Positive Operator Cocycles in Shrinking Corridors

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Abstract

We characterize convergence in trace norm for positive operator sequences whose cocycle subadditivity is imposed only on nearly equal block lengths. For a positive trace-preserving map on trace class, every normalized sequence converges precisely when the map is strongly mean ergodic and the scalar corridor has divergent square-width integral. We also obtain an exact robustness criterion: for a nonincreasing normalized trace-error profile, universal convergence holds precisely when the same two conditions hold and the error profile is integrable on logarithmic scale. A positive stationary oscillation proves necessity, including profiles with jumps. The scalar width and summability principles have classical or previously established origins; the operator transfer uses recurrent-support absorption, compact positive orbits and a regularized root-fidelity energy that pays order slack linearly by trace. The criterion includes absorbing channels with nonfaithful stationary states. Finite-dimensional selfadjoint sequences satisfy a complementary trace criterion, and unitary dynamics admit an exact classification by pure-point spectrum and corridor width. All corridor inequalities are required in both orders, with measurable nonincreasing width and eventual access to adjacent integer pairs.

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1 Restricted subadditivity and the convergence question

Let $\mathcal{H} \neq \{0\}$ be separable and let $\mathcal{S}_1(\mathcal{H})$ denote its trace class. Let Φ be a bounded positive trace-preserving linear map on $\mathcal{S}_1(\mathcal{H})$. We study sequences satisfying

$$A_{n+m} \leq A_n + \Phi^n(A_m) \tag{1}$$

for sufficiently large ordered pairs with

$$|m - n| \leq \min(m, n) w(\log \min(m, n)). \tag{2}$$

Here $w : [T, \infty) \to [0, 1]$ is measurable and nonincreasing, and $nw(\log n) \geq 2$ eventually. The order is Loewner order; both orders of every admissible pair are required. Write $X_n = A_n/n$ and

$$\int_T^\infty w(t)^2 \, dt = \infty. \tag{3}$$

Our main theorem identifies precisely the maps for which every positive corridor sequence has convergent slopes: strong mean ergodicity of Φ, together with (3), is necessary and sufficient. A property tested by additive averages therefore controls the larger class of ordered restricted-subadditive inputs.

The same question has an exact answer for accumulated order errors. For a finite-valued nonincreasing profile $\epsilon : [1,\infty) \to [0,\infty)$, allow the positive part of each order defect to have trace at most $(n+m)\epsilon(n+m)$. Theorem 3.1 shows that universal convergence is equivalent to the same map and width conditions together with

$$\int_1^\infty \epsilon(t)\,\frac{dt}{t} < \infty.$$

Thus the map controls escape of mass, the corridor controls legal comparisons, and the logarithmic error integral controls accumulation across scales. For example, profiles $(\log(et))^{-p}$ with $p \geq 0$ have the exact threshold $p > 1$. This robustness conclusion keeps the errors in their original operator directions.

Scalar input and operator transfer. The scalar corridor theorem [9] supplies the width threshold. Eventual real subadditivity on (2) and (3) imply $a_n/n \to L \in \mathbb{R}\cup\{-\infty\}$. If the square integral is finite, it supplies a nonnegative linearly bounded counterexample with nonconvergent slopes. The integer condition admits the balanced and adjacent splits used below.

Our operator argument first extracts a stationary density faithful on the recurrent support. Mean ergodicity forces absorption into that support; balanced legal expansions transfer compactness and absorption to arbitrary corridor slopes. A regularized root-fidelity energy prices positive order slack linearly by trace. Its strict convexity on the limiting support forces asymptotic invariance, and invariant observables determine the limit. This includes transient absorption when a faithful stationary density on the entire space is unavailable.

Classical predecessors and the restricted-pair increment. Derriennic and Krengel [2, Theorems 4.6 and 5.2] are direct antecedents. Their Theorem 4.6 transfers convergence of positive additive averages to all-pair positive subadditive processes in uniformly monotone Banach lattices. Its first-input domination and lattice weak-compactness argument are unavailable for a shrinking corridor. Their Theorem 5.2 gives commutative L^1 convergence for a positive contraction with a strictly positive invariant element; its fixed-small-block concatenations likewise require pairs excluded by (2). Trace-class Loewner order is not a Banach-lattice order. Jajte [4] develops noncommutative subadditive ergodic theory with all-pair inequalities for trace-preserving automorphisms. The present contribution is the restricted-pair noncommutative transfer: balanced legal leaves, absorption and trace-priced fidelity recover convergence in the original trace norm, with strong mean ergodicity as the exact map condition.

The commutative Markov lift in [10] gives near-diagonal L^p convergence under its uniform-integrability criterion. Its balanced-leaf and strict-convexity synchronization motivate the present extension; trace-class order and nonfaithful stationary support require the confinement and fidelity arguments developed here.

Füredi and Ruzsa [3] study scalar restricted subadditivity and defects. Kulikov and Shao [6] prove a Banach-valued norm version of Fekete's lemma and analyze comparable-pair difficulties. Untwisted finite matrix order also scalarizes through quadratic forms and polarization [7]. The twisted restricted inequality (1) is handled here by confinement and trace-paid order slack.

Operator tools and consequences. Positive trace-preserving fidelity monotonicity is classical [12]. Absorption into recurrent domains, normal ergodic projections and stationary-support decompositions are established themes in quantum Markov dynamics [1, Section 4]. Our proof supplies the elementary merely-positive-map support argument needed for the corridor transfer. Related T-admissible processes [8] have a different prescribed input structure.

