A Positive Corner Limit for Inverse Half-Powers of Vámos Quartics

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Abstract

For each fixed parameter $0 < \lambda \le 64/27$, we determine an anisotropic corner limit of the inverse Laplace distribution of a stable Vámos quartic raised to the power $-1/2$. The limit is an explicit positive density with arcsine slope sections. Convergence holds jointly in all scaled variables, including the variable in which the exact partial inverse has a logarithmic singularity. The proof uses a modified Bessel partial inverse, differentiation to remove logarithmic counterterms, and quantitative bounds for inversion on growing frequency regions. We also determine the first correction in the same joint distribution topology. The result assumes no positivity of the full inverse distribution.

1 Statement and conventions

Put $e_1 = a+b+c+d$, $e_3 = abc+abd+acd+bcd$, and

$$h_\lambda(a,b,c,d) = 4e_1e_3 + \lambda a^2b^2. \tag{1}$$

We use the real stability of (1) in the range $0 < \lambda \le 64/27$: it has no zero when all four arguments have positive real parts. The sharp parameter classification is established in [4]. Stability is an input here; no classification argument is needed in this note.

The square root is chosen positive on the positive real orthant and then continued to that tube. Let $\mathcal{F}_\lambda$ be the inverse Laplace distribution of $h_\lambda^{-1/2}$, with convention

$$\mathcal{L}f(a,b,c,d) = \langle f, e^{-aY_a - bY_b - cY_c - dY_d}\rangle.$$

Inverse transforms on vertical lines have a factor $(2\pi)^{-1}$ in each Fourier variable. These inverses are understood distributionally, after exponential weighting on a fixed vertical contour. Their existence follows also from the polynomial bounds proved below.

Set

$$\alpha = a+c, \qquad \tau = Y_c, \qquad x = Y_a - Y_c, \qquad \beta = Y_b, \quad \delta = Y_d.$$

The coordinate change has absolute determinant one. Write $F_\lambda(\tau,\beta,\delta,x)$ for the inverse in these coordinates. For $\varepsilon > 0$, its pullback by the invertible linear scaling below is denoted

$$F_{\lambda,\varepsilon}(\tau,\beta,\delta,z) = \varepsilon F_\lambda(\tau,\varepsilon\beta,\varepsilon\delta,\varepsilon z).$$

This notation includes the distributional Jacobian convention.

Define

$$k_\pm = \frac{4 \pm 2\sqrt{\lambda}}{4-\lambda}, \tag{2}$$

$$Q_0(B,D) = 4(B+D)^2 - \lambda B^2 = (4-\lambda)(B+k_-D)(B+k_+D), \tag{3}$$

$$f_\lambda(\beta,\delta) = \frac{1}{\pi\sqrt{4-\lambda}\sqrt{(\delta-k_-\beta)(k_+\beta-\delta)}} \tag{4}$$

on the wedge $\beta > 0$, $k_-\beta < \delta < k_+\beta$, and set $f_\lambda = 0$ off its closure. Values on the boundary do not affect its locally integrable distribution.

Theorem 1. *For every fixed $0 < \lambda \le 64/27$,*

$$F_{\lambda,\varepsilon} \longrightarrow \frac{f_\lambda(\beta,\delta)}{2\pi\tau} \quad \text{in } \mathcal{D}'(\mathbb{R}_z \times (0,\infty)_\tau \times \mathbb{R}^2_{\beta,\delta}) \qquad (\varepsilon \downarrow 0). \tag{5}$$

The right side is a positive locally integrable density, with Lebesgue dependence in z. In particular there is no additional term supported at $z = 0$ in this joint limit.

At the endpoint, $k_\pm = (27 \pm 12\sqrt{3})/11$ and the prefactor in (4) is $\sqrt{27/44}/\pi$. Conditioning the limiting density on any fixed $\beta > 0$, and normalizing its δ mass, gives the arcsine law for δ/β on $[k_-, k_+]$. This is a statement about the limiting density.

