Polynomial Recovery and Masking under Weak Exponential Tails

Scott Schubr
Luminary Oddities, Countryside, Illinois, United States
scottschubr@luminaryoddities.com

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Abstract

We determine the worst polynomial degree needed to detect a negative atom under local variation and weak exponential-moment bounds. For a tail scale $H(x) = x/W(x)$ with a controlled growth ratio, put $\Psi(t) = 1 + \int_1^t \mathrm{d}x/[xW(x)]$. When Ψ diverges, the worst detecting degree satisfies $\Psi(\max\{1, D_W(R)\}) = \Theta(R^{1/q})$ for every fixed local exponent $q > 0$. The same estimates yield the full-line origin Christoffel order $\lambda_n \asymp \Psi(n)^{-q}$ for the canonical density proportional to $|x|^{q-1}e^{-bH(|x|)}$. Finite-clock masking and logarithmic critical scales follow within this framework. We also construct one smooth nondecreasing profile with $H(x)/\log x \to \infty$ and divergent Ψ, for which the canonical density masks the atom at every degree for every fixed local exponent $q > 0$ and density rate $b > 0$. Thus divergent clock and existence of all moments do not replace growth control. The proofs combine classical quasianalytic propagation and localization with explicit moment and interpolation estimates, retaining signed and singular backgrounds and precise fixed-parameter quantifiers.

Keywords. Moment problems; Christoffel functions; weighted polynomial approximation; quasianalyticity; signed measures.

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1 Polynomial detection and the integrated tail clock

How many moments are needed to certify that a signed measure is not positive? For $\mu = -\delta_0 + \nu$, a polynomial $P(x) = \sum_{j=0}^N a_j x^j$ has a strictly negative square integral precisely when its coefficient vector is a negative direction of the Hankel moment matrix

$$\mathsf{H}_N(\mu) = \left(\int x^{i+j} \, \mathrm{d}\mu(x) \right)_{i,j=0}^{N}.$$

We ask how the first such degree depends on a tail budget and local variation control near the negative atom. The background may approach the origin arbitrarily closely. Detection is formulated through exact moment forms.

Local variation control preserves the negative atom at zero. The tail budget governs whether finite detection is uniformly possible and how rapidly the worst required degree grows. We express that dependence by one integrated quantity under a growth-ratio condition on the tail profile.

Fix $q, c, C, L, r_0 > 0$. Let $W : [0, \infty) \to [1, \infty)$ be continuous and nondecreasing, with $W(0) = 1$, and put

$$H(x) = \frac{x}{W(x)}, \qquad \Psi(t) = 1 + \int_1^t \frac{\mathrm{d}x}{xW(x)} \quad (t \geq 1).$$

For $R \geq 1$, let $\mathcal{M}_W(R)$ be the class of real signed Borel measures $\mu = -\delta_0 + \nu$ satisfying

$$\mathrm{TV}(\nu) \leq R, \qquad \int_{\mathbb{R}} e^{cH(|x|)}\, \mathrm{d}|\nu|(x) \leq CR, \qquad |\nu|([-r, r]) \leq LRr^q \quad (0 < r \leq r_0). \tag{1}$$

The local condition holds at every declared radius and forces $\nu(\{0\}) = 0$, while allowing the background support to accumulate at zero. The background may be signed, atomic away from zero, singular or absolutely continuous. Write $\mathcal{P}_N$ for ordinary polynomial degree at most N, and define

$$n_1(\mu) = \inf\{N \in \mathbb{Z}_{\geq 0} : \exists P \in \mathcal{P}_N(\mathbb{R}),\ \int P^2\, \mathrm{d}\mu < 0\}, \qquad D_W(R) = \sup_{\mu \in \mathcal{M}_W(R)} n_1(\mu),$$

with $\inf \varnothing = \infty$. The polynomial square has degree at most $2N$. Negativity is strict: a null square is not a negative direction.

Theorem 1.1 (Integrated-clock recovery and masking)**.** *Suppose fixed $A \geq 1$ and $0 \leq \delta < 1$ satisfy*

$$\frac{W(y)}{W(x)} \leq A(y/x)^\delta \qquad (y \geq x \geq 1). \tag{2}$$

If Ψ is unbounded, $D_W(R)$ is finite for every $R \geq 1$, and

$$\Psi(\max\{1, D_W(R)\}) = \Theta(R^{1/q}) \qquad (R \to \infty).$$

If $\Psi(\infty) < \infty$, there are one fixed positive atomless probability law τ, a fixed $\eta > 0$ and $R_0 \geq 1$ such that $-\delta_0 + \eta R\tau \in \mathcal{M}_W(R)$ and

$$-|P(0)|^2 + \eta R \int |P|^2\, \mathrm{d}\tau \geq 0 \quad \text{for every polynomial } P \text{ and every } R \geq R_0.$$

Thus $D_W(R) = \infty$ for every $R \geq R_0$ in the finite-clock case. All constants may depend on W and the fixed class data; only R varies.

The theorem matches the original arbitrary-signed upper class with lower examples from a single positive reference law. The masking conclusion concerns every degree simultaneously. It is stronger than a sequence of difficult examples whose background changes with degree. The growth-ratio assumption is one-sided. Section 4 includes a profile with infinitely many corners and two distinct dilation subsequential limits. Section 5 derives the logarithmic regimes from the same theorem.

For orientation, $W = 1$ gives the exponential-tail baseline $\log D_W(R) = \Theta(R^{1/q})$, whereas $W(x) = \log(e + x)$ gives $\log\log D_W(R) = \Theta(R^{1/q})$. If $W(x) = \log(e + x)^\gamma$ with $\gamma > 1$, the integrated quantity is bounded and one law masks all degrees. These are fixed-profile statements, without uniformity as γ approaches one. Section 5 supplies the complete logarithmic cases.

The lower examples as Christoffel functions. For a positive probability τ, define

$$\lambda_N(\tau;0)=\inf\left\{\int |P|^2\,\mathrm{d}\tau : \deg P\le N,\ P(0)=1\right\}.$$

The degree-N square form of $-\delta_0+\eta R\tau$ is nonnegative exactly when $\eta R\lambda_N(\tau;0)\ge 1$. Thus the lower construction is an origin-evaluation estimate for a positive weight, a standard extremal problem in orthogonal-polynomial theory. The uniform upper assertion concerns all backgrounds in (1), rather than only this positive model. Corollary 4.2 makes the consequence explicit: for the normalized density τ_b proportional to $|x|^{q-1}e^{-bH(|x|)}$, $\lambda_n(\tau_b;0)\asymp\Psi(n)^{-q}$. This includes a power singularity at the origin when $0<q<1$.

