Polynomial inertia of signed measures with negative intervals

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Luminary Oddities

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Abstract

Let μ be a finite real signed measure with a finite absolute exponential moment at rate $c > 0$. Suppose its restriction to each of finitely many disjoint bounded intervals is at most a strictly negative constant times Lebesgue measure. On the real polynomials of degree at most N, we prove that the negative index of the original square-integral form is at least $(cL/\pi^2 + o(1)) \log N$, where L is the total interval length. The coefficient is optimal over this class, as an infimum, by a classical Meixner–Pollaczek model. Arbitrary signed and singular exterior mass is allowed. The construction preserves the whole negative span, permits rational polynomial bases, and gives the corresponding bound in each parity for even measures. An absolute $\exp(c|x|^p)$ moment, $p > 1$, instead gives an explicit lower bound of order $N^{1-1/p}$; its power is optimal for $p = 2$. A credited Fourier-gap construction explains why $p < 1$ can allow a negative interval invisible to every polynomial square. Finally, a conditional application to the original positive-width Riemann-theta form converts a supplied negative interval, or a hypothetical off-axis zero with local residual bounds, into logarithmically growing negative polynomial inertia.

1 The signed-band problem

For a finite real signed Borel measure μ with all absolute moments, write $\mathcal{P}_N = \{P \in \mathbb{R}[x] : \deg P \le N\}$ and

$$q_\mu(P, Q) = \int_{\mathbb{R}} PQ \, d\mu, \qquad i_-(\mu, N) = \operatorname{ind}_-(q_\mu|_{\mathcal{P}_N}).$$

The negative index is the largest dimension of a subspace on which $q_\mu(P, P) < 0$ for every nonzero P. We study a fixed negative interval while the permitted degree and the dimension of a negative subspace grow. Both the functional and the polynomial algebra are unchanged throughout.

Theorem 1.1 (Critical exponential tail)**.** *Suppose $c > 0$ and*

$$J = \int_{\mathbb{R}} e^{c|x|} \, d|\mu|(x) < \infty. \tag{1}$$

Let $I_1, \dots, I_k$ be disjoint bounded open intervals, and suppose

$$\mu|_{I_\ell} \le -\gamma_\ell \, dx, \qquad \gamma_\ell > 0, \tag{2}$$

in measure order. If $L = \sum_\ell |I_\ell|$, then

$$\liminf_{N \to \infty} \frac{i_-(\mu, N)}{\log N} \ge \frac{cL}{\pi^2}. \tag{3}$$

The negative spaces may have bases in $\mathbb{Q}[x]$. If μ is even and all $I_\ell \subset (0,\infty)$, the same bound holds separately in each polynomial parity, with L counting only these positive-side intervals. The coefficient c/π^2 is optimal over the stated class; this means an infimum of admissible model coefficients, not attainment at a critical moment endpoint.

Condition (2) allows additional negative singular mass inside the intervals. No positivity, density, support or smoothness assumption is made on the exterior. In the even case the reflected intervals are present automatically. The per-parity coefficient is thus half the whole-space coefficient associated with both sides of a symmetric negative region.

Qualitative moment determinacy and polynomial approximation explain eventual detection of a negative interval. Our subject is the quantitative degree–index relation with the original absolute-measure error paid for a growing-dimensional common polynomial space. The critical approximation used below is the classical Meixner–Pollaczek generating-function mechanism, already used for original signed-measure degree certificates in [6]. We give its proof here. The growing-span step combines a full-period trigonometric reserve, a sublinear common Fourier-bandwidth overhead and a full mixed-Gram error bound. The complementary geometry and concentration results of [8] concern fixed requested negative index and the cost of resolving isolated or concentrating negative mass. Here the bands are fixed and the requested index grows with the degree. Their approximation tools and ours have common classical ancestry; the quantifiers and receiving conclusions are different.

The matching example uses classical Christoffel–Darboux kernel asymptotics from [2, Theorem 1.2]. The cited asymptotic and Meixner–Pollaczek identities are classical. The stronger-tail bound in Section 5 is constructive rather than sharp in its constant. The subexponential obstruction in Section 6 records the classical nonquasianalytic boundary. The theta application in Section 7 concerns the original positive-width square form, whereas the distinct localization results of [9, 10] concern a correction form and retain strictly positive original Q on their constructed channels.

