Polynomial synthesis and zero ideals for the Riemann xi-function

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Luminary Oddities

October 25, 2026

Abstract

Let $F(z) = \xi((1+iz)/2)/4$. On every fixed horizontal strip of half-width $y > 1$, we approximate arbitrary holomorphic zero-compatible Dirichlet targets by ordinary polynomial multiples of F in the same boundary norm, without assuming a larger-strip extension. For all positive widths we also prove synthesis with a strict strip margin. Convergence retains the derivative energy of the entire product, an anchored value, and the original real-axis absolute-potential weights. A paired-product minimum-modulus estimate controls initially large quotients; Gaussian regularization, Fourier truncation and a Meixner–Pollaczek expansion then give a constructive return to polynomials. The zero-free collars outside the unconditional zero strip permit a same-width Sobolev Fourier passage. In one fixed Hilbert space at each $y > 1$, and in the compatible projective space over all finite strips, the closure of the polynomial multiples is exactly the full zero ideal. The potential weights themselves force cancellation at real zeros. The remaining quotient has complex dimension equal to total nonreal zero multiplicity when that number is finite, and infinite dimension otherwise. This gives an F-dependent density characterization equivalent to the Riemann hypothesis. A singular-inner countermodel explains why fixed-width decay and interior zero jets alone do not imply a general synthesis theorem.

1 The joint approximation problem

Define

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad F(z) = \tfrac{1}{4}\xi((1+iz)/2), \quad S_b = \{z : |\Im z| < b\}.$$

The function F is entire, even and real on $\mathbb{R}$. For $y > 0$ set

$$E_y[f] = \frac{1}{8\pi y}\int_{-y}^{y}\int_{\mathbb{R}} |(Ff)'(x+iv)|^2 \,\mathrm{d}x\,\mathrm{d}v, \tag{1}$$

$$\gamma_y(x) = \frac{\partial_y |F(x+iy)|^2}{8y}, \qquad Q_y^{\mathrm{abs}}[f] = \frac{1}{2\pi}\int_{\mathbb{R}} |\gamma_y(x)||f(x)|^2 \,\mathrm{d}x. \tag{2}$$

Their continuous endpoint definitions are

$$E_0[f] = \frac{1}{4\pi}\int_{\mathbb{R}} |(Ff)'|^2 \,\mathrm{d}x, \qquad \gamma_0 = (F'^2 - FF'')/4.$$

For real multipliers the corresponding signed form is $Q_y[f] = (2\pi)^{-1}\int \gamma_y(x) f(x)^2 \,\mathrm{d}x$. In the potential integral the multiplier is evaluated on the real axis; its value at $x+iy$ is not substituted. Put

$$W_b(x) = \sup_{0\le v\le b} |\gamma_v(x)|.$$

All anchored errors below include $|(Ff)(0)|$. The positive joint norm allows complex targets; signed conclusions use real targets and the usual polarization.

Theorem 1.1 (Synthesis with a strip margin)**.** *Fix $0 < y < Y < \infty$. Let U be holomorphic on S_Y, suppose $\int_{S_Y} |U'|^2 \, \mathrm{d}A < \infty$, and suppose that every zero of F in S_Y is a zero of U of at least the same multiplicity. Supply removable values so that $f = U/F$ is holomorphic on S_Y. If $Q_y^{\mathrm{abs}}[f] < \infty$, there are polynomials P_n such that*

$$E_y[P_n - f] + Q_y^{\mathrm{abs}}[P_n - f] + |(FP_n)(0) - U(0)|^2 \longrightarrow 0.$$

The same sequence can control any finite collection of absolute potential forms at widths at most y, provided each target integral is finite. If instead $\int W_y |f|^2 < \infty$, it can control the W_y weighted norm and hence all those potential forms uniformly. With a reserved energy width strictly between y and Y, one also has uniform E_v convergence for $0 \le v \le y$.

Real targets admit real polynomials and arbitrary finite rational rounding. Either parity is preserved. On a fixed finite-dimensional target span there is one linear synthesis map with arbitrarily small operator error after its finite basis constant is paid.

For fixed $y > 1$, Theorem 4.1 removes the larger-strip assumption altogether. Its zero-free boundary collars and a full-boundary Gaussian Dirichlet lemma supply same-width synthesis. Corollary 4.5 then identifies the full zero ideal in one fixed Hilbert space.

The main point is the arbitrary zero-compatible target class, rather than a new general polynomial-density principle. Earlier work [1] constructs a receiver for a prescribed entire class formed from a critical cosh multiplier and a bandlimited input. It also gives quantitative moving-width families. The present theorem allows division by the actual F on a larger strip and controls the resulting quotient before regularization. It does not inherit that earlier growing-family degree rate.

Classical strip Hardy polynomial density is developed by Bakan–Kaijser [2]; complex-analytic approximation for the double-sided exponential weight, including its polynomial coefficient machinery, is developed by Bizeul–Klartag [3]. Mhaskar [4] treats weighted entire approximation on unbounded cones with radial exponential weights. Our recipient combines an F divisor, product-derivative energy and the original absolute-potential weights; the quotient may initially be only strip-holomorphic. These earlier theorems do not immediately supply that joint-norm conclusion. Minimum-modulus estimates, Gaussian regularization, contour shifts and the classical Meixner–Pollaczek generating formula are established ingredients.

