Positive Affine Eigenfunctions Above the First Eigenvalue on Nonconvex Pentagons

Scott Schubr
Luminary Oddities

October 20, 2026

Abstract

We construct bounded connected nonconvex planar pentagons for which the affine 2-Laplacian has a strictly positive Dirichlet eigenfunction at an eigenvalue strictly above its first variational eigenvalue. The same domain therefore supports positive affine eigenfunctions at two distinct values. The construction gives a negative answer, for $n = p = 2$, to the general-domain positive-eigenfunction question of Haddad, Jiménez and Montenegro. The mechanism is an affine-isotropic saddle formed by pinching a pentagon into two right isosceles triangles. A symmetry-preserving diagonal normalization has positive curvature, while a shear couples the ground state to a nearly coincident odd mode. We prove the sharp limit $(\lambda_2-\lambda_1)\partial_{qq}\lambda_1 \to -32768/81$ at the isotropic positions. An area-preserving apex slide has the corresponding limit $-50\pi^4$. The saddles persist under small vertex perturbations, including examples with nonparallel convex-hull sides. Consequences include distinct isotropic positions on one linear orbit and failure of positive-definite-deformation log-convexity outside convexity. We further preserve the saddle under smooth inner exhaustion and use a symmetry-balanced product construction to obtain positive affine 2-eigenfunctions at two distinct eigenvalues on smooth connected nonconvex domains in every dimension $n \geq 2$.

Keywords: affine Laplacian; positive eigenfunction; Dirichlet eigenvalue; spectral shape optimization; nonconvex polygon; second variation.

2020 Mathematics Subject Classification: 35P30, 35P15, 49Q10.

1 Introduction

For the ordinary Dirichlet Laplacian on a connected domain, a positive eigenfunction belongs to the first eigenvalue. The affine Laplacian has a different structure: its coefficient matrix depends on the global gradient distribution of its eigenfunction. In their development of affine spectral inequalities, Haddad, Jiménez and Montenegro [4, Section 7(1), second bullet] ask whether the first affine eigenvalue is the only eigenvalue admitting a positive eigenfunction. Their question is posed for bounded open domains in dimension at least two. We give a negative answer at exponent two, first through planar pentagons and then through smooth nonconvex domains in every dimension $n \geq 2$.

Throughout, $\lambda_k(\Omega)$ denotes the kth ordinary Dirichlet eigenvalue, with multiplicity. The affine operator and its first variational eigenvalue $\Lambda_{\mathcal{A},1}$ are defined with their precise normalization in Section 2. Positivity means strict positivity in the interior. The planar construction uses bounded connected simple nonconvex Lipschitz pentagons. Section 6 constructs bounded connected nonconvex domains with C^∞ boundary in every dimension $n \geq 2$.

Theorem 1.1. *There exists such a pentagon $\Omega \subset \mathbb{R}^2$ and functions $u, w \in H^1_0(\Omega)$, strictly positive in Ω, satisfying*

$$\mathcal{L}_{\mathcal{A}}u = \lambda_1(\Omega)u, \qquad \mathcal{L}_{\mathcal{A}}w = \Lambda_{\mathcal{A},1}(\Omega)w, \qquad \Lambda_{\mathcal{A},1}(\Omega) < \lambda_1(\Omega).$$

The examples may be chosen so that neither pair of opposite sides of their quadrilateral convex hull is parallel and

$$1 - \frac{|\Omega|}{|\operatorname{conv}\Omega|} > \frac{1}{4}.$$

The proof constructs a domain whose ordinary ground-state energy tensor is scalar, so that its ordinary ground state also satisfies the affine equation. A strictly negative affine second variation then places this eigenvalue above the first affine value. The required critical domain is obtained from two almost separated triangular lobes. Their nearly coincident even and odd modes create a negative inverse-gap term in the shear Hessian. The diagonal normalization is handled within the even sector, whose next spectral gap stays positive. This separation supplies an exact isotropic saddle at every sufficiently small positive height.

Function-level affine normalization and determinant energies are established in [5, 8]; the bounded-domain quadratic reduction is explicit in [7, Proposition 2.6]. The first-eigenvalue theory and inequalities of [4, 6] concern minimizers and do not identify every positive eigenfunction with a minimizer. For convex bodies, Faifman and Polterovich [3, Theorem 1.1, Theorem 2.4 and Proposition 2.6] prove positive-definite-deformation log-convexity, uniqueness of the minimizing linear position, and its energy-isotropy characterization. Our construction exhibits the failure of isotropy sufficiency and log-convexity when convexity is removed.

Section 2 reduces the counterexample to an isotropic saddle. Section 3 proves the spectral and response convergence needed at the pinching limit. Section 4 establishes the saddle and the two sharp curvature laws. Section 5 gives the nonparallel continuation and completes the polygon proof. Section 6 proves the smooth all-dimensional extension. The C^2 convergence of the affine-parameter eigenvalue, rather than ordinary eigenvalue continuity alone, preserves the saddle; reflection and transverse rotation symmetry reduce the dimensional isotropy repair to two diagonal equations.

2 Affine energy and linear images

For $0 \neq u \in H_0^1(\Omega)$, put

$$E_u = \int_\Omega \nabla u \nabla u^T \, dx.$$

This matrix is positive definite. Indeed, a zero directional energy would make the zero extension of u constant on almost every line in that direction. Bounded support then forces $u = 0$. In the planar $p = 2$ normalization of [4], the affine energy is

$$\mathcal{E}_2(u) = 2\sqrt{\pi}\left(\int_{S^1} (\xi^T E_u \xi)^{-1} \, d\xi\right)^{-1/2}, \qquad \mathcal{E}_2(u)^2 = 2\sqrt{\det E_u}. \tag{1}$$

To verify the second identity, diagonalize E_u and substitute $\tan\theta$ in the angular integral. The integral equals $2\pi/\sqrt{\det E_u}$. The determinant formulation and its relation to linear normalization are established affine-energy theory; see [5, 8, 7] and the references therein. We give the elementary planar reductions to fix all constants.

