Positive heat-flow observations on spherical shells: cutoffs, optimal preparation and concentration

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Abstract

We study nonnegative initial temperatures whose Dirichlet heat evolution has one constant signed outward normal derivative on both components of a fixed spherical shell at a prescribed time. In every dimension at least two, the admissible times form an interval $(0, \tau)$ with a finite, unattained endpoint characterized by a boundary heat-kernel ratio. Finite schedules satisfy an exact interior convex-hull criterion, and a datum-independent terminal interval contains at most one observation. With unit ground-state projection, every weighted minimum-L^q preparation, $1 < q < \infty$, is unique at every admissible time and is determined by a monotone real scalar moment equation. In the quadratic case its cost increases strictly. As the observation time approaches the cutoff, we obtain sharp constants throughout the fixed L^q scale, including separate endpoint arguments. The boundary law $H(1-\sqrt{\delta}z, \tau-\delta)/\delta \to \beta(z^2-6)$ fixes the concentration scale and limiting profiles. Every asymptotically optimal L^2 preparation has the same strong rescaled profile, quantitative ground-weighted total-variation stability, no scaled second-moment loss, and asymptotic angular symmetry. For every fixed $1 < q < \infty$, weighted near-minimizers have quantitative ground-probability stability with no second-moment loss, and physical near-minimizers have a universal strong rescaled profile including angular symmetry. Quadratic transport convergence holds, while exactly feasible examples exclude a universal higher-moment conclusion. All the norm infima are unchanged by restriction to smooth, strictly positive, fully compatible data. At the L^1 endpoint the ground-weighted probabilities concentrate at $\sqrt{6}$ in quadratic Wasserstein distance. At the cap endpoint the preparations converge strongly in every finite L^r to a filled layer, including physical angular rigidity; an exactly feasible counterexample excludes general strong L^∞ convergence. The arguments distinguish the shell-specific observation geometry from the classical positivity and moment-optimization tools used to analyze it.

1 Introduction

Overdetermined heat problems often ask what geometry is forced by a spatially constant boundary flux. Here we fix the geometry and allow the initial temperature to vary. This leads to a different quantitative question: at which times can positive initial data produce the observation, and what preparation cost is required as the last possible time is approached? The distinction between a prescribed uniform initial datum and a tailored nonnegative one is essential. We work on a shell with both boundary components included, using the signed outward normal derivative with one common value, rather than allowing a separate constant on each component.

There are three linked parts to the answer. First, the ratio of two one-dimensional boundary heat kernels determines the exact latest time. Its spatial order and temporal endpoint contraction yield a sharp existence/nonexistence theorem, including nonattainment at the cutoff. Second, several requested times impose simultaneous positive moments. An interior convex-hull criterion describes feasibility exactly; a uniform terminal simplicity theorem shows why separate feasibility need not imply joint feasibility. Third, normalizing the

ground-state projection makes the preparation problem coercive and produces an exact optimizer and sharp singular costs. The singularity is governed by a quadratic contact of the normalized flux kernel with zero and a linear zero of the ground state.

The cutoff and kernel law are proved in Section 2. Schedules and the terminal obstruction are treated in Section 3. Section 4 gives the unique weighted finite-q preparation at every admissible time, and strict cost growth in the quadratic case. Section 5 gives sharp weighted costs in every fixed norm. The stability analysis then identifies the concentration of all asymptotically optimal L^2 preparations, rather than just an explicit recovery sequence. Section 7 extends strong profile convergence to $1 < q < \infty$ and identifies the smooth infima for all norms. Section 8 proves quantitative probability-tail control and the physical angular profile for every finite strictly convex power, and shows that no higher transport order is universally forced. Section 9 completes the endpoint concentration picture, including the obstruction to strong L^∞ convergence. All constants and limits fix the dimension and shell. The data remain in L^2 even when a different norm is minimized.

Relation to earlier work

The overdetermined parabolic literature includes the symmetry work of Alessandrini–Garofalo and Alessandrini [4, 5, 6], and the later level-surface discussion of Sakaguchi [7]. The 1989–1991 originals have not been available for direct examination in preparing this work. The results below are self-contained statements for tailored nonnegative data on a fixed shell.

The positivity mechanism is classical. Determinants of one-dimensional diffusion kernels and their total positivity are treated by Karlin–McGregor and Karlin [1, 2]; Schmidt's work [3] is a particularly relevant predecessor concerning parabolic initial and boundary kernels. We give the determinant, boundary-limit and strict-endpoint arguments needed here explicitly. Schmidt's original likewise has not been directly available, so its precise theorem implications for the endpoint argument remain a historical comparison question.

The optimization and moment tools also have established antecedents. Interior moment representations by regular densities are part of moment theory; the concrete smooth-bump construction used below can be compared with [9]. Convex entropy-like minimization and positive minimum-L^2 moment reconstruction appear in [8, 11]. Compactly supported power profiles and sharp moment–entropy inequalities are classical [14, 12, 13]. The positive-part representer, Hilbert projection gap and beta integral are supporting tools. The purpose here is to connect the exact shell observation limit to global optimal preparation, sharp heat-specific singular costs and stability under the stated normalization. Saturation under a pointwise cap and its quantitative stability also have classical variational precedents; compare the fixed-volume bathtub estimate in [16, Section 2.8, Proposition 24]. Our direct argument treats the moving heat scale and the ground state's boundary zeros. For general convex-moment Bregman identities with nonnegative correction terms, see Csiszár–Matúš [15].

The physical interpretation is fixed throughout: the unknown is an initial temperature, not a time-dependent boundary control. Rotational averaging reduces feasibility and norm infima to radial data, but does not imply that every feasible datum is radial. Angular rigidity below is a consequence of asymptotically minimal cost.

2 The shell kernels and the exact cutoff

Fix $N \geq 2$ and $a > 0$, and put

$$b = a + 1, \qquad p = \frac{N-1}{2}, \qquad \Omega = \{x \in \mathbb{R}^N : a < |x| < b\}, \qquad \ell = (b/a)^p.$$

We use the generator Δ, zero Dirichlet boundary values, and the *signed outward* normal derivative. An observation at $T > 0$ means that $\partial_\nu u(\cdot, T)$ has one constant value on the entire boundary, including both components. Initial data are nonnegative and nonzero in $L^2(\Omega)$. Positive-time smoothing defines these observations even when the initial datum has no boundary trace.

For radial data set $s = r - a$ and $v(s,t) = r^p u(r,t)$. Then

$$v_t = -Lv, \qquad L = -\frac{d^2}{ds^2} + V(s), \quad V(s) = \frac{p(p-1)}{(a+s)^2}, \qquad v(0,t) = v(1,t) = 0. \tag{1}$$

The operator is the unitary radial transform of the positive Dirichlet Laplacian. In particular its eigenvalues satisfy $0 < \lambda_1 < \lambda_2 < \cdots$. Write ϕ_n for real orthonormal eigenfunctions with $\phi_n'(0) > 0$, and $\phi = \phi_1 > 0$ in $(0,1)$. Let $k(t,s,z)$ be the kernel of $P_t = e^{-tL}$, and define

$$\begin{aligned} &h_L(s,t) = \partial_z k(t,s,0), \qquad && h_R(s,t) = -\partial_z k(t,s,1), \\ &D_{LL}(t) = \partial_s \partial_z k(t,0,0), \qquad && D_{LR}(t) = -\partial_s \partial_z k(t,0,1), \qquad D_{RR}(t) = \partial_s \partial_z k(t,1,1). \end{aligned} \tag{2}$$

All three D's and both interior boundary kernels are positive, by the strong maximum principle and the Hopf lemma. These facts hold also when $N = 2$ and $V < 0$: the potential is bounded on this fixed interval and can be shifted by a constant in comparison arguments. Symmetry gives

$$h_L'(0) = D_{LL}, \quad h_L'(1) = -D_{LR}, \quad h_R'(0) = D_{LR}, \quad h_R'(1) = -D_{RR}.$$

Define the smoothly extended quotients and the imbalance kernel by

$$Q = \frac{h_R}{h_L}, \qquad m(t) = \frac{D_{LR}}{D_{LL}}, \quad M(t) = \frac{D_{RR}}{D_{LR}}, \qquad F_t = \ell h_L(\cdot,t) - h_R(\cdot,t), \qquad H_t = F_t/\phi. \tag{3}$$

Here and below a prime on a spatial function denotes its spatial derivative; time derivatives of m, M are indicated explicitly or by the argument.

Lemma 2.1 (Spatial order and endpoint contraction)**.** *For $t > 0$, $Q_s(s,t) > 0$ on $(0,1)$. The functions m and M are strictly increasing and strictly decreasing in time, respectively. If $c_a = -\phi'(1)/\phi'(0)$, then*

$$m(t) < c_a < M(t), \qquad \lim_{t\to\infty} m(t) = \lim_{t\to\infty} M(t) = c_a < \ell, \qquad \lim_{t\downarrow 0} M(t) = \infty.$$

Proof. We include the classical determinant argument to specify the boundary limits being used; the underlying total-positivity mechanism is that of Karlin–McGregor and parabolic kernel theory [1, 2, 3]. On the triangle $s_1 < s_2$ the antisymmetric product kernel

$$k(t,s_1,z_1)k(t,s_2,z_2) - k(t,s_1,z_2)k(t,s_2,z_1), \qquad z_1 < z_2,$$

is nonnegative. Indeed, antisymmetrically extend nonnegative smooth compact data from the triangle to the square. Product evolution restricts to a Dirichlet problem on the triangle with potential $V(s_1) + V(s_2)$, and comparison preserves positivity. Testing all such data proves the kernel inequality. Divide by $z_1(1 - z_2)$ and pass to $z_1 \downarrow 0, z_2 \uparrow 1$. This yields $h_L(s_1)h_R(s_2) - h_R(s_1)h_L(s_2) \geq 0$, hence $Q_s \geq 0$.

Strictness requires an additional argument. Cancellation of the common potential gives

$$Q_t = Q_{ss} + 2\frac{h_L'}{h_L} Q_s.$$

The nonnegative function $Z = Q_s$ solves a linear parabolic equation with drift $2h_L'/h_L$ and zero-order coefficient $2\partial_s(h_L'/h_L)$. These coefficients are bounded on every compact interior positive-time cylinder. An interior zero would, by the strong maximum principle and overlapping cylinders, force Z to vanish everywhere at an earlier positive time. But h_L and h_R cannot be proportional: in their eigenfunction expansions the coefficient ratio $-\phi_n'(1)/\phi_n'(0)$ is positive for $n = 1$ and negative for $n = 2$. Thus $Q_s > 0$.

For $h > 0$, $h_i(\cdot, t + h) = P_h h_i(\cdot, t)$. Differentiating at the right endpoint shows that $M(t + h)$ is the average of $Q(z, t)$ with positive weight $[-\partial_s k(h, 1, z)]h_L(z, t)$. It is strictly less than $M(t)$. The left endpoint similarly gives $m(t+h) > m(t)$. Spectral dominance gives their common limit c_a. To justify derivative-series convergence, min–max gives $\lambda_n = n^2\pi^2 + O(1)$, and the eigenfunction equation and Sobolev estimates give polynomial bounds on endpoint derivatives. The exponentially decaying series therefore converge with the required derivatives on every positive-time slab, and their rescaled tails vanish as $t \to \infty$.

Set $\psi_n(r) = r^{-p}\phi_n(r - a)$. The radial equation gives

$$\frac{d}{dr}\frac{\psi_n'^2 + \lambda_n\psi_n^2}{2} = -\frac{N-1}{r}\psi_n'^2. \tag{4}$$

Integration from a to b is strictly negative. For $n = 1$ it gives $a^{-p}\phi'(0) > b^{-p}[-\phi'(1)]$, hence $c_a < \ell$.