Theorem 1.2 (One smooth profile masking all canonical rates)**.** *There is one nondecreasing $W\in C^\infty([0,\infty))$, $W(0)=1$, $W\ge 1$, such that $H(x)=x/W(x)$ satisfies*

$$H(x)/\log x\longrightarrow\infty,\qquad \Psi(t)=1+\int_1^t\frac{dx}{xW(x)}\longrightarrow\infty,$$

and, for every $q>0$ and $b>0$, the canonical probability law

$$d\tau_{b,q}(x)=Z_{b,q}^{-1}|x|^{q-1}e^{-bH(|x|)}dx$$

has all moments and $\inf_n\lambda_n(\tau_{b,q};0)>0$. In particular this same W works simultaneously for every fixed $q,c,C,L,r_0>0$ and every $b>c$: there exist $\eta>0$ and R_0 (depending on these data and b) such that $-\delta_0+\eta R\tau_{b,q}\in\mathcal{M}_W(R)$ masks every polynomial square for every $R\ge R_0$. No uniformity in these constants as the parameters vary is asserted.

The same smooth profile therefore invalidates a divergent-clock recovery criterion throughout the canonical density family when growth control is absent. Its construction and full interpolation proof appear in Section 6.

Proof mechanism. Local mass control supplies a compact-frequency test that is itself a square and has strictly negative integral. The upper bound follows by approximating its square root in $L^2(\delta_0+|\nu|)$, using classical Bang propagation and Fourier residual duality; we give this reduction in Section 4. We also retain a self-contained finite-moment argument: strict Gaussian regularization and Jacobi realization produce a positive measure with the same first $2N$ moments, whose Fourier discrepancy contradicts the negative test. The concave primitive $F(x)=\int_0^x \mathrm{d}u/W(u)$ gives the needed moment majorant, and a change of variables identifies its reciprocal-sum estimate with Ψ.

For the lower bound, a compact Chebyshev multiplier at radius $n/\Psi(n)$ has degree $O(n)$. A singular-power Jacobi evaluation bound converts it into an original full-line norm estimate for one reference density proportional to $|x|^{q-1}e^{-bH(|x|)}$, with $b>c$. The same clock controls the difficult finite degrees and, when bounded, the simultaneous all-degree masker. Shared tools are proved once before the main argument.

Related work and the scope of the synthesis. Classical moment and weighted-approximation theory contains the determinacy boundary, origin Christoffel estimates and fast-decreasing polynomial constructions [1, 2, 3, 4, 5]. Bang's level crossing [6], finite Jacobi moment realizations and the singular-power Jacobi kernel are established tools. Koehler and Wu [7] develop quantitative approximation under Carleman conditions and the Fourier projection-residual argument. Applied to $\delta_0+|\nu|$, that argument already handles signed or singular backgrounds; signedness alone does not require a new approximation principle.

Bauldry's generalized-Freud estimates [8] and their Kasuga–Sakai extension [9] are important power-singular precedents. The Freud-exponent definition in Bauldry's author chapter [10, Chapter I, §2] assumes convexity and $Q'(2x)/Q'(x) > c_0 > 1$. Kasuga–Sakai's condition (0.1) requires $1 < A \le (xQ')'/Q' \le B$. These imply superlinear exponent growth, unlike $H \le x$. The quadratic substitution to a half-line power weight does not remove this boundary: the standard half-line class in [11, Definition 1.1] requires $tQ'(t)/Q(t) > 1/2$ uniformly, whereas our transformed exponent is at most a constant times $\sqrt{t}$.

Levin–Lubinsky's slow-Freud theory [1] remains the relevant $q = 1$ precedent. Uniform interior kernel estimates also transfer routinely to $q > 1$ lower bounds by deleting an origin interval of radius comparable to $1/\Psi(n)$. The contribution here is the matched quantitative law for the complete stated class, including the singular range $0 < q < 1$, together with a direct self-contained argument and the explicit smooth-profile obstruction. Classical smoothing and moment-problem theory supply the nonsmooth-profile reduction and qualitative masking mechanism. Section 4 constructs a comparable smooth exponent; Corollary 4.2 expresses the quantitative consequence in standard extremal terms. Section 6 shows why the uncontrolled-profile extension fails for the canonical density itself.

The preceding recovery papers [12, 13, 14] study sign certificates, collision barriers, growing negative index and logarithmically strengthened exponential tails. The present weak-tail problem has different tail assumptions; its exponential-tail baseline is part of that lineage. The logarithmic and iterated-logarithmic formulas below are consequences of the single theorem, rather than separate approximation mechanisms.

Appendix A records the finite-order approximation estimate used in the direct upper reduction. Bang propagation and Fourier residual duality are inherited. Optimizing equation (3.71) of [7] already gives the corresponding stretched-power-in-degree rate; the appendix states the finite-order form in the normalization used here.

2 Finite propagation and signed moment recovery

We use the Fourier convention $\widehat{\sigma}(t) = \int e^{itx}\, \mathrm{d}\sigma(x)$. The following finite form of Bang's level-crossing argument is classical in origin; see [6]. Its terminal derivative requires only continuity.

Lemma 2.1 (Finite Bang propagation)**.** *Let $f \in C^n([-S,S];\mathbb{C})$, $n \ge 1$, and let $M_0, \dots, M_n > 0$ have nondecreasing quotients $a_j = M_j/M_{j-1}$. Suppose $|f^{(j)}| \le M_j$ for $0 \le j \le n$ and $f^{(j)}(0) = 0$ for $j < n$. If an integer k satisfies $0 \le k < n$ and*

$$\sum_{j=k+1}^{n} a_j^{-1} > eS,$$

then $|f(t)| \le M_0 e^{-k}$ for $|t| \le S$.

Proof. Put $b_j = |f^{(j)}|/(e^j M_j)$ and $B = \max_{0\le j\le n} b_j$. Then $B(0) \le e^{-n}$. On a level strip $e^{-l} < B < e^{-(l-1)}$, only indices $j \le l-1$ can maximize: every b_j with $j \ge l$ is at most e^{-l}. Thus B is locally Lipschitz on this strip, although b_n need only be continuous. At almost every point there,

$$|B'| \le \max_{j \text{ active}} \frac{|f^{(j+1)}|}{e^j M_j} \le e a_l B, \qquad |(\log B)'| \le e a_l.$$

This uses the modulus derivative inequality, valid also for complex functions. Crossing from e^{-l} to $e^{-(l-1)}$ costs length at least $(ea_l)^{-1}$. More precisely, before the first hitting of the upper level

choose the last hitting of the lower level. Integrate on interior level intervals and pass to their endpoints. If B reached e^{-k} within distance S, the disjoint crossings for $l = n, \dots, k+1$ would have total length at least $e^{-1} \sum_{l=k+1}^{n} a_l^{-1} > S$. Apply the same argument to $f(-t)$ for negative t. No derivative of order $n+1$ is used. □

Lemma 2.2 (Finite positive moment realization)**.** *Let L be a real linear functional on polynomials through degree $2N$, $N \geq 1$, with $L[P^2] > 0$ for every nonzero real polynomial of degree at most N. There is a positive measure τ, supported on at most $N+1$ points, such that $\int x^j \, \mathrm{d}\tau = L[x^j]$ for $0 \leq j \leq 2N$.*

Proof. In the inner product $L[PQ]$, form monic orthogonal polynomials $A_0, \dots, A_N$, with squared norms $h_j > 0$. On their span define J by the recurrence for xA_j when $j < N$. Give its last diagonal entry any real value; its last off-diagonal entries are prescribed by symmetry. The usual three-term recurrence follows directly from orthogonality: xA_j is orthogonal to A_i for $i < j-1$, and its coefficient of A_{j-1} is h_j/h_{j-1}. Hence J is selfadjoint in the diagonal metric $(h_0, \dots, h_N)$, and agrees with multiplication by x on polynomials of degree below N. This is the classical Jacobi mechanism; compare [3]. For $j \leq N$, $J^j 1 = x^j$. Split $s = i + j \leq 2N$ with $i, j \leq N$. Selfadjointness gives

$$\langle 1, J^s 1 \rangle = \langle J^i 1, J^j 1 \rangle = L[x^s].$$

The spectral measure of this finite selfadjoint operator at 1 proves the assertion. The free last diagonal avoids requiring a moment of order $2N+1$. □

Remark 2.3. Positive semidefiniteness alone does not suffice here. The moments $(1, 0, 0, 0, 1)$ have positive semidefinite Hankel matrix $\mathrm{diag}(1, 0, 1)$, but a positive representing measure with second moment zero must also have fourth moment zero. We therefore regularize strictly in the next argument.