The positive-sign affine 2-Laplacian is

$$\mathcal{L}_{\mathcal{A}} u = -\operatorname{div}(M_u \nabla u), \qquad M_u = \sqrt{\det E_u}\, E_u^{-1}. \tag{2}$$

Precisely, $\mathcal{L}_{\mathcal{A}} u = \nu u$ means

$$\int_\Omega \nabla z^T M_u \nabla u \, dx = \nu \int_\Omega uz \, dx \qquad (z \in H_0^1(\Omega)). \tag{3}$$

The matrix is independent of x but depends on the entire function u. This is therefore a nonlinear, nonlocal eigenvalue problem. The factor in (2) follows from $\frac{1}{2} D(\mathcal{E}_2^2)(u)[z] = \sqrt{\det E_u} \int \nabla z^T E_u^{-1} \nabla u$. Its first variational eigenvalue is

$$\Lambda_{\mathcal{A},1}(\Omega) = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{2\sqrt{\det E_u}}{\|u\|_2^2}.$$

Lemma 2.1 (Linear-orbit reduction)**.** *For a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^2$,*

$$\Lambda_{\mathcal{A},1}(\Omega) = \min_{F \in \mathrm{SL}(2,\mathbb{R})} \lambda_1(F\Omega). \tag{4}$$

At every minimizing image the normalized ordinary ground state has energy tensor $(\lambda_1/2)I$.

Proof. If $v = u \circ F^{-1}$ and $\det F = 1$, then

$$\|v\|_2 = \|u\|_2, \qquad \int_{F\Omega} |\nabla v|^2 = \operatorname{tr}(E_u F^{-1} F^{-T}).$$

The arithmetic–geometric mean inequality for positive matrices gives

$$\inf_{F \in \mathrm{SL}(2,\mathbb{R})} \operatorname{tr}(E_u F^{-1} F^{-T}) = 2\sqrt{\det E_u}, \qquad F = \frac{E_u^{1/2}}{(\det E_u)^{1/4}}$$

as one attaining choice. Interchanging the two infima over functions and matrices proves the infimum identity in (4).

For attainment, enclose Ω in a ball of radius R. If the largest singular value of $F \in \mathrm{SL}(2,\mathbb{R})$ is σ, then $F\Omega$ lies in a strip of width $2R/\sigma$. The one-dimensional Dirichlet Poincaré inequality, applied to the zero extension and integrated along the strip, gives

$$\lambda_1(F\Omega) \geq \frac{\pi^2 \sigma^2}{4R^2}. \tag{5}$$

Left rotations do not change the eigenvalue. Polar decomposition thus reduces minimization to positive definite determinant-one matrices; (5) makes their sublevel sets compact. The pulled energy form depends continuously on the matrix, so the minimum is attained.

For the normalized ordinary ground state v on any image U, the first variation under e^{tB}, $\operatorname{tr} B = 0$, is

$$\frac{d}{dt} \lambda_1(e^{tB}U)\bigg|_{t=0} = -2 \operatorname{tr}(E_v \operatorname{sym} B). \tag{6}$$

This follows by differentiating the fixed-domain energy matrix $e^{-tB} e^{-tB^T}$. At a minimum all these derivatives vanish, so E_v is scalar; its trace is $\lambda_1(U)$. □

We call U *affine-isotropic* when its normalized ordinary ground state satisfies $E_v = (\lambda_1(U)/2)I$. This is exactly stationarity under determinant-one linear maps, by (6).

Proposition 2.2. *Suppose Ω is affine-isotropic and has a determinant-one affine deformation with zero first eigenvalue derivative and strictly negative second derivative. Then its ordinary positive ground state is an affine eigenfunction at $\lambda_1(\Omega) > \Lambda_{\mathcal{A},1}(\Omega)$. Moreover, Ω has a positive affine eigenfunction at $\Lambda_{\mathcal{A},1}(\Omega)$.*

Proof. Isotropy gives $M_u = I$, so the ordinary ground-state equation is (3) with $\nu = \lambda_1(\Omega)$. The strictly negative second derivative gives an image with smaller ordinary first eigenvalue; (4) yields the strict inequality.

Let F_0 attain (4), let $v > 0$ be the normalized ground state on $F_0\Omega$, and write $\Lambda = \Lambda_{\mathcal{A},1}(\Omega)$. Lemma 2.1 gives $E_v = (\Lambda/2)I$. For $w = v \circ F_0$ on Ω,

$$E_w = (\Lambda/2) F_0^T F_0, \qquad M_w = F_0^{-1} F_0^{-T}.$$

Changing variables in the ordinary weak equation for v gives $\mathcal{L}_{\mathcal{A}} w = \Lambda w$. Both functions are strictly positive in the interior by the ordinary ground-state maximum principle. □

3 The pinching limit and derivative convergence

Consider the simple pentagons

$$P_d = \mathrm{int}[(-1,0),(1,0),(1,1),(0,d),(-1,1)], \qquad 0 < d < 1. \tag{7}$$

They are connected and nonconvex. Their area is $1 + d$ and their convex hull is the rectangle of area 2. As $d \downarrow 0$ the domains pinch into the two open triangles

$$L = \{(x,y) : -1 < x < 0,\ 0 < y < -x\}, \qquad R = \{(-x,y) : (x,y) \in L\}.$$

Their closures meet at the origin. Define

$$D_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}, \qquad S_q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \qquad H_d(s,q) = \lambda_1(D_s S_q P_d).$$

On the fixed space $H_0^1(P_d)$ the corresponding mass is the usual L^2 product and the energy form is

$$a_{s,q}(z,z) = \int_{P_d} \left[e^{-2s} z_x^2 + e^{2s}(z_y - q z_x)^2\right]. \tag{8}$$

In particular $a_s = a_{s,0}$ is invariant under $x \mapsto -x$.