Finally let $C = \|V\|_\infty$ and let k_0 denote the free interval Dirichlet heat kernel. Comparison yields $e^{-Ct}k_0 \le k \le e^{Ct}k_0$. Dividing by the appropriate two endpoint distances and taking their limits yields the same comparison for each positive mixed inward derivative; we do not differentiate an ordered inequality arbitrarily. Consequently

$$e^{-2Ct}M_0(t) \le M(t) \le e^{2Ct}M_0(t). \tag{5}$$

For completeness the interval image kernel is

$$k_0(t, s, z) = \sum_{j\in\mathbb{Z}}[G_t(s - z + 2j) - G_t(s + z + 2j)], \qquad G_t(x) = (4\pi t)^{-1/2}e^{-x^2/(4t)}.$$

Differentiate this normally convergent Gaussian series at the endpoints. The nearest image distances are 0 for $D_{RR,0}$ and ± 1 for $D_{LR,0}$; the remaining distances start at 2 and 3, respectively. Thus

$$\begin{aligned} D_{RR,0}(t) &= (4\pi t)^{-1/2}t^{-1}[1 + O(t^{-1}e^{-1/t})],\\ D_{LR,0}(t) &= (4\pi t)^{-1/2}t^{-1}(t^{-1} - 2)e^{-1/(4t)}[1 + O(e^{-2/t})]. \end{aligned}$$

The tails obey these estimates by summing the Gaussian factors times their quadratic derivative polynomials. Consequently $M_0(t) \sim te^{1/(4t)}/(1 - 2t)$ as $t \downarrow 0$. In particular $M_0(t) \to \infty$, which proves the last assertion. □

Theorem 2.2 (Exact positive-data cutoff)**.** *There is a unique $\tau = \tau_N(a) > 0$ such that $M(\tau) = \ell$. There exists a nonnegative nonzero datum in $L^2(\Omega)$ with an observation at T if and only if $0 < T < \tau$. Every such time is realized by smooth radial data, strictly positive in the interior and satisfying all Dirichlet heat compatibility conditions. The cutoff is not attained.*

Proof. The existence and uniqueness of τ follow from Lemma 2.1. For radial initial data v, the physical inner and outer derivatives are

$$-a^{-p}\int_0^1 h_L(s, T)v(s)\,ds, \qquad -b^{-p}\int_0^1 h_R(s, T)v(s)\,ds.$$

Thus equality is precisely $\int vF_T = 0$. Since $F_T = h_L(\ell - Q)$ and $m(T) < c_a < \ell$, the kernel is positive throughout $(0,1)$ when $T \geq \tau$, including at $T = \tau$. Its vanishing limiting coefficient at the outer endpoint at cutoff cannot support nonzero L^2 mass. When $T < \tau$, strict spatial order implies that F_T has one zero, being positive before it and negative after it.

Choose $g \geq 0$ smooth and compactly supported in the negative region. The ground moment is

$$\int_0^1 \phi F_T \, ds = e^{-\lambda_1 T}[\ell\phi'(0) + \phi'(1)] > 0.$$

Hence $v = \phi + \gamma g$, with $\gamma = -\int \phi F_T / \int g F_T > 0$, gives exactly zero imbalance. It is strictly positive inside. The physical datum $r^{-p}v(r-a)$ is smooth; compact bumps do not change any boundary compatibility condition inherited from the ground eigenfunction. In precise terms $\Delta^j u_0 = 0$ on $\partial\Omega$ for every integer $j \geq 0$. Its common outward flux is strictly negative by Hopf.

For arbitrary data average over $SO(N)$. Rotations commute with Dirichlet heat evolution and the normal derivative, preserve nonnegative nonzero mass, and are transitive on each boundary sphere. A whole-boundary observation is preserved by averaging. Radial exclusion therefore excludes all data at forbidden times; radial constructions give the converse. □

Proposition 2.3 (An explicit three-dimensional root and large shells)**.** *For $N = 3$ let $x = \pi^2 t$ and*

$$E(x) = \sum_{\substack{n \geq 1 \\ n \text{ even}}} n^2 e^{-n^2 x}, \qquad O(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} n^2 e^{-n^2 x}.$$

Then $M(t) = (O+E)/(O-E)$, and the unique cutoff is specified by $(2a+1)E(\pi^2\tau) = O(\pi^2\tau)$. In every fixed dimension,

$$\tau_N(a) = \frac{\log(8a/p)}{3\pi^2} + o(1) \quad (a \to \infty). \tag{6}$$

Proof. When $p = 1$, $V = 0$ and the normalized eigenfunctions are $\sqrt{2}\sin(n\pi s)$. Their endpoint derivative products give the displayed ratio; its denominator is positive by Hopf. Substitution of $\ell = (a+1)/a$ gives the stated equation. For general fixed N, $\|V\|_\infty = O(a^{-2})$, $\ell = 1 + p/a + O(a^{-2})$, and $M_0(t) = 1 + 8e^{-3\pi^2 t} + O(e^{-6\pi^2 t})$ as $t \to \infty$. At $t_\pm = [\log(8a/p) \pm \varepsilon]/(3\pi^2)$, (5) changes this expansion by only $O(\log(a)/a^2)$. Thus $M(t_-) > \ell > M(t_+)$ for large a. Monotonicity traps τ between these times; then let $\varepsilon \downarrow 0$. □

2.1 Strict endpoint derivatives and the boundary law

Lemma 2.4. *For every $t > 0$, $m'(t) > 0$ and $M'(t) < 0$.*

Proof. Strict order in time alone would not establish this assertion. At the right endpoint write $y = 1 - s$, $l(y,t) = h_L(1-y,t) = yg(y,t)$, and $q(y,t) = h_R(1-y,t)/l(y,t)$. Boundary smoothness and the equation give $l_{yy}(0,t) = 0$, so $g > 0$, $g_y(0,t) = 0$. With $d = g_y/g = O(y)$,

$$q_t = q_{yy} + (2/y + 2d)q_y, \qquad q_y(0,t) = 0. \tag{7}$$

The potential is generally nonsymmetric, so no even extension of g is assumed. Write $q = q_0 + q_2 y^2 + q_3 y^3 + O(y^4)$. The left side of (7) has no term linear in y; the right side has linear coefficient $12q_3$. Hence $q_3 = 0$. For $A = q_y/y$ we obtain

$$A_t = A_{yy} + (4/y + 2d)A_y + (2d/y + 2d_y)A, \qquad A_y(0,t) = 0.$$

The radial extension $\widehat{A}(z,t) = A(|z|,t)$ to a small ball in $\mathbb{R}^5$ is C^2 in space and C^1 in time. It solves a uniformly parabolic equation with bounded continuous drift $2[d(|z|,t)/|z|]z$

and zero-order coefficient $2d/y + 2d_y$, extended continuously at zero. Indeed the cancellation $q_3 = 0$ gives precisely the regularity needed for this extension; higher evenness is unnecessary. Lemma 2.1 gives $A < 0$ off the center and $A \le 0$ at the center. The strong maximum principle, after a bounded zero-order shift, excludes a zero at the center at positive time. Thus $A(0,t) < 0$. Taking the endpoint limit in (7) gives $M'(t) = 3A(0,t) < 0$. At the left endpoint the same construction has $q_y > 0$, and the minimum principle gives $m'(t) > 0$. □

Put $\kappa = -\phi'(1) > 0$. The next law is the local input for all the cost and stability results; the constants depend on the fixed shell.

Lemma 2.5 (Universal boundary law)**.** *There is $\beta > 0$ such that, as $\delta \downarrow 0$, uniformly for bounded $z \ge 0$,*

$$\frac{H_{\tau-\delta}(1-\sqrt{\delta}z)}{\delta} \longrightarrow \beta(z^2-6), \qquad \frac{\phi(1-\sqrt{\delta}z)}{\sqrt{\delta}} \longrightarrow \kappa z. \tag{8}$$

Moreover there are $c_0, C_0, \delta_0 > 0$ with

$$H_{\tau-\delta}(1-y) \ge c_0 y^2 - C_0\delta \quad (0 \le y \le 1,\ 0 < \delta < \delta_0). \tag{9}$$

Proof. Use the preceding notation at $t = \tau$, and set $c = g(0,\tau)/\kappa$. Equation (7) gives $M'(\tau) = 3q_{yy}(0,\tau) < 0$. Since $H = (h_L/\phi)(\ell - q)$, set $\beta = -cq_{yy}(0,\tau)/2 > 0$. At the cutoff $H(1) = 0$, its coefficient of y^2 is β, and its time derivative at the endpoint is $-cM'(\tau) = 6\beta$. Taylor expansion at $(y,t) = (0,\tau)$ proves (8). Here $\phi(1-y) = \kappa y + O(y^3)$ by its eigenfunction equation. At cutoff H is positive on $[0,1)$ and has just the specified quadratic zero at 1. Compactness gives $H_\tau(1-y) \ge c_0 y^2$ after decreasing c_0 if necessary. A bound on $\partial_t H$ on a compact positive-time slab proves (9). □

3 Schedules and a terminal simplicity interval

The one-time criterion does not determine simultaneous feasibility. The following exact criterion is useful even when the convex hull cannot be described by a closed formula in the time nodes.

Theorem 3.1 (Finite schedules)**.** *For distinct positive times $t_1, \dots, t_m$, define on $[0,1]$*

$$\Gamma(s) = (H_{t_1}(s), \dots, H_{t_m}(s)).$$

The schedule is realized by a nonnegative nonzero $L^2(\Omega)$ datum if and only if

$$0 \in \operatorname{int}_{\mathbb{R}^m} \operatorname{conv} \Gamma([0,1]). \tag{10}$$

Equivalently no nonzero $\alpha \in \mathbb{R}^m$ has $\alpha \cdot \Gamma(s) \ge 0$ on $[0,1]$. Whenever feasible, the schedule is realized by smooth radial strictly interior-positive data satisfying all compatibility conditions.

Proof. First all spectral coefficients

$$b_n = \ell\phi_n'(0) + \phi_n'(1) \tag{11}$$

are positive. Indeed (4), valid also for sign-changing eigenfunctions, gives $\ell\phi_n'(0) > |\phi_n'(1)|$. Consequently $F_t = \sum_{n\ge1} b_n e^{-\lambda_n t}\phi_n$. The functions H_{t_j} are linearly independent: an identically vanishing combination, multiplied by ϕ and paired with ϕ_n, would give $\sum_j \alpha_j e^{-\lambda_n t_j} = 0$ for all n. The first m equations have invertible exponential interpolation matrix. For completeness a nonzero combination of m exponentials with distinct real exponents has at most

$m-1$ distinct real zeros: divide by one exponential, differentiate and use Rolle's theorem inductively. This proves invertibility.

Every nonzero such combination is real analytic and has a null zero set in $(0,1)$. Analyticity here can be checked for this particular potential: V extends holomorphically to a neighborhood of $[0,1]$ avoiding $-a$. The analytic eigenfunction ODE gives continuation bounds polynomial in λ_n times $e^{C\sqrt{1+\lambda_n}}$ on a smaller complex neighborhood. The factors $e^{-\lambda_n t}$ imply normal convergence for positive times. Division by ϕ, whose endpoint zeros are simple, has removable endpoint singularities. Thus H_t is analytic near the closed interval. This also justifies the level-set nullity used later.

For a nonnegative nonzero radial datum v, normalize the probability $d\mu = v\phi\,ds/\int v\phi$. The requested schedule is equivalent to $\int \Gamma\,d\mu = 0$. If (10) fails, separation gives nonzero α with $\alpha\cdot\Gamma \geq 0$ on the curve. Its null zero set makes its integral against this absolutely continuous nonzero probability strictly positive, a contradiction. The same separation argument proves the asserted equivalent condition. Boundary membership in the closed convex hull is therefore insufficient.