Theorem 2.4 (A general signed moment criterion)**.** *Let $q, L, r_0, K > 0$, $R \geq 1$, and $\mu = -\delta_0 + \nu$ be a finite real signed measure such that*

$$\mathrm{TV}(\nu) \leq R, \qquad |\nu|([-r, r]) \leq LRr^q \quad (0 < r \leq r_0).$$

Fix $N \geq 1$. Let $M_0 = 1, M_1, \dots, M_N > 0$ be log-convex, and let $a > 0$ satisfy, for $1 \leq j \leq N$,

$$\int |x|^{2j} \, \mathrm{d}|\nu| \leq KRa^{2j} M_j^2, \qquad \int |x|^{2j} \, \mathrm{d}\gamma \leq a^{2j} M_j^2,$$

where γ is standard Gaussian probability. Choose an integer ℓ with $\beta = 2\ell > q$, and put

$$B_0 = r_0^{-q} + \frac{\beta L}{\beta - q}, \quad T = \max\{1, r_0^{-1}, (16 B_0 R)^{1/q}\}, \quad S = 2\ell T,$$

$$A_0 = 8(1 + \sqrt{K+1})R, \qquad k = \lceil \log(16 A_0) \rceil.$$

If $k < N$ and

$$\sum_{j=k+1}^{N} \frac{M_{j-1}}{M_j} > eaS, \tag{3}$$

there is a real polynomial P of degree at most N with $\int P^2 \, \mathrm{d}\mu < 0$.

Proof. Set $g(x) = \operatorname{sinc}(Tx)^{2\ell}$, with $\operatorname{sinc}(u) = \sin(u)/u$ and its continuous value at zero. Then $0 \le g \le \min(1, (T|x|)^{-\beta})$. Layer cake gives

$$\int g \, \mathrm{d}|\nu| \le R(Tr_0)^{-\beta} + LRT^{-q} \int_0^1 u^{-q/\beta} \, \mathrm{d}u \le B_0 RT^{-q} \le 1/16.$$

Indeed on $u \ge (Tr_0)^{-\beta}$ the relevant sublevel radius is at most r_0; below it use total mass R. Consequently $\int g \, \mathrm{d}\mu \le -15/16$. Moreover $g(x) = \int e^{itx} \, \mathrm{d}\pi(t)$ for a positive probability π on $[-S, S]$: convolve 2ℓ uniform probabilities on $[-T, T]$. Suppose all degree-N real squares have nonnegative μ integral. Fix $0 < \eta \le 1/16$. The functional of $\mu + \eta\gamma$ is strictly positive on nonzero squares. Lemma 2.2 supplies a positive finite atomic τ with the same moments through $2N$. Its mass is at most R, since $\nu(\mathbb{R}) - 1 + \eta \le R$. For $1 \le j \le N$, its matched even moments and Cauchy–Schwarz give

$$\int |x|^j \, \mathrm{d}\tau \le R\sqrt{K+1}\, a^j M_j.$$

Likewise $\int |x|^j \, \mathrm{d}|\nu| \le R\sqrt{K}\, a^j M_j$, and the Gaussian contribution is at most $a^j M_j$. Thus

$$F(t) = \mu + \widehat{\eta\gamma} - \tau(t)$$

is C^N, has zero jets through order N, and satisfies $|F^{(j)}| \le A_0 a^j M_j$ for $0 \le j \le N$; at order zero use $\mathrm{TV}(\mu) \le 1 + R$. Lemma 2.1 and (3) yield $|F| \le A_0 e^{-k} \le 1/16$ on $[-S, S]$. Fubini therefore gives $|\int g \, \mathrm{d}(\mu + \eta\gamma - \tau)| \le 1/16$. Its signed value, however, is at most $-15/16 + \eta \le -7/8$, because $g \ge 0$ and $\tau \ge 0$. This contradiction proves strict negativity for the original measure. □

3 Compact localization and origin evaluation

The lower construction uses two shared classical ingredients: compact fast-decreasing polynomial factors and the exact Jacobi origin kernel. We include their normalizations for every $q > 0$; see also [4, 5, 3].

3.1 A compact dyadic multiplier

Lemma 3.1. *Let $Q : [0, \infty) \to [0, \infty)$ be continuous and nondecreasing, with $Q(0) = 0$. For $A > 0$, $J \ge 1$ and $d_j = A2^{-j}$, there is a real polynomial S with $S(0) = 1$ such that*

$$|S(x)| \le \max\{2, e^{Q(d_J)/2}\} e^{-Q(|x|)/2} \quad (|x| \le A), \tag{4}$$

$$\deg S \le \sum_{j=1}^{J} 2^{j-1} Q(A2^{-(j-1)}) + 2J. \tag{5}$$

Proof. Put $m_j = \lceil AQ(d_{j-1})/(4d_j) \rceil$ and

$$C_j(x) = \frac{T_{m_j}((A^2 + d_j^2 - 2x^2)/(A^2 - d_j^2))}{T_{m_j}((A^2 + d_j^2)/(A^2 - d_j^2))}, \qquad S = \prod_{j=1}^{J} C_j.$$

For $m_j = 0$ take $C_j = 1$. On $[-A, A]$, $|C_j| \le 1$. For $d_j \le |x| \le A$, its numerator has modulus at most one and

$$\operatorname{arcosh} \frac{A^2 + d_j^2}{A^2 - d_j^2} = \log \frac{A + d_j}{A - d_j} \ge \frac{2d_j}{A}.$$

The identity $T_m(\cosh u) = \cosh(mu)$ gives $|C_j(x)| \le 2e^{-2m_j d_j/A} \le 2e^{-Q(d_{j-1})/2}$. On $d_j \le |x| \le d_{j-1}$ this bounds S by $2e^{-Q(|x|)/2}$. On the inner cell use $|S| \le 1$ and $Q(|x|) \le Q(d_J)$. Finally $\deg S \le 2\sum m_j$ gives (5). □

3.2 Singular-power evaluation with exact normalization

Lemma 3.2. *For every $q > 0$ there is $c_q > 0$ such that every polynomial U of degree at most M obeys*

$$\int_{-A}^{A} |U(x)|^2 |x|^{q-1} \, \mathrm{d}x \geq c_q \left(\frac{A}{M+1} \right)^q |U(0)|^2.$$