2 A negative space with uniform Gram reserve

Choose closed intervals $J_\ell \Subset I_\ell$ of lengths $L_\ell > 0$, centers a_ℓ, and real $\chi_\ell \in C_c^3(I_\ell)$ equal to one on J_ℓ. For a positive integer K put

$$d_\ell = \left\lfloor \frac{KL_\ell}{2\pi} \right\rfloor, \qquad D = 2\sum_\ell d_\ell.$$

For $1 \le j \le d_\ell$ use the two targets

$$\chi_\ell(x)\cos\frac{2\pi j(x-a_\ell)}{L_\ell}, \qquad \chi_\ell(x)\sin\frac{2\pi j(x-a_\ell)}{L_\ell}. \tag{4}$$

Enumerate them as $u_1,\dots,u_D$. Orthogonality over the complete period J_ℓ and measure order give, for every $v \in \mathbb{R}^D$,

$$\int \left(\sum_j v_j u_j\right)^2 d\mu \le -\lambda\|v\|_2^2, \qquad \lambda = \frac{1}{2}\min_\ell \gamma_\ell L_\ell > 0. \tag{5}$$

Indeed distinct intervals have disjoint supports; within one interval the portion outside J_ℓ contributes a nonpositive square integral. The fixed cutoff equals one on a whole period, so the negative margin does not deteriorate with D.

Let $\|u_j\|_{L^2(|\mu|)} \le M_0$ for all targets. Such an M_0 is fixed independently of K, because their absolute values are uniformly bounded and $|\mu|(\mathbb{R}) \le J$. If p_j approximate u_j with $\|p_j - u_j\|_{L^2(|\mu|)} \le \eta_D$, then Cauchy–Schwarz gives

$$\left| q_\mu\left(\sum v_j p_j, \sum v_j p_j\right) - q_\mu\left(\sum v_j u_j, \sum v_j u_j\right)\right| \le D\eta_D(2M_0 + \eta_D)\|v\|_2^2. \tag{6}$$

This follows from the operator bounds $M_0\sqrt{D}$ and $\eta_D\sqrt{D}$ for the target and error maps into $L^2(|\mu|)$. Set

$$\eta_D = \frac{\eta_0}{D}, \qquad \eta_0 = \min\left(1, \frac{\lambda}{8(2M_0+1)}\right). \tag{7}$$

For $D \ge 1$, the right side of (6) is at most $\lambda\|v\|_2^2/8$. Therefore the polynomial span remains strictly negative and has dimension D. This pays every mixed entry, including those arising after approximation.

If μ is even and the intervals are positive, replace each target by $u_j(x) + (-1)^\varepsilon u_j(-x)$, where $\varepsilon \in \{0,1\}$ is the chosen parity. The two sides have disjoint supports. The raw reserve doubles, the number of targets stays D, and only fixed norm constants change. Hence it suffices to give the common approximation in the full space, preserving parity whenever requested.

Use the Fourier convention

$$\widehat{\chi}(t) = \frac{1}{2\pi}\int \chi(x)e^{-\mathrm{i}tx}\,dx, \qquad \chi(x) = \int \widehat{\chi}(t)e^{\mathrm{i}tx}\,dt.$$

Three integrations by parts show

$$\int_{|t|>u} |\widehat{\chi}(t)|\,dt \le \frac{\|\chi'''\|_1}{2\pi u^2}, \qquad u > 0. \tag{8}$$

The Fourier transform is integrable, for instance by splitting at $|t| = 1$ and using the second derivative. Multiplication by a sine or cosine shifts the transform by carrier frequencies of absolute value at most K without increasing its total variation. Reflection changes only a fixed factor. Consequently there are constants C_0, Λ, independent of K, D and target number, such that truncation to $[-T, T]$, where

$$T = K + U\sqrt{D}, \tag{9}$$

produces real functions $f_j(x) = \int_{-T}^{T} e^{\mathrm{i}tx}\,d\Lambda_j(t)$ satisfying

$$\|\Lambda_j\|_{\mathrm{TV}} \le \Lambda, \qquad \|f_j - u_j\|_{L^2(|\mu|)} \le \frac{\sqrt{J}C_0}{U^2 D}. \tag{10}$$

Fix U large enough that this error is at most $\eta_D/4$. Symmetric frequency truncation preserves reality and the selected parity.