Conversely, choose finitely many interior points s_i whose image vectors have a convex hull containing a neighborhood of zero. One way to do so is to represent small positive and negative coordinate vectors by finite convex combinations, and then perturb any endpoint nodes slightly inward. For narrow nonnegative compact smooth bumps g_i normalized by $\int g_i\phi = 1$, the moment vectors $q_i = (\int g_i F_{t_j})_{j=1}^m$ remain close to $\Gamma(s_i)$, so their convex hull still contains a neighborhood of zero. Put $r = (\int \phi F_{t_j})_{j=1}^m$. For sufficiently small $\eta > 0$ choose $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, with $\sum_i \alpha_i q_i = -\eta r$. Then $v = \eta\phi + \sum_i \alpha_i g_i$ has all imbalance moments zero, is strictly positive inside, and has all compatibility conditions. Rotation averaging and radial realization complete the passage to physical data. The regular-density construction is a classical interior moment mechanism, for which see also [9]; the proof above supplies the specific nondegeneracy and exact correction needed here. □

Theorem 3.2 (Uniform terminal simplicity)**.** *For this fixed shell there is $t_0 \in (0,\tau)$ such that every nonnegative nonzero initial datum has at most one whole-boundary observation in $[t_0,\infty)$. At any such time the rotationally averaged imbalance is transverse. In particular two distinct times in (t_0,τ) are each individually feasible but jointly infeasible.*

Proof. At $t=\tau$, $H_t(s) > 0$ for $s < 1$, $H_\tau(1) = 0$, and $\partial_t H_t(1)|_{t=\tau} = 6\beta > 0$ by Lemma 2.5. Thus $\partial_t H$ is positive near the outer endpoint. On the compact complement H_τ has a positive minimum, so choosing $C > 0$ sufficiently large gives $\partial_t H + CH > 0$ on all of $[0,1]$ at time τ. Continuity supplies $t_0 < \tau$ and $\eta > 0$ such that $\partial_t H + CH \geq \eta$ on $[0,1]\times[t_0,\tau]$. For a radial nonnegative nonzero v write $B(t) = \int vF_t$. Integration against $v\phi$ yields

$$B'(t) + CB(t) \geq \eta \int_0^1 v\phi\,ds > 0.$$

Hence $e^{Ct}B(t)$ is strictly increasing and has at most one zero in this interval; at a zero $B'(t) > 0$. The cutoff theorem excludes all later times. Rotation averaging extends exclusion to arbitrary data. The transversality assertion concerns this averaged imbalance, not constancy of the flux at nearby times. Finally Theorem 2.2 realizes each individual time below τ, proving the last assertion. □

4 The exact optimal normalized preparation

Fix the shell and notation of the preceding sections. All integrals without displayed limits are over $(0,1)$, and $\|\cdot\|_2$ denotes its Lebesgue L^2 norm. The ground state is normalized by

$\|\phi\|_2 = 1$. For $0 < T < \tau$ set

$$\mathcal{A}_T = \left\{ v \in L^2(0,1) : v \geq 0, \quad \int v\phi \, ds = 1, \quad \int v F_T \, ds = 0 \right\}, \qquad E(T) = \inf_{v \in \mathcal{A}_T} \|v\|_2^2 .$$

The normalization is essential: multiplication of an initial temperature by a positive constant preserves its observation time. Nonnegative minimum-norm moment problems and their convex geometry are classical; see, for example, [8, 11]. We give direct proofs of the particular optimizer, its scalar parameter, and the heat-semigroup cost inequality used below.

4.1 A monotone scalar equation for every strictly convex power

For a fixed continuous positive weight w and $1 < q < \infty$, write

$$\mathcal{E}_{q,w}(T) = \inf_{v \in \mathcal{A}_T} \int_0^1 w v^q \, ds, \qquad \alpha = \frac{1}{q-1}, \qquad \omega_{q,w} = w^{-\alpha} \phi^{\alpha+1}.$$

The admissible class remains L^2; the objective may be infinite for an individual datum. Define the nonnegative-cone Bregman remainder

$$B_q(x,y) = x^q - y^q - q y^{q-1}(x-y), \qquad x, y \geq 0.$$

Its use below is a concrete instance of classical convex-moment Pythagorean identities with a correction term [15, Lemma 4.16 and Theorem 7.10].

Theorem 4.1 (Global weighted power optimum)**.** *For every $0 < T < \tau$, the scalar function*

$$\Psi(\theta, T) = \int_0^1 \omega_{q,w} H_T \, (1 + \theta H_T)_+^{\alpha} \, ds \tag{12}$$

is continuous and strictly increasing in $\theta \in \mathbb{R}$, with opposite infinite limits at the two ends of $\mathbb{R}$. Let $\theta = \theta_{q,w}(T)$ be its unique zero, and set

$$V_{q,w,T} = w^{-\alpha} \left(\phi(1 + \theta H_T)\right)_+^{\alpha},$$
$$\mathcal{Z}_{q,w}(T) = \int_0^1 \phi V_{q,w,T} \, ds > 0, \qquad v_{q,w,T} = V_{q,w,T} / \mathcal{Z}_{q,w}(T).$$

Then $v_{q,w,T}$ is the unique minimizer in $\mathcal{A}_T$, and

$$\mathcal{E}_{q,w}(T) = \mathcal{Z}_{q,w}(T)^{1-q}. \tag{13}$$

With $\mathcal{Z} = \mathcal{Z}_{q,w}(T)$ and $g = q\mathcal{Z}^{1-q}\phi(1 + \theta H_T)$, every finite-cost $v \in \mathcal{A}_T$ satisfies

$$\int_0^1 w v^q \, ds - \mathcal{E}_{q,w}(T) = \int_0^1 w B_q(v, v_{q,w,T}) \, ds + \int_{\{1+\theta H_T \leq 0\}} (-g) v \, ds. \tag{14}$$

The root, normalizer and minimum are continuous in $T \in (0, \tau)$; the optimizer is continuous there as a map into $C([0,1])$. The sign of θ is opposite to that of $\Psi(0,T)$, with $\theta = 0$ when $\Psi(0,T) = 0$.

Proof. For each real h, the function $h\,(1 + \theta h)_+^{\alpha}$ is nondecreasing in θ. For $h > 0$ its positive-part factor increases; for $h < 0$ that factor decreases and is multiplied by a negative number. For any $\theta_1 < \theta_2$, both affine factors are positive when $0 < |h|$ is sufficiently small. Since H_T is continuous and changes sign through an interior zero, an open positive-measure subinterval has such nonzero values of H_T and positive $\omega_{q,w}$. The integrand increases strictly there. It

is nondecreasing elsewhere, proving strict increase of Ψ without differentiating a fractional power at an active boundary. Boundedness on compact parameter sets gives joint continuity by dominated convergence. The same theorem gives

$$\lim_{\theta\to+\infty}\frac{\Psi(\theta,T)}{\theta^{\alpha}}=\int_{\{H_T>0\}}\omega_{q,w}H_T^{\alpha+1}\,ds>0,$$
$$\lim_{\theta\to-\infty}\frac{\Psi(\theta,T)}{|\theta|^{\alpha}}=-\int_{\{H_T<0\}}\omega_{q,w}|H_T|^{\alpha+1}\,ds<0.$$

These limits prove existence and uniqueness of the real zero. An interior neighborhood of the zero of H_T has $1+\theta H_T>0$, so $\mathcal{Z}>0$. The displayed normalized candidate is continuous, bounded and in L^2, has unit ground projection, and has zero flux moment by (12).

On the active set, $g=qwv_{q,w,T}^{q-1}$; on its complement the candidate is zero and $g\le 0$. Both moment constraints imply $\int g(v-v_{q,w,T})\,ds=0$. Expanding B_q and separating the inactive set yields (14). Its two right-hand terms are nonnegative, and strict convexity gives uniqueness. Infinite-cost data cannot improve the bounded candidate. Since $\alpha q=\alpha+1$,

$$\begin{aligned}\int_0^1 wV_{q,w,T}^{q}\,ds&=\int_0^1\omega_{q,w}\,(1+\theta H_T)_+^{\alpha+1}\,ds\\&=\mathcal{Z}+\theta\Psi(\theta,T)=\mathcal{Z}.\end{aligned}$$

Dividing by $\mathcal{Z}^q$ proves (13).

For continuity fix T_0 and its root θ_0. At $\theta_0-\varepsilon$ and $\theta_0+\varepsilon$, strict increase gives opposite strict signs. Joint continuity preserves these signs for T close to T_0, trapping its unique root between these two values. Thus $\theta(T)\to\theta_0$. Uniform spatial continuity of H_T on a compact positive-time neighborhood, continuity of the positive-part power, and positivity of the limiting normalizer now give uniform convergence of V and the normalized candidate, and convergence of the normalizer and value. The root's sign follows from strict increase. □

Remark 4.2. The scalar parameter is allowed to have either sign. For arbitrary q,w, a representation forced to use a positive threshold c would not follow from the hypotheses. The stronger differentiability and cost growth proved next concern the unweighted quadratic case.

4.2 Quadratic preparation and strict time growth

Theorem 4.3. *For every $0<T<\tau$, there is exactly one $c(T)>0$ satisfying*

$$J_T(c):=\int\phi^2H_T\,(c-H_T)_+\,ds=0. \tag{15}$$

With

$$Z(T)=\int\phi^2\,(c(T)-H_T)_+\,ds,\qquad v_T=\frac{\phi\,(c(T)-H_T)_+}{Z(T)},$$

one has $Z(T)>0$, and v_T is the unique minimizer in $\mathcal{A}_T$. Moreover,

$$E(T)=\frac{c(T)}{Z(T)}>1,\qquad \|v\|_2^2-E(T)=\|v-v_T\|_2^2+\frac{2}{Z(T)}\int\phi\,(H_T-c(T))_+\,v\,ds\quad(v\in\mathcal{A}_T). \tag{16}$$

The infimum is unchanged if one restricts to smooth data strictly positive in the interior and satisfying all Dirichlet heat compatibility conditions.

Proof. Write $H = H_T$. The preceding kernel analysis gives both signs for H, spatial analyticity and nonconstancy, and

$$M_1 := \int \phi^2 H \, ds = \big(\ell\phi'(0) + \phi'(1)\big)e^{-\lambda_1 T} > 0.$$

In particular every level set of H has measure zero. In Theorem 4.1 with $q = 2, w = 1$, this mean equals $\Psi(0,T) > 0$, so the unique root θ is negative. Set $c = -1/\theta > 0$. The relation $(1 - H/c)_+ = (c - H)_+ / c$ proves (15), the optimizer formula and uniqueness of its positive scalar root. The general gap and cost formulas reduce to (16). Cauchy–Schwarz gives cost at least one; equality would force $v = \phi$, whose flux moment is $M_1 > 0$. Hence $E(T) > 1$.

For the differentiability assertion below it is useful to note

$$J_T(0) = -\int_{\{H<0\}} \phi^2 H^2 \, ds < 0, \qquad J_T'(c) = \int_{\{H<c\}} \phi^2 H \, ds,$$

J_T' is nondecreasing for $c > 0$: increasing a positive level only adds nonnegative values of H. Thus J_T is convex on the positive half-line. For $c > \max H$,

$$J_T(c) = cM_1 - \int \phi^2 H^2 \, ds \longrightarrow +\infty.$$

At the positive root convexity gives

$$J_T'(c) \geq \frac{J_T(c) - J_T(0)}{c} > 0.$$

The last statement permits a nonsmooth positive-part optimizer. Choose one smooth, strictly interior-positive compatible datum $w \in \mathcal{A}_T$, whose existence follows from the cutoff construction. Choose nonnegative $g_n \in C_c^\infty(0,1)$ converging in L^2 to v_T. For fixed $0 < \varepsilon < 1$ put $z_n = (1-\varepsilon)g_n + \varepsilon w$. Its two moments converge to $(1, 0)$. Take two fixed nonnegative compactly supported smooth bumps b_1, b_2 near points with different H values. For sufficiently small supports the columns

$$\begin{pmatrix} \int b_i \phi \, ds \\ \int b_i F_T \, ds \end{pmatrix}, \qquad i = 1, 2,$$

are independent: the ratios of their second to first entries approach those distinct values. Solve this fixed two-dimensional system for corrections $\alpha_n b_1 + \gamma_n b_2$ restoring both moments of z_n exactly. The coefficients tend to zero. On the compact union of bump supports, w has a positive minimum, so for sufficiently large n the correction is bounded in absolute value by $\varepsilon w/2$. The corrected functions remain strictly positive in the interior. Compactly supported corrections preserve every boundary compatibility condition. First letting $n \to \infty$, then $\varepsilon \downarrow 0$, proves equality of infima. □

Proposition 4.4. *The functions c, Z, E are C^1 on $(0,\tau)$, and $T \mapsto v_T$ is C^1 as a map into $L^2(0,1)$. For $0 < T_1 < T_2 < \tau$,*

$$E(T_1) - 1 \leq e^{-2(\lambda_2 - \lambda_1)(T_2 - T_1)}\big(E(T_2) - 1\big). \tag{17}$$

In particular $E'(T) \geq 2(\lambda_2 - \lambda_1)(E(T) - 1) > 0$.

Proof. Smooth positive-time dependence of H_T and nullity of its level sets imply that $J_T(c)$ is C^1 near each positive root. Indeed the derivatives of its positive-part integrand exist off $\{H_T = c\}$ and are locally bounded, so dominated convergence applies, also to continuity of the derivatives. Since $\partial_c J_T(c(T)) > 0$, the C^1 implicit-function theorem gives $c(T) \in C^1$.

The same argument gives $Z \in C^1$. Difference quotients of $\phi\,(c(T) - H_T)_+$ converge almost everywhere off its null zero level and are bounded by a fixed multiple of ϕ on a compact time neighborhood. Dominated convergence in L^2 proves the asserted differentiability; continuity of the derivatives follows in the same way. Analyticity across changes of the active set is not needed.