The projective application below identifies all remaining zero-jet obstructions in this precise topology. The synthesis/codimension theme also occurs in the recent step-Hardy and Dirichlet-multiplier work of span_of_dilates [5]. Its algebra and ambient space differ from ordinary polynomials and the entire projective space used here; its open infinite-zeta synthesis problem is not settled by the present construction.

2 Actual-function estimates and quotient growth

Lemma 2.1 (Fixed-strip inputs)**.** *The actual function F has a paired product*

$$F(z) = F(0) \prod_{\rho \in \mathcal{R}_+} (1 - z^2/\rho^2),$$

where one member of each $\pm$ pair is included with multiplicity. For large R,

$$n(R) \le CR \log(R+2), \qquad \sum_{|\rho| > R} |\rho|^{-2} \le C \frac{\log(R+2)}{R}.$$

For every fixed $b < \infty$ and fixed integer $j \geq 0$, there are $C_{b,j}, p_{b,j}$ with

$$|F^{(j)}(x+iv)| \leq C_{b,j}(1+|x|)^{p_{b,j}} e^{-a|x|}, \quad |v| \leq b, \quad a = \pi/8. \tag{3}$$

In particular W_b is continuous and has an integrable envelope $C_b(1+|x|)^{p_b} e^{-2a|x|}$.

Proof. The classical order-one Hadamard product for the completed zeta function, paired using evenness, has the stated form; no residual linear exponential survives the even symmetry. The unconditional zero-count bound is standard [6, 7]; summation by parts gives its displayed inverse-square tail. Complex zeros and arbitrary multiplicities are retained.

For completeness the horizontal decay in (3) can be obtained directly. On a fixed enlarged z strip the real part of $s = (1+iz)/2$ lies in a fixed bounded interval, while $\Im s = x/2$. Euler–Maclaurin with a fixed number of Bernoulli terms exceeding the left edge, and a truncation comparable to $2+|x|$, bounds $\zeta(s)$ by a fixed polynomial in $2+|x|$. Its remainder contains only a fixed rising factorial and an integral of a fixed negative power, so this statement is uniform in that bounded real interval. Vertical-strip Stirling bounds $\Gamma(s/2)$ by a polynomial times $e^{-\pi|x|/8}$. The quadratic completed-function prefactor costs another fixed polynomial. Cauchy's formula on a fixed radius in the enlarged strip gives each fixed derivative with the same exponential factor. Compact x is covered by the entire completed function; separate gamma or zeta poles are not bounded there.

For fixed real x, $A_x(v) = |F(x+iv)|^2$ is even in v and $A_x'(0) = 0$. Hence

$$\gamma_v(x) = \frac{1}{8}\int_0^1 A_x''(tv)\,\mathrm{d}t$$

including $v = 0$. The $j \leq 2$ envelopes prove the asserted uniform bound for W_b. Joint continuity, followed by a supremum over a compact parameter interval, proves continuity. The endpoint $\gamma_0 = (F'^2 - FF'')/4$ follows by differentiating at real x. □

Lemma 2.2 (Dirichlet growth)**.** *If U is holomorphic on S_Y and $\int_{S_Y} |U'|^2 < \infty$, then on every strict closed substrip $|\Im z| \leq c < Y$,*

$$U(x+iv) = o(\sqrt{|x|}) \quad (|x| \to \infty)$$

uniformly in v.

Proof. Choose $d > 0$ with $c + d < Y$. The analytic submean inequality on radius-d disks, integrated horizontally, gives

$$\sup_{|v|\leq c} \int_{\mathbb{R}} |U'(x+iv)|^2\,\mathrm{d}x \leq \frac{2}{\pi d}\int_{S_Y} |U'|^2\,\mathrm{d}A.$$

The same calculation over $|x| > R$ is bounded by the strip derivative energy over $|\Re z| > R - d$, and tends to zero uniformly in v. The values $U(iv)$ are bounded on the compact central segment. Integrating horizontally and applying Cauchy–Schwarz first gives $O(1+\sqrt{|x|})$. For the sharper claim split the primitive at a fixed large positive or negative R. The compact part contributes $o(\sqrt{|x|})$ as $|x| \to \infty$; the remaining increment divided by $\sqrt{|x|}$ is bounded by the square root of the uniformly small exterior derivative energy. Then let $R \to \infty$. □

Lemma 2.3 (Quotient growth with a width reserve)**.** *Under the zero-compatibility hypotheses of Theorem 1.1, for each $b < Y$ there is C_b such that*

$$\sup_{|v|\leq b} |f(x+iv)| \leq C_b \exp\Big(C_b(1+|x|)\log^2(3+|x|)\Big). \tag{4}$$

Fixed derivatives have the same type of bound on narrower strips.