Inverse powers and their positive representations are treated systematically in [5]. Hyperbolic localization and Lorentz inverse transforms are classical [1, 2]. Our purpose is the concrete anisotropic limit (5); the proof pays explicitly for the logarithmic partial inverse and the exchange of asymptotics with distributional inversion.

2 The exact partial inverse

Throughout the proof, λ is fixed. Constants may depend on it and on fixed positive real parts of contour variables. With $S = b + d$, define

$$Q = 4S(\alpha + S) - \lambda b^2, \quad P = 4\alpha S(\alpha + S), \quad H = 4bd\alpha(\alpha + S), \quad m = \frac{P}{2Q}, \quad r^2 = \frac{H}{Q} + m^2. \tag{6}$$

Direct substitution gives

$$h_\lambda(a, b, \alpha - a, d) = -Qa^2 + Pa + H = Q\{r^2 - (a - m)^2\}. \tag{7}$$

Lemma 1. *On $\operatorname{Re}\alpha, \operatorname{Re} b, \operatorname{Re} d > 0$ the two roots in a can be labeled $a_- = m - r$ and $a_+ = m + r$, holomorphically, so that*

$$\operatorname{Re} a_- \le 0, \qquad \operatorname{Re} a_+ \ge \operatorname{Re}\alpha, \qquad \operatorname{Re} r \ge \tfrac{1}{2}\operatorname{Re}\alpha. \tag{8}$$

The actual partial inverse in a is

$$K(\alpha, b, d; x) = \frac{e^{mx}}{\pi\sqrt{Q}}\, K_0(|x|r), \tag{9}$$

on the entire real x axis, with its canonical locally integrable logarithmic singularity at zero.

Proof. The form $Q = 4(S + \alpha/2)^2 - \lambda b^2 - \alpha^2$ is Lorentzian, and its positive orthant lies in one future cone, since $\lambda < 4$. For a Lorentz quadratic q and a future timelike v,

$$\operatorname{Re}\sqrt{q(v + i\eta)} \ge \sqrt{q(v)}. \tag{10}$$

Indeed, in a Lorentz rest frame write $v = (t, 0)$ and $\eta = (u, w)$. Then $q(v+i\eta) = t^2 - u^2 + |w|^2 + 2itu$. If $t^2 + u^2 - |w|^2 > 0$, squaring the required modulus inequality gives an extra $4t^2|w|^2 \ge 0$; otherwise it is immediate. The positive-real-part square root proves (10). Consequently Q never vanishes in the tube.

For real positive α, b, d, the roots lie below zero and above α, respectively: the leading coefficient is negative and the polynomial is positive at both 0 and α. Stability excludes a root in $0 < \operatorname{Re} a < \operatorname{Re} \alpha$ at every complex parameter in the tube. Along a path from real parameters, roots cannot cross this strip or escape to infinity because $Q \neq 0$. They cannot collide, since their real parts are separated by $\operatorname{Re} \alpha > 0$. This gives the two global holomorphic labels and (8).

For real positive parameters shift the inversion line to $\operatorname{Re} a = m$. There is no root between the lines; horizontal closing integrals tend to zero since the integrand is $O(|\operatorname{Im} a|^{-1})$ there. On the midpoint line the Fourier transform to invert is $Q^{-1/2}(\xi^2 + r^2)^{-1/2}$. The Gaussian integral representation of its inverse gives $K_0(r|x|)/(\pi\sqrt{Q})$; equivalently use the classical integral formulas in [3, §10.32]. This is a distributional identity on all of $\mathbb{R}$, with the locally integrable $\log|x|$ singularity, so it has no unspecified delta term.

