Polynomial inertia of signed measures with negative intervals

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Abstract

Let μ be a finite real signed measure with a finite absolute exponential moment at rate $c > 0$. Suppose its restriction to each of finitely many disjoint bounded intervals is at most a strictly negative constant times Lebesgue measure. On the real polynomials of degree at most N, we prove that the negative index of the original square-integral form is at least $(cL/\pi^2 + o(1)) \log N$, where L is the total interval length. The coefficient is optimal over this class, as an infimum, by a classical Meixner–Pollaczek model. Arbitrary signed and singular exterior mass is allowed. The construction preserves the whole negative span, permits rational polynomial bases, and gives the corresponding bound in each parity for even measures. A finite absolute Gaussian moment gives the second sharp coefficient $L\sqrt{2c}/\pi$ on the $\sqrt{N}$ scale. More generally, an absolute $\exp(c|x|^p)$ moment, $p > 1$, gives a lower bound of order $N^{1-1/p}$, with optimal power for every $p > 1$. A credited Fourier-gap construction explains why $p < 1$ can allow a negative interval invisible to every polynomial square. Finally, a conditional application to the original positive-width Riemann-theta form converts a supplied negative interval, or a hypothetical off-axis zero with local residual bounds, into logarithmically growing negative polynomial inertia.

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1 The signed-band problem

For a finite real signed Borel measure μ with all absolute moments, write $\mathcal{P}_N = \{P \in \mathbb{R}[x] : \deg P \le N\}$ and

$$q_\mu(P, Q) = \int_{\mathbb{R}} PQ \, d\mu, \qquad i_-(\mu, N) = \operatorname{ind}_-(q_\mu|_{\mathcal{P}_N}).$$

The negative index is the largest dimension of a subspace on which $q_\mu(P, P) < 0$ for every nonzero P. We study a fixed negative interval while the permitted degree and the dimension of a negative subspace grow. Both the functional and the polynomial algebra are unchanged throughout. For $\varepsilon \in \{0, 1\}$, put

$$\mathcal{P}_N^\varepsilon = \{P \in \mathcal{P}_N : P(-x) = (-1)^\varepsilon P(x)\}, \qquad i_-(\mu, N, \varepsilon) = \operatorname{ind}_-(q_\mu|_{\mathcal{P}_N^\varepsilon}).$$

Theorem 1.1 (Critical exponential tail)**.** *Suppose $c > 0$ and*

$$J = \int_{\mathbb{R}} e^{c|x|} \, d|\mu|(x) < \infty. \tag{1}$$

Let $I_1, \dots, I_k$ be disjoint bounded open intervals, and suppose

$$\mu|_{I_\ell} \le -\gamma_\ell \, dx, \qquad \gamma_\ell > 0, \tag{2}$$

in measure order. If $L = \sum_\ell |I_\ell|$, then

$$\liminf_{N\to\infty} \frac{i_-(\mu, N)}{\log N} \ge \frac{cL}{\pi^2}. \tag{3}$$

The negative spaces may have bases in $\mathbb{Q}[x]$. If μ is even and all $I_\ell \subset (0, \infty)$, the same bound holds separately in each polynomial parity, with L counting only these positive-side intervals. The coefficient c/π^2 is optimal over the stated class; this means an infimum of admissible model coefficients, not attainment at a critical moment endpoint.

Condition (2) allows additional negative singular mass inside the intervals. The exterior may have arbitrary signed or singular mass subject to the absolute exponential-moment hypothesis. In the even case the reflected intervals are present automatically. The per-parity coefficient is thus half the whole-space coefficient associated with both sides of a symmetric negative region.

A second sharp endpoint holds under a Gaussian absolute moment: Theorem 5.4 gives $L\sqrt{2c}/\pi$ on the $\sqrt{N}$ scale. Its proof uses the classical Hermite generating function and Mehler formula to preserve the same common negative space.

Qualitative moment determinacy and polynomial approximation explain eventual detection of a negative interval. Our subject is the quantitative degree–index relation with the original absolute-measure error paid for a growing-dimensional common polynomial space. The critical approximation used below is the classical Meixner–Pollaczek generating-function mechanism, already used for original signed-measure degree certificates in [1]. We give its proof here. The growing-span step combines a full-period trigonometric reserve, a sublinear common Fourier-bandwidth overhead and a full mixed-Gram error bound. Bizeul and Klartag [2, Theorem 1 and Lemma 6] study logarithmic L^2 polynomial approximation for the double-sided exponential weight using Meixner–Pollaczek generating functions. Here the receiving statement controls a growing common negative span in the original absolute-measure norm, including singular measures satisfying the stated tail budget.

The fixed-index geometry and concentration results of [3] concern isolated or concentrating negative mass. Their growing-index extension [4] treats shrinking negative probability packets with a containing-interval gap, as well as separate gapless single-atom classes. Here the negative intervals remain fixed and the sharp index coefficient is measured against ordinary logarithmic degree. The related Carleman moment hierarchy [5] receives the common-span construction under weaker moment budgets and gives matching orders; the present exponential benchmark identifies the sharp class coefficient. These papers share classical approximation tools while retaining their different hypotheses and quantitative conclusions.

The matching examples use classical Christoffel–Darboux kernel asymptotics from [6, Theorem 1.2] and the local zero-count conversion of [7]. These positive-measure results supply the model asymptotics, rather than the universal signed-measure lower bound. The cited asymptotic and Meixner–Pollaczek identities are classical. The general stronger-tail bound in Section 5 has a constructive constant; the Gaussian case is sharpened to its class infimum. The subexponential obstruction in Section 6 records the classical nonquasianalytic boundary. The theta application in Section A concerns the original positive-width square form, whereas the distinct localization results of [8, 9] concern a correction form and retain strictly positive original Q on their constructed channels.

The proof follows the quantitative dependencies. Section 2 constructs the common negative space and its uniform Gram reserve. The critical approximation then determines the logarithmic

degree budget, and Section 4 proves infimum sharpness using classical local kernels. Sections 5 and 6 treat stronger and weaker tails. The conditional original-theta receiver is placed in Appendix A.