Proof. Choose $b < c < Y$. For dyadic large R, remove open disks of radius R^{-2} about all zeros with $|\rho| \le 4R$. On the enlarged annulus $R/2 \le |z| \le 3R$ outside these disks, each paired factor with $|\rho| \le 4R$ has modulus at least $(16R^6)^{-1}$, since both $|z-\rho|$ and $|z+\rho|$ are at least R^{-2}. There are at most $CR\log R$ such factors. For $|\rho| > 4R$ one has $|z^2/\rho^2| \le 9/16$ and

$$\log|1 - z^2/\rho^2| \ge -C|z|^2/|\rho|^2.$$

The total far debit is at most $CR\log R$ by Lemma 2.1. Consequently, off the disks,

$$|F(z)| \ge \exp(-CR\log^2 R). \tag{5}$$

This is the classical small-exceptional-set minimum-modulus mechanism; see, for example, [8, Section3.1].

The sum of all removed radii is $O(\log R/R)$, so any connected component of their union has diameter at most twice this sum. A component meeting S_b at large real height therefore stays in S_c. Its boundary is outside the open disks. The enlarged annulus pays components meeting the original annulus sides as well. Lemma 2.2 bounds U on those boundaries, and (5) bounds the quotient there by $C(1+\sqrt{R})\exp(CR\log^2 R)$. The quotient is holomorphic throughout each such component because every zero is canceled to full multiplicity. The maximum principle extends this estimate inside. Overlapping dyadic annuli cover both horizontal tails; compact points are covered by continuity. Cauchy's formula with a further small width reserve gives the derivative versions. □

Remark 2.4. All estimates involving $1/F$ are made away from its zeros. The actual finite-order paired product, rather than just decay on a single horizontal strip, is essential to the argument controlling a previously uncontrolled quotient.

3 Constructive return to ordinary polynomials

3.1 Gaussian regularization in the original norm

Put $f_\epsilon(z) = e^{-\epsilon z^2} f(z)$ and $U_\epsilon = e^{-\epsilon z^2} U$. The exponent in (4) is subquadratic, so f_ϵ and every fixed derivative are rapidly decreasing horizontally on each strict substrip. This uses Cauchy's formula for derivatives and $|e^{-\epsilon(x+iv)^2}| = e^{-\epsilon x^2 + \epsilon v^2}$.

As $\epsilon \downarrow 0$,

$$(U_\epsilon - U)' = (e^{-\epsilon z^2} - 1)U' - 2\epsilon z e^{-\epsilon z^2} U.$$

The first term tends to zero in strip $L^2(S_y)$ by dominated convergence. For the second, the little-o statement of Lemma 2.2 gives $|U(x+iv)|^2 = o(|x|)$ uniformly for $|v| \le y$. After the compact part is removed, its squared norm is bounded by an arbitrarily small constant times

$$\epsilon^2 \int_{\mathbb{R}} (x^2+y^2)|x| e^{-2\epsilon x^2}\,\mathrm{d}x,$$

which is bounded uniformly for $0 < \epsilon \le 1$. The compact contribution tends to zero. Thus

$$E_y[f_\epsilon - f] \longrightarrow 0. \tag{6}$$

The anchored value is unchanged. For each declared absolute potential integral, the error on the real axis is $(e^{-\epsilon x^2} - 1)f(x)$, so dominated convergence applies. Exactly the same argument works with W_y, if its target integral is finite. It does not infer that stronger integrability from a single width's hypothesis.

3.2 Fourier truncation with a width reserve

Choose $y < b < c < Y$ and fix $\epsilon > 0$. The regularized quotient is Schwartz uniformly on the closed strip of width c. With $\widehat{h}(k) = \int h(x)e^{-ikx}\,\mathrm{d}x$, contour shifting to heights $\pm b$ gives

$$|\widehat{f}_\epsilon(k)| \le A_{\epsilon,b}e^{-b|k|}.$$

Rectangular contour sides vanish because the Gaussian dominates (4). Define

$$f_{\epsilon,H}(z) = \frac{1}{2\pi}\int_{-H}^{H} \widehat{f}_\epsilon(k)e^{ikz}\,\mathrm{d}k.$$

Fourier inversion and its differentiated version yield

$$\sup_{|\Im z|\le y}\left(|f_{\epsilon,H} - f_\epsilon| + |f'_{\epsilon,H} - f'_\epsilon|\right) \le CA_{\epsilon,b}(1+H)e^{-(b-y)H}. \tag{7}$$

The full F, F' envelopes of Lemma 2.1 turn a uniform C^1 error into strip product-derivative error on the complete horizontal line. The integrable W_y envelope pays every required absolute-potential error by the uniform real-axis value error. The anchor is paid by the same estimate. In particular exterior horizontal tails are included; no relative error near an F zero is used.

3.3 Shifted Meixner–Pollaczek bounds

Use the classical basis and reference weight

$$q_n(z) = P_n^{(1/2)}(z/4;\pi/2), \qquad \omega(x) = \frac{1}{4\cosh(\pi x/4)}.$$

The q_n are orthonormal for $\omega(x)\,\mathrm{d}x$. Their generating function and recurrence are [7, Sections18.19,18.22,18.23]

$$\sum_{n\ge 0} q_n(z)t^n = (1-it)^{-1/2+iz/4}(1+it)^{-1/2-iz/4}, \tag{8}$$

$$xq_n(x) = 2(n+1)q_{n+1}(x) + 2nq_{n-1}(x). \tag{9}$$

Connections between these polynomials and ξ expansions have long been studied; see Kuznetsov and Romik [9, 10]. The global complex-shift bound below is the earlier receiver's elementary polynomial-cost mechanism [1]; a proof is included to make the present construction independent of that manuscript's supplements.