3 Critical exponential approximation and logarithmic inertia

Lemma 3.1 (Original-measure MP approximation)**.** *Assume (1). Put $h = \pi/c$ and $q_n(x) = P_n^{(1/2)}(x/h; \pi/2)$. If $f(x) = \int_{-T}^{T} e^{\mathrm{i}tx}\,d\Lambda_0(t)$ is real and $\|\Lambda_0\|_{\mathrm{TV}} \le \Lambda$, then there is a real polynomial of degree at most n with $L^2(|\mu|)$ error at most*

$$e\sqrt{J}\Lambda Q_n(r), \quad r = \tanh(hT/2), \quad Q_n(r) = r^{n+1}\left(\frac{n+2}{1-r} + \frac{r}{(1-r)^2}\right). \tag{11}$$

For even $|\mu|$, an assigned parity of f can be preserved.

Proof. The classical generating function [1, 18.23.7] is

$$\sum_{n=0}^{\infty} q_n(x) z^n = (1 - \mathrm{i}z)^{-1/2+\mathrm{i}x/h}(1 + \mathrm{i}z)^{-1/2-\mathrm{i}x/h}, \qquad |z| < 1, \tag{12}$$

with branches analytic at zero. For $|z| = \rho < 1$, the Cayley ratio $(1 - \mathrm{i}z)/(1 + \mathrm{i}z)$ lies in the right half-plane, its argument having absolute value at most $2 \arctan \rho < \pi/2$. Since $|1 + z^2| \geq 1 - \rho^2$, Cauchy's estimate gives

$$|q_n(x)| \leq \rho^{-n}(1 - \rho^2)^{-1/2} e^{2 \arctan(\rho)|x|/h}.$$

For $n \geq 1$, choose $\rho = n/(n + 1)$; the first two factors are at most $e\sqrt{n + 1}$. Including $q_0 = 1$, we have

$$|q_n(x)| \leq e\sqrt{n + 1}\, e^{c|x|/2}. \tag{13}$$

Substitution $z = \mathrm{i} \tanh(ht/2)$ gives

$$e^{\mathrm{i}tx} = \operatorname{sech}(ht/2) \sum_{n=0}^{\infty} q_n(x)[\mathrm{i} \tanh(ht/2)]^n. \tag{14}$$

Hermitian symmetrization of the Fourier measure preserves f and its variation bound. Integrating the coefficients gives real coefficients bounded in modulus by Λr^n. The weighted-uniform tail is bounded by $e\Lambda \sum_{j>n}(j + 1)r^j = e\Lambda Q_n(r)$, using (13). This summable bound also justifies integration of the series. Squaring and integrating its exponential envelope proves (11) on the original absolute measure, including singular measures. For even $|\mu|$, parity projection is contractive and does not increase degree or error. □

Lemma 3.2 (Rational coefficients)**.** *A real polynomial can be approximated to arbitrary positive $L^2(|\mu|)$ error by a rational polynomial of the same degree. Under (1), only c, J and the coefficients are needed for this rounding. An assigned parity can be preserved.*

Proof. The exponential series gives $\int |x|^{2j}\, d|\mu| \leq J(2j)!/c^{2j}$. For coefficient changes δ_j, the norm error is therefore at most

$$\sqrt{J} \sum_{j=0}^{n} |\delta_j| \frac{\sqrt{(2j)!}}{c^j}.$$

Approximate the finitely many allowed coefficients by rationals to make this sum smaller than the desired tolerance. □

Proof of the lower bound in Theorem 1.1. Use the targets of Section 2 and fix all their constants before K grows. Write $H = hT$ and define

$$A = \max\left(1, \frac{4e\sqrt{J}\Lambda}{\eta_D}\right), \quad S = 4 + 2H + 2 \log A, \quad n = \left\lceil e^H S^2 \right\rceil. \tag{15}$$

Since $1 - r = 2/(e^H + 1) \geq e^{-H}$, $r^{n+1} \leq e^{-S^2}$ and $n + 2 \leq 4e^H S^2$. Thus

$$Q_n(r) \leq (4S^2 + 1)e^{2H - S^2} \leq e^{2H - S^2/2} \leq A^{-1}.$$

Here $\log(4S^2 + 1) \leq S^2/2$ for $S \geq 4$, and $S^2/2 \geq 2H + \log A$. Lemma 3.1 approximates every f_j with error at most $\eta_D/4$. Round with additional error at most $\eta_D/2$ using Lemma 3.2. Together with (10), the total per-target error is at most η_D. Equations (5)–(6) prove strict negativity on a D-dimensional rational polynomial span.