2 A negative space with uniform Gram reserve

Choose closed intervals $J_\ell \Subset I_\ell$ of lengths $L_\ell > 0$, centers a_ℓ, and real $\chi_\ell \in C_c^3(I_\ell)$ equal to one on J_ℓ. For a positive integer K put

$$d_\ell = \left\lfloor \frac{KL_\ell}{2\pi} \right\rfloor, \qquad D = 2\sum_\ell d_\ell.$$

For $1 \le j \le d_\ell$ use the two targets

$$\chi_\ell(x)\cos\frac{2\pi j(x-a_\ell)}{L_\ell}, \qquad \chi_\ell(x)\sin\frac{2\pi j(x-a_\ell)}{L_\ell}. \tag{4}$$

Enumerate them as $u_1, \dots, u_D$. Orthogonality over the complete period J_ℓ and measure order give, for every $v \in \mathbb{R}^D$,

$$\int \left(\sum_j v_j u_j\right)^2 d\mu \le -\lambda\|v\|_2^2, \qquad \lambda = \frac{1}{2}\min_\ell \gamma_\ell L_\ell > 0. \tag{5}$$

Indeed distinct intervals have disjoint supports; within one interval the portion outside J_ℓ contributes a nonpositive square integral. The fixed cutoff equals one on a whole period, so the negative margin does not deteriorate with D.

Let $\|u_j\|_{L^2(|\mu|)} \le M_0$ for all targets. Such an M_0 is fixed independently of K, because their absolute values are uniformly bounded and $|\mu|(\mathbb{R}) \le J$. If p_j approximate u_j with $\|p_j - u_j\|_{L^2(|\mu|)} \le \eta_D$, then Cauchy–Schwarz gives

$$\left| q_\mu\left(\sum v_j p_j, \sum v_j p_j\right) - q_\mu\left(\sum v_j u_j, \sum v_j u_j\right)\right| \le D\eta_D(2M_0 + \eta_D)\|v\|_2^2. \tag{6}$$

This follows from the operator bounds $M_0\sqrt{D}$ and $\eta_D\sqrt{D}$ for the target and error maps into $L^2(|\mu|)$. Set

$$\eta_D = \frac{\eta_0}{D}, \qquad \eta_0 = \min\left(1, \frac{\lambda}{8(2M_0+1)}\right). \tag{7}$$

For $D \ge 1$, the right side of (6) is at most $\lambda\|v\|_2^2/8$. Therefore the polynomial span remains strictly negative and has dimension D. This pays every mixed entry, including those arising after approximation.

If μ is even and the intervals are positive, replace each target by $u_j(x) + (-1)^\varepsilon u_j(-x)$, where $\varepsilon \in \{0,1\}$ is the chosen parity. The two sides have disjoint supports. The raw reserve doubles, the number of targets stays D, and only fixed norm constants change. Hence it suffices to give the common approximation in the full space, preserving parity whenever requested.

Use the Fourier convention

$$\widehat{\chi}(t) = \frac{1}{2\pi}\int \chi(x)e^{-\mathrm{i}tx}\,dx, \qquad \chi(x) = \int \widehat{\chi}(t)e^{\mathrm{i}tx}\,dt.$$

Three integrations by parts show

$$\int_{|t|>u} |\widehat{\chi}(t)|\,dt \le \frac{\|\chi'''\|_1}{2\pi u^2}, \qquad u > 0. \tag{8}$$

The Fourier transform is integrable, for instance by splitting at $|t| = 1$ and using the second derivative. Multiplication by a sine or cosine shifts the transform by carrier frequencies of absolute value at most K without increasing its total variation. Reflection changes only a fixed factor. Consequently there are constants C_0, Λ, independent of K, D and target number, such that truncation to $[-T, T]$, where

$$T = K + U\sqrt{D}, \tag{9}$$

produces real functions $f_j(x) = \int_{-T}^{T} e^{\mathrm{i}tx}\, d\Lambda_j(t)$ satisfying

$$\|\Lambda_j\|_{\mathrm{TV}} \le \Lambda, \qquad \|f_j - u_j\|_{L^2(|\mu|)} \le \frac{\sqrt{J}C_0}{U^2 D}. \tag{10}$$

Fix U large enough that this error is at most $\eta_D/4$. Symmetric frequency truncation preserves reality and the selected parity.

3 Critical exponential approximation and logarithmic inertia

Lemma 3.1 (Original-measure MP approximation)**.** *Assume (1). Put $h = \pi/c$ and $q_n(x) = P_n^{(1/2)}(x/h; \pi/2)$. If $f(x) = \int_{-T}^{T} e^{\mathrm{i}tx}\, d\Lambda_0(t)$ is real and $\|\Lambda_0\|_{\mathrm{TV}} \le \Lambda$, then there is a real polynomial of degree at most n with $L^2(|\mu|)$ error at most*

$$e\sqrt{J}\Lambda Q_n(r), \quad r = \tanh(hT/2), \quad Q_n(r) = r^{n+1}\left(\frac{n+2}{1-r} + \frac{r}{(1-r)^2}\right). \tag{11}$$

For even $|\mu|$, an assigned parity of f can be preserved.

Proof. The classical generating function [10, 18.23.7] is

$$\sum_{n=0}^{\infty} q_n(x) z^n = (1 - \mathrm{i}z)^{-1/2 + \mathrm{i}x/h}(1 + \mathrm{i}z)^{-1/2 - \mathrm{i}x/h}, \qquad |z| < 1, \tag{12}$$

with branches analytic at zero. For $|z| = \rho < 1$, the Cayley ratio $(1 - \mathrm{i}z)/(1 + \mathrm{i}z)$ lies in the right half-plane, its argument having absolute value at most $2 \arctan \rho < \pi/2$. Since $|1 + z^2| \ge 1 - \rho^2$, Cauchy's estimate gives

$$|q_n(x)| \le \rho^{-n}(1 - \rho^2)^{-1/2} e^{2 \arctan(\rho)|x|/h}.$$

For $n \ge 1$, choose $\rho = n/(n+1)$; the first two factors are at most $e\sqrt{n+1}$. Including $q_0 = 1$, we have

$$|q_n(x)| \le e\sqrt{n+1}\, e^{c|x|/2}. \tag{13}$$

Substitution $z = \mathrm{i}\tanh(ht/2)$ gives

$$e^{\mathrm{i}tx} = \operatorname{sech}(ht/2) \sum_{n=0}^{\infty} q_n(x)[\mathrm{i}\tanh(ht/2)]^n. \tag{14}$$