Proof. Put $a = q/2$. For the weight v^{a-1} on $[0,1]$, the Rodrigues polynomials

$$R_j(v) = \frac{v^{1-a}}{j!} \frac{\mathrm{d}^j}{\mathrm{d}v^j} [v^{j+a-1}(v-1)^j]$$

are orthogonal, have value $(-1)^j (a)_j / j!$ at zero, leading coefficient $(j+a)_j / j!$, and squared norm $1/(2j+a)$. Repeated integration by parts proves these assertions: all boundary terms at zero have a positive remaining power a, and the order-j zero at one makes the other boundary terms vanish. The norm is $[(j+a)_j / j!] \, \mathrm{B}(j+a, j+1) = 1/(2j+a)$. Thus the exact evaluation kernel through degree l is

$$K_l(0,0) = \sum_{j=0}^{l} (2j+a) \left[\frac{(a)_j}{j!} \right]^2 = \frac{(l+a)^2}{a} \left[\frac{(a)_l}{l!} \right]^2 \leq C_q (l+1)^q.$$

The equality telescopes, including $l = 0$, and the fixed-parameter gamma ratio gives the bound. Split U into its even and odd parts, orthogonal for the even weight. The odd part vanishes at zero. Substitution $v = x^2/A^2$ makes the even norm A^q times the Jacobi norm above, through $l = \lfloor M/2 \rfloor$. Cauchy–Schwarz in that orthogonal basis proves the assertion, for example with $c_q = C_q^{-1}$. No value of the density at zero is divided by. □

4 Proof of the integrated-clock theorem

We prove Theorem 1.1 under its declared hypotheses on W. Throughout this section put $H(x) = x/W(x)$ and $\Psi(t) = 1 + \int_1^t \mathrm{d}x/[xW(x)]$ for $t \geq 1$. All constants below are fixed before R varies.

4.1 Inverse moments and the upper clock

Define

$$F(x) = \int_0^x \frac{\mathrm{d}u}{W(u)}, \qquad C_0 = AW(1) + \frac{A}{1-\delta}, \qquad p = \frac{1}{1-\delta}.$$

The function F is continuously differentiable, strictly increasing and concave, and $F(x) \leq x$. Monotonicity gives $F(x) \geq H(x)$. For $x \geq 1$, the Potter bound gives

$$\int_1^x \frac{\mathrm{d}u}{W(u)} \leq \frac{Ax^\delta}{W(x)} \int_1^x u^{-\delta} \, \mathrm{d}u \leq \frac{A}{1-\delta} H(x).$$

The remaining integral on $[0,1]$ is at most one, and $1/H(x) \leq AW(1)x^{\delta-1} \leq AW(1)$. Consequently

$$H(x) \leq F(x) \leq C_0 H(x) \quad (x \geq 1). \tag{6}$$

Also $H(x) \geq x^{1-\delta}/[AW(1)]$, so F tends to infinity and has an inverse h. We have $h(t) \geq t$. For $y \geq x \geq 1$, (6) and the Potter bound imply

$$\frac{F(y)}{F(x)} \geq \frac{1}{AC_0} \left(\frac{y}{x} \right)^{1-\delta}.$$

Thus, with $K = (AC_0)^p$,

$$\frac{h(v)}{h(u)} \le K(v/u)^p \qquad (v \ge u \ge 1). \tag{7}$$

These estimates use continuity, monotonicity and the power-ratio bound for W.

For each fixed $b > 0$ there is K_b such that

$$\sup_{x \ge 0} x^j e^{-bH(x)} \le [K_b h(j)]^j \qquad (j \ge 1). \tag{8}$$

Indeed for $x \le h(j)$ this follows without the exponential factor. For $x = h(j)y$, $y \ge 1$, the preceding growth bound and (6) give

$$H(x) \ge \frac{j}{AC_0^2} y^{1/p}.$$

The supremum of $y \exp[-by^{1/p}/(AC_0^2)]$ on $[1, \infty)$ is finite, proving (8).

Set $M_0 = 1$ and $M_j = \prod_{l=1}^{j} h(l)$. Its quotients $h(j)$ increase, so it is log-convex. Equation (7) gives

$$M_j \ge K^{-j}(j!/j^j)^p h(j)^j \ge (Ke^p)^{-j} h(j)^j. \tag{9}$$

For any original class member, (8) at $2j$ gives

$$\int |x|^{2j} \, \mathrm{d}|\nu| \le CR[K_c h(2j)]^{2j} \le CR[K_c K 2^p h(j)]^{2j}.$$

Combining this with (9), there is a fixed scale $a_0 > 0$ for which these moments are at most $CRa_0^{2j} M_j^2$. Enlarge a_0 to cover standard Gaussian even moments, bounded by $(2j)^j$, using $h(j) \ge j$. Thus Theorem 2.4 applies, with its $K = C$ and $a = a_0$.

In its notation, $k = O(\log(R+2))$ and its bandwidth S is at most $C_* R^{1/q}$ for a fixed C_*. The sufficient condition in Theorem 2.4 is

$$\sum_{j=k+1}^{N} \frac{1}{h(j)} > ea_0 S. \tag{10}$$

Since $1/h$ decreases, a change of variable $t = F(x)$ yields

$$\begin{aligned} \sum_{j=k+1}^{N} \frac{1}{h(j)} &\ge \int_{k+1}^{N+1} \frac{\mathrm{d}t}{h(t)} = \int_{h(k+1)}^{h(N+1)} \frac{\mathrm{d}x}{xW(x)} \\ &\ge \Psi(N+1) - \Psi(h(k+1)). \end{aligned} \tag{11}$$

Moreover (7) and $W \ge 1$ give

$$\Psi(h(k+1)) \le 1 + \log(Kh(1)) + p \log(k+1) =: E_k.$$

When Ψ diverges, choose N to be the least integer with

$$\Psi(N) \ge 2ea_0 C_* R^{1/q} + 2E_k.$$

For large R this has $N > k$, since $\Psi(k+1) \le 1 + \log(k+1)$ whereas the target grows like $R^{1/q}$. Equations (11) and the choice of N imply (10), with strict inequality. Theorem 2.4 gives $D_W(R) \le N$. For any fixed $R \ge 1$, divergence of the terminal clock likewise permits a finite N

satisfying (10); hence finiteness holds for every R, while the displayed order is asymptotic. The integer overshoot is at most

$$\Psi(N) - \Psi(N-1) \le \log \frac{N}{N-1} \le \log 2.$$

Since $E_k = O(\log\log(R+3))$, it follows that

$$\Psi(\max\{1, D_W(R)\}) \le B_1 R^{1/q} \tag{12}$$

for a fixed B_1 and all sufficiently large R.