Let $h = T_2 - T_1$ and $P_h = e^{-hL}$. If $v \in \mathcal{A}_{T_2}$, then $w = e^{\lambda_1 h} P_h v$ is nonnegative, has unit ground projection, and is feasible at T_1. The last assertion follows because its heat evolution at T_1 equals $e^{\lambda_1 h}$ times the evolution from v at T_2. Writing $v = \phi + v_\perp$ gives

$$\|w\|_2^2 - 1 \le e^{-2(\lambda_2 - \lambda_1)h}(\|v\|_2^2 - 1).$$

Apply this to $v = v_{T_2}$ and minimize at T_1 to obtain (17). Differentiation yields the stated bound. Here $\lambda_2 - \lambda_1$ is the radial interval spectral gap; no identification with the full shell's angular spectral gap is involved. □

Remark 4.5. The exact positive set of the optimizer is $\{H_T < c(T)\}$. Its boundary consists of finitely many level points, and it contains the negative-H_T region. No single-interval description is needed. A transverse interior level crossing gives a derivative corner. If $H_T < c(T)$ throughout the interior, then $v_T = (c\phi - F_T)/Z$ is smooth, strictly interior-positive and all-order compatible; an interior zero precludes strict positivity. Equality of the smooth positive infimum with $E(T)$ does not imply attainment in that smaller class.

5 Sharp costs in every L^q norm

Write $T = \tau - \delta$. We retain the same two moment constraints and the L^2 admissible class $\mathcal{A}_T$, allowing an infinite objective value. For continuous $w > 0$ on $[0, 1]$ set

$$E_{q,w}(\delta) = \inf_{v \in \mathcal{A}_{\tau-\delta}} \int w v^q \, ds \quad (1 \le q < \infty).$$

For $1 < q < \infty$, $E_{q,w}(\delta) = \mathcal{E}_{q,w}(\tau - \delta)$ from Theorem 4.1. For continuous $A > 0$ define

$$E_{\infty,A}(\delta) = \inf_{v \in \mathcal{A}_{\tau-\delta}} \operatorname*{ess\,sup}_{0<s<1} \frac{v(s)}{A(s)}.$$

The preceding cutoff construction ensures finite-cost admissible data. Positive-part power profiles in moment-constrained optimization have classical antecedents, including the moment–entropy inequalities of [12, 13] and the quadratic compactly supported profile of [14]. The constants below arise from the heat endpoint law and the present ground-projection normalization. We prove the endpoint cases directly rather than interchange limits in q and δ.

Theorem 5.1. *For fixed $1 < q < \infty$ put $\alpha = 1/(q-1)$ and*

$$\begin{aligned} R_q^2 &= \frac{6(3\alpha + 4)}{\alpha + 2} = \frac{6(4q-1)}{2q-1}, \\ I_q &= \int_0^{R_q} z^{\alpha+1}(R_q^2 - z^2)^\alpha \, dz = \frac{R_q^{3\alpha+2}}{2} B\left(\frac{\alpha+2}{2}, \alpha + 1\right), \\ f_q(z) &= \frac{\big(z(R_q^2 - z^2)\big)_+^\alpha}{\kappa I_q} \quad (z \ge 0), \\ C_q &= \int_0^\infty f_q^q \, dz = \frac{12(\alpha+1)}{\alpha+2} \kappa^{-q} I_q^{1-q}. \end{aligned}$$

Then

$$\lim_{\delta\downarrow 0} \delta^{q-1/2} E_{q,w}(\delta) = w(1) C_q. \tag{18}$$

At the two endpoints,

$$\lim_{\delta\downarrow 0} \sqrt{\delta} E_{1,w}(\delta) = \frac{w(1)}{\kappa\sqrt{6}}, \qquad \lim_{\delta\downarrow 0} \delta E_{\infty,A}(\delta) = \frac{1}{6\kappa A(1)}. \tag{19}$$

In particular the optimal unweighted L^q norm has order $\delta^{-1+1/(2q)}$, where $1/\infty = 0$. These limits are for fixed q, dimension and shell.

We first record explicitly the boundary estimates used throughout the proof. Put $y = 1-s$. The strict endpoint law gives, at $(y,T) = (0,\tau)$,

$$\frac{h_R}{h_L}(1-y,T) = M(T) + \frac{1}{2} Q_{yy}(0,T) y^2 + o(y^2), \qquad M'(\tau) = 3Q_{yy}(0,\tau) < 0.$$

Here $Q(y,T) = (h_R/h_L)(1-y,T)$. If $c_* = \lim_{y\downarrow 0} h_L(1-y,\tau)/\phi(1-y) > 0$ and $\beta = -c_* Q_{yy}(0,\tau)/2 > 0$, then $H = (h_L/\phi)(\ell - Q)$ gives

$$\frac{H_{\tau-\delta}(1-\sqrt{\delta}z)}{\delta} \longrightarrow \beta(z^2-6), \qquad \frac{\phi(1-\sqrt{\delta}z)}{\sqrt{\delta}} \longrightarrow \kappa z \tag{20}$$

uniformly on bounded nonnegative z intervals. At the cutoff $H_\tau > 0$ on $[0,1)$ and has a quadratic zero of coefficient β at 1. Compactness and a bounded positive-time derivative therefore give fixed positive constants c_0, C_0, δ_0 such that

$$H_{\tau-\delta}(1-y) \geq c_0 y^2 - C_0\delta \quad (0 \leq y \leq 1,\ 0 < \delta < \delta_0). \tag{21}$$

In particular every region $H_{\tau-\delta} < C\delta$, for fixed C, lies in $y \leq R\sqrt{\delta}$ for fixed R. This global localization is what permits all rescaled integrations below.

Proof of Theorem 5.1 for $1 < q < \infty$. A beta-integral calculation gives

$$\frac{\int_0^R z^{\alpha+3}(R^2-z^2)^\alpha\, dz}{\int_0^R z^{\alpha+1}(R^2-z^2)^\alpha\, dz} = \frac{R^2(\alpha+2)}{3\alpha+4}.$$

Thus the displayed R_q ensures

$$\kappa \int_0^\infty z f_q(z)\, dz = 1, \qquad \int_0^\infty z(z^2-6) f_q(z)\, dz = 0. \tag{22}$$

Since $\alpha q = \alpha + 1$, another beta recursion gives

$$\int_0^{R_q} z^{\alpha+1}(R_q^2 - z^2)^{\alpha+1}\, dz = \frac{R_q^2(2\alpha+2)}{3\alpha+4} I_q.$$

This proves the stated expression for C_q.

For the lower bound write $q' = q/(q-1) = \alpha + 1$. For $v \geq 0$ and real g, weighted Young duality says

$$w v^q \geq g v - (q-1) q^{-q'} w^{-\alpha} (g)_+^{q'}. \tag{23}$$

Choose $D, B_0 > 0$ by

$$\kappa D\beta = q w(1) (\kappa I_q)^{-(q-1)}, \qquad B_0 = D\beta(R_q^2 - 6),$$

and put, with $H = H_{\tau-\delta}$,

$$g_\delta = \phi\delta^{-(q-1/2)}(B_0 - DH/\delta).$$

Every feasible v satisfies $\int g_\delta v\, ds = B_0\delta^{-(q-1/2)}$. By (21), $(g_\delta)_+$ is supported on a fixed compact interval in $z = y/\sqrt{\delta}$. On that interval (20) implies, with positive parts understood,

$$\delta^{q-1} g_\delta(1 - \sqrt{\delta}z) \longrightarrow qw(1) f_q(z)^{q-1}.$$

Since $w(1 - \sqrt{\delta}z) \to w(1)$ uniformly there, the scaled conjugate integral in (23) tends to $(q-1)w(1)C_q$. Multiplying the limiting dual polynomial by f_q and using (22) gives $B_0 = qw(1)C_q$. Integration of (23), uniformly over all feasible v, proves

$$\liminf_{\delta\downarrow 0} \delta^{q-1/2} E_{q,w}(\delta) \geq B_0 - (q-1)w(1)C_q = w(1)C_q.$$

No weak-limit preservation of moments has been used.

For recovery let ρ vary in a small fixed neighborhood of $\rho_q = R_q^2 - 6 > 0$, set $c = \beta\delta\rho$, and define

$$V_{\delta,\rho} = w^{-\alpha}\,(\phi(c - H))_+^\alpha\,, \qquad J_\delta(\rho) = \int \phi H V_{\delta,\rho}\, ds.$$

Its support is uniformly localized by (21). After dividing by a positive power of δ and a positive constant independent of ρ, its moment converges uniformly in that neighborhood to

$$\int_0^{\sqrt{\rho+6}} z^{\alpha+1}(z^2 - 6)(\rho + 6 - z^2)^\alpha\, dz.$$

The ratio calculation above shows that this has the sign of $\rho - \rho_q$ and a unique zero there. Uniform convergence of positive powers holds also for $0 < \alpha < 1$, by uniform continuity on bounded intervals. The intermediate value theorem supplies exact roots $\rho_\delta \to \rho_q$. Normalize

$$v_\delta = \frac{V_{\delta,\rho_\delta}}{\int \phi V_{\delta,\rho_\delta}\, ds}.$$

These are bounded nonnegative L^2 data with both moments satisfied exactly. The denominator has positive leading constant, and rescaling gives

$$\delta v_\delta(1 - \sqrt{\delta}z) \longrightarrow f_q(z)$$

uniformly on a common compact support. Hence $\delta^{q-1/2}\int w v_\delta^q\, ds \to w(1)C_q$, proving the upper bound. □

Proof of the $q = 1$ assertion. Put $B_1 = w(1)/(\kappa\sqrt{6})$, $D_1 = B_1/(12\beta)$ and

$$g_\delta = \phi\delta^{-1/2}(B_1 - D_1 H/\delta).$$

Its positive part is again uniformly localized. On its rescaled support,

$$\frac{g_\delta(1 - \sqrt{\delta}z)}{w(1 - \sqrt{\delta}z)} \longrightarrow \frac{z(18 - z^2)}{12\sqrt{6}}.$$

The positive part of this cubic has maximum one, attained at $z = \sqrt{6}$. Consequently

$$L_\delta = \max\left(1,\ \sup_{s\in[0,1]} \frac{g_\delta(s)}{w(s)}\right) \longrightarrow 1.$$

The pointwise inequality $wv \geq g_\delta v / L_\delta$, for $v \geq 0$, and the two constraints imply

$$\int wv \, ds \geq \frac{B_1 \delta^{-1/2}}{L_\delta}.$$

For the opposite bound choose two nonnegative smooth bumps in z, supported respectively just below and just above $\sqrt{6}$, with both supports a fixed small distance away from that point. Transform them to $s = 1 - \sqrt{\delta} z$ and normalize each to have unit ground projection. Their H moments have opposite signs for small δ. Their convex combination with the unique zero-moment weight is therefore exactly feasible. For supports within ε of $\sqrt{6}$, each normalized weighted L^1 cost, multiplied by $\sqrt{\delta}$, is $B_1 + O(\varepsilon) + o(1)$. Indeed its cost divided by its ground projection is the weighted average of w/ϕ, and (20) applies uniformly on these supports. Let $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$. This uses genuine bounded bump data, rather than an atom on a null level set, and proves the desired upper bound. □

Proof of the $q = \infty$ assertion. If $v \leq MA$, take $c = 6\beta\delta$. Feasibility implies

$$c = \int v\phi(c - H) \, ds \leq M \int A\phi \, (c - H)_+ \, ds.$$

The global localization and boundary expansion give

$$\begin{aligned} \int A\phi \, (c - H)_+ \, ds &\sim A(1)\kappa\beta\delta^2 \int_0^{\sqrt{12}} z(12 - z^2) \, dz \\ &= 36A(1)\kappa\beta\delta^2. \end{aligned}$$

This proves the lower bound $\liminf \delta M \geq 1/(6\kappa A(1))$.

For recovery use $v = MA$ on the outer layer $1 - h < s < 1$ and zero elsewhere. Its unnormalized observation moment is

$$K_\delta(h) = \int_{1-h}^{1} A\phi H \, ds.$$

Uniformly for bounded R, $h = \sqrt{\delta} R$ gives

$$\delta^{-2} K_\delta(\sqrt{\delta} R) \longrightarrow A(1)\kappa\beta \frac{R^2(R^2 - 12)}{4}.$$

Signs on either side of $\sqrt{12}$ and continuity supply roots $h_\delta/\sqrt{\delta} \to \sqrt{12}$. Choose $M_\delta = (\int_{1-h_\delta}^{1} A\phi \, ds)^{-1}$. The layer data satisfy both constraints exactly and

$$\delta M_\delta \longrightarrow \frac{1}{6\kappa A(1)}.$$

They are bounded and in L^2 for each δ, so they are admissible. The constructions are discontinuous. □

Corollary 5.2. *For physical initial temperatures on the shell, impose the unit ground projection on the rotational average as above. For finite q, the optimal qth power of the physical L^q norm has leading constant*

$$|S^{N-1}| \, b^{p(2-q)} C_q$$

when $1 < q < \infty$, and $|S^{N-1}| b^p / (\kappa\sqrt{6})$ when $q = 1$, with the respective powers of δ from Theorem 5.1. The physical L^∞ constant is $1/(6\kappa b^p)$.