Hermitian symmetrization of the Fourier measure preserves f and its variation bound. Integrating the coefficients gives real coefficients bounded in modulus by Λr^n. The weighted-uniform tail is bounded by $e\Lambda \sum_{j>n}(j+1)r^j = e\Lambda Q_n(r)$, using (13). This summable bound also justifies integration of the series. Squaring and integrating its exponential envelope proves (11) on the original absolute measure, including singular measures. For even $|\mu|$, parity projection is contractive and does not increase degree or error. □

Lemma 3.2 (Rational coefficients)**.** *A real polynomial can be approximated to arbitrary positive $L^2(|\mu|)$ error by a rational polynomial of the same degree. Under (1), only c, J and the coefficients are needed for this rounding. An assigned parity can be preserved.*

Proof. The exponential series gives $\int |x|^{2j}\, d|\mu| \le J(2j)!/c^{2j}$. For coefficient changes δ_j, the norm error is therefore at most

$$\sqrt{J}\sum_{j=0}^{n} |\delta_j| \frac{\sqrt{(2j)!}}{c^j}.$$

Approximate the finitely many allowed coefficients by rationals to make this sum smaller than the desired tolerance. □

Proof of the lower bound in Theorem 1.1. Use the targets of Section 2 and fix all their constants before K grows. Write $H = hT$ and define

$$A = \max\left(1, \frac{4e\sqrt{J}\Lambda}{\eta_D}\right), \quad S = 4 + 2H + 2\log A, \quad n = \left\lceil e^H S^2 \right\rceil. \tag{15}$$

Since $1 - r = 2/(e^H + 1) \ge e^{-H}$, $r^{n+1} \le e^{-S^2}$ and $n + 2 \le 4e^H S^2$. Thus

$$Q_n(r) \le (4S^2 + 1)e^{2H - S^2} \le e^{2H - S^2/2} \le A^{-1}.$$

Here $\log(4S^2 + 1) \le S^2/2$ for $S \ge 4$, and $S^2/2 \ge 2H + \log A$. Lemma 3.1 approximates every f_j with error at most $\eta_D/4$. Round with additional error at most $\eta_D/2$ using Lemma 3.2. Together with (10), the total per-target error is at most η_D. Equations (5)–(6) prove strict negativity on a D-dimensional rational polynomial span.

For fixed inner intervals, put $L' = \sum L_\ell$. Then

$$D = \frac{L'}{\pi}K + O(1), \qquad T = K + O(\sqrt{K}), \qquad \log n(K) = hK + O(\sqrt{K}) + O(\log K). \tag{16}$$

Indeed $\log A = O(\log K)$ because $\eta_D^{-1} = O(D)$. No exponential dependence on D is hidden in a fixed norm or cutoff constant. The degree choice is eventually increasing; alternatively its running maximum has the same asymptotic. For $n(K) \le N < n(K + 1)$, monotonicity gives $i_-(\mu, N) \ge D(K)$, while consecutive logarithmic degree costs have ratio tending to one. Hence the lower limit for all integer N is at least $L'/(\pi h)$. Let the fixed inner lengths increase to the original lengths after taking the lower limit. The parity construction already preserves its target count and proves the stated per-parity version. □

Corollary 3.3 (Open negative sets)**.** *If $\mu = g(x)dx$ is even with continuous real g and satisfies (1), then in each parity*

$$\liminf_{N\to\infty} \frac{i_-(\mu, N, \varepsilon)}{\log N} \ge \frac{c}{\pi^2}|\{x > 0 : g(x) < 0\}|.$$

The right side may be infinite.

Proof. Exhaust the open negative set by finite unions of compact inner intervals. On each such interval continuity supplies a negative upper bound. Apply Theorem 1.1 with its constants fixed during the limit, then take the supremum of the total interval lengths. □

4 Sharpness from classical local kernels

Let $w(x) = \operatorname{sech}(\pi x)$, a probability density, and fix $a > 0$. Consider

$$d\mu_a(x) = (x^2 - a^2)w(x)\,dx. \tag{17}$$

The half-parameter MP polynomials $q_j = P_j^{(1/2)}(x; \pi/2)$ are orthonormal for $w(x)dx$ and satisfy the classical recurrence [10, Sections 18.19 and 18.22]

$$xq_j = a_j q_{j+1} + a_{j-1}q_{j-1}, \qquad a_j = (j+1)/2, \quad a_{-1} = 0.$$

Let J_N be the $(N+1)$-dimensional Jacobi compression and e_N its last coordinate vector. The original signed Gram matrix is exactly

$$A_N = J_N^2 - a^2 I + a_N^2 e_N e_N^T. \tag{18}$$

This is the full Gram of xP: the single q_{N+1} component missing from the finite compression contributes precisely the last rank-one term. Since it is positive,

$$|i_-(\mu_a, N) - \#\{x \in (-a, a) : q_{N+1}(x) = 0\}| \le 1. \tag{19}$$

Strict endpoints are not counted.

For clarity we state and specialize the external asymptotic used here. For these MP polynomials, Theorem 1.2 of [6] gives

$$\frac{K_N(x,x)}{\log N} \longrightarrow \frac{1}{\pi w(x)}, \qquad K_N(x,x) = \sum_{j=0}^{N} q_j(x)^2, \tag{20}$$

locally uniformly on $\mathbb{R}$. Its period-one reference coefficients are $\alpha_j = 1, \beta_j = 0$. The relevant ratio and reciprocal sequences are $a_{n-1}/a_n = n/(n+1)$, $b_n/a_n = 0$, and $1/a_n = 2/(n+1)$. They are bounded and have summable first differences, as required by the Stolz D_1 hypothesis. The Carleman sum diverges and the limiting transfer has trace zero, strictly in $(-2, 2)$. The reference equilibrium density at zero is $1/(2\pi)$ and the theorem's clock is $\rho_N = \sum_{j=0}^{N} 1/a_j = 2H_{N+1} \sim 2\log N$. These observations give exactly (20); the source also explicitly includes the MP case after that theorem.

The following classical conversion is a special case of [7, arXiv version, Lemma 3.1 and Corollary 3.2]. We include the argument to specify the normalization used here.

