Symmetry breaking and positive eigenfunctions for the affine p-Laplacian

Scott Schubr
Luminary Oddities

November 2026

Abstract

We study the Dirichlet eigenvalue problem for the affine p-Laplacian of Haddad, Jiménez and Montenegro. For every $p > 1$ and every dimension $n \geq 2$ we answer in the negative their questions on the simplicity of the first affine eigenvalue and on the uniqueness of the eigenvalue admitting a positive eigenfunction. The explicit smooth shell $\Omega_n = \{1 < |x| < 1+2^{-12n}\}$ carries, for every $p > 1$, a continuum of distinct normalized positive first affine eigenfunctions, while its radial ordinary first p-eigenfunction is a positive affine eigenfunction at a strictly larger eigenvalue; the pth roots of the two eigenvalues differ by a factor at most $2^{7+1/n-4n} \leq 2^{-1/2}$. The proof combines a two-sided comparison of the first affine eigenvalue with equal-volume and inscribed ellipsoids, a radial Poincaré estimate, and a characterization of the functions whose frozen affine anisotropy is Euclidean. For $p = 2$, building on the linear-position reduction of Schindler and Tintarev, we classify all affine eigenfunctions as linear pullbacks of Dirichlet eigenfunctions with scalar energy tensor, identify the eigenvalues carrying positive eigenfunctions with the critical values of the real-analytic function $T \mapsto \lambda_1(T\Omega)$, and prove that on every bounded connected domain the first affine eigenvalue is isolated in the affine spectrum, with locally finitely many eigenvalues carrying positive eigenfunctions. Finally, a planar pentagon pinched into two triangular lobes yields affine-isotropic saddles with trivial affine symmetry group, positive affine eigenfunctions above the first eigenvalue, and exact inverse-gap curvature asymptotics. These saddles show that the isotropy characterization and the log-convexity established by Faifman and Polterovich for convex bodies fail for these nonconvex domains.

Keywords: affine p-Laplacian; positive eigenfunction; simplicity; spectral gap; symmetry breaking; Dirichlet eigenvalue; linear position.

2020 Mathematics Subject Classification: 35P30, 35P15, 49Q10, 52A40.

1 Introduction

For the Dirichlet p-Laplacian on a bounded connected domain, the first eigenvalue is simple and it is the only eigenvalue with a positive eigenfunction. Haddad, Jiménez and Montenegro [4] introduced the affine p-Laplacian $\Delta_p^{\mathcal{A}}$, the Euler–Lagrange operator of Zhang's affine L^p energy, proved that its first variational eigenvalue $\lambda_{1,p}^{\mathcal{A}}(\Omega)$ is attained, and identified its first eigenfunctions with the minimizers of the affine Rayleigh quotient. The operator is quasilinear and nonlocal: its coefficients depend on the full directional gradient profile of the unknown function. For a bounded open set $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, and $p > 1$, they raise the following first issues of the spectral theory of $\Delta_p^{\mathcal{A}}$ [4, Section 7(1)]:

- "Is it possible to characterize all eigenfunctions corresponding to $\lambda_{1,p}^{\mathcal{A}}(\Omega)$? Is the first affine eigenvalue simple?"
- "Is $\lambda_{1,p}^{\mathcal{A}}(\Omega)$ the unique eigenvalue which admits positive eigenfunction?"
- "Does $\Delta_p^{\mathcal{A}}$ have any spectral gap?"

This paper gives the following answers.

1. For every $p > 1$ and $n \geq 2$, one explicit smooth shell has a continuum of distinct normalized positive first affine eigenfunctions and a positive affine eigenfunction at a strictly larger eigenvalue (Theorem A in Section 3). The first affine eigenvalue is therefore not simple, and positivity does not single out the first eigenvalue, for every exponent.

2. For $p = 2$, every affine eigenfunction is a linear pullback of a Dirichlet eigenfunction whose energy tensor is scalar. The positive eigenfunctions correspond exactly to the critical points of $T \mapsto \lambda_1(T\Omega)$ on positive-definite determinant-one matrices, and the first eigenfunctions to its minimizers (Theorem B1). This characterizes the first eigenfunctions at $p = 2$.

3. For $p = 2$ and every bounded connected Ω, the first affine eigenvalue is an isolated point of the affine spectrum, and in every bounded interval only finitely many eigenvalues carry positive eigenfunctions (Theorem B2).

4. A planar pentagon with trivial affine symmetry group carries positive affine 2-eigenfunctions at two distinct eigenvalues; it is an affine-isotropic saddle with exact inverse-gap curvature asymptotics (Section 5).

Along the way, Proposition A2 sharpens the rigidity statement [4, Theorem 7(a)]: equality of the first affine and ordinary p-eigenvalues is governed by constant directional energy profile of the ordinary ground state, and at $p = 2$ it holds exactly in minimizing linear position (Remark 3.1).

The mechanism

Rotation-invariant functions on a shell are radial, and on radial functions the affine energy equals the ordinary gradient norm [4, Proposition 3]. The ordinary first p-eigenfunction of a shell is radial, so it solves the affine equation at the ordinary first eigenvalue. Proposition A2 explains this precisely: the frozen affine anisotropy of a function is Euclidean exactly when its directional L^p energy profile is constant on the sphere. On the other hand, the first affine eigenvalue is controlled from above by the largest inscribed ellipsoid (Lemma A1). A shell of thickness δ contains an ellipsoid of volume comparable to $\delta^{(n+1)/2}$, which bounds $\lambda^{\mathcal{A}}_{1,p}(\Omega_\delta)^{1/p}$ by a multiple of $\delta^{-(n+1)/(2n)}$, while $\lambda^{1/p}_{1,p}$ is at least a multiple of δ^{-1}, uniformly in p. This quantitative comparison places the first affine eigenvalue strictly below the ordinary one, so every positive first affine eigenfunction is nonradial, and its rotation orbit is a continuum.

At $p = 2$ the affine energy is $n(\det E_u)^{1/n}$, where $E_u = \int \nabla u \nabla u^T$, and the problem reduces to the linear orbit of the domain. This reduction is due to Schindler and Tintarev [10, Lemma 2.1], who introduced the affine Laplacian; the operator $\Delta^{\mathcal{A}}_2$ of [4] coincides with theirs. We develop it into a classification of all affine eigenfunctions and into the spectral gap theorem, whose proof combines analytic perturbation theory [6], coercivity of the orbit, the Łojasiewicz gradient inequality [8] and a uniform second-eigenvalue gap. A short Schur-lemma argument (Proposition C) shows that at $p = 2$ any irreducible symmetry group forces non-simplicity as soon as the ordinary first eigenvalue exceeds the affine one.

The symmetric examples use the symmetry group in an essential way. The pentagon construction of Section 5 produces a positive affine eigenfunction above the first eigenvalue on a domain whose only affine self-map is the identity. Its critical position is a saddle of $T \mapsto \lambda_1(T\Omega)$ created by a closing odd spectral gap between two almost separated lobes; the shear curvature has exact inverse-gap asymptotics. For convex bodies, Faifman and Polterovich [3] proved log-convexity of the first eigenvalue along positive-definite deformations, uniqueness of the minimizing linear position, and its characterization by energy isotropy. The pentagon saddles show that the isotropy characterization and the log-convexity hold in the convex class and fail for these nonconvex domains. Combined with [3], the classification of Theorem B1 gives uniqueness of the positive affine 2-eigenfunction on convex domains (Remark 6.1).

The spectral gap theorem is a statement about the quadratic case, where the eigenvalues carrying positive eigenfunctions form the critical-value set of a real-analytic function on a finite-dimensional manifold. The non-simplicity and positive-eigenvalue results hold for every $p > 1$.

Related work

The determinant form of the quadratic affine energy and its relation to linear normalization are part of established affine energy theory [5, 10, 9]; the bounded-domain quadratic reduction is explicit in [9, Proposition 2.6]. Langharst and Roysdon [7] develop first-eigenvalue and isoperimetric inequalities for higher-order affine p-Laplace operators. Throughout, $\lambda_{k,p}(\Omega)$ denotes the kth ordinary Dirichlet p-eigenvalue and $\lambda_k = \lambda_{k,2}$ the kth Dirichlet Laplace eigenvalue, with multiplicity. Positivity of a function means strict positivity in the interior of its domain.