Proof. For radial data $u_0(r) = r^{-p}v(r-a)$ the Jacobian gives exactly

$$\|u_0\|_{L^q(\Omega)}^q = |S^{N-1}| \int_0^1 (a+s)^{p(2-q)} v(s)^q \, ds,$$

and $\|u_0\|_\infty = \operatorname{ess\,sup} v(s)/(a+s)^p$. Apply the theorem with these weights. Rotational averaging preserves an existing whole-boundary constant-flux observation and its normalized ground projection, and Jensen's inequality decreases each physical norm. Radial recoveries supply the matching infima among all nonnegative, possibly nonradial data. No L^2 isometry has been used for $q \neq 2$. □

Remark 5.3. For $q = 2$, $R_2^2 = 14$ and $I_2 = 2\,14^{5/2}/15$, so

$$C_2 = \frac{30}{7\kappa^2 14^{3/2}}, \qquad f_2(z) = \frac{15}{2\kappa 14^{5/2}} z \left(14 - z^2\right)_+ .$$

Theorem 4.3 also shows that the same sharp squared-L^2 infimum holds over smooth strictly positive compatible preparations. Proposition 7.1 below extends equality of the smooth strictly positive compatible infima to every norm considered here. The two endpoint recoveries differ: the L^1 sequence concentrates near a single scaled point, while the L^∞ recovery fills a scaled layer. Neither is inferred by exchanging parameter limits.

6 Stability and the profile of every near-minimizer

Fix the shell and dimension throughout this section. Write $t_\delta = \tau - \delta$, $E_\delta = E(t_\delta)$, and let v_δ^* denote the unique normalized nonnegative least-L^2 initial datum at time t_δ. The ground function satisfies $\|\phi\|_2 = 1$ and $\kappa = -\phi'(1) > 0$. All interval norms and integrals below use Lebesgue measure. The cutoff expansion and global estimate established above are

$$\delta^{-1} H_{t_\delta}(1 - \sqrt{\delta} z) \longrightarrow \beta(z^2 - 6), \qquad \delta^{-1/2}\phi(1 - \sqrt{\delta} z) \longrightarrow \kappa z, \tag{24}$$

$$H_{t_\delta}(1-y) \geq c_0 y^2 - C_0 \delta \qquad (0 \leq y \leq 1), \tag{25}$$

where $\beta, c_0, C_0 > 0$, and the limits are uniform on every bounded nonnegative z-interval. The $q = 2$ recovery construction in the proof of Theorem 5.1, together with the scalar-root uniqueness in Theorem 4.3, identifies the optimizer and gives

$$v_\delta^* = \frac{\phi\,(c_\delta - H_{t_\delta})_+}{Z_\delta}, \qquad Z_\delta = \int_0^1 \phi^2\,(c_\delta - H_{t_\delta})_+ \, ds, \qquad \frac{c_\delta}{\beta\delta} \longrightarrow 8. \tag{26}$$

In particular, with $R_* = \sqrt{14}$,

$$Z_\delta \sim \kappa^2 \beta \delta^{5/2} \frac{2R_*^5}{15}, \qquad E_\delta \sim \frac{30}{7\kappa^2 14^{3/2}} \delta^{-3/2}. \tag{27}$$

These formulas fix all rescaling constants used here.

For any admissible v, define functions on $[0, \infty)$ by

$$W_\delta(z) = \kappa\delta v(1 - \sqrt{\delta} z), \tag{28}$$

$$P_\delta(z) = \sqrt{\delta}\, v(1 - \sqrt{\delta} z)\phi(1 - \sqrt{\delta} z), \tag{29}$$

for $0 < z < \delta^{-1/2}$, and set both functions equal to zero elsewhere. Use superscript $*$ for the rescalings of v_δ^*. The normalization $\int v\phi = 1$ makes P_δ a probability density.

Theorem 6.1 (Near-minimizer stability)**.** *Put*

$$W(z) = \frac{15z\,(14 - z^2)_+}{2\,14^{5/2}}, \qquad P(z) = zW(z), \qquad \varepsilon_\delta = \kappa^2\delta^{3/2}\big(\|v\|_2^2 - E_\delta\big).$$

For every admissible v,

$$\|W_\delta - W_\delta^*\|_2^2 \le \varepsilon_\delta. \tag{30}$$

There are constants $C, \delta_0 > 0$, depending on the fixed shell and its dimension, such that for $0 < \delta < \delta_0$,

$$\|P_\delta - P_\delta^*\|_1 \le C\sqrt{\varepsilon_\delta}. \tag{31}$$

Moreover $W_\delta^* \to W$ strongly in $L^2(0,\infty)$. For any family of admissible data $v = v_\delta$, the following are equivalent:

$$\delta^{3/2}\,\|v_\delta\|_2^2 \longrightarrow \frac{30}{7\kappa^2 14^{3/2}}, \tag{32}$$

$$W_\delta \longrightarrow W \quad \text{strongly in } L^2(0,\infty). \tag{33}$$

Whenever these conditions hold,

$$P_\delta \longrightarrow P \quad \text{in } L^1(0,\infty), \qquad \int_0^\infty z^2 P_\delta(z)\,dz \longrightarrow 6. \tag{34}$$

Proof. Define the signed function

$$g_\delta = \frac{\phi(c_\delta - H_{t_\delta})}{Z_\delta}, \qquad (g_\delta)_+ = v_\delta^*.$$

For every admissible v, the two constraints give $\int g_\delta v = c_\delta/Z_\delta = E_\delta$. Expanding the squared distance therefore gives the exact identity

$$\|v\|_2^2 - E_\delta = \|v - v_\delta^*\|_2^2 + \frac{2}{Z_\delta}\int_0^1 \phi\,(H_{t_\delta} - c_\delta)_+\,v\,ds. \tag{35}$$

Both terms on the right are nonnegative. Changing variables in its first term proves (30).

By (25) and $c_\delta = O(\delta)$, the support of v_δ^* lies in $1-s \le R\sqrt{\delta}$, for a fixed R. Equations (24)–(27) then imply uniform convergence of W_δ^* to W on a common compact interval containing all their supports. Hence this convergence is strong in L^2. The gap inequality and (27) show that (32) implies (33). Conversely, strong convergence gives convergence of the squared norms; since

$$\|W_\delta\|_2^2 = \kappa^2\delta^{3/2}\,\|v\|_2^2\,, \qquad \int_0^\infty W(z)^2\,dz = \frac{30}{7\,14^{3/2}},$$

it implies (32).

We next prove the probability and moment claims using the second term in (35). Increase the fixed R, if necessary, so that $v_\delta^* = 0$ when $y = 1 - s \ge R\sqrt{\delta}$, and

$$H_{t_\delta}(1-y) - c_\delta \ge c_1 y^2 \quad (y \ge R\sqrt{\delta})$$

for some $c_1 > 0$. This follows from the global lower bound by absorbing its $O(\delta)$ terms into $c_0 y^2/2$. Writing $d\mu = v\phi\,ds$, the gap identity and $Z_\delta \le C_1\delta^{5/2}$ yield

$$\int_{y \ge R\sqrt{\delta}} y^2\,d\mu \le \frac{Z_\delta}{2c_1}(\|v\|_2^2 - E_\delta).$$

After rescaling and absorbing the fixed κ into the constant,

$$\int_R^\infty z^2 P_\delta(z)\,dz \le C_2\varepsilon_\delta. \tag{36}$$

On the compact core, $\phi(1-y) \le C_3 y$, and Cauchy–Schwarz gives

$$\begin{aligned}\|P_\delta - P_\delta^*\|_{L^1(0,R)} &\le \left(\int_0^R \left[\frac{\phi(1-\sqrt{\delta}z)}{\kappa\sqrt{\delta}}\right]^2 dz\right)^{1/2} \|W_\delta - W_\delta^*\|_2 \\ &\le C_4\sqrt{\varepsilon_\delta}.\end{aligned}$$

On the tail, $P_\delta^* = 0$, so (36) bounds the L^1 difference by $C_5\varepsilon_\delta$. For $\varepsilon_\delta \le 1$ this proves (31); for larger gaps use the trivial L^1 bound two between probability densities.

For a family satisfying (32), $\varepsilon_\delta \to 0$. Uniform convergence on the common compact support also gives $P_\delta^* \to zW$ in L^1, with convergence of its second moment. Thus (31) proves the first part of (34). On $[0,R]$, the difference of second moments is bounded by $R^2\,\|P_\delta - P_\delta^*\|_1$; the tail moment tends to zero by (36). Finally, direct polynomial integration gives

$$\int_0^\infty zW(z)\,dz = 1, \qquad \int_0^\infty z^3 W(z)\,dz = 6.$$

This proves the second part without assuming that unbounded moments are preserved by weak convergence. □

The L^1 convergence in (34) is equivalently convergence in total variation, with the total-variation distance equal to half the L^1 distance. No convergence rate for the optimizer's limiting profile is required in this theorem.

Corollary 6.2 (Angular rigidity of physical near-minimizers)**.** *Let $u_\delta \ge 0$ be physical initial data on $\Omega = \{a < |x| < a+1\}$, with a whole-boundary equal-flux observation at $\tau - \delta$. Let $\bar{u}_\delta$ be its rotational average and impose $\int_0^1 (a+s)^p \bar{u}_\delta(a+s)\phi(s)\,ds = 1$. If*

$$\delta^{3/2}\,\|u_\delta\|^2_{L^2(\Omega)} \longrightarrow |S^{N-1}|\frac{30}{7\kappa^2 14^{3/2}},$$

then the functions

$$U_\delta(\omega, z) = \kappa\delta(a+1-\sqrt{\delta}z)^p u_\delta\big((a+1-\sqrt{\delta}z)\omega\big), \quad 0 < z < \delta^{-1/2},$$

extended by zero, converge strongly to $W(z)$ in $L^2(S^{N-1}\times(0,\infty), d\sigma\,dz)$.

Proof. Rotational averaging is the orthogonal projection onto radial functions. Put $\bar{v}_\delta(s) = (a+s)^p\bar{u}_\delta(a+s)$. It is admissible, and

$$\|u_\delta\|_2^2 - |S^{N-1}|E_\delta = \|u_\delta - \bar{u}_\delta\|_2^2 + |S^{N-1}|\big(\|\bar{v}_\delta\|_2^2 - E_\delta\big).$$

Both terms are nonnegative. Multiplication by $\delta^{3/2}$ shows that each tends to zero. Theorem 6.1 applies to $\bar{v}_\delta$, while the change of variables $r = a+1-\sqrt{\delta}z$ shows that the squared norm of the rescaled angular difference equals $\kappa^2\delta^{3/2}\,\|u_\delta - \bar{u}_\delta\|_2^2$. The asserted strong convergence follows. □

7 Smooth preparations and stability in the other norms

The next proposition is a density and finite-moment correction argument. It identifies the smooth admissible class for all the cost limits above. The subsequent theorem describes every asymptotic near-minimizer in the strictly convex norm range; its general convexity mechanism is classical [10].

Proposition 7.1 (Equality of smooth infima). *Fix $0 < T < \tau$. Restrict $\mathcal{A}_T$ to smooth functions strictly positive at every interior point and satisfying every Dirichlet heat compatibility condition. This restriction leaves unchanged the infimum of $\int_0^1 wv^q\,ds$, for every $1 \le q < \infty$ and continuous positive w, and the infimum of $\operatorname{ess\,sup} v/A$, for every continuous positive cap A. Thus all the sharp cost limits of Theorem 5.1 hold in this smaller class as infimum statements.*

Proof. Choose one smooth, strictly interior-positive compatible feasible background h, and two fixed nonnegative compact smooth bumps b_1, b_2 whose moment columns $(\int b_i\phi, \int b_i F_T)$ are independent. Their existence was proved in the smooth-approximation part of Theorem 4.3.

