $[a,\infty)$. Thus (33) gives locally uniform convergence of the heat transforms. In (34), the elementary Laplace integral of $v^{-1/2}$ and the transverse heat expansion give (35).

15 The first threshold and simultaneous limits

We now assume that the bottom threshold a belongs to only one arm and is simple, with real L^2-normalized transverse eigenfunction ϕ. Suppose $\phi(y_0) \neq 0$. Denote its aperture restriction by $f_\epsilon = \phi|_{K_\epsilon}$. Let $\delta > 0$ be less than half the gap to the next threshold. On the real segment $-a-\delta \le z \le -a$, let $R_\epsilon(z)$ be the aperture form obtained by omitting the lowest channel, including the contact term. The other channels have positive shifted energies on this segment. Set

$$\alpha_\epsilon(z) = \langle f_\epsilon, R_\epsilon(z)^{-1} f_\epsilon\rangle, \qquad c_\epsilon = \alpha_\epsilon(-a).$$

At $z=-a$, $R_\epsilon(-a) = M_\epsilon(-a)$, since the removed root vanishes.

Proposition 15.1. *The coefficient at the bottom threshold obeys*

$$c_\epsilon = \epsilon^{d+1}|\phi(y_0)|^2\mathcal{T}(K_0)\big(1+O(\epsilon)\big). \tag{50}$$

Uniformly for $0 < v \le \delta$ and small ϵ, the relative density satisfies

$$\nu'_\epsilon(a+v) = \frac{c_\epsilon}{2\pi\sqrt{v}}\big(1+O(\epsilon v)\big). \tag{51}$$

Along every joint limit $\epsilon \downarrow 0$, $t\to\infty$,

$$D_\epsilon(t) \sim \frac{\epsilon^{d+1}|\phi(y_0)|^2\mathcal{T}(K_0)}{2\sqrt{\pi t}}\, e^{-at}. \tag{52}$$

Let $\beta > 0$ be fixed and $\eta = a - \mu > 0$, where μ is the chemical potential. The relative mean occupation of the ideal Bose gas satisfies

$$\mathcal{N}_{\epsilon,\beta}(a-\eta) = \frac{c_\epsilon}{2\beta\sqrt{\eta}} + O_\beta(\epsilon^{d+1}), \tag{53}$$

uniformly for small ϵ and η. In particular, along every simultaneous limit $\epsilon, \eta \downarrow 0$,

$$\mathcal{N}_{\epsilon,\beta}(a-\eta) \sim \frac{\epsilon^{d+1}|\phi(y_0)|^2\mathcal{T}(K_0)}{2\beta\sqrt{\eta}}.$$

Proof. We have $W_\epsilon^* f_\epsilon = \epsilon^{d/2}\phi(y_0 + \epsilon\,\cdot)$. By (45) at $z = -a$ and interior C^1 regularity of ϕ, the rescaled inverse quadratic form converges to $|\phi(y_0)|^2\mathcal{T}(K_0)$ with error $O(\epsilon)$. Positivity of its limit gives (50).

The derivative $R_\epsilon'(z)$ is a bounded operator of norm $O(1)$ uniformly on the real segment, since all its transverse levels are separated from zero. The same segment has $\|R_\epsilon(z)^{-1}\| = O(\epsilon)$ by bounded perturbation from $z = -a$. Differentiation of the inverse gives $|\alpha_\epsilon'(z)| = O(\epsilon^2\|f_\epsilon\|^2) = O(\epsilon^{d+2})$. On the upper boundary at $z = -a - v$, the only open channel adds $i\sqrt{v}\,|f_\epsilon\rangle\langle f_\epsilon|$ to $R_\epsilon(z)$. Inverting this rank-one addition and substituting in (43) gives

$$\nu_\epsilon'(a+v) = \frac{\alpha_\epsilon(z)/(2\sqrt{v}) - \sqrt{v}\,\alpha_\epsilon'(z)}{\pi(1 + v\alpha_\epsilon(z)^2)}. \tag{54}$$

In detail, $J_\epsilon^* f_\epsilon = (1 - i\sqrt{v}\,\alpha_\epsilon)^{-1}R_\epsilon^{-1}f_\epsilon$. In the first trace of (43), $-\dot{U} = R_\epsilon'(z)$, giving the term $-\sqrt{v}\,\alpha_\epsilon'(z)/(1 + v\alpha_\epsilon^2)$. The second trace gives the real part of $\langle f_\epsilon, J_\epsilon f_\epsilon\rangle/(2\sqrt{v})$, which is the other term of (54). There is no threshold atom by Section 14. Since $\alpha_\epsilon(z) = c_\epsilon + O(\epsilon^{d+2}v)$ and $c_\epsilon \asymp \epsilon^{d+1}$, the relative band estimate (51) follows.