For fixed inner intervals, put $L' = \sum L_\ell$. Then

$$D = \frac{L'}{\pi}K + O(1), \qquad T = K + O(\sqrt{K}), \qquad \log n(K) = hK + O(\sqrt{K}) + O(\log K). \tag{16}$$

Indeed $\log A = O(\log K)$ because $\eta_D^{-1} = O(D)$. No exponential dependence on D is hidden in a fixed norm or cutoff constant. The degree choice is eventually increasing; alternatively its running maximum has the same asymptotic. For $n(K) \le N < n(K+1)$, monotonicity gives $i_-(\mu, N) \ge D(K)$, while consecutive logarithmic degree costs have ratio tending to one. Hence the lower limit for all integer N is at least $L'/(\pi h)$. Let the fixed inner lengths increase to the original lengths after taking the lower limit. The parity construction already preserves its target count and proves the stated per-parity version. □

Corollary 3.3 (Open negative sets)**.** *If $\mu = g(x)dx$ is even with continuous real g and satisfies (1), then in each parity*

$$\liminf_{N\to\infty} \frac{i_-(\mu, N, \varepsilon)}{\log N} \ge \frac{c}{\pi^2}|\{x > 0 : g(x) < 0\}|.$$

The right side may be infinite.

Proof. Exhaust the open negative set by finite unions of compact inner intervals. On each such interval continuity supplies a negative upper bound. Apply Theorem 1.1 with its constants fixed during the limit, then take the supremum of the total interval lengths. □

4 Sharpness from classical local kernels

Let $w(x) = \operatorname{sech}(\pi x)$, a probability density, and fix $a > 0$. Consider

$$d\mu_a(x) = (x^2 - a^2)w(x)\,dx. \tag{17}$$

The half-parameter MP polynomials $q_j = P_j^{(1/2)}(x; \pi/2)$ are orthonormal for $w(x)dx$ and satisfy the classical recurrence [1, Sections 18.19 and 18.22]

$$xq_j = a_j q_{j+1} + a_{j-1}q_{j-1}, \qquad a_j = (j+1)/2, \quad a_{-1} = 0.$$

Let J_N be the $(N+1)$-dimensional Jacobi compression and e_N its last coordinate vector. The original signed Gram matrix is exactly

$$A_N = J_N^2 - a^2 I + a_N^2 e_N e_N^T. \tag{18}$$

This is the full Gram of xP: the single q_{N+1} component missing from the finite compression contributes precisely the last rank-one term. Since it is positive,

$$|i_-(\mu_a, N) - \#\{x \in (-a, a) : q_{N+1}(x) = 0\}| \le 1. \tag{19}$$

Strict endpoints are not counted.

For clarity we state and specialize the external asymptotic used here. For these MP polynomials, Theorem 1.2 of [2] gives

$$\frac{K_N(x,x)}{\log N} \longrightarrow \frac{1}{\pi w(x)}, \qquad K_N(x,x) = \sum_{j=0}^{N} q_j(x)^2, \tag{20}$$

locally uniformly on $\mathbb{R}$. Its period-one reference coefficients are $\alpha_j = 1, \beta_j = 0$. The relevant ratio and reciprocal sequences are $a_{n-1}/a_n = n/(n+1)$, $b_n/a_n = 0$, and $1/a_n = 2/(n+1)$. They are bounded and have summable first differences, as required by the Stolz D_1 hypothesis. The Carleman sum diverges and the limiting transfer has trace zero, strictly in $(-2, 2)$. The reference equilibrium density at zero is $1/(2\pi)$ and the theorem's clock is $\rho_N = \sum_{j=0}^{N} 1/a_j = 2H_{N+1} \sim 2\log N$. These observations give exactly (20); the source also explicitly includes the MP case after that theorem.