The main criterion yields faithful-state and reducing-corner cases and absorbing channels beyond both sufficient hypotheses. We also give a finite-dimensional selfadjoint convergence criterion and an exact unitary spectral/width classification. Explicit examples exhibit the roles of trace preservation and infinite-dimensional escape.

2 A mean-ergodic characterization

Say that Φ is strongly mean ergodic if $C_N A = N^{-1}\sum_{j<N}\Phi^j A$ converges in trace norm for every $A \in \mathcal{S}_1$. Write $\mathrm{UC}(\Phi, w)$ for trace-norm convergence of every positive corridor slope A_n/n.

Theorem 2.1. *For w as in Section 1, $\mathrm{UC}(\Phi, w)$ holds if and only if Φ is strongly mean ergodic and (3) holds. Every limit F is positive, satisfies $\Phi F = F$, and has trace $\lim_n \operatorname{Tr} X_n$.*

2.1 Stationary support and confinement

Positivity and trace preservation give $\|\Phi H\|_1 \le \operatorname{Tr} H_+ + \operatorname{Tr} H_- = \|H\|_1$ for selfadjoint H. For $B \ge 0$ and an orthogonal projection R we use

$$\|B - RBR\|_1 \le 2\sqrt{\operatorname{Tr} B \operatorname{Tr}((I-R)B)}. \tag{4}$$

Indeed $B - RBR = (I-R)B + RB(I-R)$; factor each term through $B^{1/2}$ and apply the Hilbert–Schmidt product inequality.

Lemma 2.2. *A faithful stationary density implies relative compactness of every positive orbit in trace norm.*

Proof. For finite spectral projections R_r of the density σ, $R_r A R_r \to A$ in trace norm and $R_r A R_r \le c_r\sigma$. The whole orbit of each compression lies in $[0, c_r\sigma]$. This interval is compact: its tails are bounded by $c_r \operatorname{Tr}((I-R_s)\sigma)$, and (4) reduces compactness to bounded finite-dimensional corners. Contraction propagates the compression approximation uniformly along the orbit. □

Lemma 2.3. *Strong mean ergodicity is equivalent to relative compactness of every positive orbit. In that case the mean projection $P = \lim_N C_N$ has range supported on $R = \operatorname{supp}\sigma$ for a stationary density σ. With $Q = I - R$,*

$$\Phi^* Q \le Q, \qquad \operatorname{Tr}(Q\Phi^j A) \longrightarrow 0 \quad (A \ge 0). \tag{5}$$

Proof. If the means converge, their limit P is positive and trace-preserving with $P\Phi = \Phi P = P$. Choose a faithful density ρ and set $\sigma = P\rho$. Its support is R. The finite ρ-spectral compressions A_r of $A \ge 0$ satisfy $A_r \le c_r\rho$ and converge to A. Hence $PA_r \le c_r\sigma$ and PA is R-supported.

Using instead spectral corners of σ proves that Φ preserves R-supported inputs. The positive unital dual satisfies $0 \le \Phi^* Q \le I$ with zero R-corner. Positivity forces zero cross corners, so $\Phi^* Q \le Q$. Thus $q_j = \operatorname{Tr}(Q\Phi^j A)$ decreases, and its Cesaro means tend to $\operatorname{Tr}(QPA) = 0$. Therefore $q_j \to 0$.

On $R\mathcal{H}$ the restricted map has faithful stationary density σ. Lemma 2.2 makes the orbit of $R\Phi^N AR$ compact. Equation (4) approximates $\Phi^N A$ by that input as $N \to \infty$; contraction propagates the approximation to all later iterates. This proves compactness of the original positive orbit.

Conversely, compact positive orbits give compact closed convex hulls for their Cesaro means. A finite epsilon-net reduces such a hull to an approximating finite-dimensional compact simplex.

Every cluster point is fixed, since $\|\Phi C_N A - C_N A\|_1 \le 2\operatorname{Tr} A/N$. Each $C_N A - A$ belongs to $\operatorname{Ran}(I-\Phi)$, and

$$\operatorname{Fix}\Phi \cap \overline{\operatorname{Ran}(I-\Phi)} = \{0\} :$$

the bounded Cesaro means vanish on the closed range and equal the identity on fixed vectors. All cluster points agree. Jordan decomposition and complex linearity extend convergence to every trace-class input. □

Lemma 2.4. *Under strong mean ergodicity and* (3)*,* (X_n) *and its full orbit family* $\{\Phi^j X_n : j \ge 0, n \ge 1\}$ *are relatively compact. Moreover*

$$\operatorname{Tr}(QX_n) \to 0, \qquad \sup_{j\ge 0} \operatorname{Tr}(Q\Phi^j X_n) \to 0. \tag{6}$$

Proof. The scalar theorem gives $\operatorname{Tr} X_n \to L \ge 0$ and bounded traces. Balanced legal splitting into consecutive leaves gives

$$X_N \le Y_N = \sum_i (k_i/N)\Phi^{b_i} X_{k_i}, \qquad K \le k_i \le 2K, \quad b_i = \sum_{h<i} k_h. \tag{7}$$

For fixed sufficiently large K every child gap is at most one. The finite union of leaf orbits has compact closure and compact closed convex hull. The positive slack satisfies

$$\|Y_N - X_N\|_1 \le \max_{K\le k\le 2K} \operatorname{Tr} X_k - \operatorname{Tr} X_N.$$

Choosing K and then N large proves compactness of the slopes. A finite positive net for the slopes, contraction and compactness of each net point's orbit give compactness of the full orbit family.