Lemma 3.1 (Polynomial shift cost)**.** *For each fixed $b < \infty$ there are an integer m_b and C_b with*

$$\left(E_b[q_n] + \int_{\mathbb{R}} W_b|q_n|^2\,\mathrm{d}x\right)^{1/2} + |F(0)q_n(0)| \le C_b(n+1)^{m_b}.$$

The same holds on any larger fixed reserved strip.

Proof. Taking the ratio of (8) at $x+iv$ and x gives

$$q_n(x+iv) = \sum_{j=0}^{n} A_j(v)q_{n-j}(x), \qquad \sum_{j\ge 0} A_j(v)t^j = \left(\frac{1+it}{1-it}\right)^{v/4}.$$

The logarithms are analytic on $|t| < 1$. On $|t| = r$ the modulus of the last expression is at most $((1+r)/(1-r))^{|v|/4}$. For $n \geq 1$ choose $r = n/(n+1)$. Cauchy's bound gives $|A_j(v)| \leq e(2n+1)^{|v|/4}$ for $j \leq n$, and orthogonality therefore gives

$$\|q_n(\cdot + iv)\|_{L^2(\omega)} \leq C_b(n+1)^{b/4+1/2}, \qquad |v| \leq b.$$

Differentiation in x multiplies the generating function by $(i/4)\log[(1-it)/(1+it)]$, whose modulus on that circle is $O(\log(n+2))$. Its coefficient bound, convolved with the same shift factor and then summed in the orthonormal basis, adds only this logarithm to the estimate for q_n'. The derivative for $n = 0$ vanishes.

By (9), multiplication by x on degree at most n has norm at most $4(n+1)$ into the next degree. Repetition bounds each fixed polynomial weight $(1+|x|)^p$ by an additional fixed power of $n+1$ in both estimates. Since $2a = \pi/4$, the squared F, F' envelopes divided by ω have only polynomial growth. The same is true of W_b/ω. These estimates pay all global integrals in the assertion, including the $1/b$ energy averaging. The endpoint value follows either from (8) at0 or its coefficient bound on the same circle. Increasing the integer power absorbs the logarithm. □

3.4 Geometric coefficient tails and completion of synthesis

Substitute $t = i\tanh(2k)$ into (8). Since $\arctan(i\tanh(2k)) = 2ik$, one obtains the exact classical identity

$$e^{ikz} = \sum_{n\geq 0} \operatorname{sech}(2k)[i\tanh(2k)]^n q_n(z). \tag{10}$$

The scalar prefactor of the generating function is $\cosh(2k)$; this accounts for the $\operatorname{sech}(2k)$ factor. Put

$$c_n = \frac{1}{2\pi}\int_{-H}^{H} \widehat{f}_\epsilon(k)\operatorname{sech}(2k)[i\tanh(2k)]^n \,\mathrm{d}k, \quad r_H = \tanh(2H) < 1,$$

where $H > 0$. Then

$$|c_n| \leq B_{\epsilon,H} r_H^n, \qquad B_{\epsilon,H} = \frac{1}{2\pi}\int_{-H}^{H} |\widehat{f}_\epsilon(k)|\,\mathrm{d}k.$$

For $P_N = \sum_{n=0}^{N} c_n q_n$, Lemma 3.1 and the triangle inequality in the positive joint norm give the complete tail bound

$$C_y B_{\epsilon,H}(1-r_H)^{-m_y-1} r_H^{N/2}. \tag{11}$$

Indeed $(n+1)^m r^{n/2} \leq C_m(1-r)^{-m}$, and the remaining geometric sum costs at most $2/(1-r)$. Local uniform convergence on every compact complex set follows from the same generating function coefficient bound there. Thus the analytic limit is exactly $f_{\epsilon,H}$ by (10) and integration over the fixed finite frequency band. The norm limit is not identified by an unsupported real-axis density argument.

First choose ϵ using (6) and the declared target integrals, then H using (7), then N using (11). This proves Theorem 1.1. To control all E_v, $0 \leq v \leq y$, perform the approximation in an energy width y_1 with $y < y_1 < Y$. Analytic submean, integrated horizontally with fixed radius less than $y_1 - y$, bounds every narrower derivative-line L^2 norm by this larger-strip error, including $v = 0$.

For real f, Fourier conjugation symmetry gives real c_n; $q_n(-z) = (-1)^n q_n(z)$ preserves either parity. Finite rational rounding adds arbitrarily small error because every finite basis norm is finite. For a fixed finite-dimensional target span choose a basis and pay its finite synthesis constant when assigning each basis tolerance; extending the receiver linearly controls the whole mixed Gram matrix. No bound uniform over all target unit balls, and no growing-family degree law, follows from this argument.

4 Synthesis at the same boundary outside the zero strip

The preceding argument is valid at every positive interior width, but requires a larger-strip extension of the target. For widths containing the unconditional zero strip with a positive margin, that extension can be removed.