Lemma 4.1 (Kernel asymptotics to local zero counts)**.** *Let orthonormal polynomials for a moment-determinate probability density $w > 0$ have Gaussian quadrature measures ν_N on the zeros of q_{N+1}. If $s_N \to \infty$ and $K_N(x,x)/s_N \to d/w(x)$ locally uniformly, with a constant $d > 0$ and continuous w, then for every fixed bounded interval I,*

$$\frac{\#\{x \in I : q_{N+1}(x) = 0\}}{s_N} \longrightarrow d|I|.$$

Proof. The positive quadrature weights are $1/K_N(x_j, x_j)$ and their sum is one. The quadrature measure matches moments through degree $2N+1$. Its eventually fixed second moment gives tightness. Every fixed higher even moment is eventually exact and bounds uniform integrability of each lower monomial. Thus any weak subsequential limit has all the moments of w; determinacy identifies it as $w(x)dx$. Hence $\nu_N \Rightarrow w(x)dx$. At every quadrature atom the exact identity gives

$$\frac{\#\{x_j \in I\}}{s_N} = \int_I \frac{K_N(x,x)}{s_N}\, d\nu_N(x).$$

The uniform kernel error on $\overline{I}$ integrates to $o(1)$ because ν_N has total mass one. The remaining integrand $d\,1_I/w$ is bounded on its compact support and has only endpoint discontinuities, of zero limiting $w(x)dx$ mass. Weak convergence proves the conclusion. □

Exponential moments imply moment determinacy in the MP example. Applying Lemma 4.1 to (20), then (19), proves

$$i_-(\mu_a, N) \sim \frac{2a}{\pi}\log N, \qquad i_-(\mu_a, N, \varepsilon) \sim \frac{a}{\pi}\log N. \tag{21}$$

For the parity assertion, the zero-diagonal J_N is bipartite between even and odd degrees. The nonzero squared singular values in its two blocks occur in pairs; the block dimensions differ by at most one. Thus their negative indices for $J_N^2 - a^2 I$ differ by at most one. The boundary addition in (18) affects only one block and changes its index by at most one. Each original parity index is therefore half the full index up to $O(1)$.

For general $\tau > 0$ replace w by $w_\tau(x) = (\tau/\pi)\,\mathrm{sech}(\tau x)$. Scaling $t = \tau x/\pi$ gives

$$i_-((x^2 - a^2)w_\tau dx, N) \sim \frac{2a\tau}{\pi^2}\log N, \tag{22}$$

and half that coefficient in each parity. Its absolute exponential moment at rate c is finite precisely when $c < \tau$.

Completion of the sharpness assertion in Theorem 1.1. The density in (22) is negative on $(-a, a)$ and has a uniform negative upper bound on every fixed inner interval. Fix an admissible moment rate c, then let $\tau \downarrow c$ and the inner interval length approach $2a$. The ratio of the model's index coefficient to its negative-band length approaches c/π^2. Any larger universal coefficient would contradict one of these models after choosing its fixed parameters sufficiently close to these limits. Each model's absolute-moment estimates hold in its stated range. The comparison does not hold the budget J, band margin or onset fixed across the model family. □

5 Stronger tails and growing power index

Theorem 5.1. *Let $p > 1$, $c > 0$, and $J_p = \int e^{c|x|^p}\, d|\mu|(x) < \infty$. Under (2), put $\alpha = 1 - 1/p$ and $L = \sum |I_\ell|$. Then*

$$\liminf_{N\to\infty} \frac{i_-(\mu, N)}{N^{1-1/p}} \geq \frac{L}{\pi}\kappa_{p,c}, \qquad \kappa_{p,c} = e^{-1+1/p}(cp/2)^{1/p}. \tag{23}$$

The same assertion holds in each parity for even μ, counting only positive-side interval lengths. Bases may have rational coefficients. The displayed constant is a constructive lower coefficient.