For large t, integration of (51) gives

$$\int_0^\delta e^{-tv}\nu_\epsilon'(a+v)\,dv = \frac{c_\epsilon}{2\sqrt{\pi t}}(1 + O(\epsilon/t) + O(e^{-\delta t/2})).$$

The weighted mass bound in Section 13 gives $D_\epsilon(1) = O(\epsilon^{d+1})$. The part of $e^{at}D_\epsilon(t)$ from $u \geq a + \delta$ is consequently $O(\epsilon^{d+1}e^{-\delta(t-1)})$ for $t \geq 1$. Dividing by the main scale $\epsilon^{d+1}t^{-1/2}$ proves (52) without a relation between t and ϵ.

The relative occupation is the cap limit at fixed $\beta, \mu < a$ of the difference of the connected and cut grand-canonical mean populations. Its measure representation is

$$\mathcal{N}_{\epsilon,\beta}(\mu) = \int \frac{d\nu_\epsilon(u)}{e^{\beta(u-\mu)} - 1} = \sum_{\ell \geq 1} e^{\ell\beta\mu}D_\epsilon(\ell\beta). \tag{55}$$

To justify the cap exchange, both capped spectra are bounded below by a by the transverse Poincaré inequality. By min–max the connected eigenvalues are no greater than their cut counterparts, since the contact term vanishes on the cut form domain. Integrating each paired eigenvalue gap against te^{-tu} bounds the capped trace difference for $t \geq 1$ by $te^{-(t-1)a}$ times the time-one difference. The time-one difference is uniformly bounded in cap length by the enclosing-cylinder heat bound. For fixed $\mu < a$, this permits summing the cap limit over the Bose series. Tonelli and (31) give the measure formula.

For $v \in (0, \delta)$, the difference between the Bose factor at energy $a + v$ and $[\beta(v + \eta)]^{-1}$ stays bounded as $\eta \to 0$. The density error in (51) contributes $O_\beta(c_\epsilon\epsilon)$ against the latter

bound. Extending its leading integral from $(0,\delta)$ to $(0,\infty)$ changes it by $O_\beta(c_\epsilon)$. The identity $\int_0^\infty v^{-1/2}(v+\eta)^{-1}\,dv = \pi/\sqrt{\eta}$ gives the leading coefficient. Above the first band, the Bose factor is bounded by $C_\beta(1+u)^{-m}$, uniformly for small η. The weighted mass bound gives an $O_\beta(\epsilon^{d+1})$ contribution there. These estimates yield (53). □

The leading singularity in (53) quantifies the change in grand-canonical mean occupancy caused by the junction at the continuum threshold.

16 Shape dependence of the leading coefficient

The shape factor $\mathcal{T}(K_0)$ is half the torsional rigidity of the exterior Dirichlet fractional Laplacian of order $1/2$. For bounded Lipschitz sets, the fractional Saint-Venant comparison and the ball torsion function follow from Brandolini, de Bonis, Ferone, Piscitelli, and Volzone [22, Proposition 3.4 and (3.10)]. We give the comparison for the arbitrary-open-set form convention used here.

The inverse form principle writes $\mathcal{T}(K_0)$ as the supremum of $2\,\mathrm{Re}\langle \mathbf{1}, g\rangle - 2q(g)$ over the fractional form domain. Replacing g by its modulus does not increase the fractional energy. For a nonnegative g, symmetric decreasing rearrangement preserves its integral and does not increase q. In the heat representation of q, the integrand is a positive multiple of $\|g\|^2 - \langle g, e^{t\Delta} g\rangle$, integrated against $t^{-3/2}\,dt$. The Gaussian Riesz rearrangement inequality increases the heat inner product, proving the energy comparison. The rearranged function is supported in the closed ball of volume $|K_0|$. An inward dilation and then mollification approximate it in the fractional form norm by compactly supported smooth tests of that ball. Maximizing thus proves

$$\mathcal{T}(K_0) \le \mathcal{T}(B_R), \qquad |B_R| = |K_0|.$$

In dimension d, Dyda's power-function formula [23, Theorem 1] gives

$$(-\Delta)^{1/2}(1-|y|^2)_+^{1/2} = \sqrt{\pi}\frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \qquad (|y|<1).$$

The power function belongs to the closed ball form domain. Its squared L^2 translation difference at displacement h is $O(|h|^2(1+|\log|h||))$: on a boundary strip of thickness $O(|h|)$ use its square-root vanishing, and away from that strip integrate the squared gradient bound $C\,\mathrm{dist}(y,\partial B_1)^{-1}$. The translation bound is integrable against the $H^{1/2}$ kernel. Inward dilation and mollification then give the requisite core approximation. Dividing the power function by twice the constant above gives the ball solution of $2Fg = \mathbf{1}$. Integrating it and scaling gives

$$\mathcal{T}(B_R) = R^{d+1}\frac{\pi^{d/2}\Gamma(d/2)}{4\,\Gamma((d+1)/2)\Gamma((d+3)/2)}. \tag{56}$$

For a transverse interval $(-R,R)$, this gives $\pi R^2/4$. Thus the equal-volume ball maximizes the common aperture factor in the full limiting measure (34).

