Polynomial Recovery and Masking under Weak Exponential Tails

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Abstract

We determine the worst polynomial degree needed to detect an isolated negative atom in a signed measure under coarse local-mass and weak exponential-moment bounds. For a tail scale $H(x) = x/W(x)$, with W continuous, nondecreasing and subject to a sublinear power-ratio bound, let $\Psi(t) = 1 + \int_1^t \mathrm{d}x/[xW(x)]$. If this clock diverges, the minimax detecting degree satisfies $\Psi(D_W(R)) = \Theta(R^{1/q})$ for every fixed local exponent $q > 0$. If it converges, one fixed atomless reference law, scaled with R, masks the negative atom from every polynomial square. No differentiability or regular variation is required. A finite positive moment realization and classical Bang propagation give the uniform signed-measure upper bound. A compact polynomial multiplier and singular Jacobi evaluation give matching fixed-law lower bounds and masking. Logarithmic and iterated-logarithmic tail regimes follow as applications, including double- and triple-exponential critical degrees. We distinguish the original signed-class recovery theorem from the classical determinacy thresholds and approximation ingredients used in its proof.

1 Polynomial detection and the integrated tail clock

An isolated negative atom can be invisible to polynomial squares of every modest degree. Local variation control prevents the background from placing an atom at the same point, but it does not by itself determine the cost of detection. The tail budget determines whether finite detection is uniformly possible and, when it is, how rapidly the worst required degree grows. We give one integrated-clock answer under a growth-ratio condition that allows nonsmooth profiles and does not assume regular variation.

Fix $q, c, C, L, r_0 > 0$. Let $W : [0, \infty) \to [1, \infty)$ be continuous and nondecreasing, with $W(0) = 1$, and put

$$H(x) = \frac{x}{W(x)}, \qquad \Psi(t) = 1 + \int_1^t \frac{\mathrm{d}x}{xW(x)} \quad (t \geq 1).$$

For $R \geq 1$, let $\mathcal{M}_W(R)$ be the class of real signed Borel measures $\mu = -\delta_0 + \nu$ satisfying

$$\mathrm{TV}(\nu) \leq R, \qquad \int_{\mathbb{R}} e^{cH(|x|)}\, \mathrm{d}|\nu|(x) \leq CR, \qquad |\nu|([-r, r]) \leq LRr^q \quad (0 < r \leq r_0). \tag{1}$$

The local condition holds at every declared radius and forces $\nu(\{0\}) = 0$. The background may be signed, atomic away from zero, singular or absolutely continuous. Write $\mathcal{P}_N$ for ordinary polynomial degree at most N, and define

$$n_1(\mu) = \inf\{N \in \mathbb{Z}_{\geq 0} : \exists P \in \mathcal{P}_N(\mathbb{R}),\ \int P^2\, \mathrm{d}\mu < 0\}, \qquad D_W(R) = \sup_{\mu \in \mathcal{M}_W(R)} n_1(\mu),$$

with $\inf \varnothing = \infty$. The polynomial square has degree at most $2N$. Negativity is strict: a null square is not a negative direction.

Theorem 1.1 (Integrated-clock recovery and masking)**.** *Suppose fixed $A \geq 1$ and $0 \leq \delta < 1$ satisfy*

$$\frac{W(y)}{W(x)} \leq A(y/x)^{\delta} \qquad (y \geq x \geq 1). \tag{2}$$

If Ψ is unbounded, $D_W(R)$ is finite for every $R \geq 1$, and

$$\Psi(\max\{1, D_W(R)\}) = \Theta(R^{1/q}) \qquad (R \to \infty).$$

If $\Psi(\infty) < \infty$, there are one fixed positive atomless probability law τ, a fixed $\eta > 0$ and $R_0 \geq 1$ such that $-\delta_0 + \eta R\tau \in \mathcal{M}_W(R)$ and

$$-|P(0)|^2 + \eta R \int |P|^2 \, \mathrm{d}\tau \geq 0 \quad \text{for every polynomial } P \text{ and every } R \geq R_0.$$

Thus $D_W(R) = \infty$ for every $R \geq R_0$ in the finite-clock case. All constants may depend on W and the fixed class data; only R varies.

The theorem matches the original arbitrary-signed upper class with lower examples from a single positive reference law. The masking conclusion concerns every degree simultaneously. It is stronger than a sequence of difficult examples whose background changes with degree. The growth-ratio assumption is one-sided. Section 4 includes a profile with infinitely many corners and two distinct dilation subsequential limits. Section 5 derives the logarithmic regimes from the same theorem.

Proof mechanism. For the upper bound, local mass control supplies a compact-frequency test whose square integral is strictly negative for the original measure. If all squares through degree N were nonnegative, a small Gaussian addition would make that finite moment functional strictly positive. A finite Jacobi realization would then give a positive measure with the same first $2N$ moments. Finite Bang propagation bounds the Fourier discrepancy between these two objects and contradicts the original negative test. The concave primitive $F(x) = \int_0^x \mathrm{d}u/W(u)$ pays the inverse moment scale, and an exact change of variables returns the propagation clock to Ψ.