For fixed K, (5) gives $\operatorname{Tr}(Q\Phi^b X_k) \le \eta$ for $b \ge J$, uniformly over the finite leaf set. Leaves with $b_i < J$ have total length at most $J + 2K$. If C bounds their traces, (7) gives

$$\operatorname{Tr}(QX_N) \le C(J+2K)/N + \eta.$$

Let $N \to \infty$ and then $\eta \to 0$. The uniform orbit assertion follows from $\Phi^* Q \le Q$. □

2.2 A support-sensitive fidelity energy

In the following lemma σ is the stationary density above, faithful on $R\mathcal{H}$.

Lemma 2.5. *For each* $\varepsilon > 0$*, the functional*

$$E(X) = -\operatorname{Tr}\sqrt{\sigma^{1/2}(X+\varepsilon\sigma)\sigma^{1/2}}, \qquad X \ge 0,$$

is continuous on trace-bounded positive sets, strictly convex on positive inputs supported on $R = \operatorname{supp}\sigma$*, and contractive in the sense* $E(\Phi X) \le E(X)$*. Moreover, if* $Y \ge X \ge 0$*, then*

$$0 \le E(X) - E(Y) \le c_\varepsilon \operatorname{Tr}(Y - X), \qquad c_\varepsilon = (2\sqrt{\varepsilon})^{-1}. \tag{8}$$

Proof. The quantity whose negative defines E is root fidelity, not squared fidelity [11, 5]. Its monotonicity needs only positivity and trace preservation [12, Sections 3.2–3.3]. Explicitly, for positive trace-class A, B,

$$\operatorname{Fid}(A,B) = \frac{1}{2}\inf_{Z>0}\bigl(\operatorname{Tr}(AZ) + \operatorname{Tr}(BZ^{-1})\bigr),$$

where Z ranges over bounded positive invertible operators. Let $\Psi = \Phi^*$, a positive unital map. Its inverse-Jensen inequality gives $\Psi(Z^{-1}) \geq \Psi(Z)^{-1}$, so

$$\begin{aligned}\operatorname{Tr}(\Phi(A)Z) + \operatorname{Tr}(\Phi(B)Z^{-1}) &\geq \operatorname{Tr}(A\Psi(Z)) + \operatorname{Tr}(B\Psi(Z)^{-1}) \\ &\geq 2\operatorname{Fid}(A,B).\end{aligned}$$

Taking the infimum proves monotonicity. The inverse-Jensen inequality for a single Z follows by restricting to its generated commutative algebra and applying the operator-convex inverse inequality. The general variational formula is valid for positive normal functionals and does not require bounded inverses of A or B. Stationarity now gives contraction of E.

Trace-norm continuity follows from the Hilbert–Schmidt square-root inequality and $\operatorname{Fid}(A,B) = \|\sqrt{A}\sqrt{B}\|_1$. Trace concavity of the square root gives convexity. It is strict on distinct sandwich arguments: the resolvent integral for the square root expresses the concavity gap as an integral of positive inverse-Jensen gaps, with equality only for equal arguments. On $R\mathcal{H}$ the sandwich map is injective because $\sigma^{1/2}$ has dense range there. Convexity on the whole space follows because the sandwich depends only on RXR. All traces in question are finite by $\operatorname{Fid}(X + \varepsilon\sigma, \sigma) \leq \sqrt{\operatorname{Tr} X + \varepsilon}$.

For completeness the linear price does not require a bounded inverse of σ. In a finite σ-spectral corner inside R, differentiate the trace square root along $X + t(Y - X)$. Its argument is at least $\varepsilon\sigma^2$. Operator monotonicity of inverse square roots therefore gives

$$\sigma^{1/2}\big[\sigma^{1/2}(X + t(Y - X) + \varepsilon\sigma)\sigma^{1/2}\big]^{-1/2}\sigma^{1/2} \leq \varepsilon^{-1/2} I.$$

The derivative is half the trace pairing with $R(Y - X)R$, whose trace is at most $\operatorname{Tr}(Y - X)$. No inverse on $R^\perp$ is used. Integration gives (8), uniformly in the corner. Sigma-spectral compression and trace-norm continuity of fidelity pass to the full space. Order monotonicity gives the lower inequality. □

2.3 Proof of the characterization

Proof of Theorem 2.1. Assume strong mean ergodicity and (3). For an admissible pair put $\alpha = n/(n+m)$ and $Y = \alpha X_n + (1-\alpha)\Phi^n X_m \geq X_{n+m}$. Price, convexity and contraction show that

$$b_n = nE(X_n) + c_\varepsilon \operatorname{Tr} A_n$$

is exactly scalar corridor-subadditive. Its normalized slopes are bounded, so the scalar theorem and trace convergence give $E(X_n) \to \ell \in \mathbb{R}$.

On the three pairs $(n,n), (n,n+1), (n+1,n)$ the trace slack tends to zero. Also

$$\begin{aligned}E(X_{n+m}) - c_\varepsilon \operatorname{Tr}(Y - X_{n+m}) &\leq E(Y) \\ &\leq \alpha E(X_n) + (1-\alpha)E(\Phi^n X_m) \\ &\leq \alpha E(X_n) + (1-\alpha)E(X_m).\end{aligned}$$

Both extremes tend to ℓ, so the actual Jensen gap vanishes. The arguments lie in the compact orbit family of Lemma 2.4; all their cluster points as $n, m \to \infty$ are R-supported by (6). With $\alpha \to 1/2$, strict convexity there yields $\|X_n - \Phi^n X_m\|_1 \to 0$ on each pair. Otherwise a convergent subsequence would supply distinct R-supported arguments with zero gap.