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References

[1] J. Behrndt, F. Gesztesy, and S. Nakamura, *Spectral shift functions and Dirichlet-to-Neumann maps*, Math. Ann. **371** (2018), 1255–1300. doi:10.1007/s00208-017-1593-4.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987.

[3] S. A. Franchino-Viñas, *Resummed heat-kernel and form factors for surface contributions: Dirichlet semitransparent boundary conditions*, J. Phys. A: Math. Theor. **56** (2023), 115202. doi:10.1088/1751-8121/acbd26.

[4] A. Gomilko, M. Haase, and Y. Tomilov, *Bernstein functions and rates in mean ergodic theorems for operator semigroups*, J. Anal. Math. **118** (2012), 545–576. doi:10.1007/s11854-012-0044-0.

[5] P. Mnev and K. Wernli, *Gluing formulae for heat kernels*, J. Geom. Phys. **216** (2025), 105594. doi:10.1016/j.geomphys.2025.105594.

[6] S. Richard and R. Tiedra de Aldecoa, *Resolvent expansions and continuity of the scattering matrix at embedded thresholds: the case of quantum waveguides*, Bull. Soc. Math. France **144** (2016), 251–277. doi:10.24033/bsmf.2713.

[7] S. Schubr, *Audibility of all odd blind modes in rapidly twisted polar-strip tubes*, Luminary Research Archive, edition 5, 2026. Archive edition.

[8] S. Schubr, *Bose gases in weakly twisted annular-sector waveguides*, Luminary Research Archive, edition 4, 2026. Archive manuscript.

[9] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005. doi:10.1090/surv/120.

[10] A. Hanyga, *On solutions of matrix-valued convolution equations, anisotropic fractional derivatives and their applications in linear and non-linear anisotropic viscoelasticity*, arXiv:2106.07946, 2021.

[11] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions: Theory and Applications*, 2nd ed., De Gruyter, Berlin, Boston, 2012. doi:10.1515/9783110269338.

[12] J. Behrndt, M. M. Malamud, and H. Neidhardt, *Scattering matrices and Weyl functions*, Proc. Lond. Math. Soc. **97** (2008), 568–598. doi:10.1112/plms/pdn016.

[13] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260. arXiv:math/0608640.

[14] P. R. Stinga and J. L. Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), 2092–2122. arXiv:0910.2569.

[15] S. B. Allan, J. H. Kim, G. Michajlyszyn, R. Nichols, and D. Rung, *Explicit Krein resolvent identities for singular Sturm–Liouville operators with applications to Bessel operators*, Oper. Matrices **14** (2020), 1043–1099. Full text.

[16] S. Demirel, *The spectral shift function and Levinson's theorem for quantum star graphs*, arXiv:1205.1772, 2012.

[17] L. Bruneau, J. Dereziński, and V. Georgescu, *Homogeneous Schrödinger operators on half-line*, arXiv:0911.5569.

[18] Z. Zellak and G. Mutlu, *Self-adjoint extensions of quantum graphs with eigenparameter-dependent vertex conditions: An operator-theoretic and symplectic geometry approach*, AIMS Math. **11** (2026), 17465–17491. doi:10.3934/math.2026714.

[19] F. W. J. Olver et al., eds., *NIST Digital Library of Mathematical Functions*, National Institute of Standards and Technology, Section 10.21.

[20] L. Abatangelo, V. Felli, and S. Terracini, *On the sharp effect of attaching a thin handle on the spectral rate of convergence*, arXiv:1307.7930v1, 2013.

[21] L. Abatangelo and R. Ognibene, *Sharp behavior of Dirichlet–Laplacian eigenvalues for a class of singularly perturbed problems*, arXiv:2301.11729v2, 2023.

[22] B. Brandolini, I. de Bonis, V. Ferone, G. Piscitelli, and B. Volzone, *First eigenvalue and torsional rigidity: Isoperimetric inequalities for the fractional Laplacian*, arXiv:2512.17400v1, 2025.

[23] B. Dyda, *Fractional calculus for power functions*, arXiv:1103.3387v2, 2011.

[24] P. Exner and S. A. Vugalter, *Asymptotic estimates for bound states in quantum waveguides coupled laterally through a narrow window*, Ann. Inst. H. Poincaré Phys. Théor. **65** (1996), 109–123. Full text.

[25] V. Felli, B. Noris, and R. Ognibene, *Eigenvalues of the Laplacian with moving mixed boundary conditions: The case of disappearing Neumann region*, arXiv:2107.03862v3, 2022.