For completeness the continuation also identifies the actual inverse. A homogeneous stable polynomial h is hyperbolic in every positive direction v, and $h(v + i\eta) = h(v)\prod_j (1 + i\mu_j)$ with real μ_j. This follows by factoring the real-rooted polynomial $t \mapsto h(\eta + tv)$, using homogeneity. Thus $|h(v + i\eta)| \geq h(v) > 0$. Applied on fixed real-part contours this gives polynomial bounds for $h^{-1/2}$ and its parameter derivatives. The partial inverse is therefore holomorphic as a distribution family. Formula (9) has the same property: the root bounds, the integral representations of K_0, K_1, and its local logarithmic singularity give polynomially bounded locally integrable families on compact x intervals. The identity for real parameters extends holomorphically to the tube, including $x = 0$. This proves the claim. □

3 Bounds for the differentiated scaling

Fix positive real parts of α, B, D, put $S = B + D$ in this section, and let

$$R = 1 + |\operatorname{Im} \alpha| + |\operatorname{Im} B| + |\operatorname{Im} D|, \quad Q_\varepsilon = \varepsilon^2 Q(\alpha, B/\varepsilon, D/\varepsilon) = Q_0 + 4\varepsilon\alpha S, \quad J_0 = \frac{4BDS}{Q_0}.$$

Write $m_\varepsilon, r_\varepsilon$ for (6) at the scaled parameters and $K_\varepsilon(z) = K(\alpha, B/\varepsilon, D/\varepsilon; \varepsilon z)$. By (10), applied with real vector $(\varepsilon \operatorname{Re} \alpha, \operatorname{Re} B, \operatorname{Re} D)$, there is $c > 0$ such that

$$|Q_0|, |Q_\varepsilon| \geq c, \quad |Q_0| \leq CR^2, \qquad cR^{-2} \leq |J_0| \leq CR^3. \tag{11}$$

The last bounds use $|B|, |D|, |S| \geq c$. The exact formulas further give

$$m_\varepsilon = \frac{2\alpha S(S + \varepsilon\alpha)}{Q_\varepsilon}, \qquad |m_\varepsilon| \leq CR^3, \qquad |\partial_\alpha m_\varepsilon| \leq CR^4, \tag{12}$$

$$\varepsilon r_\varepsilon^2 = \frac{4\alpha BD(S + \varepsilon\alpha)}{Q_\varepsilon} + \varepsilon m_\varepsilon^2 = \alpha J_0 + O(\varepsilon R^6). \tag{13}$$

Differentiating the error in (13) gives $O(\varepsilon R^7)$. Since $|\alpha J_0| \geq cR^{-2}$, its relative error is $O(\varepsilon R^8)$ and its α derivative is $O(\varepsilon R^9)$. On the region $R \leq \varepsilon^{-1/256}$, for small enough ε,

$$\partial_\alpha \log r_\varepsilon = \frac{1}{2\alpha} + O(\varepsilon R^9), \qquad |\varepsilon z r_\varepsilon| \leq C\sqrt{\varepsilon}\,|z| R^2. \tag{14}$$

Lemma 2. *For every $Z < \infty$,*

$$\varepsilon^{-1}\partial_\alpha K_\varepsilon(z) \longrightarrow -\frac{1}{2\pi\alpha\sqrt{Q_0}} \tag{15}$$

as distributions in the three contour frequencies and $z \in (-Z, Z)$. The frequency variables can be tested by Schwartz functions, uniformly over compactly supported smooth z tests.

Proof. Put $u = \varepsilon|z|r_\varepsilon$. The exact differentiated expression, using $K_0' = -K_1$, is

$$\varepsilon^{-1}\partial_\alpha K_\varepsilon = \frac{e^{\varepsilon z m_\varepsilon}}{\pi\sqrt{Q_\varepsilon}}\left[\left(\varepsilon z\partial_\alpha m_\varepsilon - \frac{\partial_\alpha Q_\varepsilon}{2Q_\varepsilon}\right)K_0(u) - \varepsilon|z|\partial_\alpha r_\varepsilon K_1(u)\right]. \tag{16}$$

The principal-log convergent series [3, §10.31] give, uniformly as $u \to 0$ in $\operatorname{Re} u > 0$,

$$K_0(u) = -\log(u/2) - \gamma + O(|u|^2(1 + |\log u|)), \qquad K_1(u) = u^{-1} + O(|u|(1 + |\log u|)).$$