Remark 4.1 (Direct reduction to positive-measure approximation). There is also a direct upper-bound route through the classical projection method in Appendix A. Put $\rho = \delta_0 + |\nu|$ and $f(x) = \operatorname{sinc}(Tx)^\ell$, with T, ℓ as in Theorem 2.4. The layer-cake estimate gives $\|f\|^2_{L^2(\rho)} \le 17/16$ and $\int f^2 \,\mathrm{d}\mu \le -15/16$. The moment estimates above and Theorem A.1 give $P = \Pi_N f$ with $\varepsilon = \|P - f\|_{L^2(\rho)} \le 1/8$ under the same clock order. Since $|\mu| \le \rho$ and projection is a contraction,

$$\left| \int (P^2 - f^2) \,\mathrm{d}\mu \right| \le \varepsilon(\|P\|_{L^2(\rho)} + \|f\|_{L^2(\rho)}) \le 2\sqrt{17/16}\,\varepsilon < 1/2.$$

Thus $\int P^2 \,\mathrm{d}\mu < 0$. This alternative works for singular backgrounds and singular finite moment matrices; the Jacobi proof above instead works directly with the finite Hankel functional and strict regularization.

4.2 A compact clock and one reference law

The literal H need not be monotone. Put $H^\uparrow(x) = \max_{0 \le s \le x} H(s)$. This is continuous, nondecreasing, and at most x. There is a fixed B_H with

$$H \le H^\uparrow \le B_H H. \tag{13}$$

For $x \le 1$, use $x \le W(1)H(x)$. For $x \ge 1$ and $1 \le s \le x$, the Potter bound gives

$$H(s)/H(x) \le A(s/x)^{1-\delta} \le A.$$

For $s < 1$, use $H(s) \le 1$ and $H(x) \ge 1/[AW(1)]$. Thus $B_H = \max\{W(1), A, AW(1)\}$ suffices for the running-maximum comparison.

For integer $n \ge 2$ set $\psi_n = \Psi(n)$, $A_n = n/\psi_n$ and $J = \lceil \log_2 n \rceil$. Because $1 \le \psi_n \le 1 + \log n \le n$, we have $1 \le A_n \le n$. Fix $b > c$ and apply Lemma 3.1 with $Q = bH^\uparrow$, radius A_n, and J factors. The central constant is at most $B_0 = \max\{2, e^{b/2}\}$, since $A_n 2^{-J} \le 1$ and $H^\uparrow \le x$. To estimate the degree, put $d_u = A_n 2^{-u}$. By (13), the main degree sum is at most

$$bB_H A_n \sum_{u=0}^{J-1} \frac{1}{W(d_u)}.$$

For $d_u \ge 2$, the intervals $[d_u/2, d_u]$ are disjoint and contained in $[1, n]$, and monotonicity gives

$$\frac{1}{W(d_u)} \le \frac{1}{\log 2} \int_{d_u/2}^{d_u} \frac{\mathrm{d}t}{tW(t)}.$$

There is at most one term with $1 \le d_u < 2$. The number with $d_u < 1$ is at most $\log_2 \psi_n + 2$, each at most one. Therefore

$$\sum_{u=0}^{J-1} \frac{1}{W(d_u)} \le \frac{\psi_n - 1}{\log 2} + \log_2 \psi_n + 3 \le C_1 \psi_n.$$

Since $A_n \psi_n = n$ and $J \le n$, the resulting polynomial satisfies

$$\begin{aligned} S_n(0) = 1, \qquad & \deg S_n \le K_1 n, \\ |S_n(x)| \le B_0 e^{-bH(|x|)/2} \quad & (|x| \le A_n). \end{aligned} \tag{14}$$

with fixed K_1. Only a compact envelope is used.

Define the single positive atomless probability law

$$Z_b = \int_{\mathbb{R}} |x|^{q-1} e^{-bH(|x|)} \, \mathrm{d}x, \qquad \mathrm{d}\tau_b(x) = Z_b^{-1} |x|^{q-1} e^{-bH(|x|)} \, \mathrm{d}x.$$

The lower bound $H(x) \ge x^{1-\delta}/[AW(1)]$ ensures a positive finite normalizer and all moments. Also

$$E_b := \int e^{cH(|x|)} \, \mathrm{d}\tau_b = Z_{b-c}/Z_b < \infty, \qquad \tau_b([-r, r]) \le \frac{2r^q}{qZ_b}.$$

Only the compact envelope on $[-A_n, A_n]$ is needed: multiplying by S_n there converts the local evaluation inequality into a lower bound for the original full-line norm. Apply Lemma 3.2 to PS_n on $[-A_n, A_n]$, using (14). For every $\deg P \le n$,

$$\begin{aligned} \int |P|^2 \, \mathrm{d}\tau_b &\ge \frac{1}{B_0^2 Z_b} \int_{-A_n}^{A_n} |PS_n|^2 |x|^{q-1} \, \mathrm{d}x \\ &\ge b_0 \Psi(n)^{-q} |P(0)|^2, \qquad b_0 = \frac{c_q}{B_0^2 Z_b (K_1 + 2)^q} > 0. \end{aligned} \tag{15}$$

Here $\deg(PS_n) + 1 \le (K_1 + 2)n$. Fix $\eta = \min\{1, C/E_b, LqZ_b/2\} > 0$. Then $\nu_R = \eta R \tau_b$ meets every original class budget for all $R \ge 1$, and its original degree-n square form is nonnegative whenever

$$\Psi(n)^q \le \eta b_0 R. \tag{16}$$

If Ψ diverges, take $t_R = (\eta b_0)^{1/q} R^{1/q}/2$ and the unique $x_R \ge 1$ satisfying $\Psi(x_R) = t_R$, for large R. Set $n_R = \lfloor x_R \rfloor \ge 2$. Then (16) holds and

$$0 \le \Psi(x_R) - \Psi(n_R) \le \log(1 + 1/n_R) \le 1.$$

Thus $D_W(R) > n_R$ and $\Psi(n_R) \ge t_R/2$ for sufficiently large R. Together with (12) this proves the asserted two-sided clock law.

If $\Psi_\infty < \infty$, (15) instead gives the degree-independent bound $\int |P|^2 \, \mathrm{d}\tau_b \ge b_0 \Psi_\infty^{-q} |P(0)|^2$. Degrees zero and one are included by using $n = 2$. For $R \ge \max\{1, \Psi_\infty^q/(\eta b_0)\}$, the same law masks every polynomial square simultaneously. This completes the proof of Theorem 1.1.

4.3 The full-line origin Christoffel scale

The two sides of Theorem 1.1 also determine a standard extremal quantity for its fixed lower weight.

Corollary 4.2. *Under the hypotheses of Theorem 1.1, fix $b > c$ and put*

$$Z_b = \int_{\mathbb{R}} |x|^{q-1} e^{-bH(|x|)}\,\mathrm{d}x, \qquad \mathrm{d}\tau_b(x) = Z_b^{-1}|x|^{q-1}e^{-bH(|x|)}\,\mathrm{d}x.$$

There are constants $0 < u \le v < \infty$, independent of n, such that

$$u\Psi(n)^{-q} \le \lambda_n(\tau_b; 0) \le v\Psi(n)^{-q} \qquad (n \ge 2).$$

Consequently, if Ψ diverges then $\lambda_n(\tau_b; 0)$ tends to zero with this order. If Ψ converges, origin evaluation remains bounded uniformly over all polynomial degrees.