Lemma 4.1 (Kernel asymptotics to local zero counts)**.** *Let orthonormal polynomials for a moment-determinate probability density $w > 0$ have Gaussian quadrature measures ν_N on the zeros of q_{N+1}. If $s_N \to \infty$ and $K_N(x,x)/s_N \to d/w(x)$ locally uniformly, with a constant $d > 0$ and continuous w, then for every fixed bounded interval I,*

$$\frac{\#\{x \in I : q_{N+1}(x) = 0\}}{s_N} \longrightarrow d|I|.$$

Proof. The positive quadrature weights are $1/K_N(x_j, x_j)$ and their sum is one. The quadrature measure matches moments through degree $2N+1$. Its eventually fixed second moment gives tightness. Every fixed higher even moment is eventually exact and bounds uniform integrability of each lower monomial. Thus any weak subsequential limit has all the moments of w; determinacy identifies it as $w(x)dx$. Hence $\nu_N \Rightarrow w(x)dx$. At every quadrature atom the exact identity gives

$$\frac{\#\{x_j \in I\}}{s_N} = \int_I \frac{K_N(x,x)}{s_N}\, d\nu_N(x).$$

The uniform kernel error on $\overline{I}$ integrates to $o(1)$ because ν_N has total mass one. The remaining integrand $d\,1_I/w$ is bounded on its compact support and has only endpoint discontinuities, of zero limiting $w(x)dx$ mass. Weak convergence proves the conclusion. □

Proposition 4.2 (Polynomial multipliers and finite Jacobi sections)**.** *Let ν be a positive measure of infinite support with all moments, q_j its real orthonormal polynomials, and J_N its Jacobi compression on degrees $0,\dots,N$. For a fixed nonzero real polynomial h of degree d, set $r=\min\{N+1,\max(d-1,0)\}$. Then*

$$A_N=\left(\int hq_iq_j\,d\nu\right)_{i,j=0}^{N}=h(J_N)+D_N,\qquad \operatorname{rank}D_N\le r,$$

where D_N is supported in its final r rows and columns. In particular,

$$|i_-(h\nu,N)-\#\{x:q_{N+1}(x)=0,\ h(x)<0\}|\le r. \tag{21}$$

If $\nu=w\,dx$ satisfies the hypotheses of Lemma 4.1 with clock s_N and constant $d_0>0$, and $E=\{h<0\}$ is bounded, then

$$\frac{i_-(h\nu,N)}{s_N}\longrightarrow d_0|E|.$$

For even ν and even h, each parity index is half the full index up to an error at most $(1+r)/2$.

Proof. The three-term recurrence gives $(J^k)_{ij}=\int x^kq_iq_j\,d\nu$, with every matrix entry a finite sum. The difference from $(J_N^k)_{ij}$ consists of tridiagonal paths that visit $N+1$ and return. A path starting at $j\le N+1-d$ requires at least d steps to leave and one more to return. Thus for $k\le d$ the difference vanishes outside the final r columns, and symmetry gives the same assertion for rows. This proves the rank bound. Intersecting a negative subspace with $\ker D_N$ loses at most r dimensions; applying this argument in both directions proves the rank-perturbation bound. The eigenvalues of J_N are the simple zeros of q_{N+1}, which proves (21). The bounded set E is a finite union of intervals. Apply Lemma 4.1 to each component and divide the rank error by s_N.

Under the parity hypotheses, J_N has block form $\left(\begin{smallmatrix}0 & B\\ B^T & 0\end{smallmatrix}\right)$. Writing $h(x)=g(x^2)$, its two polynomial blocks are $g(BB^T)$ and $g(B^TB)$. Their nonzero singular-value squares coincide, and the block dimensions differ by at most one; irreducibility of J_N excludes additional zero multiplicity. Their negative indices differ by at most one. The parity-preserving correction has block ranks whose sum is at most r, proving the stated error for each original parity index. □

Exponential moments imply moment determinacy in the MP example. Applying Lemma 4.1 to (20), then (19), proves

$$i_-(\mu_a,N)\sim\frac{2a}{\pi}\log N,\qquad i_-(\mu_a,N,\varepsilon)\sim\frac{a}{\pi}\log N. \tag{22}$$

The parity assertion also follows from Proposition 4.2, since both the base measure and the quadratic multiplier are even.

For general $\tau>0$ replace w by $w_\tau(x)=(\tau/\pi)\operatorname{sech}(\tau x)$. Scaling $t=\tau x/\pi$ gives

$$i_-((x^2-a^2)w_\tau dx,N)\sim\frac{2a\tau}{\pi^2}\log N, \tag{23}$$

and half that coefficient in each parity. Its absolute exponential moment at rate c is finite precisely when $c<\tau$.

Completion of the sharpness assertion in Theorem 1.1. The density in (23) is negative on $(-a, a)$ and has a uniform negative upper bound on every fixed inner interval. Fix an admissible moment rate c, then let $\tau \downarrow c$ and the inner interval length approach $2a$. The ratio of the model's index coefficient to its negative-band length approaches c/π^2. Any larger universal coefficient would contradict one of these models after choosing its fixed parameters sufficiently close to these limits. Each model's absolute-moment estimates hold in its stated range. The comparison does not hold the budget J, band margin or onset fixed across the model family. □

5 Stronger tails and growing power index

Theorem 5.1. *Let $p > 1$, $c > 0$, and $J_p = \int e^{c|x|^p}\, d|\mu|(x) < \infty$. Under (2), put $\alpha = 1 - 1/p$ and $L = \sum |I_\ell|$. Then*

$$\liminf_{N\to\infty} \frac{i_-(\mu, N)}{N^{1-1/p}} \geq \frac{L}{\pi}\kappa_{p,c}, \qquad \kappa_{p,c} = e^{-1+1/p}(cp/2)^{1/p}. \tag{24}$$

The same assertion holds in each parity for even μ, counting only positive-side interval lengths. Bases may have rational coefficients. The displayed constant is a constructive lower coefficient. At $p = 2$, Theorem 5.4 improves it to the sharp coefficient $\sqrt{2c}$ in place of $\kappa_{2,c}$.

Proof. For real x, t, the real-segment integral Taylor remainder yields

$$\left| e^{\mathrm{i}tx} - \sum_{j=0}^{N} \frac{(\mathrm{i}tx)^j}{j!} \right| \leq \frac{|tx|^{N+1}}{(N+1)!}.$$

Set $n = N + 1$. Maximizing $u^{2n}e^{-cu^p}$ gives

$$\int |x|^{2n}\, d|\mu| \leq J_p \left(\frac{2n}{cpe}\right)^{2n/p}.$$

Since $n! \geq (n/e)^n$, the original $L^2(|\mu|)$ Taylor error is bounded by

$$\sqrt{J_p}\left(\frac{B_{p,c}|t|}{n^\alpha}\right)^n, \qquad B_{p,c} = \kappa_{p,c}^{-1}. \tag{25}$$

Minkowski integration gives an additional factor at most Λ for a real Fourier measure of variation Λ supported in $[-T, T]$. Hermitian symmetry makes the integrated polynomial real.