2 The affine p-energy

Fix $n \geq 2$ and $1 < p < \infty$, and let σ be the rotation-invariant probability measure on S^{n-1}. For a bounded open set $U \subset \mathbb{R}^n$, functions in $W_0^{1,p}(U)$ are extended by zero. For $0 \neq u \in W_0^{1,p}(U)$ and $\xi \in S^{n-1}$ put $d_u(\xi) = \|\partial_\xi u\|_p$, where $\partial_\xi u = \xi \cdot \nabla u$, and

$$A_p = \int_{S^{n-1}} |\xi_1|^p \, d\sigma(\xi), \qquad \mathcal{E}_p(u) = A_p^{-1/p} \Big(\int_{S^{n-1}} d_u(\xi)^{-n} \, d\sigma(\xi) \Big)^{-1/n}, \qquad \mathcal{R}_p(u) = \frac{\mathcal{E}_p(u)^p}{\|u\|_p^p}. \tag{1}$$

This is Zhang's energy in the normalization of [4, (3)], rewritten with the probability measure: the constant $c_{n,p}$ of [4] equals $A_p^{-1/p} |S^{n-1}|^{1/n}$. The first affine eigenvalue is

$$\lambda_{1,p}^{\mathcal{A}}(U) = \inf_{0 \neq u \in W_0^{1,p}(U)} \mathcal{R}_p(u),$$

and $\lambda_{1,p}(U)$ is the corresponding infimum of $\|\nabla u\|_p^p / \|u\|_p^p$. Linewise Poincaré inequalities for zero extensions give $d_u \geq c\|u\|_p > 0$, and Minkowski's inequality gives $|d_u(\xi) - d_u(\eta)| \leq |\xi - \eta| \, \|\nabla u\|_p$; thus d_u is positive and Lipschitz on the sphere. The frozen norm of u is

$$H_u(v)^p = A_p^{n/p} \mathcal{E}_p(u)^{n+p} \int_{S^{n-1}} d_u(\xi)^{-n-p} |\xi \cdot v|^p \, d\sigma(\xi), \qquad v \in \mathbb{R}^n, \tag{2}$$

which is H_f^p of [4, Section 1] in our measure normalization. Differentiating (1) gives $\frac{1}{p} D(\mathcal{E}_p^p)(u)[\phi] = \int_U H_u(\nabla u)^{p-1} \nabla H_u(\nabla u) \cdot \nabla \phi$. Following [4, (8)], a nonzero $u \in W_0^{1,p}(U)$ is an *affine p-eigenfunction* at ν if

$$\int_U H_u(\nabla u)^{p-1} \nabla H_u(\nabla u) \cdot \nabla \phi \, dx = \nu \int_U |u|^{p-2} u \phi \, dx \qquad (\phi \in W_0^{1,p}(U)), \tag{3}$$

that is, $\Delta_p^{\mathcal{A}} u = \nu |u|^{p-2} u$ weakly. Testing with $\phi = u$ and using $\int_U H_u(\nabla u)^p = \mathcal{E}_p(u)^p$ shows $\nu = \mathcal{R}_p(u)$. We call u *normalized* if $\|u\|_p = 1$.

We use the following results of [4].

- [4, Theorem 3]: for bounded open U, the infimum $\lambda_{1,p}^{\mathcal{A}}(U)$ is attained in $W_0^{1,p}(U)$.
- [4, Theorem 4]: u is an affine eigenfunction at $\lambda_{1,p}^{\mathcal{A}}(U)$ if and only if it minimizes $\mathcal{R}_p$; $\lambda_{1,p}^{\mathcal{A}}(U)$ is the smallest real Dirichlet eigenvalue of $\Delta_p^{\mathcal{A}}$; first eigenfunctions are bounded, of class $C^{1,\alpha}$ in U, and do not change sign when U is connected.
- [4, Theorem 5] (affine Faber–Krahn): if E is an ellipsoid with $|E| = |U|$, then $\lambda_{1,p}^{\mathcal{A}}(U) \geq \lambda_{1,p}^{\mathcal{A}}(E)$, with equality exactly for ellipsoids.
- [4, Theorem 6(a)]: $\lambda_{1,p}^{\mathcal{A}}(U) \leq \lambda_{1,p}(U)$.
- [4, Proposition 3]: $\Delta_p^{\mathcal{A}} f = \Delta_p f$ for radial f.

Lemma 2.1 (Affine covariance)**.** *Let $A \in \mathrm{GL}(n, \mathbb{R})$, $c \in \mathbb{R}^n$, and $v(x) = u(A^{-1}(x - c))$. Then $\|v\|_p = |\det A|^{1/p} \|u\|_p$ and $\mathcal{E}_p(v) = |\det A|^{1/p - 1/n} \mathcal{E}_p(u)$. Consequently $\mathcal{R}_p(v) = |\det A|^{-p/n} \mathcal{R}_p(u)$ and $\lambda_{1,p}^{\mathcal{A}}(AU + c) = |\det A|^{-p/n} \lambda_{1,p}^{\mathcal{A}}(U)$.*

Proof. Put $M = A^{-1}$. Since $\partial_\xi v(x) = (M\xi)\cdot\nabla u(A^{-1}(x-c))$, a change of variables gives

$$d_v(\xi) = |\det A|^{1/p}|M\xi|\, d_u(M\xi/|M\xi|).$$

For a positive continuous f on S^{n-1}, $|B|\int f\,d\sigma$ is the volume of the star body K_f with radial function $f^{1/n}$, and $\xi\mapsto f(M\xi/|M\xi|)^{1/n}|M\xi|^{-1}$ is the radial function of $M^{-1}K_f$. Hence

$$\int_{S^{n-1}} f(M\xi/|M\xi|)\,|M\xi|^{-n}\,d\sigma(\xi) = |\det M|^{-1}\int_{S^{n-1}} f\,d\sigma.$$

Applying this to $f = d_u^{-n}$ gives $\int d_v^{-n}\,d\sigma = |\det A|^{1-n/p}\int d_u^{-n}\,d\sigma$, which is the energy identity. The remaining assertions follow. □

Lemma 2.2 (Strict positivity)**.** *Let U be bounded, open and connected. A nonnegative affine p-eigenfunction at a value $\nu > 0$ is strictly positive in U.*

Proof. By [4, Proposition 5] with the constant weight ν, $u\in L^s(U)$ for all $s<\infty$; [4, Proposition 4] (or Sobolev embedding when $p>n$) then gives $u\in L^\infty(U)\cap C^{1,\alpha}_{\mathrm{loc}}(U)$. For the frozen field $a_u(v) = \nabla_v(H_u(v)^p/p)$, the positive upper and lower bounds of d_u and the rotation invariance of σ give the structure conditions $c|v|^{p-2}|\eta|^2\le\eta^T Da_u(v)\eta$ and $|Da_u(v)|\le C|v|^{p-2}$ of [4, Section 4.3, (T.1)–(T.2)], and hence $a_u(v)\cdot v\ge c|v|^p$ and $|a_u(v)|\le C|v|^{p-1}$. The function u is a nonnegative weak supersolution of the quasilinear equation $-\operatorname{div} a_u(\nabla w)=0$, since $-\operatorname{div} a_u(\nabla u) = \nu u^{p-1}\ge 0$. The weak Harnack inequality for such operators [11] and connectedness give $u>0$ in U; this replaces the strong maximum principle used in the proof of [4, Theorem 4]. □

3 All exponents: an explicit shell

3.1 Ellipsoid bounds and Euclidean anisotropy

For a bounded open Ω let $\varepsilon(\Omega)$ be the supremum of the volumes of open ellipsoids contained in Ω, and let B be the open unit ball.

Lemma A1. *For every bounded open $\Omega\subset\mathbb{R}^n$ and $p>1$,*

$$\lambda^{\mathcal{A}}_{1,p}(B)\Big(\frac{|B|}{|\Omega|}\Big)^{p/n}\le\lambda^{\mathcal{A}}_{1,p}(\Omega)\le\lambda^{\mathcal{A}}_{1,p}(B)\Big(\frac{|B|}{\varepsilon(\Omega)}\Big)^{p/n},\qquad \lambda^{\mathcal{A}}_{1,p}(B)\le\lambda_{1,p}(B). \tag{4}$$

Both bounds are equalities when Ω is an ellipsoid.

Proof. If $E=c+TB$ is an ellipsoid, Lemma 2.1 gives

$$\lambda^{\mathcal{A}}_{1,p}(E) = |\det T|^{-p/n}\lambda^{\mathcal{A}}_{1,p}(B) = \lambda^{\mathcal{A}}_{1,p}(B)\,(|B|/|E|)^{p/n}.$$

The lower bound is the affine Faber–Krahn inequality [4, Theorem 5] applied with a ball of volume $|\Omega|$. If $E\subset\Omega$, then $W^{1,p}_0(E)\subset W^{1,p}_0(\Omega)$ by zero extension, so $\lambda^{\mathcal{A}}_{1,p}(\Omega)\le\lambda^{\mathcal{A}}_{1,p}(E)$; taking the infimum over inscribed ellipsoids gives the upper bound. The last inequality is [4, Theorem 6(a)]. □

By [4, proof of Theorem 7(a)] (ball case), the first affine and ordinary eigenvalues of a ball coincide; only the inequality in (4) is used below.