For the lower bound, a compact Chebyshev multiplier at radius $n/\Psi(n)$ has degree $O(n)$. A singular-power Jacobi evaluation bound converts it into an original full-line norm estimate for one reference density proportional to $|x|^{q-1}e^{-bH(|x|)}$, with $b > c$. The same clock controls the difficult finite degrees and, when bounded, the simultaneous all-degree masker. Shared tools are proved once before the main argument.

Related work and contribution. The preceding recovery papers [1, 2, 3] study polynomial sign certificates, collision barriers, growing negative index and logarithmically strengthened exponential tails. The present question concerns weak tails that need not supply a positive exponential moment. Its conclusion is a uniform theorem for the full signed class (1), not merely an asymptotic formula for a regular positive weight.

Classical moment and weighted-approximation theory already contains the qualitative determinacy boundary and regular $q = 1$ origin-kernel scales [10, 7, 6]. Bang's level crossing [4], finite Jacobi moment realizations [6], fast-decreasing polynomial localization [8, 9], and the compact Jacobi kernel are established ingredients. The contribution here is their fully paid passage to matching recovery and fixed-law masking in (1), for every $q > 0$ under (2), without smoothness or regular variation.

Koehler and Wu [5] develop quantitative approximation under Carleman conditions and Fourier projection-residual duality. The finite-order approximation consequence in Section 6 couples that duality to classical terminal-level Bang propagation. It improves their displayed reciprocal-sum elementary comparator; optimizing their earlier general complex estimate also recovers the corresponding stretched-power-in-degree rate. The distinct reusable consequence is retained after the principal signed-measure theorem and its applications.

2 Finite propagation and signed moment recovery

We use the Fourier convention $\widehat{\sigma}(t) = \int e^{itx}\, \mathrm{d}\sigma(x)$. The following finite form of Bang's level-crossing argument is classical in origin; see [4]. Its terminal derivative requires only continuity.

Lemma 2.1 (Finite Bang propagation)**.** *Let $f \in C^n([-W, W]; \mathbb{C})$, $n \geq 1$, and let $M_0, \dots, M_n > 0$ have nondecreasing quotients $a_j = M_j/M_{j-1}$. Suppose $|f^{(j)}| \leq M_j$ for $0 \leq j \leq n$ and $f^{(j)}(0) = 0$ for $j < n$. If an integer k satisfies $0 \leq k < n$ and*

$$\sum_{j=k+1}^{n} a_j^{-1} > eW,$$

then $|f(t)| \leq M_0 e^{-k}$ for $|t| \leq W$.

Proof. Put $b_j = |f^{(j)}|/(e^j M_j)$ and $B = \max_{0 \leq j \leq n} b_j$. Then $B(0) \leq e^{-n}$. On a level strip $e^{-l} < B < e^{-(l-1)}$, only indices $j \leq l-1$ can maximize: every b_j with $j \geq l$ is at most e^{-l}. Thus B is locally Lipschitz on this strip, although b_n need only be continuous. At almost every point there,

$$|B'| \leq \max_{j \text{ active}} \frac{|f^{(j+1)}|}{e^j M_j} \leq e a_l B, \qquad |(\log B)'| \leq e a_l.$$

This uses the modulus derivative inequality, valid also for complex functions. Crossing from e^{-l} to $e^{-(l-1)}$ costs length at least $(ea_l)^{-1}$. More precisely, before the first hitting of the upper level choose the last hitting of the lower level. Integrate on interior level intervals and pass to their endpoints. If B reached e^{-k} within distance W, the disjoint crossings for $l = n, \dots, k+1$ would have total length at least $e^{-1} \sum_{l=k+1}^{n} a_l^{-1} > W$. Apply the same argument to $f(-t)$ for negative t. No derivative of order $n+1$ is used. □

Lemma 2.2 (Finite positive moment realization)**.** *Let L be a real linear functional on polynomials through degree $2N$, $N \geq 1$, with $L[P^2] > 0$ for every nonzero real polynomial of degree at most N. There is a positive measure τ, supported on at most $N+1$ points, such that $\int x^j\, \mathrm{d}\tau = L[x^j]$ for $0 \leq j \leq 2N$.*

Proof. In the inner product $L[PQ]$, form monic orthogonal polynomials $A_0, \dots, A_N$, with squared norms $h_j > 0$. On their span define J by the recurrence for xA_j when $j < N$. Give its last diagonal entry any real value; its last off-diagonal entries are prescribed by symmetry. The usual three-term recurrence follows directly from orthogonality: xA_j is orthogonal to A_i for $i < j-1$, and its coefficient of A_{j-1} is h_j/h_{j-1}. Hence J is selfadjoint in the diagonal metric $(h_0, \dots, h_N)$, and agrees with multiplication by x on polynomials of degree below N. This is the classical Jacobi mechanism; compare [6]. For $j \leq N$, $J^j 1 = x^j$. Split $s = i + j \leq 2N$ with $i, j \leq N$. Selfadjointness gives

$$\langle 1, J^s 1 \rangle = \langle J^i 1, J^j 1 \rangle = L[x^s].$$

The spectral measure of this finite selfadjoint operator at 1 proves the assertion. The free last diagonal avoids requiring a moment of order $2N+1$. □

Remark 2.3. Positive semidefiniteness alone does not suffice here. The moments $(1,0,0,0,1)$ have positive semidefinite Hankel matrix $\mathrm{diag}(1,0,1)$, but a positive representing measure with second moment zero must also have fourth moment zero. We therefore regularize strictly in the next argument.