Use the same fixed-period targets and common Fourier truncations as in Section 2; their total mass estimates use $|\mu|(\mathbb{R}) \leq J_p$. Fix any $0 < \kappa < \kappa_{p,c}$ and set $K = \kappa(N+1)^\alpha$ (integer rounding has no effect). Then $D = KL'/\pi + O(1)$ and $T = K + O(\sqrt{K})$, so

$$\frac{B_{p,c}T}{(N+1)^\alpha} \longrightarrow \frac{\kappa}{\kappa_{p,c}} < 1.$$

Thus for some fixed $r < 1$ and every sufficiently large integer N, the common Taylor error is at most $\Lambda\sqrt{J_p}r^{N+1}$, eventually smaller than $\eta_D/4$. The Fourier error is at most $\eta_D/4$. Rational rounding of the same-degree coefficients costs at most $\eta_D/2$, because

$$\left\| \sum_j \delta_j x^j \right\|_{L^2(|\mu|)} \leq \sqrt{J_p} \sum_j |\delta_j| \left(\frac{2j}{cpe}\right)^{j/p},$$

where the $j = 0$ factor is one. The complete Gram estimate therefore preserves all D negative directions. Reality and parity are retained, as before. Divide D by N^α, then let κ increase to $\kappa_{p,c}$ and the fixed inner interval lengths increase to the original lengths after the lower limit. This proves (24). □

Proposition 5.2 (Gaussian matching model). *For $w_\tau(x) = \sqrt{\tau/\pi}e^{-\tau x^2}$ and $a > 0$,*

$$i_-((x^2 - a^2)w_\tau dx, N) \sim \frac{2a\sqrt{2\tau}}{\pi}\sqrt{N}.$$

In each parity the coefficient is half this value. Hence the power $\sqrt{N}$ in Theorem 5.1 cannot be increased universally for $p = 2$.

Proof. The orthonormal Hermite recurrence has $a_n = \sqrt{(n+1)/(2\tau)}$, $b_n = 0$. The original signed compression is again exactly (18). The same period-one specialization of [6, Theorem 1.2] applies: the ratio $\sqrt{n/(n+1)}$ and reciprocal $\sqrt{2\tau/(n+1)}$ are bounded monotone sequences with summable first differences, the limiting trace is zero and Carleman's sum diverges. The clock is $\rho_N \sim 2\sqrt{2\tau N}$, yielding locally uniformly

$$\frac{K_N(x,x)}{\sqrt{N}} \longrightarrow \frac{\sqrt{2\tau}}{\pi w_\tau(x)}.$$

Gaussian moment determinacy and Lemma 4.1 give the local zero count. The rank-one boundary comparison and bipartite parity argument prove the assertion. The absolute exponential-square moment is finite for $c < \tau$, including the extra quadratic density factor. □

The Gaussian case admits a sharper common approximation than Taylor truncation. It yields the sharp universal coefficient.

Lemma 5.3 (Gaussian original-measure approximation). *Suppose $J_2 = \int e^{cx^2}\, d|\mu|(x) < \infty$, where $c > 0$. Let $f(x) = \int_{-T}^{T} e^{itx}\, d\Lambda_0(t)$ be real with $\|\Lambda_0\|_{\mathrm{TV}} \le \Lambda$. Put $n = N + 1$ and $A = T^2/(2c)$. If $0 < A < n$, there is a real polynomial p of degree at most N with*

$$\|f - p\|_{L^2(|\mu|)} \le \sqrt{J_2}\Lambda E_N(T),$$
$$E_N(T) = \frac{\sqrt{e}\,(N+2)^{1/4}e^{-A/2}A^{(N+1)/2}}{\sqrt{(N+1)!}(1 - r_N)},$$
$$r_N = \left(\frac{N+3}{N+2}\right)^{1/4}\sqrt{\frac{A}{N+2}} < 1.$$

For $T = 0$ the error is zero. Assigned parity can be preserved when $|\mu|$ is even. For every fixed $s \in (0,1)$, uniformly over $T^2 \le 2cs(N+1)$ the quantity $E_N(T)$ is bounded by $C_s(N+2)^{1/4}\exp[-(N+1)(s - 1 - \log s)/2]$.

Proof. Use physicists' Hermite polynomials and put $q_j(x) = H_j(\sqrt{c}\,x)/\sqrt{2^j j!}$. Mehler's formula [10, Eq. 18.18.28] gives, for $0 < r < 1$,

$$\sum_{j\ge 0} q_j(x)^2 r^j = (1 - r^2)^{-1/2}\exp\left(\frac{2cx^2 r}{1+r}\right).$$

For $j \ge 1$ choose $r = j/(j+1)$; since $r^{-j/2} \le \sqrt{e}$ and $(1 - r^2)^{-1/4} \le (j+1)^{1/4}$,

$$|q_j(x)| \le \sqrt{e}\,(j+1)^{1/4}e^{cx^2/2}.$$

The same bound holds for $j = 0$. The classical Hermite generating function gives the exact expansion

$$e^{itx} = e^{-t^2/(4c)} \sum_{j\ge 0} q_j(x) \frac{(it/\sqrt{2c})^j}{\sqrt{j!}}.$$

Writing $a = t^2/(2c)$, the modulus of its jth coefficient is $e^{-a/2}a^{j/2}/\sqrt{j!}$. For $j \ge n > A$ this is increasing in $a \in [0, A]$. After multiplication by $(j+1)^{1/4}$, the ratio of successive majorants is $((j+2)/(j+1))^{1/4}\sqrt{A/(j+1)}$, decreasing in j and at most r_N for $j \ge N+1$. Therefore the tail after degree N is at most $E_N(T)e^{cx^2/2}$, uniformly in $|t| \le T$. Integration against Λ_0 and then against $|\mu|$ gives the claim. Reality and parity follow by symmetrizing the integrated polynomial. For the last assertion, $r_N \le \sqrt{s}$ (indeed $n(n+2)^{1/2} < (n+1)^{3/2}$). Also $n! \ge (n/e)^n$ and $e^{-A/2}A^{n/2}$ is increasing for $A < n$, whence

$$\frac{e^{-A/2}A^{n/2}}{\sqrt{n!}} \le \exp[-n(s - 1 - \log s)/2].$$

One can take $C_s = \sqrt{e}/(1-\sqrt{s})$. □

Theorem 5.4 (Sharp Gaussian-tail coefficient)**.** *Under the negative-interval hypotheses of Theorem 1.1, replace its tail assumption by $J_2 = \int e^{cx^2}\, d|\mu| < \infty$. Then*

$$\liminf_{N\to\infty} \frac{i_-(\mu, N)}{\sqrt{N}} \ge \frac{L\sqrt{2c}}{\pi}.$$

For even μ, the same lower bound holds separately in each parity, with L counting the positive-side intervals. The negative spaces may have rational polynomial bases. The coefficient $\sqrt{2c}/\pi$ is optimal over this class in the infimum sense.