Theorem 4.1 (Fixed-boundary synthesis)**.** *Fix one finite $y > 1$. Let U be holomorphic on S_y, with $\int_{S_y} |U'|^2 < \infty$, and cancel every F zero to full multiplicity. Put $f = U/F$ after removals. If $Q_y^{\mathrm{abs}}[f] < \infty$, ordinary polynomials satisfy*

$$E_y[P_n - f] + Q_y^{\mathrm{abs}}[P_n - f] + |F(0)P_n(0) - U(0)|^2 \longrightarrow 0.$$

No larger-strip extension of U is assumed. Finite collections of absolute-potential forms are retained when their target integrals are finite. If $\int W_y|f|^2 < \infty$, all original potential forms for $0 \le v \le y$ are controlled uniformly. Energy convergence can also be made uniform for $0 \le v \le y$. Reality, parity and finite rational rounding are preserved.

Every actual F zero lies in $|\Im z| < 1$, by the unconditional critical-strip statement for ζ [7, Section25.10]. Thus the gap $y - 1$ leaves zero-free boundary collars. This is not an assumption of RH and does not settle the same-boundary problem for $0 < y \le 1$.

Lemma 4.2 (Full-boundary Gaussian convergence)**.** *For any holomorphic U on S_y with $U' \in L^2(S_y)$, $U_\epsilon = e^{-\epsilon z^2}U$ satisfies*

$$\int_{S_y} |(U_\epsilon - U)'|^2 \,\mathrm{d}A \longrightarrow 0, \qquad U_\epsilon(0) = U(0).$$

Proof. One cannot assume a uniform pointwise bound at the boundary. Instead, vertical integration gives for almost every $x \in [-1, 1]$ and $|v| < y$

$$|U(x + iv)|^2 \le 2|U(x)|^2 + 2y \int_{-y}^{y} |U'(x + it)|^2 \,\mathrm{d}t.$$

The compact real-axis values are bounded, so U is L^2 on the full-height rectangle. Choose one finite vertical slice there. Fubini and the horizontal fundamental theorem give a locally absolutely continuous function $V(x) = U(x+i\,\cdot)$ valued in $H = L^2(-y, y)$, with $V'(x) = U'(x+i\,\cdot) \in L^2(\mathbb{R}; H)$. Integration first gives $\|V(x)\|_H = O(1 + \sqrt{|x|})$. Splitting the primitive at a fixed large positive or negative R and using the small exterior derivative norm improves this to

$$\|V(x)\|_H^2 = o(|x|) \quad (|x| \to \infty). \tag{12}$$

This is an integrated-height bound, not pointwise boundary control. Decompose the Gaussian derivative error exactly as before. The U' term tends to zero by dominated convergence. For the other term, (12) bounds its exterior squared integral by an arbitrarily small constant times

$$\epsilon^2 e^{2\epsilon y^2} \int_{\mathbb{R}} (x^2 + y^2)|x| e^{-2\epsilon x^2} \,\mathrm{d}x,$$

uniformly bounded for $0 < \epsilon \le 1$. Compact contributions vanish. This proves the lemma without requiring $U \in L^2(S_y)$. □

Write $H^1_{\mathrm{hol}}(S_y)$ for the holomorphic first-order Sobolev space with squared norm $\int_{S_y}(|g|^2 + |g'|^2)\,\mathrm{d}A$. This notation does not mean Hardy H^1.

Lemma 4.3 (Regularized division up to the boundary)**.** *Under the zero hypotheses of Theorem 4.1, $f_\epsilon = e^{-\epsilon z^2}U/F$ belongs to $H^1_{\mathrm{hol}}(S_y)$ for every $\epsilon > 0$.*

Proof. Choose $1 < b < y$, with intermediate bands reserved between b and y. On the central closed band of width b, the proof of Lemma 2.3 and then Cauchy's estimate give $|f| + |f'| \le C\exp(C(1+|x|)\log^2(3+|x|))$. The Gaussian makes both square-integrable there.

On the collars $b \le |\Im z| < y$, the distance from every zero of F is at least $b-1$. The paired-product proof of (5) therefore applies uniformly without any exceptional disks, giving

$$|F(z)|^{-1} \le C\exp(C(1+|x|)\log^2(3+|x|)).$$

The global fixed-strip bound on F' also controls $(1/F)' = -F'/F^2$ there. Differentiating the regularized quotient and absorbing every subquadratic exponent and Gaussian derivative polynomial gives

$$|f_\epsilon(z)|^2 + |f_\epsilon'(z)|^2 \le C_\epsilon e^{-\epsilon x^2}(|U(z)|^2 + |U'(z)|^2).$$

Its U integral is finite by (12); its U' integral was assumed finite. All zeros in the central band are removed with full multiplicity. This proves the same-width Sobolev membership. □

Lemma 4.4 (Same-width Fourier cutoff)**.** *For $g \in H^1_{\mathrm{hol}}(S_y)$ there is a middle-line Fourier coefficient $a(k)$ with*

$$\|g\|^2_{H^1_{\mathrm{hol}}(S_y)} = \frac{1}{2\pi}\int_{\mathbb{R}}(1+k^2)S_y(k)|a(k)|^2\,\mathrm{d}k, \quad S_y(k) = \begin{cases}\sinh(2yk)/k, & k \ne 0,\\ 2y, & k = 0.\end{cases} \tag{13}$$

The entire functions $g_H(z) = (2\pi)^{-1}\int_{-H}^{H} a(k)e^{ikz}\,\mathrm{d}k$ converge to g in this same full-strip norm.