Theorem 2.4 (A general signed moment criterion)**.** *Let $q, L, r_0, K > 0$, $R \geq 1$, and $\mu = -\delta_0 + \nu$ be a finite real signed measure such that*

$$\mathrm{TV}(\nu) \leq R, \qquad |\nu|([-r,r]) \leq LRr^q \quad (0 < r \leq r_0).$$

Fix $N \geq 1$. Let $M_0 = 1, M_1, \ldots, M_N > 0$ be log-convex, and let $a > 0$ satisfy, for $1 \leq j \leq N$,

$$\int |x|^{2j} \, \mathrm{d}|\nu| \leq KRa^{2j}M_j^2, \qquad \int |x|^{2j} \, \mathrm{d}\gamma \leq a^{2j}M_j^2,$$

where γ is standard Gaussian probability. Choose an integer ℓ with $\beta = 2\ell > q$, and put

$$B_0 = r_0^{-q} + \frac{\beta L}{\beta - q}, \quad T = \max\{1, r_0^{-1}, (16B_0R)^{1/q}\}, \quad W = 2\ell T,$$

$$A_0 = 8(1 + \sqrt{K+1})R, \qquad k = \lceil \log(16A_0) \rceil.$$

If $k < N$ and

$$\sum_{j=k+1}^{N} \frac{M_{j-1}}{M_j} > eaW, \tag{3}$$

there is a real polynomial P of degree at most N with $\int P^2 \, \mathrm{d}\mu < 0$.

Proof. Set $g(x) = \mathrm{sinc}(Tx)^{2\ell}$, with $\mathrm{sinc}(u) = \sin(u)/u$ and its continuous value at zero. Then $0 \leq g \leq \min(1, (T|x|)^{-\beta})$. Layer cake gives

$$\int g \, \mathrm{d}|\nu| \leq R(Tr_0)^{-\beta} + LRT^{-q} \int_0^1 u^{-q/\beta} \, \mathrm{d}u \leq B_0 RT^{-q} \leq 1/16.$$

Indeed on $u \geq (Tr_0)^{-\beta}$ the relevant sublevel radius is at most r_0; below it use total mass R. Consequently $\int g \, \mathrm{d}\mu \leq -15/16$. Moreover $g(x) = \int e^{itx} \, \mathrm{d}\pi(t)$ for a positive probability π on $[-W, W]$: convolve 2ℓ uniform probabilities on $[-T, T]$. Suppose all degree-N real squares have nonnegative μ integral. Fix $0 < \eta \leq 1/16$. The functional of $\mu + \eta\gamma$ is strictly positive on nonzero squares. Lemma 2.2 supplies a positive finite atomic τ with the same moments through $2N$. Its mass is at most R, since $\nu(\mathbb{R}) - 1 + \eta \leq R$. For $1 \leq j \leq N$, its matched even moments and Cauchy–Schwarz give

$$\int |x|^j \, \mathrm{d}\tau \leq R\sqrt{K+1}\, a^j M_j.$$

Likewise $\int |x|^j \, \mathrm{d}|\nu| \leq R\sqrt{K}\, a^j M_j$, and the Gaussian contribution is at most $a^j M_j$. Thus

$$F(t) = \mu + \widehat{\eta\gamma} - \tau(t)$$

is C^N, has zero jets through order N, and satisfies $|F^{(j)}| \leq A_0 a^j M_j$ for $0 \leq j \leq N$; at order zero use $\mathrm{TV}(\mu) \leq 1 + R$. Lemma 2.1 and (3) yield $|F| \leq A_0 e^{-k} \leq 1/16$ on $[-W, W]$. Fubini therefore gives $|\int g \, \mathrm{d}(\mu + \eta\gamma - \tau)| \leq 1/16$. Its signed value, however, is at most $-15/16 + \eta \leq -7/8$, because $g \geq 0$ and $\tau \geq 0$. This contradiction proves strict negativity for the original measure. □

3 Compact localization and origin evaluation

The lower construction uses two shared classical ingredients: compact fast-decreasing polynomial factors and the exact Jacobi origin kernel. We include their normalizations for every $q > 0$; see also [8, 9, 6].