Lemma 3.1 (Spectral convergence)**.** *The eigenvalues of a_s on P_d converge, with multiplicity, to those of the direct sum on $L \cup R$. The convergence is uniform for s in any fixed compact interval. Near $s = 0$, let $u_{d,s}$ be the normalized even positive ground state and $v_{d,s}$ the normalized lowest odd mode, positive on the left half. For all sufficiently small d, their eigenvalues are the first two full eigenvalues. Their difference $\delta_{d,s} > 0$ tends to zero, while the gap from the first to the third eigenvalue stays bounded below. Their zero extensions converge strongly in H^1, also along varying $s \to s_*$, to*

$$2^{-1/2}(\psi_{s_*}^L + \psi_{s_*}^R), \qquad 2^{-1/2}(\psi_{s_*}^L - \psi_{s_*}^R),$$

respectively. Here ψ_s^L is the normalized ground state of a_s on L, extended by zero, and ψ_s^R is its reflection.

Proof. Zero-extend all functions to a fixed bounding rectangle B. Every bounded-energy sequence in $H_0^1(P_d)$ has a weak $H_0^1(B)$ and strong $L^2(B)$ subsequence. Its limit vanishes off $\overline{L} \cup \overline{R}$. On each open straight edge of either triangle the global H^1 function is zero on the exterior side, so its interior trace is zero. The finitely many endpoints have zero boundary measure and do not change this trace condition. The Lipschitz trace characterization therefore puts the restrictions in $H_0^1(L)$ and $H_0^1(R)$. Conversely, each member of that direct-sum space belongs, by zero extension and domain inclusion, to every $H_0^1(P_d)$. Thus the limiting variational space is exactly the direct sum; the touching point imposes no extra condition.

The limiting eigenfunctions give the minmax upper bounds. For the lower bounds, take normalized first-k eigenfunctions along a subsequence. Compactness preserves their L^2 orthonormality. Testing against the limiting space identifies their limit equations; minmax then gives the reverse bounds. The same argument applies to the even and odd subspaces, using symmetric or antisymmetric tests. Equality of limiting energies improves weak convergence to strong H^1 convergence in a simple sector.

The coefficients of a_s are uniformly elliptic on compact s intervals. Apply the preceding argument to every sequence $s_j \to s_*$ to obtain the compact-parameter assertions. The single-lobe ground state is simple; its next eigenvalue is separated locally in s. Each parity sector has one copy of this lobe spectrum in the limit. The full first eigenvalue is simple because P_d is connected, and its positive eigenfunction is even. Consequently the lowest odd mode is precisely the second full mode for small d, and the third-mode separation follows. □

The argument is a direct variational-space convergence proof in this geometry; general domain-convergence theory can be found in [2]. All limits below are qualitative. No rate for $\delta_{d,s}$ is needed.

Lemma 3.2 (Second variation and a uniform remainder)**.** *Let a_t, b_t be analytic symmetric pulled forms on a fixed H_0^1 space, with uniformly coercive energy, and let u be a simple ground state normalized by $b_0(u,u) = 1$. At a critical parameter with $b_0'' = 0$, put*

$$f(z) = a_0'(u,z) - \lambda b_0'(u,z).$$

For a b_0-orthonormal eigenbasis (u_k),

$$\lambda'' = a_0''(u,u) - 2\sum_{k\geq 2} \frac{|f(u_k)|^2}{\lambda_k - \lambda}. \tag{9}$$

If $a_0''(u,u)$ and the energy-dual norm of f are bounded uniformly, the first eigenvalue is bounded, and $\lambda_3 - \lambda_1$ is bounded below, then, writing $v = u_2$ and $\delta = \lambda_2 - \lambda_1$,

$$\lambda'' = -\frac{2|f(v)|^2}{\delta} + O(1). \tag{10}$$

Proof. Analytic form perturbation applies because the ground eigenvalue is simple and the embedding is compact. Differentiate its weak equation and normalization. Since $\lambda' = 0$, the component of the derivative eigenfunction in direction u_k, $k \geq 2$, is $-f(u_k)/(\lambda_k - \lambda)$. Its component along u does not contribute because $f(u) = \lambda' = 0$. A second differentiation gives (9). This is the usual reduced-resolvent formula, here applied to pulled volume forms; it does not require corner H^2 regularity or a boundary Hessian formula.

The tail is bounded as a whole: for $k \geq 3$ the numbers $\lambda_k/(\lambda_k - \lambda_1)$ are bounded uniformly, and

$$\sum_{k\geq 2} \frac{|f(u_k)|^2}{\lambda_k} \leq \|f\|_{a_0^*}^2.$$

Separating $k = 2$ proves (10). □

Lemma 3.3 (Even-sector C^2 convergence)**.** *Let $g_d(s) = \lambda_1(D_s P_d)$ and $g(s) = \lambda_1(D_s L)$. Then $g_d \to g$ locally in C^2 near $s = 0$.*

Proof. All functions in this proof are pulled to P_d or $L \cup R$ and zero-extended to B. Let u_d denote the normalized even ground state at s. Its normalized derivative $w_d = \partial_s u_d$ satisfies $(w_d, u_d)_2 = 0$ and, for even $z \perp u_d$,

$$(a_s - \lambda_d b)(w_d, z) = -a_s'(u_d, z). \tag{11}$$

By Lemma 3.1, the gap to the next *even* level stays positive. On the even orthogonal complement, $a_s - \lambda_d b$ is therefore uniformly coercive relative to a_s: the spectral factor is $1 - \lambda_d/\lambda_{\text{next,even}}$. The right side is uniformly bounded in the energy-dual norm, so w_d is bounded in H^1.

Take a subsequence with $d \to 0$ and $s \to s_*$. The first lemma gives strong H^1 convergence of u_d, so $a_s'(u_d, \cdot)$ converges in H^1-dual norm. The weak limit w lies in the limiting even space and is orthogonal to the limiting normalized even ground state u. For any even limiting test $z \perp u$, the recovery tests $z_d = z - (z, u_d)_2 u_d$ belong to the varying space and converge strongly to z. Passing to the limit in (11) identifies w as the unique limiting orthogonal response. Testing (11) with w_d gives

$$a_s(w_d, w_d) - \lambda_d \|w_d\|_2^2 = -a_s'(u_d, w_d).$$

The right side converges by dual-norm convergence and weak convergence; compactness gives strong L^2 convergence. Equality of limiting energies then gives strong H^1 convergence of the responses.