Proof. The middle-line trace is L^2 by the one-dimensional Sobolev trace estimate in v, integrated over x. Cauchy–Riemann and horizontal Fourier transformation give, in distributions, $\partial_v\widehat{g}(k,v) = -k\widehat{g}(k,v)$. On each bounded frequency interval this is a bounded-coefficient Hilbert-space ODE, so $\widehat{g}(k,v) = a(k)e^{-kv}$. Exhaust frequency intervals to obtain the representation globally. Plancherel and Tonelli give (13), since $\int_{-y}^{y} e^{-2kv}\,\mathrm{d}v = S_y(k)$ and g' contributes k^2. This is the classical strip Paley–Wiener mechanism; compare the weighted Bergman representation and converse in [11, Theorem2.1]. The formula is written here to specify its exact normalization. On each compact frequency interval a is L^1 by Cauchy–Schwarz, so g_H is entire. Its squared error in (13) is exactly the omitted tail integral, which tends to zero. No spare width or exponential rate is required. □

Proof of Theorem 4.1. Gaussian convergence in the original energy follows from Lemma 4.2, and in the declared real potential integrals from dominated convergence. Apply Lemma 4.4 to the quotient of Lemma 4.3. Boundedness of F, F' on the whole fixed strip transfers the H^1_{hol} error to the product-derivative energy. The middle-line trace estimate, and boundedness of W_y, pay every required absolute-potential norm. Analytic submean on a central disk pays the anchor. Each finite-band target has the exact coefficients and geometric tail of (10)–(11), with $\widehat{f}_\epsilon$ replaced by a on $[-H, H]$. Thus its polynomial return holds at this SAME width. Choose ϵ, then H, then degree. For uniform energies, fix $0 < y_0 < y$. For $v \le y_0$, integrated submean bounds derivative-line error by the total strip error; for $y_0 \le v \le y$ the normalization $1/v$ is bounded and the total area error suffices. Reality, parity and rounding follow from the same coefficient arguments as before. □

4.1 One fixed Hilbert space

Let X_y consist of holomorphic U on S_y with

$$\|U\|_{X_y}^2 = |U(0)|^2 + \int_{S_y} |U'|^2 \,\mathrm{d}A + \int_{\mathbb{R}} W_y |U/F|^2 \,\mathrm{d}x < \infty.$$

It initially imposes no nonreal zero constraints. This is a Hilbert space: an anchored derivative Cauchy sequence converges uniformly on strict compact subsets by submean, its full-strip derivative L^2 limit is the derivative of that holomorphic limit, and its weighted quotient limit is identified off the discrete real zeros. These observations give completeness of the polarized positive norm without a boundary trace identity for the original U.

Corollary 4.5 (Fixed-strip ideal and codimension)**.** *For every fixed $y > 1$, the closure in X_y of FP, with P polynomial, is exactly the subspace of U canceling all actual F zeros to full multiplicity. Its complex Hilbert codimension is total nonreal-zero multiplicity if that number is finite, and infinite otherwise. In particular polynomial multiples are dense in this explicitly F-dependent Hilbert space if and only if RH holds.*

Proof. Continuity of interior zero jets gives one closure inclusion; Theorem 4.1 with W_y gives the reverse. The weight W_y bounds below a positive constant near each real F zero, by its small positive-width Taylor expansion; thus local integrability forces full real-zero jets. For finitely many nonreal zeros z_j, the principal parts of prescribed jets divided by F yield a rational R with canceled poles such that $U = FR$ attains those jets. Its real quotient is bounded and its derivative has finite strip energy by the actual F envelopes. It therefore belongs to X_y. These independent onto finite jet maps give the codimension assertion; the fuller local construction is spelled out in Theorem 5.3 below. Every actual zero is interior since $y > 1$. Density is equivalent to absence of nonreal zeros. □

The fixed-space statement does not imply a theorem at arbitrary widths $0 < y \le 1$: actual zeros could approach such a chosen boundary, and the target need not extend past it. The strict-margin receiver remains available there. The next section records the compatible all-finite-strip formulation.

5 A projective zero ideal and its codimension

Let X be the complex vector space of entire functions U for which, for every integer $m \ge 1$,

$$p_m(U)^2 = |U(0)|^2 + \int_{S_m} |U'|^2 \,\mathrm{d}A + \int_{\mathbb{R}} W_m(x) |U(x)/F(x)|^2 \,\mathrm{d}x < \infty. \tag{14}$$

The quotient on the real axis is assigned arbitrary values at the discrete real zeros; those assignments do not change the integral. No nonreal zero condition is imposed in the definition of X. The norms increase with m and define a countable locally convex topology. It depends explicitly on the actual F and its original potential weights.