Proof. For real x, t, the real-segment integral Taylor remainder yields

$$\left| e^{\mathrm{i}tx} - \sum_{j=0}^{N} \frac{(\mathrm{i}tx)^j}{j!} \right| \leq \frac{|tx|^{N+1}}{(N+1)!}.$$

Set $n = N + 1$. Maximizing $u^{2n}e^{-cu^p}$ gives

$$\int |x|^{2n}\, d|\mu| \leq J_p \left(\frac{2n}{cpe}\right)^{2n/p}.$$

Since $n! \geq (n/e)^n$, the original $L^2(|\mu|)$ Taylor error is bounded by

$$\sqrt{J_p}\left(\frac{B_{p,c}|t|}{n^\alpha}\right)^n, \qquad B_{p,c} = \kappa_{p,c}^{-1}. \tag{24}$$

Minkowski integration gives an additional factor at most Λ for a real Fourier measure of variation Λ supported in $[-T, T]$. Hermitian symmetry makes the integrated polynomial real.

Use the same fixed-period targets and common Fourier truncations as in Section 2; their total mass estimates use $|\mu|(\mathbb{R}) \leq J_p$. Fix any $0 < \kappa < \kappa_{p,c}$ and set $K = \kappa(N+1)^\alpha$ (integer rounding has no effect). Then $D = KL'/\pi + O(1)$ and $T = K + O(\sqrt{K})$, so

$$\frac{B_{p,c}T}{(N+1)^\alpha} \longrightarrow \frac{\kappa}{\kappa_{p,c}} < 1.$$

Thus for some fixed $r < 1$ and every sufficiently large integer N, the common Taylor error is at most $\Lambda\sqrt{J_p}r^{N+1}$, eventually smaller than $\eta_D/4$. The Fourier error is at most $\eta_D/4$. Rational rounding of the same-degree coefficients costs at most $\eta_D/2$, because

$$\left\| \sum_j \delta_j x^j \right\|_{L^2(|\mu|)} \leq \sqrt{J_p} \sum_j |\delta_j| \left(\frac{2j}{cpe}\right)^{j/p},$$

where the $j = 0$ factor is one. The complete Gram estimate therefore preserves all D negative directions. Reality and parity are retained, as before. Divide D by N^α, then let κ increase to $\kappa_{p,c}$ and the fixed inner interval lengths increase to the original lengths after the lower limit. This proves (23). □

Proposition 5.2 (Optimal Gaussian power)**.** *For $w_\tau(x) = \sqrt{\tau/\pi}e^{-\tau x^2}$ and $a > 0$,*

$$i_-((x^2 - a^2)w_\tau dx, N) \sim \frac{2a\sqrt{2\tau}}{\pi}\sqrt{N}.$$

In each parity the coefficient is half this value. Hence the power $\sqrt{N}$ in Theorem 5.1 cannot be increased universally for $p = 2$.

Proof. The orthonormal Hermite recurrence has $a_n = \sqrt{(n+1)/(2\tau)}$, $b_n = 0$. The original signed compression is again exactly (18). The same period-one specialization of [2, Theorem 1.2] applies: the ratio $\sqrt{n/(n+1)}$ and reciprocal $\sqrt{2\tau/(n+1)}$ are bounded monotone sequences with summable first differences, the limiting trace is zero and Carleman's sum diverges. The clock is $\rho_N \sim 2\sqrt{2\tau N}$, yielding locally uniformly

$$\frac{K_N(x,x)}{\sqrt{N}} \longrightarrow \frac{\sqrt{2\tau}}{\pi w_\tau(x)}.$$

Gaussian moment determinacy and Lemma 4.1 give the local zero count. The rank-one boundary comparison and bipartite parity argument prove the assertion. The absolute exponential-square moment is finite for $c < \tau$, including the extra quadratic density factor. □

At $p = 2$ the constructive coefficient in (23) is $\sqrt{c/e}$, whereas the model coefficient approaches $\sqrt{2c}$ per unit-length factor as $\tau \downarrow c$. We prove optimality of the power, not of the Taylor constant.

6 The subexponential boundary

The following is the classical Fourier-gap, nonquasianalytic obstruction, included to delimit the preceding receiving theorems. Compact Gevrey bump constructions and their Fourier decay are standard, for example [3, Appendix A].