Remark 3.1 (The rigidity statement of [4]). The rigidity statement [4, Theorem 7(a)] asserts that, for $p>1$, $\lambda^{\mathcal{A}}_{1,p}(\Omega)=\lambda_{1,p}(\Omega)$ only when Ω is a ball. Its proof rests on the implication from $\mathcal{E}_p(f)=\|\nabla f\|_p$ to radial symmetry of f; by Proposition A2 below, this equality characterizes constant directional profile $\xi\mapsto\|\partial_\xi f\|_p$. The correct form of the rigidity statement therefore reads: $\lambda^{\mathcal{A}}_{1,p}(\Omega)=\lambda_{1,p}(\Omega)$ exactly when the positive ordinary ground state minimizes $\mathcal{R}_p$ and has constant directional profile. At $p=2$, equality holds exactly when Ω is in minimizing linear position (Lemma 4.1(c)). Combined with the preprint [3, Corollary 2.11], by which every convex body with irreducible symmetry group is in minimizing linear position, this gives equality for the square, and more generally for every such convex body. The quantitative comparison of Theorem A is therefore what places $\lambda^{\mathcal{A}}_{1,p}(\Omega_n)$ strictly below $\lambda_{1,p}(\Omega_n)$.

Proposition A2. *Let U be bounded and open, $p > 1$, and $0 \neq u \in W_0^{1,p}(U)$. The following are equivalent:*

(i) $H_u(v) = \kappa|v|$ *for some* $\kappa > 0$ *and all* v*;*

(ii) $\xi \mapsto \|\partial_\xi u\|_p$ *is constant on* S^{n-1}*;*

(iii) $H_u(v) = |v|$ *for all* v*, and* $\mathcal{E}_p(u) = \|\nabla u\|_p$*.*

In this case (3) *for* u *is the weak form of* $-\Delta_p u = \nu|u|^{p-2}u$*. In particular,* u *is an affine* p*-eigenfunction exactly when it is an ordinary Dirichlet* p*-eigenfunction, with the same eigenvalue.*

Proof. Write $M_q = (\int d_u^q \, d\sigma)^{1/q}$ for the power means of d_u, so that $\mathcal{E}_p(u) = A_p^{-1/p} M_{-n}$. Integrating $|\xi \cdot \nabla u|^p$ over $U \times S^{n-1}$ in both orders gives $M_p^p = A_p \|\nabla u\|_p^p$, and averaging (2) over $v \in S^{n-1}$ gives

$$\int_{S^{n-1}} H_u(v)^p \, d\sigma(v) = A_p^{n/p+1} \mathcal{E}_p(u)^{n+p} M_{-n-p}^{-n-p} = \Big(\frac{M_{-n}}{M_{-n-p}}\Big)^{n+p}. \tag{5}$$

(ii)⇒(iii): if $d_u \equiv d_0$, then $\mathcal{E}_p(u) = A_p^{-1/p} d_0$ and (2) gives $H_u(v)^p = A_p^{-1} \int |\xi \cdot v|^p d\sigma = |v|^p$; moreover $\|\nabla u\|_p^p = A_p^{-1} M_p^p = A_p^{-1} d_0^p = \mathcal{E}_p(u)^p$. (iii)⇒(i) is trivial. (i)⇒(ii): Fubini gives $\int_U H_u(\nabla u)^p \, dx = A_p^{n/p} \mathcal{E}_p(u)^{n+p} \int d_u^{-n} d\sigma = \mathcal{E}_p(u)^p$. With (i) this reads $\kappa^p \|\nabla u\|_p^p = \mathcal{E}_p(u)^p$, so $\kappa^p = (M_{-n}/M_p)^p \leq 1$. By (5), $\kappa^p = (M_{-n}/M_{-n-p})^{n+p} \geq 1$. Hence $\kappa = 1$ and $M_{-n} = M_p$; equality in the power-mean inequality for the continuous function d_u forces d_u to be constant.

When $H_u = |\cdot|$, the left side of (3) is $\int |\nabla u|^{p-2} \nabla u \cdot \nabla \phi$, which proves the last assertions. □

Condition (ii) says that the polar L^p projection body $\{\xi : \|\partial_\xi u\|_p \leq 1\}$ of u is a Euclidean ball.

Remark 3.2 (Symmetry and Euclidean anisotropy). Let a closed group $G \subset O(n)$ preserve a connected U, and let φ_p be the positive ordinary first p-eigenfunction. By simplicity, $\varphi_p \circ g = \varphi_p$ for $g \in G$, hence d_{φ_p} is G-invariant. If G is transitive on S^{n-1}, as $SO(n)$ is, condition (ii) holds for every $p > 1$. For $p = 2$, $d_u(\xi)^2 = \xi^T E_u \xi$ with $E_u = \int \nabla u \nabla u^T$, and (ii) holds whenever G acts irreducibly, by Schur's lemma. For $p = 2k$ and $n = 2$, $d_u(\xi)^{2k}$ is a homogeneous polynomial of degree $2k$, hence a trigonometric polynomial of degree at most $2k$ on the circle; invariance under the rotation by $2\pi/m$ with $m > 2k$ removes every nonconstant frequency, so (ii) holds for the cyclic group C_m. For exponents $p \notin 2\mathbb{N}$ the profile d_u^p is no longer a polynomial, and the construction below uses the full rotation group.

3.2 The shell theorem

For $0 < \delta \leq 1$ put $\Omega_\delta = \{x \in \mathbb{R}^n : 1 < |x| < 1 + \delta\}$.

Theorem A. *For every integer* $n \geq 2$ *set*

$$\delta_n = 2^{-12n}, \qquad \Omega_n = \Omega_{\delta_n} = \{x \in \mathbb{R}^n : 1 < |x| < 1 + 2^{-12n}\}, \qquad r_n = 2^{7+1/n-4n}.$$

The domain Ω_n *is bounded, connected and nonconvex, with* C^∞ *boundary. For every* $p \in (1, \infty)$*,*

$$\lambda_{1,p}^{\mathcal{A}}(\Omega_n)^{1/p} \leq r_n \, \lambda_{1,p}(\Omega_n)^{1/p}, \qquad r_n \leq 2^{-1/2}. \tag{6}$$

On Ω_n *there is a continuum of distinct normalized strictly positive affine* p*-eigenfunctions at* $\lambda_{1,p}^{\mathcal{A}}(\Omega_n)$*, namely the* $SO(n)$*-orbit of any one of them. The normalized positive ordinary first* p*-eigenfunction of* Ω_n *is radial and is a strictly positive affine* p*-eigenfunction at* $\lambda_{1,p}(\Omega_n) > \lambda_{1,p}^{\mathcal{A}}(\Omega_n)$*.*

Lemma 3.3 (Ordinary lower bound)**.** *For* $p > 1$ *and* $0 < \delta \leq 1$*,* $\lambda_{1,p}(\Omega_\delta)^{1/p} \geq (1 + \delta)^{-(n-1)/p} \delta^{-1} \geq 2^{-(n-1)} \delta^{-1}$*.*

Proof. Let $u \in C_c^\infty(\Omega_\delta)$ and $\theta \in S^{n-1}$. The profile $w(r) = u(r\theta)$ vanishes near $r = 1$, so $|w(t)| \le \int_1^t |w'| \le \delta^{1-1/p}\|w'\|_{L^p(1,1+\delta)}$ and $\int_1^{1+\delta} |w|^p\, dr \le \delta^p \int_1^{1+\delta} |w'|^p\, dr$. Since $1 \le r^{n-1} \le (1+\delta)^{n-1}$ and $|w'(r)| \le |\nabla u(r\theta)|$,

$$\int_1^{1+\delta} |u(r\theta)|^p r^{n-1}\, dr \le (1+\delta)^{n-1}\delta^p \int_1^{1+\delta} |\nabla u(r\theta)|^p r^{n-1}\, dr.$$

Integrating over θ gives $\|u\|_p^p \le (1+\delta)^{n-1}\delta^p \|\nabla u\|_p^p$, which extends to $W_0^{1,p}(\Omega_\delta)$ by density. The final inequality uses $(1+\delta)^{(n-1)/p} \le 2^{n-1}$. □

Lemma 3.4 (Inscribed ellipsoid)**.** *Let $c = (1+\delta/2)e_1$ and $T = \operatorname{diag}(\delta/4, \sqrt{\delta}/2, \dots, \sqrt{\delta}/2)$. Then $\overline{c+TB} \subset \Omega_\delta$, and $|c+TB| = D|B|$ with $D = \det T = 2^{-(n+1)}\delta^{(n+1)/2}$. Consequently*

$$\lambda_{1,p}^{\mathcal{A}}(\Omega_\delta)^{1/p} \le \lambda_{1,p}(B)^{1/p} D^{-1/n} \le 2^{1+n/p} D^{-1/n} \le 2^{n+2+1/n}\delta^{-(n+1)/(2n)} \qquad (p > 1). \tag{7}$$