Lemma 5.1. *The space X, with the norms p_m, is a Fréchet space. Finite derivative evaluations at every complex point are continuous.*

Proof. Each p_m is a norm: zero derivative energy makes an entire U constant, and the anchor then makes it zero. A metric for the countable topology is $\sum_{m\ge1} 2^{-m} \min(1, p_m(U - V))$. Suppose a

sequence is Cauchy in every p_m. For each compact set, choose a strictly wider strip. Submean applied to derivative differences bounds their uniform values on that compact by the larger-strip derivative L^2 norm. Integrating from0 and using the anchor proves local uniform convergence to an entire U. The derivative L^2 limit on each S_m agrees with U' by local uniform convergence of holomorphic derivatives. The real quotient also has a weighted L^2 limit. On every compact real interval avoiding F zeros, uniform convergence identifies that limit with U/F; exhaust the real line apart from the discrete null set. Thus convergence holds in every p_m. Cauchy's formula on compact disks proves continuity of finite derivative evaluations. □

Theorem 5.2 (Exact projective closure)**.** *Define*

$$I_F = \{U \in X : \operatorname{ord}_z U \geq \operatorname{ord}_z F \text{ for every complex zero } z \text{ of } F\}.$$

Then

$$\overline{\{FP : P \text{ polynomial}\}}^{X} = I_F. \tag{15}$$

Every $U \in X$ already vanishes to full multiplicity at every real zero of F.

Proof. All polynomial multiples belong to X by Lemma 2.1. Their full zero jets are preserved under X convergence by Lemma 5.1, so the closure is contained in the closed space I_F. Conversely take $U \in I_F$. At stage n, apply Theorem 1.1 with smaller width n, larger width $n+2$, and the W_n integral from (14). Choose its tolerance so that the unnormalized derivative error, the W_n error and the anchor are each less than $1/n$. Since $S_j \subset S_n$ and $W_j \leq W_n$ for $j \leq n$, this same polynomial sequence converges in every fixed p_j.

For the final assertion, let a real zero r have multiplicity $k \geq 1$, and write $F(r+iy) = c(iy)^k + O(y^{k+1})$, $c \neq 0$. Then

$$\gamma_y(r) = \frac{k|c|^2}{4} y^{2k-2} + O(y^{2k-1}) > 0$$

for sufficiently small positive y. For each $m \geq 1$ choose one such $y \leq m$; continuity bounds W_m below by a positive constant on a real neighborhood of r. If U vanished to order $l < k$, its quotient square would be comparable to $|x-r|^{-2(k-l)}$, which is not locally integrable. Membership in X excludes this. The real-zero conditions are therefore consequences of the original weights, not additional definitions. □

Theorem 5.3 (Off-axis codimension)**.** *If F has exactly $N < \infty$ nonreal zeros counting multiplicity, then $\dim_{\mathbb{C}}(X/I_F) = N$. If there are infinitely many nonreal zeros, the quotient is infinite-dimensional. In particular,*

$$\{FP : P \text{ polynomial}\} \text{ is dense in } X \quad \Longleftrightarrow \quad \text{RH}. \tag{16}$$

This is an F-dependent density characterization.

Proof. Every finite collection of nonreal zero-jet conditions is jointly onto. Let distinct zeros z_j have multiplicities k_j, and prescribe arbitrary jets of U of orders below k_j. At z_j take the principal part of the prescribed Taylor polynomial divided by F. Sum these finitely many principal parts to obtain a rational R with poles only at those points, of orders at most k_j, and no polynomial part. Then $U = FR$ is entire. Other principal parts are holomorphic at z_j, so multiplication by F makes them vanish to order k_j there; they do not alter the assigned lower jets.

The ambient-space membership is also paid. The poles stay a positive distance from the real axis, so R is bounded on $\mathbb{R}$ and is $O(1/|x|)$ at infinity. Hence every W_m quotient integral is finite. On a fixed horizontal strip, R, R' have bounded denominators for sufficiently large $|\Re z|$; the actual F, F'

exponential envelopes imply finite derivative energy of FR. All pole neighborhoods are compact and removable. Thus $FR \in X$.

If these are all nonreal zeros, the onto map $X \to \mathbb{C}^N$ has kernel I_F, since real-zero jets are automatic. This proves the finite dimension formula. If nonreal zeros are infinite, onto finite jet maps of arbitrarily large dimension prove infinite dimension.

If all zeros are real then $X = I_F$ and Theorem 5.2 proves density. Conversely, for any nonreal zero z_0 of exact order k the entire function $U = F/(z - z_0)^k$ belongs to X by the same argument and has a nonzero value at z_0. Continuous evaluation there obstructs its approximation by FP. Finally, under $s = (1 + iz)/2$, all F zeros are real exactly when all nontrivial zeta zeros have real part $1/2$. This proves (16) without proving either equivalent condition. □

Corollary 5.4 (All finite strips and conditional sign transfer)**.** *An entire U with finite derivative energy on every finite strip, full F-zero jets, and all quotient W_m integrals finite admits one polynomial sequence converging in every p_m. For an eligible real multiplier with strictly negative original Q_y, polynomial negative witnesses follow.*

Proof. The first statement is the diagonal construction already given. For the second, convergence in the absolute-potential weighted L^2 norm and Cauchy–Schwarz control the difference of the signed quadratic values. Strict negativity then persists. □

The codimension step is elementary once synthesis and the original real-zero integrability mechanism are available. It counts total nonreal multiplicity in a complex projective space; no separate real or parity count is implicit. This topology is not the step-Hardy space of [5], nor a single boundary Dirichlet Hilbert space. The unknown exclusion of nonreal zeros remains exactly the missing RH deduction.