Proposition 6.1. *For every $0 < p < 1$ there is a real even continuous signed density g negative on an open interval such that*

$$\int e^{c|x|^p} |g(x)|\, dx < \infty \quad \text{for every } c > 0, \qquad \int P(x)^2 g(x)\, dx > 0$$

for every nonzero real polynomial P. Thus its polynomial negative index is zero at every degree.

Proof. Choose $p < r < 1$ and $\beta = r/(1-r)$. The standard bump $\exp[-(1-t^2)^{-\beta}]$ on $(-1,1)$, extended by zero, has Gevrey order $1/r$. Here is the relevant estimate. Near an endpoint at distance d, a complex Cauchy circle of radius εd, with ε fixed sufficiently small depending on β, gives

$$|\eta_0^{(n)}(t)| \le n!(\varepsilon d)^{-n} e^{-c_0 d^{-\beta}}.$$

Maximizing in d gives $CA^n(n!)^{1+1/\beta} = CA^n(n!)^{1/r}$. In the interior a fixed analytic disk suffices. Each fixed derivative vanishes at the endpoints, so extension by zero is smooth. Translate, rescale and reflect the bump to obtain a nonzero nonnegative real even η supported in $[-2,-1] \cup [1,2]$, with the same Gevrey order. Its inverse Fourier transform v is real and even. Repeated integration by parts gives $|v(x)| \le CA^n(n!)^{1/r}|x|^{-n}$; choosing n comparable to $(|x|/(2A))^r$ proves

$$|v(x)| \le Ce^{-b|x|^r}.$$

Consequently v has all the stated absolute p-moments, and Fourier differentiation shows that every polynomial moment of v vanishes, because η vanishes near zero. On the other hand $v(0) > 0$ and $\int v = 0$, so continuity and integrability force $v < 0$ on a nonempty open interval. Take $g = v + \epsilon e^{-x^2}$ with $\epsilon > 0$ sufficiently small to preserve a negative value. All moments of v vanish, and hence

$$\int P^2 g = \epsilon \int P^2 e^{-x^2} > 0$$

for every nonzero real polynomial P. □

7 A conditional receiver for the original theta form

Let

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad F(z) = \tfrac{1}{4}\xi((1+\mathrm{i}z)/2).$$

The even entire function F is real on the real line. For $y > 0$ define

$$g_y(x) = \frac{\partial_y |F(x+\mathrm{i}y)|^2}{8y}, \qquad Q_y(P) = \frac{1}{2\pi}\int_{\mathbb{R}} P(x)^2 g_y(x)\,dx. \tag{25}$$

The point of this application is quantitative original-square inertia after a negative interval is supplied. The qualitative modulus/zero strategy is classical; see Sondow–Dumitrescu [4], as well as the attribution and half-plane discussion in [5].

We use the following previously established closed-strip envelope from the complete physical-form estimates of [7]:

$$|g_y(x)| \le 21200(1+|x|/2)^7 e^{-\pi|x|/4}, \qquad 0 \le y \le 1. \tag{26}$$

At $y = 0$ the density is defined by its removable even limit. To identify the constant and normalization, the underlying common strip bounds for F, F', F'' have constants $28, 200, 1600$; differentiation gives $|\partial_y^2|F|^2| \le 2(200^2 + 28\cdot 1600) = 169600$ times the same seventh-power exponential envelope. Evenness in y gives $\partial_y|F|^2 = 0$ at zero and divides this bound by eight in (25). This explains (26); its global analytic theta estimates are an inherited input. In particular for $a_0 = \pi/4$ and $0 < c < a_0$, a conservative absolute moment budget for $g_y(x)dx/(2\pi)$ is

$$J_c \le 42400 \sum_{j=0}^{7} \binom{7}{j} \frac{j!}{2^j (a_0 - c)^{j+1}}. \tag{27}$$

Corollary 7.1 (Supplied negative theta intervals)**.** *Fix $0 < y < 1$. If $g_y < 0$ on finitely many disjoint bounded positive-side intervals I_ℓ, then, in each polynomial parity,*

$$\liminf_{N\to\infty} \frac{i_-(Q_y, N, \varepsilon)}{\log N} \ge \frac{1}{4\pi}\sum_{\ell} |I_\ell|. \tag{28}$$

The negative spaces may have rational polynomial bases.