Finally,

$$g_d' = a_s'(u_d, u_d), \qquad g_d'' = a_s''(u_d, u_d) + 2a_s'(u_d, w_d).$$

These converge to the corresponding derivatives on the limiting even space, whose first eigenvalue is g. The subsequence argument with varying s proves local uniform convergence of both derivatives. The closing odd gap is never inverted in this argument. □

4 An affine-isotropic saddle

Translate L by $(1,0)$ to the right isosceles triangle $T = \{(X,Y) : X,Y > 0,\ X+Y < 1\}$. Its normalized ground state is

$$\phi(X,Y) = 2\big[\sin(\pi X)\sin(2\pi Y) + \sin(2\pi X)\sin(\pi Y)\big], \qquad \lambda_* = 5\pi^2. \tag{12}$$

The factorization $\phi = 4\sin(\pi X)\sin(\pi Y)(\cos(\pi X)+\cos(\pi Y))$ shows positivity in T. Odd reflection across $X+Y=1$ identifies the triangle eigenfunctions with the antisymmetric square modes. In particular its first two eigenvalues are $5\pi^2$ and $10\pi^2$, and direct integration gives $\int_T \phi^2 = 1$.

Lemma 4.1 (Diagonal nondegeneracy)**.** *For $g(s) = \lambda_1(D_s L)$,*

$$g'(0) = 0, \qquad g''(0) \geq \frac{28}{5}\pi^2 > 0.$$

Proof. Leg interchange gives $g'(0) = 0$. The operator derivative on T is $A'\phi = 2(\phi_{XX} - \phi_{YY})$. Its L^2 norm is $6\pi^2$, and it is orthogonal to ϕ. Indeed, with $f = \sin(\pi X)\sin(2\pi Y)$ and $h = \sin(2\pi X)\sin(\pi Y)$, $\phi_{XX} - \phi_{YY} = 6\pi^2(f-h)$ and $\int_T (f-h)^2 = 1/4$. The direct second energy term is $4\lambda_* = 20\pi^2$. The gap $5\pi^2$ and (9) therefore give

$$g''(0) \geq 20\pi^2 - \frac{2(6\pi^2)^2}{5\pi^2} = \frac{28}{5}\pi^2.$$

□

Theorem 4.2 (Isotropic saddles and shear curvature)**.** *There exist $d_0 > 0$ and numbers $s_d \to 0$ for $0 < d < d_0$ such that $\Omega_d = D_{s_d} P_d$ is affine-isotropic. At $(s_d, 0)$ the Hessian of H_d has one positive and one negative eigenvalue, with*

$$H_{d,ss} > 0, \qquad H_{d,sq} = 0, \qquad H_{d,qq} < 0. \tag{13}$$

If $\delta_{d,s} = \lambda_2(D_s P_d) - \lambda_1(D_s P_d)$, then

$$\lim_{d \downarrow 0} \delta_{d,s_d} H_{d,qq}(s_d, 0) = -\frac{32768}{81}. \tag{14}$$

The hull deficit $1 - |\Omega_d|/|\operatorname{conv}\Omega_d|$ tends to $1/2$.

Proof. Choose a small fixed interval about zero on which $g'' > 0$ and g' has opposite endpoint signs. Lemma 3.3 gives a unique root s_d of g_d' in this interval for every sufficiently small d, with $s_d \to 0$ and $g_d''(s_d) > 0$. Reflection gives zero cross energy on $D_s P_d$, while $g_d' = 2(E_y - E_x)$, using physical energies on $D_s P_d$. Hence $E_x = E_y$ and $E_{xy} = 0$ at s_d: the domain is exactly affine-isotropic.

Differentiate (8) in q at $q = 0$. Since reflection takes q to $-q$, $H_{d,q}(s,0) = H_{d,sq}(s,0) = 0$. For the normalized even mode $u = u_{d,s}$ and an arbitrary test z the forcing is

$$f_q(z) = -e^{2s}\int_{P_d} (u_x z_y + u_y z_x), \qquad a_{qq}(u,u) = 2e^{2s}\int_{P_d} u_x^2. \tag{15}$$

Both the direct term and the energy-dual norm of the forcing are bounded uniformly for s near zero. The third-mode gap is bounded below by Lemma 3.1.

Put $K(s) = \int_L (\psi_s^L)_x (\psi_s^L)_y$, in the pulled coordinates of that lemma. It is continuous in s by strong H^1 dependence. Even/odd convergence, with the odd mode positive on the left, gives

$$f_q(v_{d,s}) \longrightarrow -2e^{2s}K(s).$$

On the right lobe the cross energy changes sign and the odd mode also changes sign, so the two lobe contributions add. At $s = 0$ direct integration of (12) yields

$$K(0) = \int_T \phi_X \phi_Y = \frac{64}{9}. \tag{16}$$

For example, the product $\phi_X\phi_Y$ is invariant under reflection across $X+Y=1$. On the unit square use

$$\int_0^1 \sin(2\pi X)\cos(\pi X)\,dX = \frac{4}{3\pi}, \qquad \int_0^1 \sin(\pi X)\cos(2\pi X)\,dX = -\frac{2}{3\pi}.$$

The two surviving cross products give a square integral $128/9$; halving gives (16). Along $s = s_d \to 0$, the forcing therefore tends to $-128/9$. Lemma 3.2 now gives

$$\delta_{d,s_d} H_{d,qq}(s_d, 0) \longrightarrow -2(128/9)^2 = -32768/81.$$

Since $\delta_{d,s_d} > 0$, the shear curvature is negative for all small d. This proves (13). Area and hull area are invariant under D_s, so the deficit is $(1-d)/2$. □

The construction also has an exactly area-preserving non-affine descent that retains five genuine sides. This provides a second sharp curvature law without a degenerating deformation field.