6 Why zero jets and one-strip decay are not sufficient

The strict strip margin in the theorem is not removed by ordinary zero compatibility. The following countermodel also shows why a general theorem must use more than entire functions with good decay on a chosen strip. It concerns a different infinite-order generator, not the actual F.

Proposition 6.1 (A singular-inner obstruction)**.** *Fix $y > 0$, $k = \pi/(4y)$ and $c = 2k$. Put*

$$F_*(z) = z^2 e^{-z^2} e^{-\cosh(cz)}, \qquad U_*(z) = z^2 e^{-z^2}.$$

These entire real even functions have precisely the same interior zero, at0 with multiplicity two. Every polynomial multiple of F_* and U_* has finite derivative energy on S_y, with Gaussian horizontal decay of every fixed derivative. Nevertheless U_* is not in the anchored strip derivative-energy closure of $\{PF_* : P \text{ polynomial}\}$.

Proof. For $z = x + iv$, $|v| \le y$, one has $\Re \cosh(cz) = \cosh(cx)\cos(cv) \ge 0$ and $|e^{-z^2}| \le e^{y^2} e^{-x^2}$. Each fixed derivative adds at most a polynomial in x and a fixed exponential in $|x|$, so all claimed derivative integrals are finite.

The conformal map $w = \tanh(kz)$ sends S_y onto the disk, with inverse $z(w) = \operatorname{artanh}(w)/k$. Dirichlet energy is conformally invariant. For an anchored disk function $h(w) = \sum_{n\ge1} a_n w^n$ it is

$$\frac{1}{\pi} \int_{\mathbb{D}} |h'|^2 \, \mathrm{d}A = \sum_{n\ge1} n|a_n|^2 \ge \|h\|^2_{H^2(\mathbb{D})} .$$

Define the nonconstant singular inner function

$$S(w) = \exp\left[-\frac{1}{2}\left\{\frac{1+w}{1-w} + \frac{1-w}{1+w}\right\}\right].$$

Both fractions have positive real part in the disk, and purely imaginary boundary values except at ± 1, so $|S| \le 1$ inside and $|S| = 1$ almost everywhere on its boundary. Moreover, $e^{-\cosh(cz(w))} = S(w)$. The function $z(w)^2 P(z(w)) e^{-z(w)^2}$ is bounded in the disk: the corresponding z remains in S_y, and the Gaussian dominates every fixed polynomial in $\Re z$. Thus every pullback of PF_* belongs to the closed Hardy subspace SH^2. Closedness follows from the boundary-modulus isometry of multiplication by S.

The pullback $h(w) = z(w)^2 e^{-z(w)^2}$ of U_* does not belong to SH^2. Indeed for $0 < r < 1$ write $t = (1+r)/(1-r)$; then $z(r) = \log t/(2k)$ and

$$\log|h(r)/S(r)| = 2\log\frac{\log t}{2k} - \frac{(\log t)^2}{4k^2} + \frac{t+t^{-1}}{2}.$$

This grows like $t/2$. Every H^2 function q instead satisfies $|q(r)| \le \|q\|_{H^2}/\sqrt{1-r^2}$, by Cauchy–Schwarz on its Taylor coefficients; its logarithmic bound is only $O(\log t)$. Hence $h/S \notin H^2$. Anchored Dirichlet convergence would imply H^2 convergence into the closed subspace SH^2, a contradiction. □

This model has infinite entire order and fails the actual paired product/count hypothesis. It also need not retain the same decay on larger strips, where $\cos(cv)$ changes sign. The role of the example is to isolate a genuine boundary/at-infinity obstruction which interior zero jets alone cannot exclude. Classical inner/outer and cyclicity questions have substantial literature; see the function-space discussion in [12].

7 Scope of the construction

The constructive synthesis proved here receives arbitrary zero-compatible targets in the stated joint norm: with a strip margin at every positive width, and at the same boundary for every fixed width $y > 1$. Its fixed-Hilbert and projective applications describe the accessible zero ideal and the remaining independent off-axis jets. No target-independent degree bound or general fixed-boundary theorem for $0 < y \le 1$ is supplied. A separate source-specific estimate would still be required to prove a Jensen or Weil sign inequality. The density equivalence of Theorem 5.3 is a description of that unresolved obstruction, not a resolution.

The accompanying exact checker verifies plane-wave coefficient identities through degree32 using Fourier-weight moments and the MP recurrence, independently of the generating-function substitution in the proof. Finite symbolic controls check scale and phase; the infinite-tail and density arguments are analytic.

Research assistance and responsibility

AI tools assisted mathematical exploration, literature discovery, proof checking, exact-control development and manuscript preparation. The author is responsible for the statements, proofs, attribution and final text.

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