Proof. For $x = c + Ty$ with $|y| \le 1$, we have $1 + \delta/4 \le x_1 \le 1 + 3\delta/4$ and $|x'|^2 = (\delta/4)|y'|^2 \le \delta/4$, where x' collects the last $n-1$ coordinates. Hence $|x| \ge x_1 > 1$ and $|x|^2 \le (1+3\delta/4)^2 + \delta/4 = (1+\delta)^2 - \delta/4 - 7\delta^2/16 < (1+\delta)^2$. This proves the containment. The first inequality in (7) is Lemma A1. For the second, the cone $f(y) = (1-|y|)_+$ satisfies $|\nabla f| = 1$ a.e. in B and $f \ge 1/2$ on $\frac{1}{2}B$, so $\|\nabla f\|_p/\|f\|_p \le |B|^{1/p}/(\frac{1}{2}(2^{-n}|B|)^{1/p}) = 2^{1+n/p}$. The last inequality uses $2^{1+n/p} \le 2^{n+1}$ and $D^{-1/n} = 2^{(n+1)/n}\delta^{-(n+1)/(2n)}$. □

Lemma 3.5 (Radial functions)**.** *If $u \in W_0^{1,p}(\Omega_\delta)$ is radial and nonzero, then $\|\partial_\xi u\|_p^p = A_p\|\nabla u\|_p^p$ for all $\xi \in S^{n-1}$; hence $H_u = |\cdot|$ and $\mathcal{E}_p(u) = \|\nabla u\|_p$. The normalized positive ordinary first p-eigenfunction φ_p of Ω_δ is radial and is an affine p-eigenfunction at $\lambda_{1,p}(\Omega_\delta)$.*

Proof. For $u(x) = w(|x|)$, $\partial_\xi u(x) = w'(|x|)\,\xi \cdot x/|x|$; polar coordinates and rotation invariance of σ give the first identity, and Proposition A2 gives the rest (compare [4, Proposition 3]). The ordinary first p-eigenfunction of a connected domain is simple, and rotations preserve Ω_δ and the ordinary equation, so $\varphi_p \circ g = \varphi_p$ for every $g \in SO(n)$. Thus φ_p is radial, and Proposition A2 identifies its ordinary equation with (3) at $\nu = \lambda_{1,p}(\Omega_\delta)$. □

Proof of Theorem A. The domain is bounded, connected and smooth; the points $\pm(1+\delta_n/2)e_1$ belong to it while their midpoint does not, so it is nonconvex. By Lemmas 3.3 and 3.4,

$$\frac{\lambda_{1,p}^{\mathcal{A}}(\Omega_\delta)^{1/p}}{\lambda_{1,p}(\Omega_\delta)^{1/p}} \le 2^{n+2+1/n}\delta^{-(n+1)/(2n)} \cdot 2^{n-1}\delta = 2^{2n+1+1/n}\delta^{(n-1)/(2n)}.$$

At $\delta = \delta_n$ the right side is $2^{2n+1+1/n-6(n-1)} = r_n$, and $7 + 1/n - 4n \le -1/2$ for $n \ge 2$. This proves (6), and in particular $\lambda_{1,p}^{\mathcal{A}}(\Omega_n) < \lambda_{1,p}(\Omega_n)$.

By [4, Theorems 3 and 4], a normalized minimizer w of $\mathcal{R}_p$ exists and is a first affine eigenfunction; replacing w by $|w|$, which has the same directional energies, we may take $w \ge 0$, and Lemma 2.2 gives $w > 0$. If w were radial, Lemma 3.5 would give $\mathcal{R}_p(w) = \|\nabla w\|_p^p \ge \lambda_{1,p}(\Omega_n) > \lambda_{1,p}^{\mathcal{A}}(\Omega_n)$, a contradiction. For $g \in SO(n)$, Lemma 2.1 shows that $w \circ g$ is again a normalized positive minimizer, hence a first eigenfunction by [4, Theorem 4]. The map $g \mapsto w \circ g$ is continuous from $SO(n)$ into $L^p(\Omega_n)$ (approximate the zero extension of w by $f \in C_c(\mathbb{R}^n)$ and use $\|w\circ g - w\circ h\|_p \le 2\|w - f\|_p + \|f\circ g - f\circ h\|_p$). Since $SO(n)$ is compact and connected and the orbit is not a point, the orbit is a compact connected metric space with at least two points; the continuous function $v \mapsto \|v - w\|_p$ maps it onto an interval $[0, \rho]$ with $\rho > 0$. Hence the orbit has the cardinality of the continuum. The final assertion is Lemma 3.5, with strict positivity of φ_p from the ordinary theory. □

Remark 3.6. The proof gives a general principle. Let a closed group $G \subset O(n)$ preserve a bounded connected Ω, and suppose that every G-invariant function has constant directional profile d_u (for instance, G transitive on S^{n-1}; at $p = 2$, G irreducible; at $p = 2k$ in the plane, $G \supseteq C_m$ with $m > 2k$; see

Remark 3.2). If Ω contains an ellipsoid of volume ε with $\lambda_{1,p}(B)(|B|/\varepsilon)^{p/n} < \lambda_{1,p}(\Omega)$, then the ordinary ground state is a positive affine p-eigenfunction at $\lambda_{1,p}(\Omega) > \lambda^{\mathcal{A}}_{1,p}(\Omega)$, and by Proposition A2 no first affine eigenfunction is G-invariant, so G moves every positive first affine eigenfunction to a distinct one. The same estimates give the additive separation $\lambda_{1,p}(\Omega_n) - \lambda^{\mathcal{A}}_{1,p}(\Omega_n) \geq (1 - r_n)2^{11n+1}$ for every $p > 1$.

4 The quadratic case

Throughout this section $p = 2$, $\Omega \subset \mathbb{R}^n$ is bounded, open and connected, and $\Lambda(\Omega) = \lambda^{\mathcal{A}}_{1,2}(\Omega)$. Let $\mathcal{P}$ be the set of symmetric positive-definite matrices of determinant one; it is a real-analytic manifold, and exp is an analytic diffeomorphism from the trace-free symmetric matrices onto $\mathcal{P}$.

4.1 Energy tensor and linear positions

For $0 \neq u \in H^1_0(\Omega)$ let $E_u = \int_\Omega \nabla u \nabla u^T$. It is positive definite, since $d_u(\xi)^2 = \xi^T E_u \xi > 0$. As $A_2 = 1/n$ and $\int (\xi^T E \xi)^{-n/2} d\sigma = (\det E)^{-1/2}$ (Lemma 2.1 with $f \equiv 1$), (1) gives

$$\mathcal{E}_2(u)^2 = n(\det E_u)^{1/n}, \qquad \tfrac{1}{2} D(\mathcal{E}_2^2)(u)[z] = \int_\Omega \nabla z^T M_u \nabla u, \qquad M_u = (\det E_u)^{1/n} E_u^{-1}. \tag{8}$$

Thus u is an affine 2-eigenfunction at ν if and only if $\int \nabla z^T M_u \nabla u = \nu \int uz$ for all $z \in H^1_0(\Omega)$. The matrix M_u is constant in x, has determinant one, and depends on the whole function u. This operator is the affine Laplacian of Schindler and Tintarev [10].

For $A \in \mathrm{SL}(n, \mathbb{R})$ put $\tilde{\Phi}(A) = \lambda_1(A\Omega)$, and let Φ be its restriction to $\mathcal{P}$. Left multiplication by orthogonal matrices does not change $\tilde{\Phi}$, and $A \mapsto (A^T A)^{1/2}$ is a submersion of $\mathrm{SL}(n, \mathbb{R})$ onto $\mathcal{P}$ through which $\tilde{\Phi}$ factors; so critical points of $\tilde{\Phi}$ in $\mathcal{P}$ and of Φ coincide. We call a domain U *affine-isotropic* when its normalized ordinary ground state v has $E_v = (\lambda_1(U)/n) I_n$.

Lemma 4.1 (Orbit lemma)**.** *(a) For a normalized ground state v of a bounded connected open set U and a trace-free matrix B,*

$$\tfrac{d}{dt} \lambda_1(e^{tB} U)\big|_{t=0} = -2 \operatorname{tr}(E_v \operatorname{sym} B).$$

Hence $T \in \mathcal{P}$ is a critical point of Φ if and only if $T\Omega$ is affine-isotropic.