Denote the three distances by r_n, s_n, t_n. Contractivity gives

$$\|X_n - \Phi^{2n} X_n\|_1 \leq 2r_n, \qquad \|X_n - \Phi^{2n+1} X_n\|_1 \leq s_n + t_n.$$

Comparison of consecutive powers yields $\|\Phi X_n - X_n\|_1 \le 2r_n + s_n + t_n \to 0$. Every cluster point is fixed. For each bounded positive observable B, P^*B is positive and Φ^*-fixed. Thus $\mathrm{Tr}((P^*B)A_n)$ is scalar corridor-subadditive with bounded nonnegative normalized slopes and has a limit. On a fixed cluster point F it equals $\mathrm{Tr}(BF)$. These observables separate selfadjoint trace class, so all cluster points agree. Compactness proves convergence; trace continuity identifies its trace.

Conversely, for each $B \ge 0$, the positive additive sequence $A_n = \sum_{j<n} \Phi^j B$ satisfies the full cocycle equality. Universal convergence implies mean ergodicity on positive inputs, hence on all trace class by linear decomposition. It supplies a stationary density $\sigma = P\rho$. If the square-width integral is finite, the inherited nonnegative scalar corridor counterexample a_n makes $A_n = a_n\sigma$ a nonconvergent positive corridor sequence. This proves both necessities for each fixed map. □

Corollary 2.6. *A positive trace-preserving map with a faithful stationary density has universal positive corridor convergence under* (3)*.*

Proof. Lemmas 2.2 and 2.3 give mean ergodicity; apply Theorem 2.1. □

3 A sharp trace-error criterion

Order errors are measured by their positive part: for a selfadjoint trace-class operator B, write $B_+ = (|B| + B)/2$. This measures the portion of the defect that violates the desired inequality. The scalar summability principle is classical nearly-subadditive theory [3, Sections 4–6]; the theorem below identifies its exact role in the fixed-map operator problem, including discontinuous normalized error profiles.

Theorem 3.1 (Map, width and error characterization)**.** *Let Φ be a bounded positive trace-preserving linear map on $\mathcal{S}_1(\mathcal{H})$, where $\mathcal{H} \ne \{0\}$ is separable. Let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing with $nw(\log n) \ge 2$ eventually, and let $\epsilon : [1, \infty) \to [0, \infty)$ be finite-valued and nonincreasing. Every positive trace-class sequence satisfying*

$$\mathrm{Tr}\Big[\big(A_{n+m} - A_n - \Phi^n A_m\big)_+\Big] \le (n+m)\epsilon(n+m) \tag{9}$$

on all sufficiently large ordered corridor pairs has convergent trace-norm slopes if and only if Φ is strongly mean ergodic, (3) *holds, and*

$$\int_1^\infty \epsilon(t)\,\frac{dt}{t} < \infty. \tag{10}$$

Under these conditions every limit is positive and Φ-fixed, with trace $\lim_n \mathrm{Tr}(A_n/n)$.

We first prove sufficiency in the equivalent form with an explicitly positive operator error. Necessity and the exact threshold follow afterward. Monotonicity gives

$$(\log 2)\epsilon(2^{j+1}) \le \int_{2^j}^{2^{j+1}} \epsilon(t)\,\frac{dt}{t} \le (\log 2)\epsilon(2^j),$$

so (10) is equivalent to convergence of the dyadic error sum.

Theorem 3.2. *Assume strong mean ergodicity and* (3)*. Let $\epsilon : [1, \infty) \to [0, \infty)$ be nonincreasing and suppose*

$$h(t) = \sum_{j \ge 0} \epsilon(2^j t) < \infty$$

for all sufficiently large t. Suppose positive A_n satisfy, on all sufficiently large ordered admissible pairs,

$$A_{n+m} \le A_n + \Phi^n A_m + D_{n,m}, \qquad D_{n,m} \ge 0, \quad \operatorname{Tr} D_{n,m} \le (n+m)\epsilon(n+m).$$

Then A_n/n converges in trace norm to a positive Φ-fixed operator, with trace $\lim_n \operatorname{Tr}(A_n/n)$.

Proof. The tail $h(t)$ tends to zero. For $n \le m \le 2n$ and $N = n + m$,

$$\begin{aligned} nh(n) + mh(m) - Nh(N) &\ge n[h(n) - h(2n)] \\ &= n\epsilon(n) \ge N\epsilon(N)/3. \end{aligned}$$

Thus $a_N \le a_n + a_m + cN\epsilon(N)$ is repaired to exact scalar corridor-subadditivity by adding $3cnh(n)$. Applied to traces, this gives $t_n = \operatorname{Tr} X_n \to L \ge 0$ and bounded traces.

Balanced expansion down to leaves in $[K, 2K]$ gives $X_N \le Y_N + Z_{N,K}$, where Y_N is the leaf sum in (7), $Z_{N,K} \ge 0$, and

$$\operatorname{Tr} Z_{N,K} \le h(K/2), \qquad K \ge 2. \tag{11}$$

To verify the bound, charge each internal-node error of size s to its descendant leaves with weights k_i/N, using that their lengths sum to s. On any leaf path the jth ancestor has size at least $2^j(K-1)+1 \ge 2^j K/2$. Each path therefore has total normalized charge at most $h(K/2)$; the leaf weights sum to one. Trace preservation pays every propagated error, including uneven tree depths.