Proposition 4.3 (Apex-slide curvature)**.** *Let $T_t(x,y) = (x + t(1-|x|), y)$, $|t| < 1$, on P_d. It maps P_d onto the pentagon with apex (t,d) and the same four outer vertices. Its area is $1+d$ for every t. For s near zero,*

$$\delta_{d,s}\ \partial_{tt}\lambda_1(D_sT_tP_d)|_{t=0} \longrightarrow -8X(s)^2, \tag{17}$$

uniformly on a sufficiently small compact s interval, where $X(s) = e^{-2s}\int_L |(\psi_s^L)_x|^2$. The first derivative is zero. At $s=0$, and also along $s=s_d$, the limit in (17) *is $-50\pi^4$.*

Proof. The map is affine on each half: its Jacobian is $\mathrm{diag}(1\pm t, 1)$. It is a continuous bi-Lipschitz homeomorphism for $|t|<1$. The bottom midpoint moves to $(t,0)$ on the same straight bottom edge, so it introduces no sixth side. The two halves have equal area, which proves area preservation. The pulled mass densities are $1\pm t$ and stiffness matrices are

$$C^{\pm}_{s,t} = \mathrm{diag}\left(\frac{e^{-2s}}{1\pm t}, e^{2s}(1\pm t)\right).$$

Their first two derivatives are bounded uniformly in d for t near zero; the mass has second derivative zero. Reflection takes t to $-t$, giving the first derivative zero.

For the even and left-positive odd modes, Lemma 3.1 gives $b'(u,v)\to 1$ and $a'(u,v)\to Y(s)-X(s)$, where $Y(s) = e^{2s}\int_L |(\psi_s^L)_y|^2$. The limiting eigenvalue is $X(s)+Y(s)$, hence

$$a'(u,v) - \lambda b'(u,v) \longrightarrow -2X(s).$$

The direct term is $2e^{-2s}\int u_x^2$, uniformly bounded. The dual forcing and third-mode gap satisfy Lemma 3.2, proving (17). At $s=0$, leg interchange gives $X(0) = 5\pi^2/2$. The compact-parameter convergence also permits $s = s_d \to 0$. □

5 Nonparallel hulls and consequences

Proposition 5.1 (Continuation of the saddle)**.** *For every sufficiently small fixed $d>0$, set*

$$Q_{d,\eta} = \mathrm{int}[(-1,0),(1,0),(1+\eta,1+\eta),(0,d),(-1,1)].$$

There exist $\eta_0(d)>0$ and smooth parameters $s(d,\eta), q(d,\eta)$, equal to $s_d, 0$ at $\eta=0$, such that

$$U_{d,\eta} = D_{s(d,\eta)}S_{q(d,\eta)}Q_{d,\eta}$$

is affine-isotropic and has an indefinite affine Hessian for $|\eta| < \eta_0(d)$. For $0<\eta<\eta_0(d)$ it can be required to have neither pair of opposite convex-hull sides parallel and hull deficit greater than $1/4$.

Proof. At fixed positive d, triangulate P_d with nondegenerate triangles. Small vertex perturbations are represented by continuous piecewise-affine maps on this fixed triangulation. Their Jacobians and inverses depend analytically on the vertex coordinates. Pulled mass and energy forms are analytic and uniformly coercive locally. The simple first eigenvalue is therefore smooth in these coordinates and in s, q. This volume-form approach is compatible with polygonal shape calculus as in [1] and requires no corner regularity beyond H^1.

Write $h_\eta(s,q) = \lambda_1(D_s S_q Q_{d,\eta})$. The Jacobian in (s,q) of the two equations $h_{\eta,s} = h_{\eta,q} = 0$ is the invertible Hessian (13) at $(s_d, 0, 0)$. The implicit-function theorem gives the asserted parameters. Their vanishing equations are exactly isotropy: the Eulerian generators of $D_s S_q$ are $\operatorname{diag}(1,-1)$ and $e^{2s}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, so (6) gives

$$h_{\eta,s} = -2(E_x - E_y), \qquad h_{\eta,q} = -2e^{2s}E_{xy}$$

on the transformed domain. The Hessian remains indefinite by continuity. Thus there is an actual critical affine path with negative second variation on every continued shape.

Denote the outer hull vertices in their order by A, B, C, D. For the displayed raw perturbation,

$$\det(B - A, D - C) = -2\eta, \qquad \det(D - A, C - B) = -\eta.$$

Both pairs are nonparallel for $\eta > 0$. The hull remains strictly convex and the reflex apex remains interior for small η; invertible affine maps preserve these properties and five genuine sides. Shoelace gives

$$|Q_{d,\eta}| = (1 + d)(1 + \eta/2), \qquad |\operatorname{conv} Q_{d,\eta}| = 2 + 3\eta/2.$$

First fix small $d < 1/2$, then decrease $\eta_0(d)$ so the deficit is greater than $1/4$. The ratio is affine invariant. □

Proof of Theorem 1.1. Use Ω_d from Theorem 4.2, or $U_{d,\eta}$ from Proposition 5.1. Each is a bounded connected nonconvex Lipschitz pentagon, is affine-isotropic, and has a strictly negative affine second variation. Proposition 2.2 supplies positive affine eigenfunctions on the same domain at the two distinct values $\Lambda_{\mathcal{A},1}(\Omega)$ and $\lambda_1(\Omega)$. The asserted hull geometry and deficit follow from the construction. □

Corollary 5.2. *The affine orbit of each constructed pentagon contains two affine-isotropic positions with different ordinary first eigenvalues. In particular these positions are not related by an orthogonal map. There also exist a constructed pentagon Ω and a nonzero symmetric trace-free matrix B such that*

$$\frac{d}{dt}\lambda_1(e^{tB}\Omega)\bigg|_0 = 0, \qquad \frac{d^2}{dt^2}\log\lambda_1(e^{tB}\Omega)\bigg|_0 < 0.$$

Proof. Lemma 2.1 supplies an isotropic orbit minimum. The saddle has a smaller neighboring value, so it differs from the minimum. Orthogonal maps preserve the eigenvalue.