Proof. The density is even and continuous. First take compact inner intervals with a uniform strictly negative bound. Apply Theorem 1.1 to the original measure $g_y(x)dx/(2\pi)$, using (27). Exhaust the interval lengths, then let $c \uparrow \pi/4$ after the N lower limit. Positive scalar normalization does not alter inertia. □

Proposition 7.2 (Conditional zero-to-band transfer)**.** *Suppose F has a zero $z_0 = a + \mathrm{i}b$, where $a > 0$ and $0 < b < 1$, of multiplicity $m \ge 1$. Write $F(z) = (z - z_0)^m H(z)$, and choose $\rho > 0$ so that H is nonzero on $|z - z_0| \le \rho$. Set*

$$h_* = \min_{|z-z_0|\le\rho} |H(z)| > 0, \qquad B = \max_{|z-z_0|\le\rho} 2|H'(z)/H(z)|.$$

Choose $\delta_0 > 0$ satisfying

$$\delta_0 \le \frac{b}{2}, \quad \delta_0 \le \frac{m}{2(B+1)}, \quad \delta_0 \le \frac{(B+1)\rho^2}{2m}, \quad \delta_0 \le \frac{(B+1)a^2}{2m}. \tag{29}$$

For every fixed $0 < \delta < \delta_0$*, put*

$$y = b - \delta, \qquad w = \sqrt{\frac{m\delta}{2(B+1)}}.$$

Then $I = (a - w, a + w) \subset (0, \infty)$ *and throughout* I

$$g_y(x) \le -\frac{mh_*^2\delta^{2m-1}}{8b} < 0. \tag{30}$$

Consequently, in each parity,

$$\liminf_{N\to\infty} \frac{i_-(Q_{b-\delta}, N, \varepsilon)}{\log N} \ge \frac{1}{2\pi}\sqrt{\frac{m\delta}{2(B+1)}}. \tag{31}$$

Proof. Write $x = a + u$ and $s = u^2 + \delta^2$. The bounds imply $w \le a/2$ and $s \le m\delta/(B+1) \le \rho^2$. Complete differentiation, retaining the residual function, gives

$$\partial_y |F(x + \mathrm{i}y)|^2 = |H|^2 s^{m-1}\left[-2m\delta + s\partial_y \log |H|^2\right].$$

Since $|\partial_y \log |H|^2| \le B$, the bracket is at most $-m\delta$. Also $s \ge \delta^2$ and $|H| \ge h_*$, so the derivative is at most $-mh_*^2\delta^{2m-1}$. Division by $8y$, with $y < b$, proves (30). Evenness supplies the reflected band, and Corollary 7.1 applied to the positive-side length $2w$ proves (31). □

No off-axis zero or negative interval of the actual theta density is supplied by these statements. In particular the conditional conclusion is not a counterexample to the Riemann hypothesis. H retains every other zero and all residual variation; neither simplicity nor a truncation of the complete theta function is assumed. The construction has no uniform onset as $\delta \downarrow 0$, and gives no joint δ–degree rate. The endpoint correction-inertia results of [9, 10] are not scalar moment-form negativity statements and are not inserted as negative-band hypotheses.

8 Reproduction and scope

The accompanying standard-library Python program `verify_mp_grams.py` checks finite original MP moment matrices against their full Jacobi-square-plus-boundary representation. It computes moments from the exact generating function $\sec(t/2)$, uses rational congruence pivots to compute inertia, checks the rank-one interlacing bound and checks both parity blocks. The prescribed run contains 65 full cases and 130 parity blocks. These exact finite checks support the normalization and boundary identity, not the asymptotic kernel theorem, arbitrary-measure lower law or actual-theta negativity. The reader instructions include ordinary and optimized Python commands; acceptance uses explicit exceptions and does not depend on assertions.

The critical coefficient is sharp under the coarse absolute-tail and negative-band hypotheses. It is not a complete asymptotic eigenvalue distribution. Certified computable band, cutoff and moment-budget inputs can make the finite construction effective, but that statement is distinct from a uniform computational complexity theorem.

Assistance disclosure

AI tools assisted mathematical development, drafting, literature comparison and internal checking. The author is responsible for the statements and presentation.

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