(b) If $\Omega \subset \{|x| < R\}$ and $\sigma_{\min}(T)$ is the smallest singular value of T, then $\Phi(T) \geq \pi^2/(4R^2 \sigma_{\min}(T)^2)$; the sublevel sets $\{T \in \mathcal{P} : \Phi(T) \leq c\}$ are compact.

(c) $\Lambda(\Omega) = \min_{\mathcal{P}} \Phi = \min_{\mathrm{SL}(n,\mathbb{R})} \tilde{\Phi}$.

Proof. (a) Pulling $e^{tB}U$ back to U preserves mass, since $\det e^{tB} = 1$, and gives the energy form $\int \nabla z^T e^{-tB} e^{-tB^T} \nabla z$; its derivative at the simple ground state is $-2 \operatorname{tr}(E_v \operatorname{sym} B)$. Skew B gives zero, and vanishing for all symmetric trace-free B means that E_v is scalar, with trace $\lambda_1(U)$. (b) $T\Omega$ lies in a slab of width $2R\sigma_{\min}(T)$, and the one-dimensional Dirichlet Poincaré inequality on lines across the slab gives the bound. A determinant-one positive matrix with bounded-below smallest eigenvalue has bounded largest eigenvalue, and Φ is continuous because the pulled-back energy forms depend continuously on T. (c) For $v = u \circ A^{-1}$, $A \in \mathrm{SL}(n, \mathbb{R})$, one has $\|v\|_2 = \|u\|_2$ and $\int |\nabla v|^2 = \operatorname{tr}(E_u A^{-1} A^{-T})$, whose infimum over $\mathrm{SL}(n, \mathbb{R})$ is $n(\det E_u)^{1/n}$ by the arithmetic–geometric mean inequality, attained at $A = (\det E_u)^{-1/(2n)} E_u^{1/2}$. Interchanging the two infima gives $\Lambda = \inf \tilde{\Phi}$, and (b) gives attainment. □

Theorem B1 (classification). *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and connected, $n \geq 2$.*

(a) A nonzero $u \in H^1_0(\Omega)$ is an affine 2-eigenfunction at ν if and only if $u = g \circ T$, where $T \in \mathcal{P}$ and g is a Dirichlet eigenfunction of $-\Delta$ on $T\Omega$ at ν whose energy tensor E_g is scalar. The matrix is then unique, $T = T_u := (\det E_u)^{-1/(2n)} E_u^{1/2}$, and $E_g = (\nu \|u\|_2^2/n) I_n$.

(b) The map $u \mapsto T_u$ is a bijection from normalized positive affine 2-eigenfunctions onto the critical points of Φ on $\mathcal{P}$; the eigenvalue of u is $\Phi(T_u)$.

(c) The first affine eigenfunctions are, up to sign, the normalized positive eigenfunctions with $T_u \in \operatorname{argmin}\Phi$. In particular, $\Lambda(\Omega)$ is simple (up to sign and scaling) if and only if Φ has a unique minimizer on $\mathcal{P}$.

Proof. (a) Let u be an eigenfunction at ν and $T = M_u^{-1/2}$, which is the displayed T_u and lies in $\mathcal{P}$. For $g = u \circ T^{-1}$ and $\tilde{z} = z \circ T^{-1}$ we have $\nabla u = T\nabla g \circ T$, so the change of variables $y = Tx$ turns $\int \nabla z^T M_u \nabla u$ into $\int_{T\Omega} \nabla\tilde{z} \cdot \nabla g$ and the eigenvalue equation into $-\Delta g = \nu g$ weakly on $T\Omega$. Moreover $E_g = T^{-1}E_u T^{-1} = (\det E_u)^{1/n} I_n$, and its trace is $\int |\nabla g|^2 = \nu\|g\|_2^2 = \nu\|u\|_2^2$. Conversely, if $T \in \mathcal{P}$ and g is an eigenfunction on $T\Omega$ with $E_g = cI_n$, then $u = g \circ T$ has $E_u = cT^2$, hence $M_u = T^{-2}$, and the same change of variables gives the affine equation; $M_u = T^{-2}$ also shows the uniqueness of T. (b) u is positive exactly when g is. A positive Dirichlet eigenfunction of the connected set $T\Omega$ is its ground state (test against the ground state), so $\nu = \lambda_1(T\Omega) = \Phi(T)$, and E_g scalar means that $T\Omega$ is affine-isotropic, that is, T is critical by Lemma 4.1(a). Conversely, the normalized ground state of $T\Omega$ at a critical T pulls back to a normalized positive eigenfunction u with $T_u = T$. (c) By [4, Theorem 4] first eigenfunctions have one sign. If $u > 0$ is one, then $\Lambda = \nu = \Phi(T_u) \geq \min\Phi = \Lambda$ by Lemma 4.1(c). Conversely, a minimizer of Φ is critical, and (b) gives an eigenfunction at $\min\Phi = \Lambda$. □

Theorem B1 extends the orbit reduction of [10, Lemma 2.1], and part (b) characterizes all positive eigenfunctions, not only the minimizers. Every affine 2-eigenvalue is therefore an ordinary Dirichlet eigenvalue $\lambda_k(T\Omega)$ of some linear image, and the eigenvalues carrying positive eigenfunctions are exactly the critical values of Φ.

Corollary 4.2 (Saddle criterion)**.** *Let Ω be affine-isotropic, and let $A(t)$ be a C^2 curve in $\mathrm{SL}(n,\mathbb{R})$ with $A(0) = I$ such that $t \mapsto \lambda_1(A(t)\Omega)$ has negative second derivative at $t = 0$. Then the ordinary ground state of Ω is a positive affine 2-eigenfunction at $\lambda_1(\Omega) > \Lambda(\Omega)$, and Ω also has a positive affine 2-eigenfunction at $\Lambda(\Omega)$.*

Proof. $T = I$ is critical by Lemma 4.1(a), so Theorem B1(b) applies. The first derivative along $A(t)$ vanishes at isotropy, so nearby images have smaller first eigenvalue, hence $\Lambda < \lambda_1(\Omega)$ by Lemma 4.1(c). The second function comes from Theorem B1(c). □

4.2 A Schur principle

Proposition C. *Let a closed subgroup $G \subset O(n)$ preserve Ω and act irreducibly on $\mathbb{R}^n$. Then the normalized ordinary ground state φ of Ω is a positive affine 2-eigenfunction at $\lambda_1(\Omega)$. If $\lambda_1(\Omega) > \Lambda(\Omega)$, then no first affine eigenfunction is G-invariant, and $\{w \circ g : g \in G\}$ contains at least two distinct normalized positive first affine eigenfunctions for every positive first eigenfunction w.*

Proof. By simplicity $\varphi \circ g = \varphi$, so $g^T E_\varphi g = E_\varphi$ for $g \in G$. The eigenspaces of the symmetric matrix E_φ are G-invariant, so E_φ is scalar by irreducibility, and Theorem B1(a) applies with $T = I$. If a first eigenfunction $w > 0$ were G-invariant, E_w would be scalar in the same way, so $T_w = I$ and w would be the ground state of Ω, giving $\Lambda = \lambda_1(\Omega)$. Since $w \circ g$ is again a normalized positive first eigenfunction by Lemma 2.1, some $w \circ g$ differs from w. □

By Theorem A, the shells Ω_n with $G = SO(n)$ satisfy the hypothesis $\lambda_1(\Omega_n) > \Lambda(\Omega_n)$.

4.3 The spectral gap

Theorem B2 (spectral gap). *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and connected, $n \geq 2$. Let $\sigma^{\mathcal{A}}(\Omega)$ be the set of affine 2-eigenvalues and $\sigma_+^{\mathcal{A}}(\Omega)$ the subset of those carrying a positive eigenfunction. Then:*

(a) Φ is real-analytic on $\mathcal{P}$, and $\sigma_+^{\mathcal{A}}(\Omega)$ is the set of critical values of Φ;

(b) $\sigma_+^{\mathcal{A}}(\Omega) \cap [\Lambda, M]$ *is finite for every* M*;*

(c) there is $\delta > 0$ *such that* $\sigma^{\mathcal{A}}(\Omega) \cap (\Lambda, \Lambda + \delta) = \varnothing$*, where* $\Lambda = \Lambda(\Omega) = \min \sigma^{\mathcal{A}}(\Omega)$.