The positive slack now gives

$$\|X_N - Y_N\|_1 \le \max_{K \le k \le 2K} t_k - t_N + 2h(K/2).$$

The compact-hull argument of Lemma 2.4 proves compactness of the slopes and their full orbit family. With R, Q from Lemma 2.3, its absorption argument gives

$$\operatorname{Tr}(QX_N) \le C(J + 2K)/N + \eta + h(K/2).$$

First let $N \to \infty$, then $\eta \to 0$, and finally $K \to \infty$. Thus (6) persists.

Fix a fidelity regularization parameter $\delta > 0$, distinct from ϵ, and use the energy E of Lemma 2.5, with $c_\delta = (2\sqrt{\delta})^{-1}$. For one pair put $Y_0 = (n/N)X_n + (m/N)\Phi^n X_m$ and $e = D_{n,m}/N$. Since $Y_0 + e \ge X_N$, order monotonicity and the price give

$$E(X_N) \le E(Y_0) + c_\delta[\operatorname{Tr} Y_0 + \operatorname{Tr} e - \operatorname{Tr} X_N].$$

Consequently $b_n = nE(X_n) + c_\delta \operatorname{Tr} A_n$ satisfies

$$b_N \le b_n + b_m + c_\delta N\epsilon(N).$$

The scalar tail repair makes this exact and proves convergence of $E(X_n)$. On the three adjacent pairs, $\operatorname{Tr} e \to 0$ and $\operatorname{Tr}(Y_0 + e - X_N) \to 0$, hence $\|X_N - Y_0\|_1 \to 0$. Continuity gives the same limiting energy for Y_0. Its Jensen gap tends to zero as in the proof of Theorem 2.1. Compactness, absorption and strict convexity on R imply asymptotic invariance of X_n.

Finally every positive invariant observable P^*B has scalar defect at most $\|B\|N\epsilon(N)$. The same tail repair gives convergence of its normalized values. Fixed cluster points are separated by these observables, proving uniqueness and full convergence. □

Proof of Theorem 3.1. If the three conditions hold, set

$$D_{n,m} = \left(A_{n+m} - A_n - \Phi^n A_m\right)_+.$$

The inequality $B \le B_+$ and (9) put the sequence within Theorem 3.2. The dyadic equivalence above supplies its error hypothesis and proves sufficiency with the stated limit.

Conversely, exact cocycles satisfy (9), so universal convergence and Theorem 2.1 force mean ergodicity and (3). In particular there is a stationary density $\sigma = P\rho$, with ρ any faithful density and P the mean projection. Suppose now that (10) fails. Define

$$\theta(x) = \frac{1}{\log 3} \int_1^x \epsilon(3t)\, \frac{dt}{t}, \qquad f(x) = 2 + \sin\theta(x), \qquad x \ge 1.$$

The phase is continuous, nondecreasing and unbounded. Its integer increments satisfy

$$0 \le \theta(n+1) - \theta(n) \le \frac{\epsilon(3)}{n \log 3} \longrightarrow 0.$$

For any sequence of target phases tending to infinity, choose the first integer at which θ exceeds that phase. The overshoot tends to zero. Applying this to phases $\pi/2 + 2\pi k$ and $3\pi/2 + 2\pi k$ proves

$$\liminf_n f(n) = 1, \qquad \limsup_n f(n) = 3.$$

Let $n \le m \le 2n$ and $N = n + m$. For $u \in [n, N]$ one has $3u \ge N$; hence monotonicity, the Lipschitz bound for sine and $N/n \le 3$ give

$$|f(N) - f(n)| \le \frac{\epsilon(N)}{\log 3} \log(N/n) \le \epsilon(N).$$

The same bound holds with m in place of n. Therefore

$$Nf(N) - nf(n) - mf(m) \le N\epsilon(N).$$

The positive sequence $A_n = nf(n)\sigma$ satisfies (9) on the entire comparable-pair window in both orders, since $\Phi\sigma = \sigma$ and $\operatorname{Tr}\sigma = 1$. Its normalized slopes have distinct limiting extrema. This contradicts universal convergence and proves necessity of (10). □

Corollary 3.3. *The same conclusion holds when the order inequality is written with a selfadjoint error $E_{n,m}$ satisfying $\operatorname{Tr}[(E_{n,m})_+] \le (n+m)\epsilon(n+m)$ and (10). In particular, for $\epsilon(t) = (\log(et))^{-p}$ with $p \ge 0$, universal convergence in this class holds exactly when the map and width conditions hold and $p > 1$.*

Proof. Replace $E_{n,m}$ by its positive part and apply Theorem 3.1. The integral test gives the asserted logarithmic threshold; the preceding stationary construction supplies necessity. □

4 Finite-dimensional positive maps

Theorem 4.1. *Let $\Phi : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be positive and trace-preserving. Let selfadjoint A_n satisfy (1) and assume (3). Then X_n converges in operator norm if and only if $\operatorname{Tr} X_n$ has a finite real limit. The limit is fixed by Φ. In particular, convergence is automatic if $A_n \ge 0$ eventually. Neither complete positivity nor unitality is required.*

Proof. For selfadjoint H, positivity and trace preservation give

$$\|\Phi H\|_1 \le \operatorname{Tr}\Phi H_+ + \operatorname{Tr}\Phi H_- = \|H\|_1.$$

Thus Φ is power-bounded on the real space of selfadjoint matrices. On this finite-dimensional space it admits an equivalent Hilbert norm $\|\cdot\|_*$ in which it is a contraction. Indeed unit-circle Jordan blocks are semisimple; choose a norm making the peripheral real rotation blocks isometric. On the stable spectral subspace put

$$\langle x, y\rangle_s = \sum_{j\ge 0}\langle \Phi^j x, \Phi^j y\rangle_0.$$

The series converges, and its squared norm decreases under Φ. Make the two subspaces orthogonal. The Cesaro means $C_J = J^{-1}\sum_{j<J}\Phi^j$ converge in operator norm to a positive trace-preserving projection P onto $\operatorname{Fix}\Phi$. Also, for a fixed c_*,

$$\|S\|_* \le c_* \operatorname{Tr} S \qquad (S \ge 0). \tag{12}$$

The ordinary Hilbert–Schmidt norm need not contract for nonunital Φ; the equivalent norm is needed.