Let N generate a negative affine shear direction at the isotropic shape. Polar decomposition writes $e^{tN} = O_t V_t$ with V_t positive definite, $\det V_t = 1$, and $V_0' = B = \operatorname{sym} N$. The paths V_t and e^{tB} have the same first velocity. Differentiating their determinant constraints twice shows that the difference of their second accelerations has trace zero. At isotropy the first variation annihilates this difference, by (6). Their second eigenvalue derivatives agree and are negative. The first derivative vanishes; division by $\lambda_1 > 0$ gives the logarithmic assertion. □

Thus the isotropy characterization and positive-definite-deformation log-convexity for convex bodies in [3] do not extend to arbitrary nonconvex domains. Corollary 5.2 gives distinct isotropic positions. The examples are affine-critical; full stationarity under all polygon vertex motions is a different condition. The existence parameters d_0 and $\eta_0(d)$ are qualitative, with no prescribed positive-height numerical instance needed for the theorem.

6 Smooth examples in every dimension

The corners and the planar dimension are not essential to the failure of positive-eigenvalue uniqueness.

Theorem 6.1. *For every integer $n \geq 2$, there exists a bounded connected nonconvex domain $\Omega \subset \mathbb{R}^n$ with C^∞ boundary and two strictly positive weak Dirichlet eigenfunctions of the affine 2-Laplacian at distinct eigenvalues. One eigenvalue is the first affine variational eigenvalue $\Lambda_{\mathcal{A},1}(\Omega)$; the other is $\lambda_1(\Omega) > \Lambda_{\mathcal{A},1}(\Omega)$.*

We first prove a derivative-convergence statement for smooth inner approximations. The variational approach is related to domain-convergence theory such as [2]; the response estimates below supply the additional parameter derivatives needed here.

Lemma 6.2 (Inner approximation with parameter derivatives)**.** *Let U be a bounded connected Lipschitz domain and let U_j be nested bounded connected smooth domains with $\overline{U_j} \subset U$ and $\bigcup_j U_j = U$. Let $C(z)$ be a smooth symmetric positive-definite matrix for z in an open subset of a finite-dimensional real space. Suppose it is uniformly elliptic on compact parameter sets. The first eigenvalues associated with*

$$a_z(v, w) = \int_U \nabla v^T C(z) \nabla w, \qquad m(v, w) = \int_U vw$$

on $H_0^1(U_j)$ converge locally in C^2 to the first eigenvalue on $H_0^1(U)$, using zero extensions throughout.

Proof. Write $V_j = H_0^1(U_j)$ and $V = H_0^1(U)$. The union of the nested spaces V_j is dense in V: every compactly supported smooth function on U has support in some U_j. Minmax, density and compact embedding show convergence of each fixed eigenvalue, with multiplicity. The same proof works along any sequence of parameters $z_j \to z$, since the coefficient matrices converge and are uniformly elliptic. A subsequence argument therefore gives uniform convergence on compact parameter sets for the first two eigenvalues.

The limiting first eigenvalue is simple by connectedness and the ground-state maximum principle. On a compact parameter set its gap to the second eigenvalue has a positive minimum. For all sufficiently large j, the approximating first gaps thus have a common positive lower bound. Denote the normalized positive ground states by u_j, u and their eigenvalues by λ_j, λ. Compactness, testing against the dense nested spaces and simplicity identify every weak limit of u_j with u. Convergence of eigenvalues and of their energy identities then gives strong H^1 convergence. This conclusion also holds along varying parameters.

For a parameter coordinate a, write $a_a = \partial_a a_z$. Simple eigenpair perturbation gives

$$\lambda_a = a_a(u, u), \qquad f_a(v) = a_a(u, v) - \lambda_a m(u, v).$$

The functional f_a annihilates u. There is a unique response r_a orthogonal to u in L^2 such that

$$(a_z - \lambda m)(r_a, v) = f_a(v) \qquad (v \in V).$$

Indeed, on the orthogonal complement,

$$(a_z - \lambda m)(v, v) \geq \left(1 - \frac{\lambda}{\lambda_2}\right) a_z(v, v).$$

Define $r_{j,a}$ analogously on V_j. The gap estimate gives uniform energy coercivity. Strong H^1 convergence of u_j implies convergence of $\lambda_{j,a} = a_a(u_j, u_j)$ and of the associated functionals $f_{j,a}$ in the dual norm of V, where the same integral formulas extend them to all of V.

The responses are bounded in H^1. To identify a weak limit, first take a test in a fixed V_k and, for $j \geq k$, subtract its L^2 projection onto u_j. Density then identifies the limiting orthogonal response as r_a. Finally,

$$a_z(r_{j,a}, r_{j,a}) - \lambda_j \|r_{j,a}\|_2^2 = f_{j,a}(r_{j,a}).$$

The right side converges by dual-norm convergence and weak convergence, and compactness gives strong L^2 convergence. Equality of limiting energies yields strong H^1 convergence of the responses.

Differentiating the normalized simple eigenpair gives $u_a = -r_a$ and

$$\lambda_{ab} = a_{ab}(u,u) - 2(a_z - \lambda m)(r_a, r_b).$$

The preceding strong convergence therefore proves convergence of all first and second coordinate derivatives. Applying the same argument to arbitrary varying parameter sequences proves local uniform convergence. □

Lemma 6.3 (Symmetric smooth exhaustion)**.** *Let U be a bounded connected Lipschitz domain, let a compact group G of orthogonal maps preserve U, and suppose $x_0 \in U$ is fixed by G. If the positive ordinary ground state ϕ is continuous on $\overline{U}$ and zero on ∂U, then U has a nested connected G-invariant smooth inner exhaustion.*

Proof. The ground state is smooth in the interior. Choose decreasing positive regular values $\epsilon_j \to 0$ below $\phi(x_0)$, and take the component U_j of $\{\phi > \epsilon_j\}$ containing x_0. Its closure is compact in U by boundary continuity. The regular-level theorem gives smooth boundary; compactness of the level hypersurface ensures that a selected component also has smooth compact boundary. Simplicity and positivity make ϕ G-invariant. Since x_0 is fixed, each U_j is invariant. The components are nested. Every $x \in U$ can be joined to x_0 by a compact path in U; the positive minimum of ϕ on that path puts x in U_j for all sufficiently large j. □

Fix a sufficiently small positive d in Theorem 4.2, and write $P = P_d$, $s_0 = s_d$ and

$$L(s,q) = \lambda_1(D_s S_q P), \qquad \ell = L(s_0, 0).$$

Thus at $(s_0, 0)$ the first derivatives vanish, $L_{ss} > 0$, $L_{sq} = 0$ and $L_{qq} < 0$. The height d stays fixed in every approximation below. In particular its first spectral gap is strictly positive.