Proof. (a) Pulling back to Ω, $\Phi(T)$ is the first eigenvalue of the form $a_T(z,z) = \int_\Omega \nabla z^T C(T) \nabla z$, $C(T) = T^{-2}$, relative to the fixed $L^2(\Omega)$ product. The map $C \mapsto (-\operatorname{div} C\nabla)^{-1} \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ is real-analytic on positive definite symmetric matrices, by a Neumann series around any point, because $-\operatorname{div} C\nabla$ depends linearly and boundedly on C. Composing with the compact embeddings, $K_T = (-\operatorname{div} C(T)\nabla)^{-1}$ is a real-analytic family of compact self-adjoint operators on $L^2(\Omega)$. Its largest eigenvalue $1/\Phi(T)$ is simple, since $T\Omega$ is connected, and isolated. The Riesz projection P_T onto it is analytic, and $1/\Phi(T) = \operatorname{tr}(K_T P_T)$ is analytic [6, Chapter VII, Section 1]. The description of $\sigma_+^{\mathcal{A}}$ is Theorem B1(b).

(b) Let $M > \Lambda$ and $K = \{T \in \mathcal{P} : \Phi(T) \le M\}$, compact by Lemma 4.1(b). By the Łojasiewicz gradient inequality [8], every $T_0 \in \mathcal{P}$ has a neighbourhood V and constants $c > 0$, $\theta \in (0,1)$ with $|\Phi(T) - \Phi(T_0)|^\theta \le c|\nabla\Phi(T)|$ on V (in the exponential chart); hence every critical point in V has the critical value $\Phi(T_0)$. The critical points in K form a compact set, covered by finitely many such neighbourhoods, so Φ has finitely many critical values in $[\Lambda, M]$.

(c) Every affine eigenvalue is $\nu = \lambda_k(T\Omega)$ for some $T \in \mathcal{P}$ and $k \ge 1$, by Theorem B1(a). If $\nu = \lambda_1(T\Omega)$, then g is a multiple of the ground state of $T\Omega$ and has one sign, so ν is a critical value of Φ; otherwise $\nu \ge \lambda_2(T\Omega)$. By (b) there is $\delta_1 > 0$ with no critical value in $(\Lambda, \Lambda + \delta_1)$. Let $K = \{\Phi \le \Lambda + 1\}$. The function $T \mapsto \lambda_2(T\Omega) - \lambda_1(T\Omega)$ is continuous by minmax and positive on K, since $T\Omega$ is connected, so it has a positive minimum δ_2 on the compact set K. For $T \in K$, $\lambda_k(T\Omega) \ge \lambda_2(T\Omega) \ge \Lambda + \delta_2$ when $k \ge 2$; for $T \notin K$, $\lambda_k(T\Omega) \ge \Phi(T) > \Lambda + 1$. Hence $\delta = \min(\delta_1, \delta_2, 1)$ works. □

5 Symmetry-free saddles

The examples of Sections 3 and 4 use a symmetry group. This section works in the plane at $p = 2$ and produces positive affine eigenfunctions above Λ on domains whose only affine self-map is the identity, together with exact curvature asymptotics. By Corollary 4.2, it suffices to construct affine-isotropic domains at which Φ has a direction of negative curvature.

5.1 Pinched pentagons and derivative convergence

Consider the simple pentagons

$$P_d = \operatorname{int}[(-1,0),(1,0),(1,1),(0,d),(-1,1)], \qquad 0 < d < 1, \tag{9}$$

which are connected and nonconvex. As $d \downarrow 0$ they pinch into the open triangles

$$L = \{(x,y) : -1 < x < 0,\ 0 < y < -x\}, \qquad R = \{(-x,y) : (x,y) \in L\},$$

whose closures meet at the origin. Put

$$D_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}, \qquad S_q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \qquad H_d(s,q) = \lambda_1(D_s S_q P_d).$$

On the fixed space $H_0^1(P_d)$ the mass is the L^2 product and the pulled-back energy form is

$$a_{s,q}(z,z) = \int_{P_d} \left[e^{-2s} z_x^2 + e^{2s}(z_y - q z_x)^2 \right]. \tag{10}$$

The form $a_s = a_{s,0}$ is invariant under $x \mapsto -x$.

Lemma 5.1 (Spectral convergence)**.** *The eigenvalues of a_s on P_d converge, with multiplicity, to those of the direct sum on $L \cup R$, uniformly for s in compact intervals. Near $s = 0$, let $u_{d,s}$ be the normalized even positive ground state and $v_{d,s}$ the normalized lowest odd mode, positive on the left half. For small d their eigenvalues are the first two eigenvalues; their difference $\delta_{d,s} > 0$ tends to zero, while the gap from the first to the third eigenvalue stays bounded below. Their zero extensions converge strongly in H^1, also along $s \to s_*$, to $2^{-1/2}(\psi^L_{s_*} \pm \psi^R_{s_*})$, where ψ^L_s is the normalized ground state of a_s on L, extended by zero, and ψ^R_s its reflection.*

Proof. Zero-extend all functions to a fixed rectangle Q. A bounded-energy sequence in $H^1_0(P_d)$ has a subsequence converging weakly in $H^1_0(Q)$ and strongly in $L^2(Q)$; its limit vanishes off $\overline{L} \cup \overline{R}$. On each open edge of either triangle the limit has zero trace from the exterior side, hence zero interior trace, and the finitely many vertices do not affect this. The Lipschitz trace characterization puts the restrictions in $H^1_0(L) \oplus H^1_0(R)$. Conversely this direct sum lies in every $H^1_0(P_d)$, so the limiting variational space is exactly the direct sum.

The limiting eigenfunctions give minmax upper bounds. For the lower bounds, take normalized first-k eigenfunctions along a subsequence; strong L^2 convergence preserves orthonormality, testing against the limiting space identifies the limit equations, and minmax gives the reverse bounds. The same applies in the even and odd subspaces. To upgrade to strong H^1 convergence in a simple sector, let $u_j \rightharpoonup u$ weakly in $H^1_0(Q)$ with $\|u_j\|_2 = \|u\|_2 = 1$ and $a_{s_j}(u_j, u_j) = \lambda_j \to \lambda = a_{s_*}(u, u)$. The coefficients of a_{s_j} converge uniformly to those of a_{s_*}, so $a_{s_*}(u_j - u, u_j - u) = a_{s_*}(u_j, u_j) - 2a_{s_*}(u_j, u) + a_{s_*}(u, u) \to 0$, and uniform coercivity gives $u_j \to u$ in H^1. Uniform ellipticity on compact s intervals and the sequential argument along $s_j \to s_*$ give the compact-parameter assertions. Each parity sector contains one copy of the single-lobe spectrum in the limit, whose ground state is simple. The full first eigenvalue is simple and even because P_d is connected, so the lowest odd mode is the second full mode for small d, and the third-mode separation follows. □

General domain-convergence theory is in [2]; the limits below are qualitative, and no rate for $\delta_{d,s}$ is needed.

Lemma 5.2 (Second variation)**.** *Let a_t, b_t be analytic symmetric pulled-back forms on a fixed H^1_0 space, with uniformly coercive energy, and let u be a simple ground state with $b_0(u, u) = 1$. At a critical parameter with $b_0' = 0$ put $f(z) = a_0'(u, z) - \lambda b_0'(u, z)$. For a b_0-orthonormal eigenbasis (u_k),*

$$\lambda'' = a_0''(u, u) - 2 \sum_{k \geq 2} \frac{|f(u_k)|^2}{\lambda_k - \lambda}. \tag{11}$$

If $a_0''(u, u)$, the energy-dual norm of f and λ are bounded and $\lambda_3 - \lambda_1$ is bounded below, then, with $v = u_2$ and $\delta = \lambda_2 - \lambda_1$,

$$\lambda'' = -\frac{2|f(v)|^2}{\delta} + O(1). \tag{12}$$

Proof. Analytic perturbation applies to the simple ground state. Differentiating the weak equation and normalization, the component of the derivative of the eigenfunction along u_k, $k \geq 2$, is $-f(u_k)/(\lambda_k - \lambda)$, and a second differentiation gives (11); no corner regularity is used. For $k \geq 3$ the ratios $\lambda_k/(\lambda_k - \lambda_1)$ are bounded, and $\sum_{k \geq 2} |f(u_k)|^2/\lambda_k \leq \|f\|^2_{a_0^*}$; separating $k = 2$ gives (12). □

Lemma 5.3 (Even-sector C^2 convergence)**.** *Let $g_d(s) = \lambda_1(D_s P_d)$ and $g(s) = \lambda_1(D_s L)$. Then $g_d \to g$ locally in C^2 near $s = 0$.*

Proof. Let u_d be the normalized even ground state at s, and $w_d = \partial_s u_d$. Then $(w_d, u_d)_2 = 0$ and, for even $z \perp u_d$,

$$(a_s - \lambda_d b)(w_d, z) = -a_s'(u_d, z). \tag{13}$$

By Lemma 5.1 the gap to the next even level stays positive, so $a_s - \lambda_d b$ is uniformly coercive on the even orthogonal complement, and w_d is bounded in H^1. Along a subsequence $d \to 0$, $s \to s_*$, strong

convergence of u_d gives dual-norm convergence of $a_s'(u_d, \cdot)$; a weak limit w of w_d is even and orthogonal to the limiting ground state u, and the recovery tests $z - (z, u_d)_2 u_d$ identify w as the unique limiting response. Testing (13) with w_d and using strong L^2 convergence gives convergence of energies, hence strong H^1 convergence of responses. Finally $g_d' = a_s'(u_d, u_d)$ and $g_d'' = a_s''(u_d, u_d) + 2a_s'(u_d, w_d)$ converge to the derivatives of g. The closing odd gap is never inverted here. □

5.2 The isotropic saddle

Translate L by $(1, 0)$ to $T_0 = \{X, Y > 0,\ X+Y < 1\}$. Its normalized ground state and first two eigenvalues are

$$\phi = 2\big[\sin(\pi X)\sin(2\pi Y) + \sin(2\pi X)\sin(\pi Y)\big], \qquad \lambda_1(T_0) = 5\pi^2, \quad \lambda_2(T_0) = 10\pi^2, \tag{14}$$

by odd reflection across $X + Y = 1$ onto antisymmetric square modes; $\phi = 4\sin(\pi X)\sin(\pi Y)(\cos \pi X + \cos \pi Y) > 0$ in T_0.