3.1 A compact dyadic multiplier

Lemma 3.1. *Let $Q : [0,\infty) \to [0,\infty)$ be continuous and nondecreasing, with $Q(0) = 0$. For $A > 0$, $J \geq 1$ and $d_j = A2^{-j}$, there is a real polynomial S with $S(0) = 1$ such that*

$$|S(x)| \leq \max\{2, e^{Q(d_J)/2}\} e^{-Q(|x|)/2} \quad (|x| \leq A), \tag{4}$$

$$\deg S \leq \sum_{j=1}^{J} 2^{j-1} Q(A2^{-(j-1)}) + 2J. \tag{5}$$

Proof. Put $m_j = \lceil AQ(d_{j-1})/(4d_j) \rceil$ and

$$C_j(x) = \frac{T_{m_j}((A^2 + d_j^2 - 2x^2)/(A^2 - d_j^2))}{T_{m_j}((A^2 + d_j^2)/(A^2 - d_j^2))}, \qquad S = \prod_{j=1}^{J} C_j.$$

For $m_j = 0$ take $C_j = 1$. On $[-A, A]$, $|C_j| \leq 1$. For $d_j \leq |x| \leq A$, its numerator has modulus at most one and

$$\operatorname{arcosh} \frac{A^2 + d_j^2}{A^2 - d_j^2} = \log \frac{A + d_j}{A - d_j} \geq \frac{2d_j}{A}.$$

The identity $T_m(\cosh u) = \cosh(mu)$ gives $|C_j(x)| \leq 2e^{-2m_j d_j/A} \leq 2e^{-Q(d_{j-1})/2}$. On $d_j \leq |x| \leq d_{j-1}$ this bounds S by $2e^{-Q(|x|)/2}$. On the inner cell use $|S| \leq 1$ and $Q(|x|) \leq Q(d_J)$. Finally $\deg S \leq 2\sum m_j$ gives (5). □

3.2 Singular-power evaluation with exact normalization

Lemma 3.2. *For every $q > 0$ there is $c_q > 0$ such that every polynomial U of degree at most M obeys*

$$\int_{-A}^{A} |U(x)|^2 |x|^{q-1} \, \mathrm{d}x \geq c_q \left(\frac{A}{M+1}\right)^q |U(0)|^2.$$

Proof. Put $a = q/2$. For the weight v^{a-1} on $[0,1]$, the Rodrigues polynomials

$$R_j(v) = \frac{v^{1-a}}{j!} \frac{\mathrm{d}^j}{\mathrm{d}v^j} [v^{j+a-1}(v-1)^j]$$

are orthogonal, have value $(-1)^j (a)_j/j!$ at zero, leading coefficient $(j+a)_j/j!$, and squared norm $1/(2j+a)$. Repeated integration by parts proves these assertions: all boundary terms at zero have a positive remaining power a, and the order-j zero at one pays that endpoint. The norm is $[(j+a)_j/j!]\,\mathrm{B}(j+a, j+1) = 1/(2j+a)$. Thus the exact evaluation kernel through degree l is

$$K_l(0,0) = \sum_{j=0}^{l} (2j+a) \left[\frac{(a)_j}{j!}\right]^2 = \frac{(l+a)^2}{a} \left[\frac{(a)_l}{l!}\right]^2 \leq C_q (l+1)^q.$$

The equality telescopes, including $l = 0$, and the fixed-parameter gamma ratio gives the bound. Split U into its even and odd parts, orthogonal for the even weight. The odd part vanishes at zero.

Substitution $v = x^2/A^2$ makes the even norm A^q times the Jacobi norm above, through $l = \lfloor M/2 \rfloor$. Cauchy–Schwarz in that orthogonal basis proves the assertion, for example with $c_q = C_q^{-1}$. No value of the density at zero is divided by. $\square$

4 Proof of the integrated-clock theorem

We prove Theorem 1.1 under its declared hypotheses on W. Throughout this section put $H(x) = x/W(x)$ and $\Psi(t) = 1 + \int_1^t \mathrm{d}x/[xW(x)]$ for $t \geq 1$. All constants below are fixed before R varies.

4.1 Inverse moments and the upper clock

Define

$$F(x) = \int_0^x \frac{\mathrm{d}u}{W(u)}, \qquad C_0 = AW(1) + \frac{A}{1-\delta}, \qquad p = \frac{1}{1-\delta}.$$

The function F is continuously differentiable, strictly increasing and concave, and $F(x) \leq x$. Monotonicity gives $F(x) \geq H(x)$. For $x \geq 1$, the Potter bound gives

$$\int_1^x \frac{\mathrm{d}u}{W(u)} \leq \frac{Ax^\delta}{W(x)} \int_1^x u^{-\delta}\,\mathrm{d}u \leq \frac{A}{1-\delta} H(x).$$

The remaining integral on $[0,1]$ is at most one, and $1/H(x) \leq AW(1)x^{\delta-1} \leq AW(1)$. Consequently

$$H(x) \leq F(x) \leq C_0 H(x) \quad (x \geq 1). \tag{6}$$

Also $H(x) \geq x^{1-\delta}/[AW(1)]$, so F tends to infinity and has an inverse h. We have $h(t) \geq t$. For $y \geq x \geq 1$, (6) and the Potter bound imply

$$\frac{F(y)}{F(x)} \geq \frac{1}{AC_0} \left(\frac{y}{x}\right)^{1-\delta}.$$

Thus, with $K = (AC_0)^p$,

$$\frac{h(v)}{h(u)} \leq K(v/u)^p \qquad (v \geq u \geq 1). \tag{7}$$

No derivative or regular-variation equivalence for W has been used.