For finite q, take a feasible v of finite cost. Positivity and boundedness of w imply $v \in L^q \cap L^2$. Truncation, cutoff near the endpoints, and convolution with a nonnegative mollifier give nonnegative $g_n \in C_c^\infty(0,1)$ converging to v in both spaces. Fix $\eta > 0$ and mix $z_n = (1-\eta)g_n + \eta h$, taking $\eta < 1$. Its moment vector tends to $(1, 0)$. Solving the fixed invertible moment system gives corrections $\alpha_n b_1 + \gamma_n b_2$, with coefficients tending to zero, which restore both moments exactly. On their compact supports the correction is eventually smaller than $\eta h/2$. The corrected functions are therefore strictly positive in the interior, smooth and compatible, with weighted L^q cost tending to that of $(1-\eta)v + \eta h$. Letting $\eta \downarrow 0$, then taking v arbitrarily close to the original infimum, proves the finite-q claim. All boundary compatibility conditions are preserved because every added function other than h is compactly supported in the open interval.

For the cap norm, take a feasible v with $M = \|v/A\|_\infty < \infty$. Cut it off near the endpoints, extend by zero, and convolve with a positive mollifier of radius r_n smaller than the cutoff distance. The resulting compact smooth $g_n \ge 0$ converge in L^1 to v, and

$$g_n(s) \le M\big(A(s) + \omega_A(r_n)\big), \qquad \|g_n/A\|_\infty \le M\left(1 + \frac{\omega_A(r_n)}{\min A}\right),$$

where ω_A is the modulus of continuity on $[0,1]$. Thus their cap norms are at most $M + o(1)$; uniform convergence to the possibly discontinuous v is not required. Their moments converge. Mix with ηh and perform the same fixed-bump correction. For fixed η, its cap cost is asymptotically at most $(1-\eta)M + \eta\,\|h/A\|_\infty$, since the corrections tend uniformly to zero and $\min A > 0$. Their positivity and compatibility follow as before. Taking the two limits proves the cap-norm assertion. □

Theorem 7.2 (Universal finite-q profile). *Fix $1 < q < \infty$, a continuous positive w, and the shell. Use f_q, C_q from Theorem 5.1. For every family $v_\delta \in \mathcal{A}_{\tau-\delta}$, the following statements are equivalent:*

$$\delta^{q-1/2}\int_0^1 wv_\delta^q\,ds \longrightarrow w(1)C_q, \tag{37}$$

$$X_\delta(z) := \delta v_\delta(1-\sqrt{\delta}z) \longrightarrow f_q(z) \quad \text{strongly in } L^q(0,\infty). \tag{38}$$

The rescaled functions are extended by zero for $z \ge \delta^{-1/2}$. The conclusion holds without a support, regularity or compactness assumption on the radial near-minimizers. Endpoint concentration has different topologies and is treated in Section 9.

Proof. Let $\alpha = 1/(q-1)$. The exact recovery construction in the proof of Theorem 5.1 gives, for sufficiently small δ,

$$v_\delta^* = \frac{w^{-\alpha}\,(\phi(c_\delta - H_{\tau-\delta}))_+^\alpha}{Z_\delta}, \qquad Z_\delta = \int_0^1 \phi w^{-\alpha}\,(\phi(c_\delta - H_{\tau-\delta}))_+^\alpha\, ds,$$

where both constraints hold exactly and $c_\delta/(\beta\delta) \to R_q^2 - 6$. For this proof the superscript $*$ denotes the weighted q-candidate. Taking $\theta = -1/c_\delta$ in Theorem 4.1 gives this same normalized candidate: its unnormalized numerator differs by the positive factor $c_\delta^{-\alpha}$. It therefore is the unique global optimum at $\tau - \delta$. In particular

$$\beta\delta\,\theta_{q,w}(\tau - \delta) \longrightarrow -\frac{1}{R_q^2 - 6}.$$

Set $H = H_{\tau-\delta}$ and $g = qZ_\delta^{1-q}\phi(c_\delta - H)$. The general gap (14) becomes

$$\int_0^1 wv^q\, ds - \int_0^1 w(v_\delta^*)^q\, ds = \int_0^1 wB_q(v, v_\delta^*)\, ds + \int_{\{H \ge c_\delta\}} (-g)v\, ds. \tag{39}$$

Both terms on the right are nonnegative. In particular this bounded candidate is the unique minimizer of the weighted q-cost in the stated L^2 admissible class.

We give the elementary convexity estimates needed for stability. Taylor's integral remainder gives

$$B_q(x, y) = q(q-1)(x-y)^2 \int_0^1 (1-t)\big(y + t(x-y)\big)^{q-2}\, dt.$$

For $1 < q < 2$, the segment is at most $x + y$, so

$$B_q(x, y) \ge \frac{q(q-1)}{2}|x-y|^2(x+y)^{q-2}. \tag{40}$$

The right side is defined as zero when $x = y = 0$; the formula at a zero endpoint follows by a limit. For $q \ge 2$, if $x \ge y$, bound the segment below by $t(x-y)$; if $x < y$, bound it below by $(1-t)(y-x)$. The respective integrals give factors one and $q-1$, hence

$$B_q(x, y) \ge |x-y|^q \qquad (q \ge 2). \tag{41}$$

These are standard positive-cone forms of uniform convexity.

Put $Y_\delta(z) = \delta v_\delta^*(1-\sqrt{\delta}z)$ and $w_\delta(z) = w(1-\sqrt{\delta}z)$ on the rescaled domain. The minimum and maximum of these weights are bounded uniformly away from zero and infinity. By (37), the sharp cost limit, and (39),

$$\int w_\delta B_q(X_\delta, Y_\delta)\, dz \longrightarrow 0.$$

For $q \ge 2$, (41) gives $\|X_\delta - Y_\delta\|_q \to 0$. For $1 < q < 2$, (40) and Hölder's inequality give

$$\begin{aligned}\int |X_\delta - Y_\delta|^q\, dz \le &\left(\int |X_\delta - Y_\delta|^2 (X_\delta + Y_\delta)^{q-2}\, dz\right)^{q/2} \\ &\times \left(\int (X_\delta + Y_\delta)^q\, dz\right)^{(2-q)/2} \longrightarrow 0.\end{aligned}$$

The second factor stays bounded because the weights have a positive minimum and both scaled costs are bounded. The recovery construction gives uniform convergence $Y_\delta \to f_q$ on a common compact support, so strong convergence follows.

Conversely, suppose (38) holds. Strong L^q convergence implies convergence of the qth powers in L^1. On every fixed compact interval, $w_\delta \to w(1)$ uniformly. Outside a fixed compact interval containing the support of f_q, the L^q mass of X_δ tends to zero, while the weights remain bounded. Splitting into these two regions proves (37). This avoids an incorrect assumption of uniform weight convergence on the entire expanding domain. □

8 Ground-probability and angular rigidity for every finite power

Strong L^q convergence in Theorem 7.2 does not by itself justify convergence of moments on the expanding scaled interval. The inactive-set term in the exact gap controls these tails. It yields weighted total variation, quadratic transport convergence, and a full physical angular profile for every fixed $1 < q < \infty$. Here W_2 denotes quadratic Wasserstein distance on $[0, \infty)$.

Fix q, w and the shell, and retain the notation of Theorem 7.2. In particular

$$v_\delta^* = \frac{w^{-\alpha}\,(\phi(c_\delta - H_\delta))_+^\alpha}{Z_\delta}, \qquad Z_\delta = \int_0^1 \phi w^{-\alpha}\,(\phi(c_\delta - H_\delta))_+^\alpha \, ds, \qquad \alpha = \frac{1}{q-1},$$

where $H_\delta = H_{\tau-\delta}$ and $c_\delta/(\beta\delta) \to R_q^2 - 6$. Here Z_δ is the normalizer of the $c_\delta - H_\delta$ representation, rather than the normalizer of the affine expression $1 + \theta H_T$ in Theorem 4.1. Define the nonnegative scaled gap

$$G_\delta(v) = \delta^{q-1/2}\left(\int_0^1 w v^q \, ds - E_{q,w}(\delta)\right).$$

For an admissible v of finite cost, put

$$X_\delta(z) = \delta v(1 - \sqrt{\delta} z), \qquad Y_\delta(z) = \delta v_\delta^*(1 - \sqrt{\delta} z), \qquad P_\delta(z) = \frac{\phi(1 - \sqrt{\delta} z)}{\sqrt{\delta}} X_\delta(z),$$

and define P_δ^* from Y_δ. All functions are extended by zero beyond $z = \delta^{-1/2}$; both probability densities have mass one.

Theorem 8.1 (Weighted probability control)**.** *Put $\gamma_q = 1/\max(q, 2)$. There are constants $C, R, \delta_0 > 0$, depending on the fixed q, w and shell, such that every admissible finite-cost v with $0 \le G_\delta(v) \le 1$ and $0 < \delta < \delta_0$ satisfies*

$$\int_R^\infty z^2 P_\delta(z)\, dz \le C G_\delta(v), \tag{42}$$

$$\int_0^\infty (1 + z^2)|P_\delta - P_\delta^*|\, dz \le C G_\delta(v)^{\gamma_q}. \tag{43}$$

Consequently every admissible asymptotic near-minimizer satisfies

$$\int_0^\infty (1 + z^2)|P_\delta - P_q|\, dz \longrightarrow 0, \qquad P_q(z) = \kappa z f_q(z). \tag{44}$$

In particular $\int z^2 P_\delta \to 6$ and $W_2(P_\delta\, dz, P_q\, dz) \to 0$.

Proof. The common compact support of Y_δ and the local expansions (20) imply

$$Z_\delta \sim K_q \delta^{1+3\alpha/2}, \qquad K_q = w(1)^{-\alpha} \kappa^{\alpha+1} \beta^\alpha I_q > 0. \tag{45}$$

Indeed the factors are $\delta^{1/2}$ from ϕ, $\delta^{3\alpha/2}$ from the powered numerator, and $\delta^{1/2}$ from ds. Uniform convergence of positive powers on a common compact interval justifies integration also for $\alpha < 1$. Therefore

$$A_\delta := q\delta^{q+1/2} Z_\delta^{1-q} \longrightarrow qK_q^{1-q} > 0. \tag{46}$$

Choose fixed $R \geq 1$ large enough that $v_\delta^* = 0$ for $1 - s \geq R\sqrt{\delta}$ and

$$H_\delta(1-y) - c_\delta \geq c_1 y^2 \quad (y \geq R\sqrt{\delta})$$

for some $c_1 > 0$. This follows from (21) and $c_\delta = O(\delta)$. The inactive-set term of (39) gives

$$G_\delta(v) \geq q\delta^{q-1/2} Z_\delta^{1-q} \int_{1-s\geq R\sqrt{\delta}} \phi(H_\delta - c_\delta) v \, ds \geq c_1 A_\delta \int_R^\infty z^2 P_\delta(z) \, dz.$$

Equation (46) proves (42). No division by the vanishing ground function at either physical boundary is needed.

The other term of the exact gap, its homogeneity, and the change of variables give

$$\int w_\delta B_q(X_\delta, Y_\delta) \, dz \leq G_\delta(v), \qquad w_\delta(z) = w(1 - \sqrt{\delta} z).$$

The weights have a fixed positive minimum and finite maximum. For $q \geq 2$, (41) gives $\|X_\delta - Y_\delta\|_q \leq C G_\delta^{1/q}$. For $1 < q < 2$, (40) and Hölder give

$$\|X_\delta - Y_\delta\|_q \leq C G_\delta^{1/2} (\|X_\delta\|_q + \|Y_\delta\|_q)^{(2-q)/2}.$$

When $G_\delta \leq 1$, both norms on the right are bounded uniformly: their weighted qth powers are the bounded scaled minimum, plus G_δ for the first one. Hence, throughout the stated range,

$$\|X_\delta - Y_\delta\|_q \leq C G_\delta^{\gamma_q}. \tag{47}$$

On $[0, R]$, $\phi(1 - \sqrt{\delta} z)/\sqrt{\delta}$ is uniformly bounded. Hölder and (47) bound the core of (43) by $C G_\delta^{\gamma_q}$. On $[R, \infty)$, $P_\delta^* = 0$, and (42) bounds the remaining weighted integral by $C G_\delta$. This proves (43).