Lemma 5.4 (Diagonal nondegeneracy)**.** *For $g(s) = \lambda_1(D_s L)$, $g'(0) = 0$ and $g''(0) \geq \frac{28}{5}\pi^2$.*

Proof. Leg interchange gives $g'(0) = 0$. The operator derivative is $A'\phi = 2(\phi_{XX} - \phi_{YY}) = 12\pi^2(f - h)$ with $f = \sin \pi X \sin 2\pi Y$, $h = \sin 2\pi X \sin \pi Y$; it is orthogonal to ϕ and $\|A'\phi\|_2 = 6\pi^2$, since $\int_{T_0}(f - h)^2 = 1/4$. The direct term is $4\lambda_1(T_0) = 20\pi^2$, and (11) with the gap $5\pi^2$ gives $g''(0) \geq 20\pi^2 - 2(6\pi^2)^2/(5\pi^2) = \frac{28}{5}\pi^2$. □

Theorem 5.5 (Isotropic saddles and exact shear asymptotics)**.** *There exist $d_0 > 0$ and numbers $s_d \to 0$ $(0 < d < d_0)$ such that $W_d = D_{s_d} P_d$ is affine-isotropic and the Hessian of H_d at $(s_d, 0)$ satisfies*

$$H_{d,ss} > 0, \qquad H_{d,sq} = 0, \qquad H_{d,qq} < 0. \tag{15}$$

With $\delta_{d,s} = \lambda_2(D_s P_d) - \lambda_1(D_s P_d)$,

$$\lim_{d \downarrow 0} \delta_{d,s_d} H_{d,qq}(s_d, 0) = -\frac{32768}{81}. \tag{16}$$

Proof. On a small interval about zero, $g'' > 0$ and g' has opposite endpoint signs. Lemma 5.3 gives a unique root s_d of g_d' there for small d, with $s_d \to 0$ and $g_d''(s_d) > 0$. Reflection gives zero cross energy on $D_s P_d$, and $g_d' = 2(E_y - E_x)$ by Lemma 4.1(a); hence $E_x = E_y$, $E_{xy} = 0$ at s_d, and W_d is affine-isotropic.

Reflection maps q to $-q$, so $H_{d,q}(s, 0) = H_{d,sq}(s, 0) = 0$. For the even mode $u = u_{d,s}$ the shear forcing and direct term are

$$f_q(z) = -e^{2s} \int_{P_d} (u_x z_y + u_y z_x), \qquad a_{qq}(u, u) = 2e^{2s} \int_{P_d} u_x^2,$$

both uniformly bounded, and the third-mode gap is bounded below by Lemma 5.1. Let $K(s) = \int_L (\psi_s^L)_x (\psi_s^L)_y$ in the pulled coordinates; it is continuous in s. Even/odd convergence gives $f_q(v_{d,s}) \to -2e^{2s} K(s)$, the two lobes contributing equally because both the cross energy and the odd mode change sign under reflection. At $s = 0$,

$$K(0) = \int_{T_0} \phi_X \phi_Y = \frac{64}{9}, \tag{17}$$

for instance by reflecting to the unit square, where the two surviving products use

$$\int_0^1 \sin(2\pi X)\cos(\pi X)\, dX = \frac{4}{3\pi}, \qquad \int_0^1 \sin(\pi X)\cos(2\pi X)\, dX = -\frac{2}{3\pi},$$

and give the square integral $128/9$. Along $s = s_d \to 0$ the forcing therefore tends to $-128/9$, and Lemma 5.2 gives $\delta_{d,s_d} H_{d,qq}(s_d, 0) \to -2(128/9)^2 = -32768/81$. Since $\delta_{d,s_d} > 0$, $H_{d,qq}(s_d, 0) < 0$ for small d. □

The construction also has an area-preserving, non-affine descent that keeps five genuine sides, with a second exact asymptotic law.

Proposition 5.6 (Apex-slide asymptotics)**.** *Let $\Theta_t(x,y) = (x + t(1-|x|), y)$, $|t| < 1$, which maps P_d onto the pentagon with apex (t,d) and the same outer vertices and preserves area. Uniformly for s in a small compact interval,*

$$\delta_{d,s}\,\partial_{tt}\lambda_1(D_s\Theta_t P_d)|_{t=0} \to -8X(s)^2, \qquad X(s) = e^{-2s}\int_L |(\psi_s^L)_x|^2, \tag{18}$$

the first t-derivative vanishes, and at $s = 0$ and along $s = s_d$ the limit equals $-50\pi^4$.

Proof. Θ_t is affine on each half with Jacobian $\mathrm{diag}(1\pm t, 1)$, a bi-Lipschitz homeomorphism for $|t| < 1$; the halves keep equal areas. The pulled-back mass densities are $1 \pm t$ and the stiffness matrices $\mathrm{diag}(e^{-2s}/(1\pm t), e^{2s}(1\pm t))$, with first two t-derivatives bounded uniformly in d, and the mass has zero second derivative. Reflection takes t to $-t$. For the even and left-positive odd modes, Lemma 5.1 gives $b'(u,v) \to 1$ and $a'(u,v) \to Y(s) - X(s)$, where $Y(s) = e^{2s}\int_L |(\psi_s^L)_y|^2$; as the limiting eigenvalue is $X + Y$, $a'(u,v) - \lambda b'(u,v) \to -2X(s)$. Lemma 5.2 gives (18), and leg interchange gives $X(0) = 5\pi^2/2$. □

5.3 Continuation to trivial symmetry

Proposition 5.7 (Continuation)**.** *For every sufficiently small $d > 0$ let*

$$Q_{d,\eta} = \mathrm{int}[(-1,0),(1,0),(1+\eta,1+\eta),(0,d),(-1,1)].$$

There exist $\eta_0(d) > 0$ and smooth $s(d,\eta), q(d,\eta)$, equal to $s_d, 0$ at $\eta = 0$, such that for $|\eta| < \eta_0(d)$ the pentagon $U_{d,\eta} = D_{s(d,\eta)}S_{q(d,\eta)}Q_{d,\eta}$ is affine-isotropic and the Hessian of $(s,q) \mapsto \lambda_1(D_sS_qQ_{d,\eta})$ there is indefinite. For $0 < \eta < \eta_0(d)$, the only affine map of $\mathbb{R}^2$ onto itself preserving $U_{d,\eta}$ is the identity, and neither pair of opposite sides of its quadrilateral convex hull is parallel.

Proof. At fixed $d > 0$ triangulate P_d by nondegenerate triangles; vertex perturbations are realized by piecewise-affine maps on this triangulation whose Jacobians depend analytically on the vertices. The pulled-back forms are analytic and locally uniformly coercive, so the simple first eigenvalue is smooth in the vertex coordinates and in (s,q); compare the polygonal shape calculus of [1]. Write $h_\eta(s,q) = \lambda_1(D_sS_qQ_{d,\eta})$. The Jacobian in (s,q) of $h_{\eta,s} = h_{\eta,q} = 0$ at $(s_d,0,0)$ is the invertible Hessian (15), so the implicit-function theorem gives the parameters. By Lemma 4.1(a), with Eulerian generators $\mathrm{diag}(1,-1)$ and $e^{2s}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, these equations read $E_x = E_y$, $E_{xy} = 0$ on the transformed domain, which is isotropy. The Hessian stays indefinite by continuity.