For each fixed $b > 0$ there is K_b such that

$$\sup_{x \geq 0} x^j e^{-bH(x)} \leq [K_b h(j)]^j \qquad (j \geq 1). \tag{8}$$

Indeed for $x \leq h(j)$ this follows without the exponential factor. For $x = h(j)y$, $y \geq 1$, the preceding growth bound and (6) give

$$H(x) \geq \frac{j}{AC_0^2} y^{1/p}.$$

The supremum of $y \exp[-by^{1/p}/(AC_0^2)]$ on $[1,\infty)$ is finite, proving (8).

Set $M_0 = 1$ and $M_j = \prod_{l=1}^j h(l)$. Its quotients $h(j)$ increase, so it is log-convex. Equation (7) gives

$$M_j \geq K^{-j}(j!/j^j)^p h(j)^j \geq (Ke^p)^{-j} h(j)^j. \tag{9}$$

For any original class member, (8) at $2j$ gives

$$\int |x|^{2j}\,\mathrm{d}|\nu| \leq CR[K_c h(2j)]^{2j} \leq CR[K_c K 2^p h(j)]^{2j}.$$

Combining this with (9), there is a fixed scale $a_0 > 0$ for which these moments are at most $CRa_0^{2j}M_j^2$. Enlarge a_0 to cover standard Gaussian even moments, bounded by $(2j)^j$, using $h(j) \geq j$. Thus Theorem 2.4 applies, with its $K = C$ and $a = a_0$.

In its notation, $k = O(\log(R+2))$ and its bandwidth W (which is distinct from the present tail function) is at most $C_* R^{1/q}$ for a fixed C_*. Denote this bandwidth by S. The receiver requires

$$\sum_{j=k+1}^{N} \frac{1}{h(j)} > ea_0 S. \tag{10}$$

Since $1/h$ decreases, a change of variable $t = F(x)$ yields

$$\begin{aligned} \sum_{j=k+1}^{N} \frac{1}{h(j)} &\geq \int_{k+1}^{N+1} \frac{\mathrm{d}t}{h(t)} = \int_{h(k+1)}^{h(N+1)} \frac{\mathrm{d}x}{xW(x)} \\ &\geq \Psi(N+1) - \Psi(h(k+1)). \end{aligned} \tag{11}$$

Moreover (7) and $W \geq 1$ give

$$\Psi(h(k+1)) \leq 1 + \log(Kh(1)) + p\log(k+1) =: E_k.$$

When Ψ diverges, choose N to be the least integer with

$$\Psi(N) \geq 2ea_0 C_* R^{1/q} + 2E_k.$$

For large R this has $N > k$, since $\Psi(k+1) \leq 1 + \log(k+1)$ whereas the target grows like $R^{1/q}$. Equations (11) and the choice of N strictly pay (10). Theorem 2.4 gives $D_W(R) \leq N$. For any fixed $R \geq 1$, divergence of the terminal clock likewise permits a finite N satisfying (10); hence finiteness holds for every R, while the displayed order is asymptotic. The integer overshoot is at most

$$\Psi(N) - \Psi(N-1) \leq \log \frac{N}{N-1} \leq \log 2.$$

Since $E_k = O(\log\log(R+3))$, it follows that

$$\Psi(\max\{1, D_W(R)\}) \leq B_1 R^{1/q} \tag{12}$$

for a fixed B_1 and all sufficiently large R.

4.2 A compact clock and one reference law

The literal H need not be monotone. Put $H^{\uparrow}(x) = \max_{0 \leq s \leq x} H(s)$. This is continuous, nondecreasing, and at most x. There is a fixed B_H with

$$H \leq H^{\uparrow} \leq B_H H. \tag{13}$$

For $x \leq 1$, use $x \leq W(1)H(x)$. For $x \geq 1$ and $1 \leq s \leq x$, the Potter bound gives

$$H(s)/H(x) \leq A(s/x)^{1-\delta} \leq A.$$

For $s < 1$, use $H(s) \leq 1$ and $H(x) \geq 1/[AW(1)]$. Thus $B_H = \max\{W(1), A, AW(1)\}$ suffices. No eventual monotonicity of H is required.

For integer $n \geq 2$ set $\psi_n = \Psi(n)$, $A_n = n/\psi_n$ and $J = \lceil \log_2 n \rceil$. Because $1 \leq \psi_n \leq 1 + \log n \leq n$, we have $1 \leq A_n \leq n$. Fix $b > c$ and apply Lemma 3.1 with $Q = bH^{\uparrow}$, radius A_n, and J factors. The

central constant is at most $B_0 = \max\{2, e^{b/2}\}$, since $A_n 2^{-J} \le 1$ and $H^{\uparrow} \le x$. To pay the degree, put $d_u = A_n 2^{-u}$. By (13), the main degree sum is at most

$$bB_H A_n \sum_{u=0}^{J-1} \frac{1}{W(d_u)}.$$

For $d_u \ge 2$, the intervals $[d_u/2, d_u]$ are disjoint and contained in $[1, n]$, and monotonicity gives

$$\frac{1}{W(d_u)} \le \frac{1}{\log 2} \int_{d_u/2}^{d_u} \frac{\mathrm{d}t}{tW(t)}.$$

There is at most one term with $1 \le d_u < 2$. The number with $d_u < 1$ is at most $\log_2 \psi_n + 2$, each at most one. Therefore

$$\sum_{u=0}^{J-1} \frac{1}{W(d_u)} \le \frac{\psi_n - 1}{\log 2} + \log_2 \psi_n + 3 \le C_1 \psi_n.$$

Since $A_n \psi_n = n$ and $J \le n$, the resulting polynomial satisfies

$$\begin{aligned} S_n(0) = 1, \qquad & \deg S_n \le K_1 n, \\ |S_n(x)| \le B_0 e^{-bH(|x|)/2} \quad & (|x| \le A_n). \end{aligned} \tag{14}$$

with fixed K_1. Only a compact envelope is used.