For near-minimizers $G_\delta \to 0$ by the sharp cost limit. The optimizer densities have common compact support and converge uniformly there to P_q. Its mass and second moment are one and six by (22). This proves (44) and second-moment convergence. To obtain the transport conclusion directly, for any probability densities P, Q keep their common mass $\min(P, Q)$ fixed, and couple the residual measures $(P - Q)_+ \, dz$ and $(Q - P)_+ \, dz$ by their normalized product. The elementary inequality $|z - y|^2 \leq 2z^2 + 2y^2$ gives

$$W_2(P \, dz, Q \, dz)^2 \leq 2 \int_0^\infty z^2 |P - Q| \, dz. \tag{48}$$

If their residual mass is zero, the measures agree and the same inequality holds. Applying it to (44) proves the result. This common-mass coupling is a standard weighted-total-variation bound. □

Theorem 8.2 (The full physical finite-power profile)**.** *Let $u_\delta \geq 0$ belong to $L^2(\Omega)$ on the fixed shell and have a whole-boundary equal-flux observation at $\tau - \delta$. Normalize the ground projection of its rotational average as in Corollary 6.2. Write $p = (N-1)/2$ and $d\bar{\sigma} = d\sigma/|S^{N-1}|$. For fixed $1 < q < \infty$, the following statements are equivalent:*

$$\delta^{q-1/2} \, \|u_\delta\|_{L^q(\Omega)}^q \longrightarrow |S^{N-1}| b^{p(2-q)} C_q, \tag{49}$$

$$X_\delta(\omega, z) := \delta(b - \sqrt{\delta} z)^p u_\delta((b - \sqrt{\delta} z)\omega) \longrightarrow f_q(z) \quad \textit{strongly in } L^q(d\bar{\sigma} \, dz). \tag{50}$$

The rescaling is extended by zero outside its defining interval. Whenever these conditions hold, the joint probability densities

$$P_\delta(\omega, z) = \frac{\phi(1-\sqrt{\delta}z)}{\sqrt{\delta}} X_\delta(\omega, z)$$

satisfy

$$\int (1+z^2)|P_\delta(\omega, z) - P_q(z)|\, d\bar{\sigma}\, dz \longrightarrow 0.$$

Their radial second moments tend to six, and their radial marginals converge to $P_q\, dz$ in W_2.

Proof. Put $V_\delta(\omega, s) = (a+s)^p u_\delta((a+s)\omega)$ and $w(s) = (a+s)^{p(2-q)}$. The exact Jacobian identity is

$$\frac{\|u_\delta\|^q_{L^q(\Omega)}}{|S^{N-1}|} = \int w(s) V_\delta(\omega, s)^q\, ds\, d\bar{\sigma}.$$

This weight is continuous and strictly positive because $a > 0$. The angular average $\bar{V}_\delta$ is admissible. Jensen implies that the physical infimum divided by $|S^{N-1}|$ is exactly $E_{q,w}(\delta)$, since radial optimizers supply equality.

There is a direct product-space form of the exact gap. Its signed dual function depends only on s, and both of its moments against $V_\delta - v^*_\delta$ vanish after angular averaging. Therefore, with

$$G^{\mathrm{phys}}_\delta = \delta^{q-1/2}\left(\frac{\|u_\delta\|^q_{L^q(\Omega)}}{|S^{N-1}|} - E_{q,w}(\delta)\right),$$

the same subtraction as in (39) gives

$$\begin{aligned} G^{\mathrm{phys}}_\delta = &\int w_\delta B_q(X_\delta, Y_\delta)\, dz\, d\bar{\sigma} \\ &+ A_\delta \int \Big(H_\delta(1-\sqrt{\delta}z)/\delta - c_\delta/\delta\Big)_+ P_\delta(\omega, z)\, dz\, d\bar{\sigma}. \end{aligned}$$

Under (49), the sharp radial cost gives $G^{\mathrm{phys}}_\delta \to 0$. The scalar Bregman inequalities used in (47) work unchanged on this product space. They yield $\|X_\delta - Y_\delta\|_{L^q(d\bar{\sigma}dz)} \to 0$; for $q < 2$ the required norm factor is bounded by the scaled physical cost. Since $Y_\delta \to f_q$ on a common compact support, this proves (50).

The inactive-set term gives the same tail bound (42), integrated also in angle. The compact core Hölder estimate likewise gives, when $G^{\mathrm{phys}}_\delta \le 1$,

$$\int (1+z^2)|P_\delta(\omega, z) - P^*_\delta(z)|\, d\bar{\sigma}\, dz \le C(G^{\mathrm{phys}}_\delta)^{\gamma_q}.$$

The optimizer convergence proves the stated joint probability and moment conclusions. Integrating angle and applying (48) proves the radial transport conclusion.

Conversely, strong convergence in (50) implies convergence of the qth powers in L^1 on the product space. The weights w_δ are bounded globally and converge uniformly to $w(1) = b^{p(2-q)}$ on compact z intervals. Outside a fixed interval containing the support of f_q, strong convergence makes the L^q mass tend to zero. Splitting into core and tail in the Jacobian identity proves (49). □

Remark 8.3. The angular part can also be separated by the exact Jensen defect

$$\int w_\delta (X^q_\delta - \bar{X}^q_\delta)\, dz\, d\bar{\sigma} = \int w_\delta B_q(X_\delta, \bar{X}_\delta)\, dz\, d\bar{\sigma},$$

because the linear Bregman term has zero angular average. At every fixed $0 < T < \tau$, the same strict Jensen inequality and Theorem 4.1 show that the physical L^q minimum, corresponding to $w(s) = (a+s)^{p(2-q)}$, is attained uniquely by the radial optimizer, up to equality almost everywhere. The theorem uses the admissible observations specified above.

Proposition 8.4 (Sharp second-moment frontier)**.** *For every fixed $r > 2$, fixed $1 < q < \infty$ and positive continuous w, there are admissible asymptotically optimal weighted-q families for which $\int z^r P_\delta(z)\,dz \to \infty$. The same assertion holds for every positive continuous cap A at the $q = \infty$ endpoint. Thus no general W_r convergence with $r > 2$ follows from near-optimality.*

Proof. Choose $1 < t < r/2$ and $m_\delta = \delta^t$. Fix a nonnegative smooth bump b on a compact physical subinterval of $(0,1)$, normalized by $\int \phi b = 1$. Its moment $h_\delta = \int \phi H_\delta b$ is bounded. For small δ its support is disjoint from the boundary-layer optimizer v_δ^*.

Add $m_\delta b$ to that optimizer. Choose two nonnegative smooth scaled bumps b_1, b_2 on disjoint compact subintervals of $(0, R_q)$, away from its endpoints. Their supports can be chosen sufficiently narrow near distinct positive points that the limiting columns

$$\begin{pmatrix} \int \kappa z b_i\,dz \\ \int \kappa\beta z(z^2-6) b_i\,dz \end{pmatrix}, \qquad i = 1, 2,$$

are independent. The ratios approach distinct values of $\beta(z^2-6)$. The exact columns use the coefficients

$$a_{0,\delta}(z) = \frac{\phi(1-\sqrt{\delta}z)}{\sqrt{\delta}}, \qquad a_{1,\delta}(z) = a_{0,\delta}(z)\frac{H_\delta(1-\sqrt{\delta}z)}{\delta}.$$

Their matrices converge to the invertible limiting matrix and therefore have uniformly bounded inverses. Solve the exact system with right-hand side $(-m_\delta, -m_\delta h_\delta/\delta)$. Its coefficients satisfy $|\alpha_{1,\delta}| + |\alpha_{2,\delta}| = O(\delta^{t-1}) \to 0$. Add the physical corrections

$$\delta^{-1}\alpha_{i,\delta} b_i((1-s)/\sqrt{\delta}), \qquad i = 1, 2.$$

These restore both moments exactly. On their supports the baseline scaled profile converges uniformly to strictly positive f_q, so the corrections preserve nonnegativity for small δ. The resulting data have finite cost and belong to L^2 for each δ.

The scaled core corrections tend to zero in L^q. The added physical bump has scaled qth-power norm

$$\int |\delta m_\delta b(1-\sqrt{\delta}z)|^q\,dz = \delta^{q-1/2} m_\delta^q \int |b|^q\,ds \longrightarrow 0.$$

The scaled profile thus still converges strongly to f_q, and Theorem 7.2 gives the sharp weighted cost. On the support of the physical bump, however, $z \geq c/\sqrt{\delta}$ for some $c > 0$. The corrections are disjoint from that support, so

$$\int z^r P_\delta(z)\,dz \geq c^r \delta^{-r/2} m_\delta \int \phi b\,ds = c^r \delta^{t-r/2} \longrightarrow \infty.$$

The limiting ground probability has compact support. Any coupling to it must transport this interior bump through a scaled distance at least $c/(2\sqrt{\delta})$ for small δ, so its rth transport cost also diverges.

For the cap endpoint start instead with the exactly feasible saturated-layer recovery in Theorem 5.1, choosing the scaled correction bumps inside $(0, \sqrt{12})$. The same moment system and positivity argument apply. Since $\min A > 0$, the corrections increase δ times the cap cost by at most $O(\delta^{t-1})$, and the physical bump increases it by at most $O(\delta m_\delta)$. The sharp lower bound therefore forces the optimal cap asymptotic. The same divergent rth moment remains. □

9 Rigidity at the two endpoint costs

The endpoint cost limits in Theorem 5.1 also determine every asymptotic near-minimizer, in different topologies at the two endpoints. The exposing cubic used for the L^1 lower bound forces concentration at one scaled point. The cap certificate used for the L^∞ lower bound forces saturation of an entire scaled layer. The latter is a complementary-slackness argument of the same general type as quantitative bathtub stability; see [16]. We give direct proofs, including the tails at both physical boundaries.

Keep the shell fixed, put $H_\delta = H_{\tau-\delta}$, and retain $\|\phi\|_2 = 1$, $\kappa = -\phi'(1) > 0$ and $\beta > 0$ from (20). For $v_\delta \in \mathcal{A}_{\tau-\delta}$, define the probability measure

$$d\mu_\delta(z) = \sqrt{\delta}\,\phi(1-\sqrt{\delta}z)v_\delta(1-\sqrt{\delta}z)\,dz, \qquad 0 < z < \delta^{-1/2}, \tag{51}$$

and extend its density by zero. Its total mass is the normalized ground projection. We use

$$\delta^{-1}H_\delta(1-\sqrt{\delta}z) \longrightarrow \beta(z^2-6), \qquad \delta^{-1/2}\phi(1-\sqrt{\delta}z) \longrightarrow \kappa z, \tag{52}$$

$$H_\delta(1-y) \geq c_0 y^2 - C_0\delta, \qquad 0 \leq y \leq 1, \tag{53}$$

where the limits are uniform on bounded z intervals. These are exactly (20) and (21). The ground function is positive in the interior and has simple zeros at both endpoints, so also

$$\phi(s) \geq c\min(s, 1-s) \qquad (0 \leq s \leq 1) \tag{54}$$

for some $c > 0$.

Theorem 9.1 (L^1 concentration of every near-minimizer)**.** *Let w be continuous and positive on $[0,1]$. If an admissible family satisfies*

$$J_\delta := \sqrt{\delta}\int_0^1 w v_\delta\,ds \longrightarrow B := \frac{w(1)}{\kappa\sqrt{6}},$$

then

$$\int_0^\infty (z-\sqrt{6})^2\,d\mu_\delta(z) \longrightarrow 0. \tag{55}$$

Equivalently, μ_δ converges to the unit atom at $\sqrt{6}$ in quadratic Wasserstein distance. In particular its second moment tends to 6. No support or regularity assumption on v_δ is needed.

Proof. Write $z_* = \sqrt{6}$, $D = B/(12\beta)$, and use the dual function from the $q = 1$ proof of Theorem 5.1:

$$g_\delta(s) = \phi(s)\delta^{-1/2}\big(B - DH_\delta(s)/\delta\big), \qquad L_\delta = \max\left(1, \sup_{s\in[0,1]}\frac{g_\delta(s)}{w(s)}\right) \longrightarrow 1.$$

For $0 < z < \delta^{-1/2}$ put

$$k_\delta(z) = \frac{g_\delta(1-\sqrt{\delta}z)}{w(1-\sqrt{\delta}z)}, \qquad d\nu_\delta(z) = \delta w(1-\sqrt{\delta}z)v_\delta(1-\sqrt{\delta}z)\,dz.$$

The mass of ν_δ is J_δ. The two exact moments give the nonnegative dual defect

$$e_\delta := J_\delta - \frac{B}{L_\delta} = \int\left(1 - \frac{k_\delta}{L_\delta}\right)d\nu_\delta \geq 0. \tag{56}$$

Near-optimality implies $e_\delta \to 0$. The global bound (53) confines every positive k_δ to a fixed interval $[0, R_0]$. On each bounded interval,

$$k_\delta(z) \longrightarrow h(z) = \frac{z(18 - z^2)}{12 z_*}, \qquad 1 - h(z) = \frac{(z - z_*)^2 (z + 2z_*)}{12 z_*}.$$

Thus h has unique maximum one at z_*. Fix $R > \max(R_0, z_* + 1)$ and define

$$\omega_\delta = \sup_{0 \le z \le R} |k_\delta(z)/L_\delta - h(z)| \longrightarrow 0, \qquad d_\delta = e_\delta + \omega_\delta J_\delta.$$

Outside $[0, R]$, the integrand in (56) is at least one. On $[0, R]$, $1 - h(z) \ge (z - z_*)^2/6$. Hence

$$\nu_\delta((R, \infty)) \le e_\delta, \qquad \int_0^R (z - z_*)^2 \, d\nu_\delta \le 6 d_\delta. \tag{57}$$

This gives a quantitative concentration estimate for the rescaled cost measure directly from its exact dual defect.