Proof. The lower estimate is (15). Suppose first that Ψ diverges. Retain the fixed $\eta > 0$ from the lower construction, so $-\delta_0 + \eta R\tau_b \in \mathcal{M}_W(R)$ for $R \ge 1$. Let B_1 be the constant in (12) for sufficiently large R, and set

$$R_n = \left(\frac{\Psi(n)}{2B_1}\right)^q.$$

For large n, this lies in the range of that estimate and

$$\Psi(\max\{1, D_W(R_n)\}) \le \Psi(n)/2 < \Psi(n).$$

Strict increase of Ψ gives $D_W(R_n) < n$. The particular positive-background model therefore has a negative square of degree at most n. Its polynomial has nonzero value at zero; normalizing that value to one gives

$$\lambda_n(\tau_b; 0) < \frac{1}{\eta R_n} = \frac{(2B_1)^q}{\eta}\Psi(n)^{-q}.$$

The finitely many remaining degrees are absorbed into a fixed upper constant, using $\lambda_n \le 1$. If $\Psi_\infty < \infty$, (15) gives $\lambda_n \ge b_0\Psi_\infty^{-q} > 0$, while $\lambda_n \le 1$ and $1 \le \Psi(n) \le \Psi_\infty$. The two-sided estimate follows in this case as well. □

This follows from the matching detection theorem. For example, for $W(x) = \log(e + x)$ it gives $\lambda_n(\tau_b; 0) \asymp (\log\log n)^{-q}$ for large n, including the singular-origin range $0 < q < 1$. The exponent q records local mass behavior; the integrated quantity records the tail scale.

4.4 A profile without regular variation

The growth hypothesis admits more than fixed logarithmic products. Fix $0 < \delta < 1$ and set $u_0 = 0$. Inductively define a continuous function g on $[0, \infty)$ by taking the value $k\delta$ on

$$[u_k, u_k + e^{k\delta}],$$

then slope δ on the following interval of length one, and setting $u_{k+1} = u_k + e^{k\delta} + 1$. Put

$$W(x) = \exp(g(\log(1 + x))).$$

This function is continuous and nondecreasing, $W(0) = 1$, and g is Lipschitz with constant δ. Hence, for $y \ge x \ge 1$,

$$\frac{W(y)}{W(x)} \le \left(\frac{1+y}{1+x}\right)^\delta \le (y/x)^\delta.$$

Its clock diverges. Under $u = \log(1 + x)$ one has $\mathrm{d}x/x = e^u\,\mathrm{d}u/(e^u - 1) \ge \mathrm{d}u$. Every full plateau after $x = 1$ contributes at least $e^{k\delta}e^{-k\delta} = 1$ to $\int \mathrm{d}x/(xW(x))$.

Nevertheless W is not regularly varying of any index. Fix $c = e^{1/2}$. Along points whose logarithmic coordinate is in the middle of a long plateau, both x and cx remain on that plateau, and $W(cx)/W(x) = 1$. At ramp starts $v_k = u_k + e^{k\delta}$, take $x_k = e^{v_k} - 1$. Then

$$\log(1 + cx_k) - v_k = \tfrac{1}{2} + \log(1 + (e^{-1/2} - 1)e^{-v_k}) \longrightarrow \tfrac{1}{2}$$

from below. The shifted coordinate remains inside the same unit ramp, so $W(cx_k)/W(x_k) \to e^{\delta/2} > 1$. These two limits preclude regular variation for this fixed dilation. There are also infinitely many corners. Theorem 1.1 gives its sharp integrated-clock recovery law for every $q > 0$ for this profile.

Remark 4.3 (Comparison with a smooth exponent). A comparable smooth exponent connects the input profile to classical slow-Freud theory. With F and C_0 from Section 4, define, for $x \geq 1$,

$$Q(x) = 1 + \int_1^x \frac{F(t)}{t}\,\mathrm{d}t.$$

Since W is continuous, Q is twice continuously differentiable on $(1, \infty)$, and

$$xQ'(x) = F(x), \qquad \frac{(xQ'(x))'}{Q'(x)} = \frac{H(x)}{F(x)} \in [C_0^{-1}, 1].$$

Moreover $Q(x) \asymp H(x)$ for $x \geq 2$: the upper estimate follows from $Q(x) \leq 1 + C_0F(x)$, while integration over $[x/2, x]$ gives $Q(x) \geq H(x)/2$, using $H(t) \geq (t/x)H(x)$. Thus the exponent has a smooth comparable representative with the large-x differential-ratio behavior used in [1]. The comparison concerns the exponents themselves. The direct proof above works with the original input profile.

5 Logarithmic and iterated-logarithmic applications

The general clock theorem gives explicit scales without repeating either the signed moment argument or the compact lower construction. All parameters below are fixed as $R \to \infty$.

Lemma 5.1 (A sufficient power-ratio condition). *Suppose $W : [0, \infty) \to [1, \infty)$ is continuous, nondecreasing, $W(0) = 1$, and is continuously differentiable on $[1, \infty)$. If $xW'(x)/W(x) \to 0$, then for every fixed $0 < \delta < 1$ there is $A \geq 1$ such that*

$$W(y)/W(x) \leq A(y/x)^{\delta} \qquad (y \geq x \geq 1).$$

Proof. Choose $X \geq 1$ such that $tW'(t)/W(t) \leq \delta$ for $t \geq X$. Integration of the logarithmic derivative proves the bound with $A = 1$ when $y \geq x \geq X$. When $1 \leq x \leq y \leq X$, monotonicity gives $W(y)/W(x) \leq W(X)/W(1)$. When $x < X < y$, split the ratio at X and use

$$\frac{W(y)}{W(x)} \leq \frac{W(X)}{W(1)}(y/X)^{\delta} \leq \frac{W(X)}{W(1)}(y/x)^{\delta}.$$

Thus $A = \max\{1, W(X)/W(1)\}$ suffices globally. □

Put $\ell(x) = \log(e + x)$ and $h(x) = \log\log(e^e + x)$. These functions are positive, increasing and smooth on $[0, \infty)$, with $\ell(0) = h(0) = 1$. Their logarithmic derivatives satisfy

$$\frac{x\ell'(x)}{\ell(x)} \longrightarrow 0, \qquad \frac{xh'(x)}{h(x)} \longrightarrow 0.$$

Consequently every fixed profile $W(x) = \ell(x)^{\gamma}$, $\gamma > 0$, and $W(x) = \ell(x)h(x)^{a}$, $a > 0$, satisfies every hypothesis of Theorem 1.1, by Lemma 5.1. The constants in its ratio bound may depend on the fixed exponent and on the chosen δ.

Corollary 5.2 (First logarithmic scale)**.** *For $W(x) = \ell(x)^\gamma$, write $D_\gamma(R) = D_W(R)$. For every fixed $q > 0$ and the fixed original signed-class budgets,*

$$\log D_\gamma(R) = \Theta\big(R^{1/[q(1-\gamma)]}\big), \qquad 0 < \gamma < 1,$$
$$\log\log D_1(R) = \Theta(R^{1/q}).$$

If $\gamma > 1$, one fixed positive atomless probability law and one fixed positive scaling constant mask the negative atom from every polynomial square, while meeting all class budgets, for every sufficiently large R. In particular $D_\gamma(R) = \infty$ there.