For the planar smooth construction, apply Lemma 6.3 to P, the reflection $x \mapsto -x$, and a point $(0, y_0)$ with $0 < y_0 < d$. Boundary continuity of the ground state holds on this polygon by its exterior-cone condition. Let P_j be the resulting exhaustion and put $L_j(s,q) = \lambda_1(D_s S_q P_j)$. Pulled affine energy forms have smooth uniformly elliptic coefficient matrices on compact parameter sets, so Lemma 6.2 applies.

Reflection gives $L_j(s,q) = L_j(s,-q)$. On a sufficiently small fixed interval about s_0, $L_s(s,0)$ has opposite endpoint signs and $L_{ss}(s,0) > 0$. Consequently there is a root $s_j \to s_0$ of $L_{j,s}(s_j, 0) = 0$, with $L_{j,ss}(s_j, 0) > 0$. Also $L_{j,qq}(s_j, 0) < 0$ for all sufficiently large j. For $\Omega_j = D_{s_j} P_j$, reflection makes $E_{xy} = 0$ and the stationarity equation gives $E_{xx} = E_{yy}$. Thus the tensor is exactly scalar, rather than merely close to scalar. Nonconvexity persists: choose two points of P whose midpoint lies in the open removed notch. The points eventually belong to P_j, whereas their midpoint never does. Invertible affine maps preserve this obstruction to convexity.

6.1 A symmetry-balanced lift

Let $n = m + 2$, $m \geq 1$, and choose an m-dimensional ball B with first eigenvalue $\mu = m\ell/2$. For $m = 1$ a ball means an interval. Such a choice is possible by inverse-square scaling. Set $W = P \times B$ and

$$F(s,q,r) = \operatorname{diag}(e^r D_s S_q, e^{-2r/m} I_m), \qquad \det F = 1.$$

Separation of variables gives

$$H(s,q,r) = \lambda_1(F(s,q,r)W) = e^{-2r} L(s,q) + e^{4r/m}\mu.$$

At $(s,q,r) = (s_0, 0, 0)$ all three first derivatives vanish, and

$$H_{ss} = L_{ss} > 0, \quad H_{sr} = 0, \quad H_{rr} = 4\ell + 16\mu/m^2 > 0, \quad H_{qq} = L_{qq} < 0.$$

Only the displayed positive diagonal block is needed.

The product positive ground state is continuous on $\overline{W}$, zero on its boundary and smooth inside. The product domain is Lipschitz. Apply Lemma 6.3 with $G = \{I, R_x\} \times O(m)$, where $R_x(x, y) = (-x, y)$, and fixed point $(0, y_0, 0)$. This gives connected smooth G-invariant inner domains W_j exhausting W, despite the nonsmooth product boundary. Set $H_j(s, q, r) = \lambda_1(F(s, q, r)W_j)$. Lemma 6.2 gives local C^2 convergence to H.

There exist $(s_j, r_j) \to (s_0, 0)$ with

$$H_{j,s}(s_j, 0, r_j) = H_{j,r}(s_j, 0, r_j) = 0.$$

For completeness, set $h_j(s, r) = H_j(s, 0, r)$ and let A be the positive-definite limiting Hessian in (s, r) at $(s_0, 0)$. In a sufficiently small ball, continuity and C^2 convergence make the derivative of $z \mapsto z - A^{-1}\nabla h_j(z)$ have norm at most $1/2$. Its displacement at the center tends to zero, so for large j it maps that closed ball to itself and has a fixed point. Shrinking the ball shows convergence of the fixed points to the center.

Put $\Omega_j = F(s_j, 0, r_j)W_j$. Its positive ordinary ground state inherits the reflection and $O(m)$ symmetries. These kill every mixed planar/transverse energy and the xy energy, and make the transverse block αI_m. The two stationarity equations, expressed in the physical coordinates of Ω_j, are

$$0 = -2(E_{xx} - E_{yy}), \qquad 0 = -2(E_{xx} + E_{yy}) + \frac{4}{m}\operatorname{tr} E_{\text{transverse}}.$$

They imply $E_{xx} = E_{yy} = \alpha$, hence $E = \alpha I_n$ exactly. Reflection also gives $H_{j,q}(s_j, 0, r_j) = 0$, and convergence gives $H_{j,qq}(s_j, 0, r_j) < 0$. This is a genuine determinant-one affine descent at Ω_j: the varying transformations relative to $F(s_j, 0, r_j)$ have determinant one. Nonconvexity follows from the same two planar points, now with transverse coordinate zero. Thus smooth isotropic domains with negative affine curvature exist in every dimension at least two.

6.2 The affine equation in general dimension

We record the dimension-dependent normalization to complete the eigenfunction conclusion. These determinant-energy and linear-position identities belong to established affine energy theory [4, 8, 7]; the following derivation fixes the constants and the attainment argument used here. For $0 \neq u \in H_0^1(\Omega)$ in $\mathbb{R}^n$, $E_u = \int \nabla u \nabla u^T$ is positive definite by the zero-extension argument in Section 2. The ellipsoid polar-volume identity gives

$$\int_{S^{n-1}} (\xi^T E_u \xi)^{-n/2}\, d\xi = \frac{|S^{n-1}|}{\sqrt{\det E_u}}.$$

In the normalization of [4], the affine energy at exponent two therefore satisfies

$$\mathcal{E}_2(u)^2 = n(\det E_u)^{1/n}, \qquad M_u = (\det E_u)^{1/n} E_u^{-1}.$$

Half the first variation is $\int \nabla z^T M_u \nabla u$, so the positive-sign weak affine operator is

$$\mathcal{L}_{\mathcal{A}} u = -\operatorname{div}(M_u \nabla u).$$

These definitions agree with Section 2 when $n = 2$.