Proof. Fix the inner intervals and targets of Section 2, with total inner length L', and fix $0 < \kappa < \sqrt{2c}$. Take an integer $K = \kappa\sqrt{N+1} + O(1)$. Their dimension is $D = L'K/\pi + O(1)$. The common Fourier truncation has variation bounded independently of K, D and bandwidth $T = K + U\sqrt{D}$, and its error is at most $\eta_D/4$, using J_2 in place of J in the total-mass bound. In particular, $T^2/[2c(N+1)] \to \kappa^2/(2c) < 1$. Choose s strictly between this limit and one. Lemma 5.3 approximates every truncated target by a degree-N polynomial with error $O_s(N^{1/4}e^{-d_sN}) < \eta_D/4$, where $d_s = (s - 1 - \log s)/2 > 0$. Rational rounding uses the remaining $\eta_D/2$: the original-measure monomial norm obeys $\|x^j\|_{L^2(|\mu|)} \le \sqrt{J_2}(j/(ce))^{j/2}$ for $j \ge 1$, so coefficient errors may be chosen arbitrarily small in this finite basis. Rounding preserves reality and parity. The full mixed-Gram bound (6) thus gives a negative space of dimension D in $\mathcal{P}_N$. Divide by $\sqrt{N}$ and let $\kappa \uparrow \sqrt{2c}$ and $L' \uparrow L$. The reflected target construction proves the parity assertion with the same argument. For sharpness use the model in Proposition 5.2 $d\mu = (x^2 - a^2)\sqrt{\tau/\pi}e^{-\tau x^2}dx$, where $\tau > c$. Its absolute c-Gaussian moment is finite and its negative index is asymptotic to $2a\sqrt{2\tau N}/\pi$. It satisfies the negative-band hypothesis on every compact inner interval of $(-a, a)$. Let those lengths increase to $2a$ and then $\tau \downarrow c$. No larger universal coefficient is possible. This varies the admissible models; it does not hold their moment budgets or their onset degrees fixed. □

Proposition 5.5 (Optimal stronger-tail powers)**.** *For every $p > 1$, $c > 0$, $a > 0$ and $\tau > c$, let*

$$w_{p,\tau}(x) = \frac{\tau^{1/p}}{2\Gamma(1 + 1/p)} e^{-\tau|x|^p}, \qquad d\mu_{p,\tau,a}(x) = (x^2 - a^2)w_{p,\tau}(x)dx,$$

and set

$$A_p = \frac{1}{2}\left[\frac{\sqrt{\pi}\Gamma(p/2)}{\Gamma((p+1)/2)}\right]^{1/p}, \quad \alpha = 1 - \frac{1}{p}.$$

These signed measures have finite absolute c-weighted p-moment, are uniformly negative on each compact inner interval of $(-a, a)$, and satisfy

$$i_-(\mu_{p,\tau,a}, N) \sim \frac{a\tau^{1/p}}{\pi A_p \alpha} N^\alpha.$$

Consequently the exponent $1 - 1/p$ in Theorem 5.1 is optimal for every $p > 1$. This does not assert that its displayed coefficient is sharp for $p \neq 2$.

Proof. The normalization makes $w_{p,\tau}dx$ a probability measure. Its Jacobi diagonal vanishes. The classical Freud recurrence asymptotic, with its summable normalized remainder and the bounded-variation hypotheses needed here, is recorded and checked for every $p \geq 1$ in [11, Section 5, Example 3]; scaling x by $\tau^{-1/p}$ gives $a_n \sim A_p \tau^{-1/p}(n+1)^{1/p}$. Specifically, for $\tau = 1$ that example has $a_n/(n+1)^{1/p} = A_p + r_n$, with $(r_n) \in \ell^1$, and proves that (a_{n+1}/a_n) and $(1/a_n)$ have bounded total variation. The inverse ratio has bounded variation as well, since the ratios converge to one. Thus the $N = 1$ periodic-modulation hypotheses of [6, Theorem 1.2] hold; Carleman's condition follows from $p \geq 1$. In the present case

$$\rho_N = \sum_{j=0}^{N} \frac{1}{a_j} \sim \frac{\tau^{1/p}}{A_p \alpha} N^\alpha, \qquad \frac{K_N(x,x)}{\rho_N} \longrightarrow \frac{1}{2\pi w_{p,\tau}(x)}$$

locally uniformly. The absolute exponential moment implies moment determinacy. Apply Lemma 4.1 to the bounded interval $(-a, a)$ to obtain a local zero count asymptotic $a\rho_N/\pi$. In the corresponding orthonormal polynomial basis, the original signed Gram matrix differs from $J_N^2 - a^2 I$ only by its one-dimensional boundary term, exactly as in (18). Its negative index differs by at most one from this zero count. This proves the asymptotic. Finally,

$$\int |x^2 - a^2| e^{-(\tau - c)|x|^p} dx < \infty,$$

and the continuous signed density is bounded above by a strictly negative constant on each compact inner interval, proving the remaining claims. □

Corollary 5.6 (Matching models on prescribed interval unions)**.** *Fix $u_1 < v_1 < \cdots < u_k < v_k$, put $h(x) = \prod_{j=1}^{k}(x - u_j)(x - v_j)$, and let $L = \sum_j (v_j - u_j)$. For the MP and Hermite probability densities*

$$w_\tau^{\mathrm{MP}}(x) = \frac{\tau}{\pi}\operatorname{sech}(\tau x), \qquad w_\tau^{\mathrm{H}}(x) = \sqrt{\frac{\tau}{\pi}} e^{-\tau x^2},$$

one has

$$i_-(h w_\tau^{\mathrm{MP}} dx, N) = \left(\frac{\tau L}{\pi^2} + o(1)\right)\log N, \qquad i_-(h w_\tau^{\mathrm{H}} dx, N) = \left(\frac{\sqrt{2\tau} L}{\pi} + o(1)\right)\sqrt{N}.$$

For symmetric interval unions the corresponding even multiplier gives half these coefficients in each parity. Thus the matching coefficients depend on total negative length, independently of interval locations.