On $R \le \varepsilon^{-1/256}$ and $|z| \le Z$, (14) and (12) make both u and $\varepsilon z m_\varepsilon$ uniformly small. The leading term in (16) is $-\partial_\alpha \log r_\varepsilon/(\pi\sqrt{Q_\varepsilon})$. The remaining terms, and the difference of this term from the right side of (15), are bounded, for almost every z, by

$$C\varepsilon R^{16}\big(1 + |\log\varepsilon| + |\log|z||\big). \tag{17}$$

Here one may obtain the powers directly from $\partial_\alpha Q_\varepsilon = 4\varepsilon S$, (12), (14), and $|u|^2 \le C\varepsilon z^2 R^4$; the exponent 16 is deliberately inessential and larger than needed. Polynomial lower bounds in (11) and $|r_\varepsilon| \ge \operatorname{Re}\alpha/2$ also bound $|\log u|$ by a constant times the logarithmic factor in (17) plus $\log R$; the latter is absorbed into the displayed polynomial.

We also need a bound outside that region. For real $v > 0$, the integral representations and elementary estimates at zero and infinity imply

$$e^v K_0(v) \le C(1 + \log^- v), \qquad e^v K_1(v) \le C(1 + v^{-1}), \quad \log^- v = \max(0, -\log v).$$

The same integral representations give $|K_j(u)| \le K_j(\operatorname{Re} u)$. The root-strip inequalities imply

$$|e^{\varepsilon z m_\varepsilon}|e^{-\varepsilon|z|\operatorname{Re} r_\varepsilon} \le 1 \quad (z \in \mathbb{R}). \tag{18}$$

For $z > 0$ use $\operatorname{Re}(m_\varepsilon - r_\varepsilon) \le 0$; for $z < 0$ use $\operatorname{Re}(m_\varepsilon + r_\varepsilon) \ge \operatorname{Re}\alpha$. The rational formulas, (11), and $|r_\varepsilon| \ge \operatorname{Re}\alpha/2$ give, for example, $|\partial_\alpha r_\varepsilon| \le C\varepsilon^{-1}R^7$. Inserting these estimates into (16) gives the coarse global bound

$$|\varepsilon^{-1}\partial_\alpha K_\varepsilon(z)| \le C\varepsilon^{-4}R^{16}(1 + |\log|z||), \qquad |z| \le Z. \tag{19}$$

In particular the K_1 term has no nonintegrable $1/|z|$ singularity: its factor $\varepsilon|z|$ cancels that denominator. The extra inverse powers of ε in (19) absorb $|\log\varepsilon|$.

Pair with a test $\phi(\operatorname{Im}\alpha, \operatorname{Im} B, \operatorname{Im} D, z)$ which is Schwartz in the first three variables and smooth with compact z support. The integral of (17) times $|\phi|$ tends to zero, since $1 + |\log|z||$ is locally integrable. On $R > \varepsilon^{-1/256}$, a Schwartz bound $|\phi| \le C_N R^{-N}$ and (19) bound the tail by $C_N\varepsilon^{-4}\varepsilon^{(N-19)/256}$. Choose $N > 1043$. The limit symbol has polynomial growth and its tail also tends to zero. This proves (15) jointly across $z = 0$. □

4 Inversion and the positive density

We finish the proof of Theorem 1. Inverse Fourier transformation on fixed vertical contours is continuous for the tested distributions in Lemma 2; the exponential weights are smooth multipliers on compact sets in the dual variables. The change $b = B/\varepsilon$, $d = D/\varepsilon$ contributes ε^2 to the inverse in B, D. Differentiation in α is multiplication by $-\tau$. Consequently the inverse of the left side of (15) is exactly

$$-\tau\,\varepsilon F_\lambda(\tau, \varepsilon\beta, \varepsilon\delta, \varepsilon z).$$

The inverse of α^{-1} is the Heaviside density $\mathbf{1}_{\tau>0}$. Thus (15) implies

$$-\tau F_{\lambda,\varepsilon} \longrightarrow -\frac{\mathbf{1}_{\tau>0}}{2\pi}\,\mathcal{L}_{B,D}^{-1}[Q_0^{-1/2}].$$

Restricting to $\tau > 0$ and dividing by the smooth nonzero multiplier $-\tau$ is legitimate in $\mathcal{D}'$. This step avoids inverting the undifferentiated logarithmic counterterms; it asserts nothing at $\tau = 0$.