Proof. For x above a fixed lower limit, $\ell(x) \sim \log x$. Integral comparison with $1/[x(\log x)^\gamma]$ therefore gives

$$\Psi(t) \asymp \begin{cases} (\log t)^{1-\gamma}, & 0 < \gamma < 1, \\ \log\log t, & \gamma = 1, \end{cases} \qquad (t \to \infty),$$

and $\Psi(\infty) < \infty$ when $\gamma > 1$. Theorem 1.1 supplies the two-sided clock law in the divergent cases. In particular its lower bound forces $D_\gamma(R) \to \infty$, so these asymptotic clock comparisons apply at the detecting degree. Taking the power $1/(1-\gamma)$ in the first case and using the second comparison directly proves the displayed formulas. The convergent case is exactly the fixed-law masking conclusion of that theorem. □

Corollary 5.3 (Second logarithmic scale)**.** *For $W(x) = \ell(x)h(x)^a$, write $E_a(R) = D_W(R)$. Then*

$$\log\log E_a(R) = \Theta\big(R^{1/[q(1-a)]}\big), \qquad 0 < a < 1,$$
$$\log\log\log E_1(R) = \Theta(R^{1/q}).$$

For $a > 1$, one fixed positive atomless probability law, scaled by a fixed multiple of R, meets all original class budgets and masks every polynomial square for all sufficiently large R; thus $E_a(R) = \infty$ there.

Proof. For sufficiently large x, $\ell(x) \sim \log x$ and $h(x) \sim \log\log x$. Compare the clock integrand with $1/[x \log x(\log\log x)^a]$. The substitution $u = \log\log x$ gives

$$\Psi(t) \asymp \begin{cases} (\log\log t)^{1-a}, & 0 < a < 1, \\ \log\log\log t, & a = 1, \end{cases}$$

whereas its integral converges for $a > 1$. Apply Theorem 1.1 and invert the divergent comparisons as above. Its finite-clock assertion supplies the same-law all-degree masker in the remaining case. □

All these scales follow by inverting the clock with the profile and class data fixed.

6 Why growth control cannot simply be removed

The growth-ratio assumption controls the relation between tail moments and the integrated quantity. Smoothness of W alone does not provide that control. The next construction keeps the same canonical positive density as Corollary 4.2.

Proof of Theorem 1.2. Choose $x_1 > e$ so large that $W_1 = x_1/\log x_1 \geq 1$. For $j \geq 2$, choose x_j recursively, writing $W_j = x_j/(\sqrt{j}\log x_j)$, so that

$$x_j \geq 4x_{j-1}, \qquad x_j \geq e^{j^2}, \qquad W_j \geq W_{j-1}, \tag{17}$$

$$\log \frac{x_j}{2x_{j-1}} \geq W_{j-1}, \tag{18}$$

$$\frac{4^j(\prod_{i<j} x_i)^2}{x_j^{j-2}} \leq 2^{-j} \qquad (j \geq 3). \tag{19}$$

These conditions involve none of the parameters q, c, C, L, r_0, b. The finitely many lower bounds recursively define the sequence, since $x/\log x$, $\log x$, and x^{j-2} tend to infinity.

Fix a smooth nondecreasing function $\theta : [0,1] \to [0,1]$ with $\theta(0) = 0$, $\theta(1) = 1$ and every positive-order derivative zero at both endpoints. On $[0, x_1]$ put $W(x) = 1 + (W_1 - 1)\theta(x/x_1)$. For each $j \geq 1$, put

$$W(x) = W_j \quad (x_j \leq x \leq x_{j+1}/2),$$

and on $[x_{j+1}/2, x_{j+1}]$ put

$$W(x) = W_j + (W_{j+1} - W_j)\theta(2x/x_{j+1} - 1).$$

The pieces match to every order, giving the asserted smooth monotone profile. Each plateau contributes at least one to $\int \mathrm{d}x/(xW(x))$, by (18); thus Ψ diverges. On a plateau, monotonicity of $x/\log x$ gives $H(x)/\log x \geq \sqrt{j}$. On the following transition, $W(x) \leq W_{j+1}$ and $x \geq x_{j+1}/2$, so

$$H(x)/\log x \geq \sqrt{j+1}/2.$$

Hence $H(x)/\log x \to \infty$ globally. In particular, for any fixed $q > 0$, the normalizer

$$Z_{a,q} = \int_{\mathbb{R}} |x|^{q-1} e^{-aH(|x|)}\,\mathrm{d}x$$

is positive and finite for every $a > 0$, and the canonical laws have all moments.

We next establish bounded evaluation using intervals within the plateaux. Put

$$I_j = [x_j, 5x_j/4], \quad w_j = x_j^{-j}, \quad Z = \sum_{j\geq 1} w_j, \qquad \sigma = Z^{-1}\sum_{j\geq 1} w_j\sigma_j,$$

where σ_j is uniform probability on I_j. Here $0 < Z < \infty$. For any $N \geq 1$, choose arbitrary $y_j \in I_j$, $1 \leq j \leq N$. Lagrange interpolation for $\deg P \leq N-1$ gives

$$P(0) = \sum_{j=1}^{N} P(y_j)\ell_{j,N}(0), \qquad \ell_{j,N}(0) = \prod_{i\neq j} \frac{-y_i}{y_j - y_i}.$$

For $i < j$, (17) gives $y_i/(y_j - y_i) \leq 2x_i/x_j$. For $i > j$, $y_j/y_i \leq (5/4)4^{-(i-j)}$. Since their sum is at most $5/12$, the elementary inequality $\prod(1-a_i) \geq 1 - \sum a_i$ bounds the future product by $12/7 < 2$. Therefore, uniformly in the selected points and N,

$$|\ell_{j,N}(0)| \leq \frac{2^j \prod_{i<j} x_i}{x_j^{j-1}}, \qquad \sum_{j=1}^{N} \frac{|\ell_{j,N}(0)|^2}{w_j} \leq B_0 := 4x_1 + 16x_1^2 + \tfrac{1}{4}.$$

The terms $j \geq 3$ are bounded by (19); the first two are finite as displayed. Weighted Cauchy–Schwarz, followed by integration over $\sigma_1 \times \cdots \times \sigma_N$, yields

$$|P(0)|^2 \leq B_0 \sum_{j=1}^{N} w_j \int |P|^2 \, \mathrm{d}\sigma_j \leq B_0 Z \int |P|^2 \, \mathrm{d}\sigma. \tag{20}$$

Taking N larger than the degree proves this simultaneously for every polynomial.

To transfer this estimate, fix any $q, b > 0$ and put $m_q = \min\{1, (5/4)^{q-1}\}$. On I_j the profile is constant, so

$$bH(x) \leq \frac{5b}{4}\sqrt{j}\log x_j, \qquad |x|^{q-1}e^{-bH(x)} \geq m_q x_j^{q-1-(5b/4)\sqrt{j}}.$$

The density of σ there is $4Z^{-1}x_j^{-j-1}$. Its ratio to this unnormalized canonical density is at most

$$\frac{4}{Zm_q} x_j^{-j-q+(5b/4)\sqrt{j}}.$$

For all sufficiently large j the exponent is negative, and the ratio is at most $4/(Zm_q)$. The finitely many earlier ratios have a finite maximum. Off these intervals the density of σ is zero. Consequently there is a finite $C_{b,q} > 0$ with $d\sigma \leq C_{b,q} d\tau_{b,q}$. Equation (20) gives

$$|P(0)|^2 \leq B \int |P|^2 \, \mathrm{d}\tau_{b,q}, \qquad B = B_0 Z C_{b,q} < \infty,$$

independently of polynomial degree. This proves the uniform lower bound for $\lambda_n(\tau_{b,q}; 0)$ for every fixed $q, b > 0$.