Taking traces and applying the scalar input gives $\operatorname{Tr} X_n \to L \in \mathbb{R}\cup\{-\infty\}$. Suppose L is finite. Fix a sufficiently large K. Balanced splitting of N until all leaves lie in $[K, 2K]$ uses child gaps at most one. Iterating (1) gives

$$X_N \le Y_N := \sum_i \frac{k_i}{N}\Phi^{b_i} X_{k_i}, \qquad K \le k_i \le 2K, \quad \sum_i k_i = N, \tag{13}$$

where b_i are the consecutive leaf starts. Trace-norm contraction bounds $\|Y_N\|_1$ uniformly in N. Hence $\lambda_{\max}(X_N)$ is bounded above; the finite lower trace bound then bounds $\lambda_{\min}(X_N)$ below. Thus (X_n) is bounded in every matrix norm.

The sequence PA_n satisfies untwisted order subadditivity since $P\Phi^j = P$. Apply the scalar input to its quadratic forms at $e_i, e_i + e_j, e_i + \mathrm{i}e_j$. Boundedness makes all limits finite, and polarization yields $PX_n \to F$, where $\Phi F = F$ and $\operatorname{Tr} F = L$.

Put $f_n = X_n - F$,

$$\delta_K = \sup_{n\ge K} |\operatorname{Tr} X_n - L|, \qquad \epsilon_K = \sup_{n\ge K}\|PX_n - F\|_*.$$

The positive slack in (13) has trace at most $2\delta_K$. Consequently

$$\|C_J f_N\|_* \le \max_{K\le k\le 2K}\|C_J f_k\|_* + 2c_*\delta_K. \tag{14}$$

Choose N_j realizing $V = \limsup_n \|f_n\|_*^2$. For $t = 0, 1$ take $r_j = \lfloor N_j/2\rfloor + t$ and $s_j = N_j - r_j$. The child gap is at most two, so both orders are admissible. With $\alpha = r_j/N_j$ and $\beta = s_j/N_j$, trace convergence and (12) give

$$f_{N_j} - \alpha f_{r_j} - \beta\Phi^{r_j} f_{s_j} \longrightarrow 0.$$

The bracket has squared norm tending to V, whereas both child norms have squared limsup at most V. The Hilbert identity

$$\alpha\|x\|_*^2 + \beta\|y\|_*^2 - \|\alpha x + \beta y\|_*^2 = \alpha\beta\|x - y\|_*^2$$

forces the parent to approach both terms. The reversed inequality also makes it approach f_{s_j}. Contractivity therefore gives $\Phi^{r_j} f_{N_j} - f_{N_j} \to 0$. The two values of t provide consecutive exponents, so

$$\|\Phi f - f\|_* \le \|f - \Phi^r f\|_* + \|\Phi^{r+1} f - f\|_*$$

implies $\Phi f_{N_j} - f_{N_j} \to 0$. For fixed J, telescoping gives $C_J f_{N_j} - f_{N_j} \to 0$. Apply (14), let $j \to \infty$, then $J \to \infty$ at fixed K, and finally $K \to \infty$. This gives

$$\limsup_j \|f_{N_j}\|_* \le \epsilon_K + 2c_*\delta_K \longrightarrow 0.$$

Thus $V = 0$ and the entire sequence converges. The converse follows from continuity of trace. For positive A_n, the scalar limit cannot be $-\infty$, and the scalar theorem excludes $+\infty$; hence it is finite and $F \ge 0$. □

Corollary 4.2. *The same conclusion holds for a positive map admitting a faithful invariant functional $\ell(A) = \mathrm{Tr}(SA)$ with $S > 0$, with $\ell(X_n)$ replacing the trace criterion.*

Proof. Set $B_n = S^{1/2} A_n S^{1/2}$ and

$$\Psi(B) = S^{1/2} \Phi(S^{-1/2} B S^{-1/2}) S^{1/2}.$$

Then Ψ is positive and trace-preserving, and congruence preserves (1). Apply Theorem 4.1 and invert the congruence. □

5 Trace tightness from reducing corners

Theorem 5.1. *Let $\Phi : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ be positive and trace-preserving. Suppose increasing finite-rank projections $P_r \to I$ strongly satisfy*

$$\Phi(P_r A P_r) = P_r \Phi(A) P_r \quad (A \in \mathcal{S}_1(\mathcal{H})). \tag{15}$$

Under (3)*, every positive trace-class sequence satisfying* (1) *has $X_n \to F$ in trace norm, with $F \ge 0$ and $\Phi F = F$.*

Proof. Taking traces and using the scalar theorem gives $\mathrm{Tr}\, X_n \to L \ge 0$ and a uniform trace bound C. The leaf inequality (13) remains valid in the original trace class. By (15) and trace preservation,

$$\mathrm{Tr}(P_r \Phi A) = \mathrm{Tr}\, \Phi(P_r A P_r) = \mathrm{Tr}(P_r A).$$

The same holds for every power and for $I - P_r$. Thus for fixed legal K,

$$\mathrm{Tr}((I - P_r) X_N) \le \max_{K \le k \le 2K} \mathrm{Tr}((I - P_r) X_k), \qquad N \ge K. \tag{16}$$

The right side tends to zero as $r \to \infty$. A finite prefix can be included, so the full sequence is uniformly trace-tight.