The change from the cost measure to the normalized probability requires care on the expanding domain. On $[0, R]$ only, put

$$a_\delta(z) = \frac{\phi(1 - \sqrt{\delta} z)}{\sqrt{\delta}\, w(1 - \sqrt{\delta} z)}, \qquad a(z) = \frac{\kappa z}{w(1)}, \qquad \theta_\delta = \|a_\delta - a\|_{L^\infty(0,R)} \longrightarrow 0.$$

There $d\mu_\delta = a_\delta \, d\nu_\delta$ and a_δ is uniformly bounded. Since J_δ stays bounded, Cauchy–Schwarz and (57) give

$$\int_0^R |z - z_*| \, d\mu_\delta \le C \sqrt{d_\delta}, \tag{58}$$

$$|\mu_\delta([0, R]) - a(z_*) J_\delta| \le C \sqrt{d_\delta} + \theta_\delta J_\delta + a(z_*) e_\delta. \tag{59}$$

For the second estimate use that a is linear and replace the mass of ν_δ on $[0, R]$ by J_δ, with error at most e_δ. As $a(z_*)B = 1$ and μ_δ has mass one, (59) implies $\mu_\delta((R, \infty)) \to 0$. Together with (58) this proves weak convergence to the atom at z_*. For example, the bounded-Lipschitz distance to that atom is at most

$$C\big(\sqrt{d_\delta} + \theta_\delta + |J_\delta - B| + e_\delta\big)$$

for small δ.

It remains to control the second moment. Enlarge the fixed R so that $c_0 z^2 - C_0 \ge c_1 z^2$ for $z \ge R$, with $c_1 > 0$. Feasibility and (53) imply

$$0 \le c_1 \int_R^\infty z^2 \, d\mu_\delta \le \int_R^\infty \frac{H_\delta(1 - \sqrt{\delta} z)}{\delta} \, d\mu_\delta = - \int_0^R \frac{H_\delta(1 - \sqrt{\delta} z)}{\delta} \, d\mu_\delta. \tag{60}$$

Uniform convergence in (52) and concentration on $[0, R]$ make the last expression tend to zero, since $\beta(z_*^2 - 6) = 0$. The tail second moment therefore vanishes, whereas the second moment on $[0, R]$ tends to $z_*^2 = 6$. Cauchy–Schwarz gives convergence of the first moment from the same tail estimate. Expanding the square proves (55). □

Remark 9.2. The measures in (51) have densities and therefore cannot converge in total variation to an atom. Theorem 9.1 asserts quadratic transport convergence, not a strong L^1 profile for the rescaled initial temperatures.

Theorem 9.3 (Saturation of the near-optimal capped layer)**.** *Let A be continuous and positive on $[0,1]$. Suppose an admissible family has finite cap costs $M_\delta = \|v_\delta/A\|_\infty$ with*

$$\delta M_\delta \longrightarrow \frac{1}{6\kappa A(1)}.$$

Then, for every fixed $1 \le r < \infty$,

$$X_\delta(z) := \delta v_\delta(1-\sqrt{\delta}z) \longrightarrow f_\infty(z) := \frac{1}{6\kappa}\mathbf{1}_{(0,\sqrt{12})}(z) \quad \text{strongly in } L^r(0,\infty). \tag{61}$$

Also μ_δ converges in total variation to the probability density

$$P_\infty(z) = \frac{z}{6}\mathbf{1}_{(0,\sqrt{12})}(z), \qquad \int z^2\, d\mu_\delta \longrightarrow 6.$$

These are implications of near-optimal cap cost. Strong convergence in finite L^r spaces alone does not control the essential supremum.

Proof. Set $c_\delta = 6\beta\delta$, $Q_\delta = \phi(c_\delta - H_\delta)$, and

$$I_\delta = \int_0^1 A\,(Q_\delta)_+\, ds, \qquad \Delta_\delta = M_\delta I_\delta - c_\delta.$$

Feasibility gives $\int Q_\delta v_\delta = c_\delta$. Consequently the pointwise cap $0 \le v_\delta \le M_\delta A$ gives the exact identity

$$\Delta_\delta = \int (Q_\delta)_+ (M_\delta A - v_\delta)\, ds + \int (-Q_\delta)_+ v_\delta\, ds \ge 0. \tag{62}$$

The endpoint calculation in Theorem 5.1 gives $I_\delta \sim 36A(1)\kappa\beta\delta^2$. Thus the assumed cap asymptotic implies $\Delta_\delta/\delta \to 0$.

On the rescaled interval put

$$q_\delta(z) = \delta^{-3/2} Q_\delta(1-\sqrt{\delta}z), \qquad m_\delta(z) = \delta M_\delta A(1-\sqrt{\delta}z).$$

Changing variables in (62) yields

$$\frac{\Delta_\delta}{\delta} = \int (q_\delta)_+ (m_\delta - X_\delta)\, dz + \int (-q_\delta)_+ X_\delta\, dz. \tag{63}$$

Both X_δ and m_δ are bounded globally by a fixed constant K. On bounded intervals,

$$m_\delta \longrightarrow m := \frac{1}{6\kappa}, \qquad q_\delta(z) \longrightarrow \kappa\beta z(12 - z^2)$$

uniformly. On each compact subinterval of $(0,\sqrt{12})$, the positive q_δ is bounded away from zero, so (63) forces $\int (m_\delta - X_\delta) \to 0$. On compact subintervals of $(\sqrt{12},\infty)$, its negative part is bounded away from zero, so $\int X_\delta \to 0$. The cap K bounds the integrals in arbitrarily short neighborhoods of 0 and $\sqrt{12}$. It follows that $X_\delta \to f_\infty$ in L^1 on every bounded interval.

Choose a fixed $R > \sqrt{12}$ large enough that

$$H_\delta(1-y) - c_\delta \ge c_2 y^2 \quad (y \ge R\sqrt{\delta})$$

for small δ and some $c_2 > 0$; this follows from (53). The second term of (62) then gives

$$\int_R^\infty z^2\, d\mu_\delta \le \frac{\Delta_\delta}{c_2\delta} \longrightarrow 0. \tag{64}$$

On $[0,R]$, the density of μ_δ is $\delta^{-1/2}\phi(1-\sqrt{\delta}z)X_\delta(z)$, which converges in L^1 to $\kappa z f_\infty(z) = P_\infty(z)$ by local L^1 convergence and (52). Equation (64) controls the remaining mass and second moment. It proves total variation convergence and

$$\lim \int z^2\, d\mu_\delta = \int_0^{\sqrt{12}} \frac{z^3}{6}\, dz = 6.$$

To prove global unweighted convergence one must also handle the zero of ϕ at the inner endpoint. Fix $h>0$. In the region $0<s<h\sqrt{\delta}$, the z interval has length h, so $\int X_\delta\, dz \le Kh$. On the remaining tail, where $1-s \ge R\sqrt{\delta}$ and $s \ge h\sqrt{\delta}$, (54) implies

$$\frac{\phi(s)}{\sqrt{\delta}} \ge c\min(h,R).$$

Therefore

$$\int_{\{z\ge R,\ s\ge h\sqrt{\delta}\}} X_\delta(z)\, dz \le \frac{\mu_\delta((R,\infty))}{c\min(h,R)} \longrightarrow 0.$$

First take the limit superior in δ, then let $h \downarrow 0$. Together with the local convergence this proves global L^1 convergence. Finally $|X_\delta - f_\infty|$ is uniformly bounded; its rth power is bounded by a fixed constant times its first power for each fixed finite r. This proves (61). □

Corollary 9.4 (Physical angular rigidity at the cap endpoint)**.** *Let $u_\delta \ge 0$ be initial data on the fixed shell $\Omega = \{a<|x|<b\}$, with whole-boundary equal-flux observation at $\tau-\delta$ and the normalized ground projection on its rotational average as in Corollary 6.2. Write $p=(N-1)/2$. If*

$$\delta\, \|u_\delta\|_{L^\infty(\Omega)} \longrightarrow \frac{1}{6\kappa b^p},$$

then the functions

$$X_\delta(\omega,z) = \delta(b-\sqrt{\delta}z)^p u_\delta((b-\sqrt{\delta}z)\omega), \qquad 0<z<\delta^{-1/2},$$

extended by zero, converge to $f_\infty(z)$ strongly in $L^r(S^{N-1}\times(0,\infty), d\sigma\, dz)$ for every fixed $1\le r<\infty$.

Proof. Use normalized surface measure first. Rotational averaging preserves the observation and gives the two exact moments for $\bar v_\delta(s) = (a+s)^p \bar u_\delta(a+s)$. Pointwise the rescaled data obey the cap

$$X_\delta(\omega,z) \le \delta\, \|u_\delta\|_\infty\, (b-\sqrt{\delta}z)^p,$$

which corresponds to $A(s)=(a+s)^p$. Apply the exact defect identity (63), integrating also in ω. Its two nonnegative integrals equal the radial-average defect, which tends to zero. The compact-core proof uses only this defect, nonnegativity and the uniform cap, so proves local L^1 convergence on the product space. The weighted-tail estimate (64) and the inner-endpoint split use the same integrated moments and cap, and therefore give global L^1 convergence there. The uniform cap again gives every finite L^r. Replacing normalized surface measure by $d\sigma$ multiplies each integral by $|S^{N-1}|$ and preserves convergence. □

Proposition 9.5 (The cap profile need not converge in L^∞)**.** *For every positive continuous A there are exactly admissible, asymptotically optimal capped families for which*

$$\liminf_{\delta\downarrow 0} \|X_\delta - f_\infty\|_\infty \ge \frac{1}{6\kappa}.$$

Proof. Start with the exactly feasible saturated-layer recovery from the $q = \infty$ proof of Theorem 5.1:

$$v_\delta^0 = M_\delta^0 A \mathbf{1}_{(1-h_\delta,1)}, \qquad \frac{h_\delta}{\sqrt{\delta}} \longrightarrow \sqrt{12}, \qquad \delta M_\delta^0 A(1) \longrightarrow m = \frac{1}{6\kappa}.$$

Choose three disjoint compact intervals J_0, J_1, J_2 strictly inside $(0, \sqrt{12})$. Choose nonnegative smooth bumps b_1, b_2 supported in J_1, J_2. Their supports can be chosen sufficiently narrow, near distinct positive points, that the two limiting moment columns

$$\begin{pmatrix} \int \kappa z b_i(z)\, dz \\ \int \kappa\beta z(z^2-6) b_i(z)\, dz \end{pmatrix}, \qquad i = 1, 2,$$

are independent: their second-to-first ratios approach distinct values of the strictly increasing function $\beta(z^2 - 6)$.

Remove the scaled baseline X_δ^0 on an interval $B_\delta \subset J_0$ of positive length $\eta_\delta \to 0$. The removed moments are $O(\eta_\delta)$ for the two rescaled moment functionals with coefficients

$$A_{0,\delta}(z) = \frac{\phi(1-\sqrt{\delta}z)}{\sqrt{\delta}}, \qquad A_{1,\delta}(z) = A_{0,\delta}(z)\frac{H_\delta(1-\sqrt{\delta}z)}{\delta}.$$

The matrix with columns $(\int A_{0,\delta} b_i, \int A_{1,\delta} b_i)$ converges to the invertible matrix above. Solve its exact linear system to restore the removed moments by adding $\alpha_\delta b_1 + \gamma_\delta b_2$, where $|\alpha_\delta| + |\gamma_\delta| = O(\eta_\delta)$. Both exact constraints are thus restored, since their rescaled values are precisely 1 and 0.

On $J_1 \cup J_2$, the scaled baseline tends uniformly to $m > 0$. The corrections therefore preserve nonnegativity for small δ. They are disjoint from B_δ, so the corrected profile remains zero on that hole. Since $\min A > 0$, their cap cost is at most $M_\delta^0 + O(\eta_\delta)/\delta$ and hence has the optimal leading constant, by the endpoint lower bound. The data are bounded and in L^2 for each δ. On the positive-measure hole, however, $f_\infty = m$ and $X_\delta = 0$. This proves the asserted essential-supremum obstruction with exact feasibility. □

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