Define the single positive atomless probability law

$$Z_b = \int_{\mathbb{R}} |x|^{q-1} e^{-bH(|x|)}\, \mathrm{d}x, \qquad \mathrm{d}\tau_b(x) = Z_b^{-1} |x|^{q-1} e^{-bH(|x|)}\, \mathrm{d}x.$$

The lower bound $H(x) \ge x^{1-\delta}/[AW(1)]$ ensures a positive finite normalizer and all moments. Also

$$E_b := \int e^{cH(|x|)}\, \mathrm{d}\tau_b = Z_{b-c}/Z_b < \infty, \qquad \tau_b([-r, r]) \le \frac{2r^q}{qZ_b}.$$

Apply Lemma 3.2 to PS_n on $[-A_n, A_n]$, using (14). For every $\deg P \le n$,

$$\begin{aligned} \int |P|^2\, \mathrm{d}\tau_b &\ge \frac{1}{B_0^2 Z_b} \int_{-A_n}^{A_n} |PS_n|^2 |x|^{q-1}\, \mathrm{d}x \\ &\ge b_0 \Psi(n)^{-q} |P(0)|^2, \qquad b_0 = \frac{c_q}{B_0^2 Z_b (K_1+2)^q} > 0. \end{aligned} \tag{15}$$

Here $\deg(PS_n) + 1 \le (K_1 + 2)n$. Fix $\eta = \min\{1, C/E_b, LqZ_b/2\} > 0$. Then $\nu_R = \eta R \tau_b$ meets every original class budget for all $R \ge 1$, and its original degree-n square form is nonnegative whenever

$$\Psi(n)^q \le \eta b_0 R. \tag{16}$$

If Ψ diverges, take $t_R = (\eta b_0)^{1/q} R^{1/q}/2$ and the unique $x_R \ge 1$ satisfying $\Psi(x_R) = t_R$, for large R. Set $n_R = \lfloor x_R \rfloor \ge 2$. Then (16) holds and

$$0 \le \Psi(x_R) - \Psi(n_R) \le \log(1 + 1/n_R) \le 1.$$

Thus $D_W(R) > n_R$ and $\Psi(n_R) \ge t_R/2$ for sufficiently large R. Together with (12) this proves the asserted two-sided clock law.

If $\Psi_\infty < \infty$, (15) instead gives the degree-independent bound $\int |P|^2\, \mathrm{d}\tau_b \ge b_0 \Psi_\infty^{-q} |P(0)|^2$. Degrees zero and one are included by using $n = 2$. For $R \ge \max\{1, \Psi_\infty^q/(\eta b_0)\}$, the same law masks every polynomial square simultaneously. This completes the proof of Theorem 1.1.

4.3 A profile without regular variation

The growth hypothesis admits more than fixed logarithmic products. Fix $0 < \delta < 1$ and set $u_0 = 0$. Inductively define a continuous function g on $[0, \infty)$ by taking the value $k\delta$ on

$$[u_k, u_k + e^{k\delta}],$$

then slope δ on the following interval of length one, and setting $u_{k+1} = u_k + e^{k\delta} + 1$. Put

$$W(x) = \exp(g(\log(1 + x))).$$

This function is continuous and nondecreasing, $W(0) = 1$, and g is Lipschitz with constant δ. Hence, for $y \geq x \geq 1$,

$$\frac{W(y)}{W(x)} \leq \left(\frac{1+y}{1+x}\right)^{\delta} \leq (y/x)^{\delta}.$$

Its clock diverges. Under $u = \log(1 + x)$ one has $\mathrm{d}x/x = e^u \,\mathrm{d}u/(e^u - 1) \geq \mathrm{d}u$. Every full plateau after $x = 1$ contributes at least $e^{k\delta}e^{-k\delta} = 1$ to $\int \mathrm{d}x/(xW(x))$.

Nevertheless W is not regularly varying of any index. Fix $c = e^{1/2}$. Along points whose logarithmic coordinate is in the middle of a long plateau, both x and cx remain on that plateau, and $W(cx)/W(x) = 1$. At ramp starts $v_k = u_k + e^{k\delta}$, take $x_k = e^{v_k} - 1$. Then

$$\log(1 + cx_k) - v_k = \tfrac{1}{2} + \log(1 + (e^{-1/2} - 1)e^{-v_k}) \longrightarrow \tfrac{1}{2}$$

from below. The shifted coordinate remains inside the same unit ramp, so $W(cx_k)/W(x_k) \to e^{\delta/2} > 1$. These two limits preclude regular variation for this fixed dilation. There are also infinitely many corners. Theorem 1.1 still gives its sharp integrated-clock recovery law for every $q > 0$. No additional named asymptotic scale is needed for this conclusion.