Proof. The negative set of h is precisely the displayed interval union. Apply Proposition 4.2 to the MP and Hermite kernel asymptotics above. The fixed polynomial factor preserves the absolute moment at every rate $c < \tau$, for $e^{c|x|}$ and e^{cx^2}, respectively. Each density has a strictly negative uniform upper bound on every compact inner subinterval. To obtain matching infima while holding prescribed bands fixed, first move the polynomial's endpoints slightly outward, so its density is uniformly negative on the original band closures. After taking the degree limit, let these enlargements shrink to zero and $\tau \downarrow c$. This gives coefficients tending to cL/π^2 and $\sqrt{2c}L/\pi$, respectively. Band margins, moment budgets and onsets may vary in this comparison. □

6 The subexponential boundary

The following is the classical Fourier-gap, nonquasianalytic obstruction, included to delimit the preceding receiving theorems. Compact Gevrey bump constructions and their Fourier decay are standard, for example [12, Appendix A].

Proposition 6.1. *For every $0 < p < 1$ there is a real even continuous signed density g negative on an open interval such that*

$$\int e^{c|x|^p}|g(x)|\,dx < \infty \quad \text{for every } c > 0, \qquad \int P(x)^2 g(x)\,dx > 0$$

for every nonzero real polynomial P. Thus its polynomial negative index is zero at every degree.

Proof. Choose $p < r < 1$ and $\beta = r/(1-r)$. The standard bump $\exp[-(1-t^2)^{-\beta}]$ on $(-1,1)$, extended by zero, has Gevrey order $1/r$. Here is the relevant estimate. Near an endpoint at distance d, a complex Cauchy circle of radius εd, with ε fixed sufficiently small depending on β, gives

$$|\eta_0^{(n)}(t)| \le n!(\varepsilon d)^{-n} e^{-c_0 d^{-\beta}}.$$

Maximizing in d gives $CA^n(n!)^{1+1/\beta} = CA^n(n!)^{1/r}$. In the interior a fixed analytic disk suffices. Each fixed derivative vanishes at the endpoints, so extension by zero is smooth. Translate, rescale and reflect the bump to obtain a nonzero nonnegative real even η supported in $[-2,-1] \cup [1,2]$, with the same Gevrey order. Its inverse Fourier transform v is real and even. Repeated integration by parts gives $|v(x)| \le CA^n(n!)^{1/r}|x|^{-n}$; choosing n comparable to $(|x|/(2A))^r$ proves

$$|v(x)| \le Ce^{-b|x|^r}.$$

Consequently v has all the stated absolute p-moments, and Fourier differentiation shows that every polynomial moment of v vanishes, because η vanishes near zero. On the other hand $v(0) > 0$ and $\int v = 0$, so continuity and integrability force $v < 0$ on a nonempty open interval. Take $g = v + \epsilon e^{-x^2}$ with $\epsilon > 0$ sufficiently small to preserve a negative value. All moments of v vanish, and hence

$$\int P^2 g = \epsilon \int P^2 e^{-x^2} > 0$$

for every nonzero real polynomial P. □

A A conditional receiver for the original theta form

Let

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad F(z) = \tfrac{1}{4}\xi((1+\mathrm{i}z)/2).$$

The even entire function F is real on the real line. For $y > 0$ define

$$g_y(x) = \frac{\partial_y |F(x+\mathrm{i}y)|^2}{8y}, \qquad Q_y(P) = \frac{1}{2\pi}\int_{\mathbb{R}} P(x)^2 g_y(x)\,dx. \tag{26}$$

For the formulation and earlier explicit estimates see [13]. The analytic moment input required below is proved here. The point of this application is quantitative original-square inertia after a negative interval is supplied. The qualitative modulus/zero strategy is classical; see Sondow–Dumitrescu [14], as well as the attribution and half-plane discussion in [15].

The following elementary analytic estimate establishes the absolute-moment hypothesis directly for the density in (26).

Lemma A.1 (Absolute moments of the theta density)**.** *For $F(z) = \xi((1+\mathrm{i}z)/2)/4$ there is a constant $C > 0$ such that*

$$|F^{(j)}(x+\mathrm{i}v)| \le C(1+|x|)^5 e^{-\pi|x|/8}, \qquad j = 0,1,2, \quad |v| \le 1. \tag{27}$$

The density in (26) extends continuously to $y = 0$ by $g_0 = (F'^2 - FF'')/4$. It is even in x, and for a constant $C_g > 0$,

$$|g_y(x)| \le C_g(1+|x|)^{10} e^{-\pi|x|/4}, \qquad 0 \le y \le 1. \tag{28}$$

Consequently its absolute exponential moment is finite at every rate $0 < c < \pi/4$, uniformly over this closed y interval.

Proof. First work on $|\Im z| \le 2$, and write $s = (1+\mathrm{i}z)/2 = \sigma + \mathrm{i}t$. Then $-1/2 \le \sigma \le 3/2$ and $t = x/2$. Euler–Maclaurin summation on $[1, N]$, followed by $N \to \infty$ initially for $\Re s > 1$, gives

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)}{2}\int_1^\infty B_2(\{u\})u^{-s-2}\,du, \qquad B_2(v) = v^2 - v + \frac{1}{6}.$$

The remainder is absolutely convergent for $\Re s > -1$, so analytic continuation gives the identity there away from $s = 1$. Since $|B_2(\{u\})| \le 1/6$, the integral has modulus at most $1/[6(\sigma+1)] \le 1/3$ on the stated strip. For $|t| \ge 1$, $|s-1| \ge |t|$, and hence $|\zeta(s)| \le C_1(1+|t|)^2$ uniformly. Uniform Stirling estimates on this fixed real strip, with $\Gamma(w+1) = w\Gamma(w)$ if needed, give

$$|\Gamma(s/2)| \le C_2|t|^{\sigma/2-1/2}e^{-\pi|t|/4}.$$

The largest polynomial exponent here is $1/4$. Using the exact product

$$F(z) = \frac{1}{8}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

therefore bounds F by $C_3(1+|x|)^5 e^{-\pi|x|/8}$ on the larger strip when $|x| \ge 2$. Entirety of ξ covers the remaining compact rectangle, including the removable factors at $s = 0, 1$. Cauchy's formula on radius-$1/2$ circles centered in $|\Im z| \le 1$ gives the same bound for the first two derivatives: the circles lie in the larger strip, and their real coordinates differ from x by at most $1/2$. These are standard Euler–Maclaurin and Stirling estimates; see also [10, Sections 25.2 and 5.11].