Finally, fix any class parameters $q, c, C, L, r_0 > 0$ and $b > c$. Then $E = \int e^{cH} \, \mathrm{d}\tau_{b,q} = Z_{b-c,q}/Z_{b,q} < \infty$, and $\tau_{b,q}([-r, r]) \leq 2r^q/(qZ_{b,q})$ for every $r > 0$. Choose

$$\eta = \min\{1, C/E, LqZ_{b,q}/2\} > 0, \qquad R_0 = \max\{1, B/\eta\}.$$

All three budgets in (1) hold for $\eta R \tau_{b,q}$. The evaluation bound gives nonnegativity of every square for $R \geq R_0$. The profile was fixed before any of these parameters were chosen.

If (2) held with a fixed $\delta < 1$, applying it between 1 and x_j would bound $x_j^{1-\delta}/(\sqrt{j}\log x_j)$ uniformly in j. But $x_j \geq e^{j^2}$ makes this expression at least $e^{(1-\delta)j^2}/j^{5/2}$ for large j, which diverges. This completes the construction. □

Bounded polynomial evaluation and sparse interpolation are classical moment-problem mechanisms. Indeterminate densities with a divergent logarithmic integral are also classical; see the discussion of converse Krein criteria in [15]. The construction above identifies these features within the present smooth monotone input-profile and canonical-density model, while preserving all declared local and tail budgets. It proves that smoothness, existence of all moments and divergence of the integrated clock do not replace growth control.

A A finite-order polynomial approximation bound

The same propagation estimate couples directly to Fourier residual duality, as used in [7]. The finite-order bound below applies the classical Bang mechanism in the normalization needed for the signed-measure theorem.

Theorem A.1 (Finite moment projection bound)**.** *Let ρ be a finite positive Borel measure on $\mathbb{R}$ with moments through order $2N$, $N \geq 1$. Let $M_0 = 1, M_1, \ldots, M_N > 0$ be log-convex and suppose*

$$\|x^j\|_{L^2(\rho)} \leq BC^j M_j \qquad (0 \leq j \leq N)$$

for $B, C > 0$. Let π be a finite complex measure of total variation V, supported on $[-T, T]$, and let $f(x) = \int e^{itx}\, \mathrm{d}\pi(t)$. Write $r = f - \Pi_N f$, where Π_N is orthogonal projection onto the image of degree-N polynomials in $L^2(\rho)$. For any integer k with $0 \leq k < N$ such that

$$\sum_{j=k+1}^{N} \frac{M_{j-1}}{M_j} > eCT, \tag{21}$$

one has $\|r\|_{L^2(\rho)} \leq BVe^{-k}$. Independently of (21),

$$\|r\|_{L^2(\rho)} \leq BV \min\{1, (CT)^N M_N/N!\}. \tag{22}$$

If π is any finite complex measure without a support restriction, the corresponding bound is

$$\|r\|_{L^2(\rho)} \leq B\left[|\pi|([-T,T])\, \varepsilon_N(T) + |\pi|(\mathbb{R} \setminus [-T,T])\right],$$

where $\varepsilon_N(T)$ is the minimum of 1, $(CT)^N M_N/N!$, and all e^{-k} allowed by (21).

Proof. The polynomial image is finite-dimensional and closed, even when its Gram matrix is singular. Thus the projection exists without strict positivity. If $r = 0$, there is nothing to prove. Otherwise define

$$F(t) = \int \overline{r(x)} e^{itx}\, \mathrm{d}\rho(x).$$

Cauchy–Schwarz and the finite moments justify N differentiations and give $|F^{(j)}(t)| \leq B\|r\|C^j M_j$. Orthogonality gives $F^{(j)}(0) = 0$ for $0 \leq j \leq N$. Lemma 2.1, applied to $F/(B\|r\|)$, yields $|F(t)| \leq B\|r\|e^{-k}$ on $[-T, T]$. The order-zero bound gives $|F| \leq B\|r\|$ everywhere. Using the inner product $\langle u, v\rangle = \int \overline{u}v\, \mathrm{d}\rho$, finite-measure Fubini and projection yield

$$\|r\|^2 = \langle r, f\rangle = \int F(t)\, \mathrm{d}\pi(t) \leq \int |F(t)|\, \mathrm{d}|\pi|(t).$$

Divide by $\|r\|$ and, for the noncompact case, split the frequency integral. Finally the integral Taylor remainder at zero gives

$$|F(t)| \leq B\|r\|C^N M_N |t|^N/N!,$$

since the first N jets vanish. The same integration proves (22) and permits the stated minimum. The argument applies to every finite positive measure with the stated moments through order $2N$. □

Remark A.2 (Representative regimes and comparison). For $M_j = \sqrt{j!}$, the clock has terms $j^{-1/2}$; taking $k = \lfloor N/2 \rfloor$ gives exponential error when T is a sufficiently small fixed multiple of $\sqrt{N}$. At fixed T, the Taylor bound is $BV(CT)^N/\sqrt{N!}$, with logarithm $-\frac{1}{2}N \log N + O(N)$ apart from $\log(BV)$. For $M_j = j!$, taking $k = \lfloor N \exp(-eCT - 2) \rfloor$ gives exponential error in N at fixed bandwidth. For $M_j = j! \prod_{l=1}^{j} \log(e+l)^{\alpha}$, the logarithmic tail sum gives the corresponding finite approximation scale. The signed-measure degree laws in Section 5 are obtained instead by evaluating the integrated clock of Theorem 1.1.

Bang's tail-sum propagation is classical [6], and the Fourier residual argument is present in [7]. Optimizing the free parameter in equation (3.71) of [7] already gives a critical bound of the form $\exp(-cN^{\delta})$. The finite-order form above supplies the approximation estimate used for the full signed class.

B Finite controls and scope

The accompanying standard-library program `verify_finite_controls.py` uses exact rational arithmetic to check finite Jacobi moment matching in the original Gram metric, the free last diagonal, the singular semidefinite obstruction and singular-power evaluation identities. The accompanying `verify_sparse_interpolation.py` additionally checks the interpolation identities and coefficient bounds used in Theorem 1.2, on finite exact node sets. These controls corroborate finite algebra; the analytic arguments above establish the all-degree and asymptotic statements.

The finite moment criterion proves an existence degree bound. Given exact original moments through order $2N$, a negative Hankel direction can be converted into polynomial coefficients. Execution uses exact or certified coefficients as inputs. Together, the matched estimate and Theorem 1.2 identify the recovery scale within the controlled class and exhibit an explicit failure of the divergent-clock criterion beyond it.

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