5 Logarithmic and iterated-logarithmic applications

The general clock theorem gives explicit scales without repeating either the moment receiver or the compact lower construction. All parameters below are fixed as $R \to \infty$.

Lemma 5.1 (A sufficient power-ratio condition)**.** *Suppose $W : [0, \infty) \to [1, \infty)$ is continuous, nondecreasing, $W(0) = 1$, and is continuously differentiable on $[1, \infty)$. If $xW'(x)/W(x) \to 0$, then for every fixed $0 < \delta < 1$ there is $A \geq 1$ such that*

$$W(y)/W(x) \leq A(y/x)^{\delta} \qquad (y \geq x \geq 1).$$

Proof. Choose $X \geq 1$ such that $tW'(t)/W(t) \leq \delta$ for $t \geq X$. Integration of the logarithmic derivative proves the bound with $A = 1$ when $y \geq x \geq X$. When $1 \leq x \leq y \leq X$, monotonicity gives $W(y)/W(x) \leq W(X)/W(1)$. When $x < X < y$, split the ratio at X and use

$$\frac{W(y)}{W(x)} \leq \frac{W(X)}{W(1)}(y/X)^{\delta} \leq \frac{W(X)}{W(1)}(y/x)^{\delta}.$$

Thus $A = \max\{1, W(X)/W(1)\}$ suffices globally. □

Put $\ell(x) = \log(e + x)$ and $h(x) = \log\log(e^e + x)$. These functions are positive, increasing and smooth on $[0, \infty)$, with $\ell(0) = h(0) = 1$. Their logarithmic derivatives satisfy

$$\frac{x\ell'(x)}{\ell(x)} \longrightarrow 0, \qquad \frac{xh'(x)}{h(x)} \longrightarrow 0.$$

Consequently every fixed profile $W(x) = \ell(x)^\gamma$, $\gamma > 0$, and $W(x) = \ell(x)h(x)^a$, $a > 0$, satisfies every hypothesis of Theorem 1.1, by Lemma 5.1. The constants in its ratio bound may depend on the fixed exponent and on the chosen δ.

Corollary 5.2 (First logarithmic scale)**.** *For $W(x) = \ell(x)^\gamma$, write $D_\gamma(R) = D_W(R)$. For every fixed $q > 0$ and the fixed original signed-class budgets,*

$$\log D_\gamma(R) = \Theta\big(R^{1/[q(1-\gamma)]}\big), \qquad 0 < \gamma < 1,$$
$$\log\log D_1(R) = \Theta(R^{1/q}).$$

If $\gamma > 1$, one fixed positive atomless probability law and one fixed positive scaling constant mask the negative atom from every polynomial square, while meeting all class budgets, for every sufficiently large R. In particular $D_\gamma(R) = \infty$ there.

Proof. For x above a fixed lower limit, $\ell(x) \sim \log x$. Integral comparison with $1/[x(\log x)^\gamma]$ therefore gives

$$\Psi(t) \asymp \begin{cases} (\log t)^{1-\gamma}, & 0 < \gamma < 1, \\ \log\log t, & \gamma = 1, \end{cases} \qquad (t \to \infty),$$

and $\Psi(\infty) < \infty$ when $\gamma > 1$. Theorem 1.1 supplies the two-sided clock law in the divergent cases. In particular its lower bound forces $D_\gamma(R) \to \infty$, so these asymptotic clock comparisons apply at the detecting degree. Taking the power $1/(1-\gamma)$ in the first case and using the second comparison directly proves the displayed formulas. The convergent case is exactly the fixed-law masking conclusion of that theorem. □

Corollary 5.3 (Second logarithmic scale)**.** *For $W(x) = \ell(x)h(x)^a$, write $E_a(R) = D_W(R)$. Then*

$$\log\log E_a(R) = \Theta\big(R^{1/[q(1-a)]}\big), \qquad 0 < a < 1,$$
$$\log\log\log E_1(R) = \Theta(R^{1/q}).$$

For $a > 1$, one fixed positive atomless probability law, scaled by a fixed multiple of R, meets all original class budgets and masks every polynomial square for all sufficiently large R; thus $E_a(R) = \infty$ there.

Proof. For sufficiently large x, $\ell(x) \sim \log x$ and $h(x) \sim \log\log x$. Compare the clock integrand with $1/[x\log x(\log\log x)^a]$. The substitution $u = \log\log x$ gives

$$\Psi(t) \asymp \begin{cases} (\log\log t)^{1-a}, & 0 < a < 1, \\ \log\log\log t, & a = 1, \end{cases}$$

whereas its integral converges for $a > 1$. Apply Theorem 1.1 and invert the divergent comparisons as above. Its finite-clock assertion supplies the same-law all-degree masker in the remaining case. □

The statements concern ordinary polynomial degree, and retain arbitrary signed, singular and atomic remainders in their upper classes. Constants may depend on all fixed class data and the displayed exponents. The results assert neither leading constants nor uniformity as an exponent approaches its critical value. The triple-exponential endpoint is an application of the integrated-clock theorem, rather than a separate quasianalytic principle.