Put $G(x,y) = |F(x+\mathrm{i}y)|^2$. The functional equation and conjugation give $F(-z) = F(z)$ and $F(\overline{z}) = \overline{F(z)}$, so G is even in both x and y. Direct differentiation yields

$$G_{yy} = 2|F'|^2 - 2\Re(F''\overline{F}), \qquad g_y(x) = \frac{1}{8}\int_0^1 G_{yy}(x,ty)\,dt.$$

The second identity follows from $G_y(x,0) = 0$. It proves the continuous extension, its stated value at zero, and (28). Integrating the envelope proves the final assertion, since $\pi/4 - c > 0$. No endpoint moment at $c = \pi/4$ is asserted. □

For the original measure $g_y(x)dx/(2\pi)$, one may in particular use the finite budget

$$J_c \le \frac{C_g}{\pi} \sum_{j=0}^{10} \binom{10}{j} \frac{j!}{(\pi/4 - c)^{j+1}}, \qquad 0 < c < \pi/4. \tag{29}$$

Here C_g is an existence constant from the proved strip estimate; an explicit numerical value is not needed for the following asymptotic conclusion.

Corollary A.2 (Supplied negative theta intervals)**.** *Fix $0 < y < 1$. If $g_y < 0$ on finitely many disjoint bounded positive-side intervals I_ℓ, then, in each polynomial parity,*

$$\liminf_{N\to\infty} \frac{i_-(Q_y, N, \varepsilon)}{\log N} \ge \frac{1}{4\pi} \sum_{\ell} |I_\ell|. \tag{30}$$

The negative spaces may have rational polynomial bases.

Proof. The density is even and continuous. First take compact inner intervals with a uniform strictly negative bound. Apply Theorem 1.1 to the original measure $g_y(x)dx/(2\pi)$, using (29). Exhaust the interval lengths, then let $c \uparrow \pi/4$ after the N lower limit. Positive scalar normalization does not alter inertia. □

Proposition A.3 (Conditional zero-to-band transfer)**.** *Suppose F has a zero $z_0 = a + \mathrm{i}b$, where $a > 0$ and $0 < b < 1$, of multiplicity $m \ge 1$. Write $F(z) = (z - z_0)^m H(z)$, and choose $\rho > 0$ so that H is nonzero on $|z - z_0| \le \rho$. Set*

$$h_* = \min_{|z-z_0|\le\rho} |H(z)| > 0, \qquad B = \max_{|z-z_0|\le\rho} 2|H'(z)/H(z)|.$$

Choose $\delta_0 > 0$ satisfying

$$\delta_0 \le \frac{b}{2}, \quad \delta_0 \le \frac{m}{2(B+1)}, \quad \delta_0 \le \frac{(B+1)\rho^2}{2m}, \quad \delta_0 \le \frac{(B+1)a^2}{2m}. \tag{31}$$

For every fixed $0 < \delta < \delta_0$, put

$$y = b - \delta, \qquad w = \sqrt{\frac{m\delta}{2(B+1)}}.$$

Then $I = (a - w, a + w) \subset (0, \infty)$ and throughout I

$$g_y(x) \le -\frac{m h_*^2 \delta^{2m-1}}{8b} < 0. \tag{32}$$

Consequently, in each parity,

$$\liminf_{N\to\infty} \frac{i_-(Q_{b-\delta}, N, \varepsilon)}{\log N} \ge \frac{1}{2\pi} \sqrt{\frac{m\delta}{2(B+1)}}. \tag{33}$$

Proof. Write $x = a + u$ and $s = u^2 + \delta^2$. The bounds imply $w \le a/2$ and $s \le m\delta/(B+1) \le \rho^2$. Complete differentiation, retaining the residual function, gives

$$\partial_y |F(x + \mathrm{i}y)|^2 = |H|^2 s^{m-1} \left[-2m\delta + s\partial_y \log |H|^2 \right].$$

Since $|\partial_y \log |H|^2| \le B$, the bracket is at most $-m\delta$. Also $s \ge \delta^2$ and $|H| \ge h_*$, so the derivative is at most $-m h_*^2 \delta^{2m-1}$. Division by $8y$, with $y < b$, proves (32). Evenness supplies the reflected band, and Corollary A.2 applied to the positive-side length $2w$ proves (33). □

The zero-to-band transfer applies at each fixed δ satisfying Proposition A.3. The residual factor H retains all other zeros and its complete local variation; the stated multiplicity is arbitrary. The degree limit is taken with this supplied band and its residual bounds fixed. The correction-inertia results of [8, 9] concern their separate correction form, while the present receiver acts on the original scalar measure in (26).

B Reproducibility and interpretation

The accompanying standard-library Python program `verify_mp_grams.py` checks finite original MP moment matrices against their full Jacobi-square-plus-boundary representation. It computes moments from the exact generating function $\sec(t/2)$, uses rational congruence pivots to compute inertia, checks the rank-one interlacing bound and checks both parity blocks. These exact finite checks corroborate the original-form normalization, boundary identity and parity splitting. The preceding analytic arguments supply the asymptotic laws and the conditional theta transfer.

The critical coefficient is sharp as an infimum under the stated absolute-tail and negative-band hypotheses. Certified band, cutoff and moment-budget inputs make the displayed finite construction effective with the error payments given in the proof. The constants and onset belong to those supplied inputs.

Declaration of generative AI and AI-assisted technologies

ChatGPT Sol 6.1 assisted mathematical research, proof development, literature discovery, drafting and verification through workflows developed in the author's AI governance research. The author reviewed and edited the material and is responsible for the statements, proofs, attribution and presentation.

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