For any bounded connected Lipschitz $\Omega \subset \mathbb{R}^n$,

$$\Lambda_{\mathcal{A},1}(\Omega) = \inf_{u \neq 0} \frac{n(\det E_u)^{1/n}}{\|u\|_2^2} = \min_{F \in \mathrm{SL}(n,\mathbb{R})} \lambda_1(F\Omega). \tag{18}$$

Indeed, under $v = u \circ F^{-1}$ the ordinary energy is $\operatorname{tr}(E_u F^{-1} F^{-T})$. Its infimum over determinant-one F is $n(\det E_u)^{1/n}$ by the matrix arithmetic–geometric mean inequality; one attaining choice is $F = E_u^{1/2}/(\det E_u)^{1/(2n)}$. Interchange the function and matrix infima.

To prove attainment, enclose Ω in a ball of radius R and write $\sigma_{\min}$ for the smallest singular value of F. The image $F\Omega$ lies in a strip of width $2R\sigma_{\min}$, and one-dimensional Poincare applied to zero extensions gives

$$\lambda_1(F\Omega) \geq \frac{\pi^2}{4R^2\sigma_{\min}^2}.$$

After quotienting left rotations, polar decomposition reduces to positive-definite determinant-one matrices. A bounded eigenvalue sublevel bounds their smallest eigenvalue below; determinant one then bounds all eigenvalues above. These sublevels are compact. Continuity of the pulled forms proves that the minimum in (18) is attained.

At a minimizing image $F_0\Omega$, stationarity under every trace-free generator gives, for its normalized positive ordinary ground state v, $E_v = (\Lambda/n)I_n$, where $\Lambda = \Lambda_{\mathcal{A},1}(\Omega)$. For $w = v \circ F_0$,

$$E_w = (\Lambda/n)F_0^T F_0, \qquad M_w = F_0^{-1}F_0^{-T}.$$

The transformed ordinary weak equation yields $\mathcal{L}_{\mathcal{A}} w = \Lambda w$, with $w > 0$ in Ω.

Proof of Theorem 6.1. Use a sufficiently advanced smooth domain Ω_j in the planar or the higher-dimensional construction. Its ordinary positive normalized ground state u has scalar tensor, hence $M_u = I_n$ and $\mathcal{L}_{\mathcal{A}} u = \lambda_1(\Omega_j)u$. The negative affine curvature supplies a determinant-one image with a strictly smaller ordinary first eigenvalue. Equation (18) implies $\Lambda_{\mathcal{A},1}(\Omega_j) < \lambda_1(\Omega_j)$, and the attained orbit minimum supplies the positive eigenfunction at the smaller value. All constructed domains are bounded, connected, nonconvex and smooth. □

The theorem is qualitative. It asserts neither a prescribed smooth boundary nor a numerical height or spectral gap. Its exponent is two; the cases $p \neq 2$ and the existence of analogous convex examples are not settled here.

A Exact auxiliary calculations

The accompanying standard-library Python program verifies the triangle normalization, the two diagonal energies, the cross energy (16), the forcing norm in Lemma 4.1, and the matrix pullbacks used above by rational arithmetic. The identities needed in the proof are

$$\int_T \phi^2 = 1, \quad \int_T \phi_X^2 = \int_T \phi_Y^2 = \frac{5}{2}\pi^2, \quad \int_T \phi_X \phi_Y = \frac{64}{9}, \quad \|2(\phi_{XX} - \phi_{YY})\|_2^2 = 36\pi^4.$$

One exact integration method expands the trigonometric products into $e^{i\pi(mX+nY)}$. Write $q = (i\pi)^{-1}$ and $E(k) = \int_0^1 e^{i\pi kX}\, dX$, so $E(0) = 1$ and $E(k) = q((-1)^k - 1)/k$ for $k \neq 0$. Then for $n \neq 0$,

$$\int_T e^{i\pi(mX+nY)}\, dX\, dY = \frac{q}{n}\big[(-1)^n E(m-n) - E(m)\big].$$

For $n = 0$, the integral is $1/2$ if $m = 0$, and otherwise $-q/m + q^2((-1)^m - 1)/m^2$. The program evaluates these finite expressions with Gaussian rational coefficients and checks the cross integral independently by separated one-dimensional integrals on the reflected square. It verifies auxiliary identities only. The limiting-space, response, existence and continuation arguments are the analytic proofs in the text.

AI assistance

AI tools assisted research, proof development, writing, and verification.

References

[1] B. Bogosel and D. Bucur, *On the polygonal Faber–Krahn inequality*, Journal de l'École polytechnique — Mathématiques **11** (2024), 19–105. doi:10.5802/jep.250.

[2] D. Daners, *Dirichlet problems on varying domains*, Journal of Differential Equations **188** (2003), 591–624. doi:10.1016/S0022-0396(02)00105-5.

[3] D. Faifman and I. Polterovich, *The Faber–Krahn position of convex bodies and Gaussian measure inequalities*, preprint, arXiv:2607.21539v2, 5 August 2026.

[4] J. Haddad, C. H. Jiménez and M. Montenegro, *From affine Poincaré inequalities to affine spectral inequalities*, Advances in Mathematics **386** (2021), article 107808. doi:10.1016/j.aim.2021.107808.

[5] Q. Huang and A.-J. Li, *Optimal Sobolev norms in the affine class*, Journal of Mathematical Analysis and Applications **436** (2016), 568–585. doi:10.1016/j.jmaa.2015.11.063.

[6] D. Langharst and M. Roysdon, *Affine isoperimetric inequalities for the first eigenvalue of the mth order Affine p-Laplace Operator*, preprint, arXiv:2512.17237v2, 25 September 2026.

[7] F. Marín Sola, *Affine variational functionals, isoperimetric problems and stability*, preprint, arXiv:2610.05403v1, 4 October 2026.

[8] I. Schindler and C. Tintarev, *Compactness properties and ground states for the affine Laplacian*, Calculus of Variations and Partial Differential Equations **57** (2018), article 48. doi:10.1007/s00526-018-1338-6.