For the remaining inverse, the gamma integral gives

$$(B + kD)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t(B+kD)}\, dt.$$

Apply this identity to the two factors in (3). Under $\beta = t_1 + t_2$, $\delta = k_- t_1 + k_+ t_2$, the Jacobian is $k_+ - k_- > 0$, and

$$t_1 t_2 = \frac{(k_+\beta - \delta)(\delta - k_-\beta)}{(k_+ - k_-)^2}.$$

The resulting density is precisely (4). It is locally integrable, including at the wedge vertex, as is also immediate from the locally integrable positive gamma density in (t_1, t_2). This establishes (5). The joint convergence proved in Lemma 2, rather than only convergence at fixed $z \neq 0$, excludes any additional distribution supported at $z = 0$. □

5 A first finite correction

The same estimates yield a finite asymptotic expansion. All derivatives below act on the binary distributions before any restriction to a section. Let $D_s = \partial_\beta + \partial_\delta$, and define

$$g_\lambda(\beta, \delta) = \frac{\sqrt{4 - \lambda}}{4\pi\lambda} \sqrt{(\delta - k_-\beta)(k_+\beta - \delta)} \tag{20}$$

on the open wedge, extended by zero outside its closure.

Theorem 2. *For each fixed $0 < \lambda \leq 64/27$, in the same joint distribution space as Theorem 1,*

$$F_{\lambda,\varepsilon} = \frac{f_\lambda}{2\pi\tau} + \varepsilon\, \frac{D_s g_\lambda - z D_s^2 g_\lambda - (z^2/2)\partial_\beta\partial_\delta D_s g_\lambda}{\pi\tau^2} + o(\varepsilon). \tag{21}$$

Boundary derivatives in this formula are distributional and need not be locally integrable densities.

Proof. Write $q = Q_0$ and $J = 4BDS/q$. Expanding the exact formulas gives

$$m_\varepsilon = \alpha A + O(\varepsilon R^N), \quad A = 2S^2/q, \qquad r_\varepsilon^2 = \frac{\alpha J}{\varepsilon} + \alpha^2 E + O(\varepsilon R^N), \quad E = \frac{4BD}{q} + \frac{4S^2(B-D)^2}{q^2},$$

for some finite N on a sufficiently slowly growing frequency ball. Here and below one may enlarge N finitely to accommodate products and derivatives. In particular

$$\log(\varepsilon|z|r_\varepsilon/2) + \gamma = \tfrac{1}{2}\log\alpha + \ell + \frac{\varepsilon\alpha E}{2J} + O(\varepsilon^2 R^N), \qquad \ell = \tfrac{1}{2}\log\varepsilon + \tfrac{1}{2}\log J + \log(|z|/2) + \gamma.$$

Logarithms are canonical holomorphic logarithms agreeing at positive real parameters; explicitly one can take $\log J = \log 4 + \log B + \log D + \log S - \log q$. These expressions are interpreted almost everywhere in z and have locally integrable logarithmic singularities there.