For positive trace-class X and an orthogonal projection P,

$$\|X - PXP\|_1 \le 2\sqrt{\mathrm{Tr}\, X \ \mathrm{Tr}((I - P)X)}. \tag{17}$$

Indeed $X - PXP = (I-P)X + PX(I-P)$, and factoring each term through $X^{1/2}$ gives the bound by the Hilbert–Schmidt product inequality. This controls off-diagonal coherence without assuming X is block diagonal.

The compressed sequence satisfies the finite-dimensional theorem for the restriction of Φ to $P_r\mathcal{H}$. Each fixed compression therefore converges. Equations (16)–(17) show that (X_n) is trace-norm Cauchy. Its limit is positive and has trace L. Writing $F_r = P_rFP_r$, fixedness of every compressed limit gives

$$P_r(\Phi F - F)P_r = 0.$$

Finite corners recover every trace-class operator in trace norm, so $\Phi F = F$. □

Condition (15) is stronger than invariance of a chosen subspace. It is an identity for the whole map and its source coupling. A numerical finite-section approximation alone does not establish it. The theorem permits coherences in A_n across the reducing sectors and requires no prior uniform-integrability hypothesis.

6 Exact unitary classification

Let $\mathcal{H} \neq \{0\}$ be separable and let U be unitary. Fix a nonincreasing measurable $w : [T, \infty) \to [0, 1]$ with $nw(\log n) \geq 2$ eventually, without yet assuming its square integral diverges. Say that $\mathrm{UC}(U, w)$ holds if every positive trace-class sequence satisfying (1) for $\Phi = \mathrm{Ad}_U$ converges after division by n in trace norm.

Theorem 6.1. *The property* $\mathrm{UC}(U, w)$ *holds if and only if* U *has a complete orthonormal eigenbasis and* $\int_T^\infty w(t)^2\,dt = \infty$. *In that case the limit commutes with* U *and has trace* $\lim_n \operatorname{Tr} X_n$.

Proof. If U has a complete eigenbasis, the projections onto its first r basis vectors commute with U and satisfy (15). Theorem 5.1 proves sufficiency.

If U lacks a complete eigenbasis, choose a unit vector x in its nonzero continuous spectral subspace. Its probability spectral measure μ on the unit circle is nonatomic. Set

$$A_n = \sum_{k=0}^{n-1} U^k|x\rangle\langle x|U^{-k}, \qquad X_n = A_n/n.$$

These operators are positive, $\operatorname{Tr} X_n = 1$, and the full cocycle equality holds for every pair. Expansion by the spectral theorem gives

$$\begin{aligned}\|X_n\|_2^2 &= \frac{1}{n^2}\sum_{k,l<n} |\langle U^k x, U^l x\rangle|^2 \\ &= \iint \left|\frac{1}{n}\sum_{k<n}(z\overline{v})^k\right|^2 d\mu(z)d\mu(v) \longrightarrow 0.\end{aligned}$$

The integrand is bounded by one and tends to zero off the diagonal; the diagonal has product measure zero because μ has no atoms. Thus $\|X_n\|_{\mathrm{op}} \to 0$. A trace-norm convergent subsequence would have limit zero but trace one, a contradiction. Therefore no corridor width can repair continuous spectral escape.

Finally suppose U is pure point and $\int w^2 < \infty$. Choose a normalized eigenvector and its rank-one projection B. Then $U^nBU^{-n} = B$. The positive scalar counterexample from the scalar input yields $A_n = a_nB$ satisfying both ordered corridor inequalities with nonconvergent normalized slopes. This proves necessity for every fixed pure-point U, not merely for the identity map. □

Corollary 6.2. *A unitary* U *is pure point if and only if for every unit vector* x *the rank-one additive means* $n^{-1}\sum_{k<n} U^k|x\rangle\langle x|U^{-k}$ *converge in trace norm.*

Proof. Use Theorem 5.1 with $w = 1$ for the forward implication and the continuous spectral construction above for the converse. □

The ordinary rank-one Cesaro characterization and its spectral tools are classical. Theorem 6.1 identifies their role in the universal restricted-pair problem. Pure point here means spectral type, not discreteness of the spectral set or compactness of a resolvent. For a selfadjoint H_0 with a complete eigenbasis, the conclusion applies to $U = e^{-itH_0}$ at each fixed t, even when H_0 has essential spectrum. The finite projections are basis-index projections; they need not be finite-rank energy cuts.

7 Absorbing channels with nonfaithful stationary states

On $\ell^2(\{0, 1, \ldots\})$ choose $p_j \in (0, 1]$, $j \geq 1$, and define Kraus operators

$$K_0 e_0 = e_0, \qquad K_0 e_j = \sqrt{1 - p_j}\, e_j, \qquad K_j = \sqrt{p_j}\, |0\rangle\langle j|.$$

Their adjoint products sum strongly to I, defining a completely positive trace-preserving map. For every $A \geq 0$,

$$\mathrm{Tr}\big((I - |0\rangle\langle 0|)\Phi^n A\big) = \sum_{j \geq 1} (1 - p_j)^n \langle e_j, A e_j\rangle \longrightarrow 0.$$

Dominated convergence and (4) imply $\Phi^n A \to \mathrm{Tr}(A)|0\rangle\langle 0|$ in trace norm. In particular the map is strongly mean ergodic and its unique stationary density is the nonfaithful ground state. Theorems 2.1 and 3.2 therefore give, respectively, exact and trace-error corridor convergence to $L|0\rangle\langle 0|$.