6 A finite-order polynomial approximation receiver

The same propagation estimate couples directly to Fourier residual duality, as used in [5]. This is a finite-order application of the classical Bang mechanism, rather than a new quasianalytic principle.

Theorem 6.1 (Finite moment projection bound)**.** *Let ρ be a finite positive Borel measure on $\mathbb{R}$ with moments through order $2N$, $N \geq 1$. Let $M_0 = 1, M_1, \dots, M_N > 0$ be log-convex and suppose*

$$\|x^j\|_{L^2(\rho)} \leq BC^j M_j \qquad (0 \leq j \leq N)$$

for $B, C > 0$. Let π be a finite complex measure of total variation V, supported on $[-T, T]$, and let $f(x) = \int e^{itx}\, \mathrm{d}\pi(t)$. Write $r = f - \Pi_N f$, where Π_N is orthogonal projection onto the image of degree-N polynomials in $L^2(\rho)$. For any integer k with $0 \leq k < N$ such that

$$\sum_{j=k+1}^{N} \frac{M_{j-1}}{M_j} > eCT, \tag{17}$$

one has $\|r\|_{L^2(\rho)} \leq BVe^{-k}$. Independently of (17),

$$\|r\|_{L^2(\rho)} \leq BV \min\{1, (CT)^N M_N/N!\}. \tag{18}$$

If π is any finite complex measure without a support restriction, the corresponding bound is

$$\|r\|_{L^2(\rho)} \leq B\left[|\pi|([-T,T])\, \varepsilon_N(T) + |\pi|(\mathbb{R} \setminus [-T,T])\right],$$

where $\varepsilon_N(T)$ is the minimum of 1, $(CT)^N M_N/N!$, and all e^{-k} allowed by (17).

Proof. The polynomial image is finite-dimensional and closed, even when its Gram matrix is singular. Thus the projection exists without strict positivity. If $r = 0$, there is nothing to prove. Otherwise define

$$F(t) = \int \overline{r(x)} e^{itx}\, \mathrm{d}\rho(x).$$

Cauchy–Schwarz and the finite moments justify N differentiations and give $|F^{(j)}(t)| \leq B\|r\|C^j M_j$. Orthogonality gives $F^{(j)}(0) = 0$ for $0 \leq j \leq N$. Lemma 2.1, applied to $F/(B\|r\|)$, yields $|F(t)| \leq B\|r\|e^{-k}$ on $[-T, T]$. The order-zero bound gives $|F| \leq B\|r\|$ everywhere. Using the inner product $\langle u, v\rangle = \int \overline{u}v\, \mathrm{d}\rho$, finite-measure Fubini and projection yield

$$\|r\|^2 = \langle r, f\rangle = \int F(t)\, \mathrm{d}\pi(t) \leq \int |F(t)|\, \mathrm{d}|\pi|(t).$$

Divide by $\|r\|$ and, for the noncompact case, split the frequency integral. Finally the integral Taylor remainder at zero gives

$$|F(t)| \leq B\|r\|C^N M_N |t|^N/N!,$$

since the first N jets vanish. The same integration proves (18) and permits the stated minimum. No density, determinacy, or infinite-moment assumption is used. □

Remark 6.2 (Representative regimes and comparison). For $M_j = \sqrt{j!}$, the clock has terms $j^{-1/2}$; taking $k = \lfloor N/2 \rfloor$ gives exponential error when T is a sufficiently small fixed multiple of $\sqrt{N}$. At fixed T, the Taylor bound is $BV(CT)^N/\sqrt{N!}$, with logarithm $-\frac{1}{2}N \log N + O(N)$ apart from $\log(BV)$. For $M_j = j!$, taking $k = \lfloor N \exp(-eCT - 2) \rfloor$ gives exponential error in N at fixed

bandwidth. For $M_j = j! \prod_{l=1}^{j} \log(e+l)^{\alpha}$, the logarithmic tail sum gives the corresponding finite approximation scale. The signed-measure degree laws in Section 5 are obtained instead by evaluating the integrated clock of Theorem 1.1.

Bang's tail-sum propagation is classical [4], and the Fourier residual argument is present in [5]. In particular, optimizing the free parameter in equation (3.71) of [5] already gives a critical bound of the form $\exp(-cN^{\delta})$; comparing only its weaker displayed example (3.73) would overstate the rate improvement. The point here is the short finite-order receiver, together with its application to the full signed class.

7 Finite controls and scope

The accompanying standard-library program `verify_finite_controls.py` uses exact rational arithmetic to check finite Jacobi moment matching in the original Gram metric, the free last diagonal, the singular semidefinite obstruction and singular-power evaluation identities. Its checks remain active under optimized Python. These controls corroborate finite algebra; the analytic arguments above establish the all-degree and asymptotic statements.

The finite moment criterion proves an existence degree bound. Given exact original moments through order $2N$, a negative Hankel direction can be converted into polynomial coefficients. Execution uses exact or certified coefficients as inputs. Leading gapless constants, profiles outside (2) and compatible lower laws for arbitrary log-convex moment sequences remain separate questions. A scalar signed-measure theorem gives no theta or spectral sign conclusion without an exact receiving representation.

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