Use the convergent expansion

$$K_0(u) = -L_u + \frac{u^2}{4}(1 - L_u) + O(|u|^4(1 + |\log u|)), \qquad L_u = \log(u/2) + \gamma,$$

and

$$\frac{e^{\varepsilon z m_\varepsilon}}{\sqrt{Q_\varepsilon}} = q^{-1/2}\{1 + \varepsilon\alpha H_0 + O(\varepsilon^2 R^N)\}, \qquad H_0 = zA - 2S/q.$$

Putting $H_1 = H_0 + Jz^2/4$, multiplication gives the explicit finite expansion

$$\begin{aligned} \frac{K_\varepsilon}{\varepsilon} &= -\frac{\log\alpha}{2\pi\sqrt{q}} - \frac{\varepsilon\alpha\log\alpha}{2\pi\sqrt{q}}\, H_1 + U_0 + \varepsilon U_1 + \mathcal{E}_\varepsilon, \\ U_0 &= -\frac{\ell}{\pi\sqrt{q}}, \qquad U_1 = \frac{\alpha}{\pi\sqrt{q}}\left(-\frac{E}{2J} + \frac{Jz^2}{4} - H_1\ell\right). \end{aligned} \tag{22}$$

The coefficients of U_0 and U_1/α are independent of α. Their inverse transforms in α are therefore supported at $\tau = 0$, even though they depend on ε and z.

We justify the remainder before discarding these terms. On $R \le \varepsilon^{-\eta}$, choose $\eta > 0$ sufficiently small that all relative Taylor errors above are small. The exact rational expressions, the lower bounds (11), and the convergent Bessel series give, for one fixed finite N,

$$|\mathcal{E}_\varepsilon| \le C\varepsilon^2 R^N(1 + |\log\varepsilon| + |\log|z||), \qquad |z| \le Z.$$

Indeed the rational Taylor remainders are of second order, and the next Bessel term is $u^4\log u = O(\varepsilon^2|z|^4R^8(1+|\log u|))$. Polynomial lower bounds on J, α, q control every reciprocal and logarithm; decreasing η accommodates the finitely many resulting powers of R. Thus the error divided by ε tends to zero when paired with a Schwartz frequency test and a compact smooth z test. Globally, (18) and the integral bound for K_0 give a bound $C\varepsilon^{-M}R^{N'}(1+|\log|z||)$ for $K_\varepsilon/\varepsilon$, for finite M, N'. The explicit subtracted terms have a bound of the same kind. On $R > \varepsilon^{-\eta}$, Schwartz decay of arbitrarily large order therefore absorbs this bound even after division by ε. This proves $\mathcal{E}_\varepsilon = o(\varepsilon)$ in the joint tested distribution topology. It uses only a finite expansion, not summation of a formal series.

On $\tau > 0$ the inverse of $\log\alpha$ is $-1/\tau$ and the inverse of $\alpha\log\alpha$ is $1/\tau^2$. For example the first identity follows by inverting $\partial_\alpha\log\alpha = \alpha^{-1}$, and the second by the Laplace differentiation rule; possible origin-supported terms do not affect this open half-line. The inverse of $K_\varepsilon/\varepsilon$ is $F_{\lambda,\varepsilon}$. Hence the first correction has binary transform

$$-\frac{H_1}{2\pi\tau^2\sqrt{q}} = \frac{S - zS^2 - (z^2/2)BDS}{\pi\tau^2}\, q^{-3/2}.$$

Gamma integrals of shape $3/2$ in the two factors of (3) give $\mathcal{L}^{-1}[q^{-3/2}] = g_\lambda$: the density before the linear change is $(4-\lambda)^{-3/2}(t_1t_2)^{1/2}/\Gamma(3/2)^2$, and the same Jacobian used above gives (20). Multiplication by B, D becomes $\partial_\beta, \partial_\delta$ on the full binary distributions. This proves (21). □

6 Scope of the conclusion

The theorem gives joint distributional convergence for each fixed positive parameter, treating the inverse throughout as a distribution. Its limiting density is positive on the stated wedge. Lower-order signed contributions can disappear in this limit, and the finite correction is understood in the same distributional topology.

Restricting a distribution to a graph such as $z = \delta - \beta$ requires additional trace or uniform wavefront estimates, together with compatibility of the restriction and limiting operations. These hypotheses identify the analytic conditions for a conditional-section asymptotic.

References

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