This example also separates the criterion from a reducing finite-corner hypothesis. If a projection P satisfied (15), then $P|0\rangle\langle 0|P$ would be fixed, so either $Pe_0 = 0$ or $e_0 \in \mathrm{Ran}\, P$. A strongly exhaustive increasing family must eventually take the latter case. For any such proper finite P, choose a unit vector $x \in \ker P$. Its excited mass gives

$$\langle e_0, \Phi(|x\rangle\langle x|) e_0\rangle = \sum_{j \geq 1} p_j |\langle e_j, x\rangle|^2 > 0,$$

contradicting (15), whose left side vanishes on this input. Hence no reducing finite-corner exhaustion exists.

The parameters p_j may tend to zero, allowing slow absorption of individual excited levels. Independently, every decreasing $r_n \to 0$ gives the exact stationary cocycle $A_n = n(L + r_n)|0\rangle\langle 0|$. These examples exhibit convergence with arbitrarily slow normalized slopes under the qualitative criterion.

8 Explicit boundaries

8.1 Unitality does not replace a faithful invariant trace

On $M_3(\mathbb{C})$ define $\Phi(A) = a_{00} I_3$. This map is completely positive, unital and idempotent, with Kraus operators $|i\rangle\langle 0|$, $i = 0, 1, 2$, but it is not trace-preserving. Let

$$V_n = \cos(\log n)\sigma_z + \sin(\log n)\sigma_x, \quad X_n = 2 \oplus (I_2 + V_n/5), \quad A_n = nX_n.$$

The Pauli matrices obey $V_n^2 = I_2$. Thus $X_n > 0$, its eigenvalues are $2, 6/5, 4/5$, and $\mathrm{Tr}\, X_n = 4$ for every n. For $n/2 \leq m \leq 2n$, put $N = n + m$ and $\alpha = n/N \in [1/3, 2/3]$. Since $\Phi^n X_m = 2I_3$,

$$\frac{A_n + \Phi^n A_m}{N} = \alpha X_n + (1 - \alpha) 2 I_3.$$

Its lower block has minimum eigenvalue $2 - 6\alpha/5 \geq 6/5$, whereas the lower block of X_N is at most $6I_2/5$. The top blocks agree. Hence the exact order inequality holds for every comparable pair, including both orders, and thus for every allowed shrinking corridor.

Nevertheless integers nearest $e^{2\pi k}$ and $e^{\pi+2\pi k}$ produce distinct limits of V_n. The sequence does not converge. It is genuinely noncommuting: $[V_n, V_m] = \sin(\log m - \log n)[\sigma_z, \sigma_x]$. Also $\Phi^*(S) = \mathrm{Tr}(S)|0\rangle\langle 0|$ admits no faithful positive fixed density. This counterexample distinguishes the invariant faithful-functional condition of Corollary 4.2 from unitality and an ordinary constant trace slope.

8.2 A channel with metastable escape

There is a normal completely positive trace-preserving map on $\mathcal{S}_1(\ell^2(\mathbb{N}))$ for which positive full-additive cocycles can fail even operator-norm convergence. Set

$$H_j = 2^{2^{j+3}}, \quad p_j = H_j^{-1}, \quad q_j = 1 - p_j,$$

and use Kraus operators $K_j = \sqrt{q_j}|j\rangle\langle j|$ and $L_j = \sqrt{p_j}|j+1\rangle\langle j|$. Their adjoint squares sum strongly to I. The map kills off-diagonal entries and evolves diagonal states by a pure-birth chain with geometric holding mean H_j at state j. The positive Kraus series converges in trace norm by monotone trace summation.

Take $B = |1\rangle\langle 1|$ and $A_n = \sum_{k<n} \Phi^k B$. The full additive identity holds and $\mathrm{Tr}\, X_n = 1$. If T_j is the arrival time at state j, then $\mathbb{E}T_j = \sum_{i<j} H_i = o(H_j)$. Therefore $\mathbb{P}(T_j \leq H_j/4) \to 1$. Independently, the chance of holding at j for at least H_j further steps is $q_j^{H_j} \geq 1/4$. At $N_j = H_j$ this gives $(X_{N_j})_{jj} \geq 1/16$ for all sufficiently large j, with rounding absorbed by the slack in the constant. Hence $\limsup \|X_n\|_{\mathrm{op}} \geq 1/16$.

For each fixed j, total expected occupation is H_j, so $(X_n)_{jj} \leq H_j/n \to 0$. All matrices are diagonal, and any operator-norm limit would be zero, contradicting the lower bound. Trace-norm convergence also fails. The channel has no nonzero normal stationary state: its fixed diagonal recursion forces every entry to vanish. This classical metastable mechanism, expressed as a channel, shows why the infinite-dimensional structural hypotheses cannot simply be discarded.

8.3 A nonunital channel within the finite theorem

The qubit amplitude-damping channel with Kraus operators $\mathrm{diag}(1, 1/2)$ and $(\sqrt{3}/2)|0\rangle\langle 1|$ is trace-preserving but nonunital. For $B = |+\rangle\langle +|$, its additive cocycle has

$$A_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 11/8 & 3/4 \\ 3/4 & 5/8 \end{pmatrix}, \quad [A_1, A_2] = \begin{pmatrix} 0 & -3/8 \\ 3/8 & 0 \end{pmatrix}.$$

Nevertheless $A_n/n \to |0\rangle\langle 0|$. This additive example exhibits noncommuting inputs and nonunital evolution within Theorem 4.1.

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