Label the vertices $A = (-1,0)$, $B = (1,0)$, $C = (1+\eta,1+\eta)$, $P = (0,d)$, $D = (-1,1)$. Then $\det(B-A, D-C) = -2\eta$ and $\det(D-A, C-B) = -\eta$, so for $\eta > 0$ no hull sides are parallel. An affine bijection preserving the pentagon permutes its vertices, preserving the boundary cycle $ABCPD$ and the unique reflex vertex P. It therefore either fixes A, B, C, and is the identity, or swaps $A \leftrightarrow B$, $C \leftrightarrow D$ and fixes P. In the second case it fixes the midpoint 0 of AB, so it is linear with $Me_1 = -e_1$, and $MC = D$ gives $Me_2 = (\eta,1)/(1+\eta)$; then $MP = d(\eta,1)/(1+\eta) \neq P$ for $\eta > 0$, a contradiction. Conjugation by D_sS_q transfers both conclusions to $U_{d,\eta}$. □

Theorem S (symmetry-free positive eigenfunctions). *For every sufficiently small $d > 0$ and $0 < \eta < \eta_0(d)$, the pentagon $U = U_{d,\eta}$ is a bounded connected nonconvex Lipschitz domain whose only affine self-map is the identity, and it carries strictly positive affine 2-eigenfunctions at the two distinct values $\Lambda(U)$ and $\lambda_1(U) > \Lambda(U)$.*

Proof. Write $U = A_0Q_{d,\eta}$ with $A_0 = D_{s(d,\eta)}S_{q(d,\eta)}$. If (a,b) is a negative direction of the indefinite Hessian of Proposition 5.7, the curve $A(t) = D_{s(d,\eta)+ta}S_{q(d,\eta)+tb}A_0^{-1}$ in $\mathrm{SL}(n,\mathbb{R})$ satisfies $A(0) = I$, and $\lambda_1(A(t)U)$ has negative second derivative at $t = 0$. Since U is affine-isotropic, Corollary 4.2 applies. The geometric properties are those of Proposition 5.7. □

Corollary 5.8 (Isotropy and log-convexity off convex sets)**.** *The linear orbit of each pentagon of Theorem 5.5 or Theorem S contains two affine-isotropic positions with different first eigenvalues, not related by an orthogonal map. There are such a pentagon* Ω *and a nonzero symmetric trace-free* B *with*

$$\frac{d}{dt}\lambda_1(e^{tB}\Omega)\Big|_0 = 0, \qquad \frac{d^2}{dt^2}\log\lambda_1(e^{tB}\Omega)\Big|_0 < 0.$$

Proof. Lemma 4.1(c) supplies an isotropic minimum, and the saddle has a smaller neighbouring value, so it differs from the minimum. For the second statement take $\Omega = W_d$ from Theorem 5.5 and $N = D_{s_d}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)D_{s_d}^{-1}$, so that $D_{s_d}S_tP_d = e^{tN}\Omega$ and $\frac{d^2}{dt^2}\lambda_1(e^{tN}\Omega)|_0 = H_{d,qq}(s_d,0) < 0$. Write $e^{tN} = O_tV_t$ with $V_t \in \mathcal{P}$ and $V_0' = B = \operatorname{sym}N$. The paths V_t and e^{tB} have the same velocity, and differentiating the determinant constraints twice shows that their accelerations differ by a trace-free matrix, which the first variation annihilates at isotropy. Hence their second eigenvalue derivatives agree and are negative; dividing by $\lambda_1 > 0$ gives the logarithmic statement. (Equivalently, at the critical point I of Φ, second derivatives along curves in $\mathcal{P}$ with equal velocity agree.) □

For convex bodies, [3, Theorem 1.1] proves strict log-convexity of $t \mapsto \lambda_1(e^{tB}K)$ and [3, Proposition 2.6] shows that affine-isotropic convex bodies are orbit minima. Corollary 5.8 shows that both properties fail for these nonconvex pentagons. The existence parameters d_0 and $\eta_0(d)$ are qualitative.

6 Convex domains

Remark 6.1 (Convex domains, combined with the preprint [3]). Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Combined with [3, Theorem 2.4 and Proposition 2.6], Theorem B1 shows that every positive affine 2-eigenfunction of Ω belongs to $\Lambda(\Omega)$ and that it is unique up to a positive factor. Indeed, for a normalized positive eigenfunction u, the convex image $T_u\Omega$ is affine-isotropic by Theorem B1(b), hence an orbit minimum by [3, Proposition 2.6], so its eigenvalue is $\Lambda(\Omega)$. If u_1, u_2 are two such functions, both T_{u_i} minimize the orbit, and [3, Theorem 2.4], uniqueness of the minimizing position up to orthogonal maps, gives an orthogonal O with $T_{u_2} = OT_{u_1}$; since $T_{u_i} \in \mathcal{P}$, uniqueness of the polar decomposition gives $O = I$, and Theorem B1(b) gives $u_2 = u_1$. Thus on convex domains Φ has a single critical point and Theorem B2 holds with $\sigma_+^{\mathcal{A}}(\Omega) = \{\Lambda(\Omega)\}$.

A Exact auxiliary identities

The proofs of Section 5 use the identities

$$\int_{T_0}\phi^2 = 1, \quad \int_{T_0}\phi_X^2 = \int_{T_0}\phi_Y^2 = \frac{5}{2}\pi^2, \quad \int_{T_0}\phi_X\phi_Y = \frac{64}{9}, \quad \|2(\phi_{XX} - \phi_{YY})\|_2^2 = 36\pi^4,$$

together with the arithmetic $20 - 2\cdot 36/5 = 28/5$, $-2(128/9)^2 = -32768/81$ and $-8(5/2)^2 = -50$. One exact method expands the trigonometric products into $e^{i\pi(mX+nY)}$. With $q = (i\pi)^{-1}$ and $E(k) = \int_0^1 e^{i\pi kX}dX$, so that $E(0) = 1$ and $E(k) = q((-1)^k - 1)/k$ for $k \neq 0$, one has for $n \neq 0$

$$\int_{T_0} e^{i\pi(mX+nY)}\,dX\,dY = \frac{q}{n}\big[(-1)^nE(m-n) - E(m)\big],$$

while for $n = 0$ the integral is $1/2$ if $m = 0$ and $-q/m + q^2((-1)^m - 1)/m^2$ otherwise. The accompanying standard-library Python program `code/verify_constants.py` evaluates these finite expressions in Gaussian rational arithmetic, checks the cross integral independently by separated one-dimensional integrals on the reflected square, and checks the matrix pullbacks used above. It also checks, in exact base-2 exponent arithmetic for $2 \le n \le 50$, the constants of Lemmas 3.3 and 3.4 and the exponent of r_n in Theorem A. It verifies these auxiliary identities; the analytic arguments are the proofs in the text.

AI assistance

AI tools assisted research, proof development, writing, and verification.

References

[1] B. Bogosel and D. Bucur, *On the polygonal Faber–Krahn inequality*, J. Éc. polytech. Math. **11** (2024), 19–105. doi:10.5802/jep.250.

[2] D. Daners, *Dirichlet problems on varying domains*, J. Differential Equations **188** (2003), 591–624. doi:10.1016/S0022-0396(02)00105-5.

[3] D. Faifman and I. Polterovich, *The Faber–Krahn position of convex bodies and Gaussian measure inequalities*, preprint, arXiv:2607.21539v2, 5 August 2026.

[4] J. Haddad, C. H. Jiménez and M. Montenegro, *From affine Poincaré inequalities to affine spectral inequalities*, Adv. Math. **386** (2021), article 107808. doi:10.1016/j.aim.2021.107808.

[5] Q. Huang and A.-J. Li, *Optimal Sobolev norms in the affine class*, J. Math. Anal. Appl. **436** (2016), 568–585. doi:10.1016/j.jmaa.2015.11.063.

[6] T. Kato, *Perturbation Theory for Linear Operators*, reprint of the 1980 edition, Classics in Mathematics, Springer, Berlin, 1995. doi:10.1007/978-3-642-66282-9.

[7] D. Langharst and M. Roysdon, *Affine isoperimetric inequalities for the first eigenvalue of the mth order affine p-Laplace operator*, preprint, arXiv:2512.17237v2, 25 September 2026.

[8] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, in: Les Équations aux Dérivées Partielles (Paris, 1962), Colloq. Internat. CNRS 117, Éditions du CNRS, Paris, 1963, 87–89.

[9] F. Marín Sola, *Affine variational functionals, isoperimetric problems and stability*, preprint, arXiv:2610.05403v1, 4 October 2026.

[10] I. Schindler and C. Tintarev, *Compactness properties and ground states for the affine Laplacian*, Calc. Var. Partial Differential Equations **57** (2018), article 48. doi:10.1007/s00526-018-1338-6.

[11] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747. doi:10.1002/cpa.3160200406.