Polynomial cores and inertia under Carleman moment bounds

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Abstract

We establish a universal threshold for polynomial approximation in a joint graph norm containing a holomorphic product derivative on a full horizontal strip, an anchor value and an original real-axis observation. For a positive log-convex moment envelope m_n, the divergence of $\sum_{n\geq 1} m_{n-1}/m_n$ guarantees a polynomial core under finite-order divisor hypotheses. When this sum converges, real-zero entire functions of order one satisfy the same envelope for their full graph and their own physical observation weights but fail to have a polynomial core. A separate signed countermodel has a negative interval while every nonzero polynomial square has positive integral. The core theorem also yields same-width synthesis for the actual Riemann ξ-function at every positive strip width, retaining every interior zero multiplicity and permitting boundary zeros. Uniform physical-potential and normalized substrip-energy convergence follow. Quantitative finite-moment graph estimates and signed-measure inertia bounds give whole negative polynomial spaces on logarithmic and iterated-logarithmic scales, with matching finite-measure Jacobi models. A double-exponential spatial kernel realizes the joint-core and negative-inertia phenomena with its actual physical weight. Classical quasianalyticity, cyclicity, Fourier support and orthogonal-polynomial results supply the underlying tools.

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1 Polynomial approximation and negative inertia

Moment bounds can control polynomial approximation without furnishing a positive quadratic form. Conversely, real zeros and positive physical potentials need not imply polynomial completeness in a joint holomorphic norm. We make these distinctions quantitative for a joint approximation norm and a real quadratic form.

For $y > 0$ put $S_y = \{z \in \mathbb{C} : |\Im z| < y\}$. Given a holomorphic divisor G, an anchor $z_* \in S_y$ and a finite positive Borel measure μ on $\mathbb{R}$, the first problem uses the graph

$$\mathbf{T}_G(f) = \big(G(z_*)f(z_*), (Gf)', f|_{\mathbb{R}}\big) \in \mathcal{H} := \mathbb{C} \oplus L^2(S_y, dA) \oplus L^2(\mu). \tag{1}$$

The polynomials are ordinary polynomials in the single complex coordinate z. Approximating only on the real axis, in a smaller strip, or in a conformal disk coordinate would give different conclusions.

The combined positive horizontal measure is

$$d\nu(x) = d\delta_0(x) + d\mu(x) + \left[\int_{-y}^{y} (|G(x+iv)|^2 + |G'(x+iv)|^2)\, dv\right] dx, \qquad M_n = \left[\int (1+|x|)^{2n} d\nu\right]^{1/2}. \tag{2}$$

The atom makes $M_n \geq 1$; the sequence is nondecreasing and log-convex.

Our universal envelope theorem, Theorem 5.1, identifies the exact threshold for a guarantee across all admissible divisors. For a positive log-convex sequence $m_0 = 1$, divergence of $\sum m_{n-1}/m_n$ implies the core conclusion whenever the actual graph moments satisfy $M_n \leq BC^n m_n$. Convergence admits order-one, real-zero entire countermodels with this full graph envelope and their own physical weights. The same dichotomy distinguishes unbounded negative polynomial index from signed negative-band countermodels that are positive on every polynomial square. The quantifier ranges over every admissible divisor satisfying the envelope.

Corollary 6.1 applies the core theorem to the actual function $F(z) = \xi((1+iz)/2)/4$ at every $y > 0$. It removes the restriction $y > 1$ in the preceding same-boundary theorem of [14], using its fixed-strip analytic estimates. Interior zero jets are prescribed, boundary zeros are permitted, and no extension of the target beyond the given strip is needed.

The supporting fixed-divisor core result, Theorem 3.3, states that if G is a nonzero finite-order entire function, G, G' are bounded on the closed strip and $\sum_{n\geq 1} M_n^{-1/n} = \infty$, then every holomorphic U for which U/G is holomorphic and

$$|U(z_*)|^2 + \int_{S_y} |U'|^2 + \int_{\mathbb{R}} |U/G|^2 d\mu < \infty$$

is a limit of GP in all three components. This is a full same-width zero-jet core, including boundary widths containing zeros. Its finite moment counterpart is Theorem 2.2.

The second problem concerns an even finite real signed measure σ and its *original* real square form

$$q_\sigma(P,Q) = \int PQ\, d\sigma, \qquad i_-(\sigma, N) = \operatorname{ind}_-(q_\sigma|_{\mathcal{P}_N(\mathbb{R})}).$$

For $\varepsilon \in \{0,1\}$ define

$$\mathcal{P}_N^\varepsilon = \{P \in \mathcal{P}_N(\mathbb{R}) : P(-x) = (-1)^\varepsilon P(x)\}, \qquad i_-(\sigma, N, \varepsilon) = \operatorname{ind}_-(q_\sigma|_{\mathcal{P}_N^\varepsilon}).$$

Negative index means strict negativity on the whole nonzero real span, not a list of negative diagonal entries. Suppose an even positive finite measure ω dominates $|\sigma|$, and the entire restriction of σ to a supplied positive-side interval is at most $-\eta\, dx$. For the scale

$$m_0(\alpha) = 1, \qquad m_n(\alpha) = n! \prod_{j=1}^{n} \log(e+j)^\alpha, \tag{3}$$

Theorem 7.2 gives, under $\|x^n\|_{L^2(\omega)} \leq BC^n m_n(\alpha)$,

$$i_-(\sigma, N, \varepsilon) \geq \begin{cases} c_\alpha (\log N)^{1-\alpha}, & 0 \leq \alpha < 1, \\ c_1 \log\log N, & \alpha = 1, \end{cases}$$

in both polynomial parities for all sufficiently large N. The bases may be rational. Theorem 8.1 matches these orders in the finite signed-measure class.

The exponent-one boundary gives the threshold for this universal guarantee. Section 4 constructs countermodels for every $\alpha > 1$. Section 9 treats $\phi(s) = \exp[-Ae^{s^2} + Bs^2]$: its actual physical observation has $n!\log(n)^{n/2}$ moment growth, while no positive exponential Fourier envelope exists. The same fixed all-depth example therefore demonstrates why the two mathematical problems above must be distinguished.

Related work. Carleman's polynomial density and the annihilator method based on quasi-analyticity are classical; de Jeu's equal-closure results give especially close general ancestry [3, 4]. The finite propagation used here is Bang's level-crossing argument [9]. Koehler–Wu [8] develop quantitative Carleman approximation and Fourier projection-residual duality. The author's current integrated-clock paper [13] already contains the scalar terminal-clock estimate, logarithmic and iterated-logarithmic thresholds and fixed-law masking.

The bounded quotient construction uses classical Dirichlet clipping and cyclicity [10, 5]; complete-Pick multiplier methods provide another route [1, 7]. The extension established here is to the ordinary-z, full-strip product-derivative and original observation norm. The signed-band paper [12] already proves the complete-period packet construction, full mixed-Gram reserve, rational/parity passage and exponential-moment logarithmic benchmark. Our weaker moment hierarchy receives that construction through the existing scalar clock. Its matching models specialize the classical Świderski–Trojan kernel theorem [15] and retain the original rank-one boundary term. The beyond-threshold convolution mechanism has classical Ingham ancestry [6, 2]; the additional statements control the specified joint norm and the divisor's own physical weight.

Part I. Joint polynomial cores and their threshold

The finite moment estimate supplies the quantitative input. The full-strip core leads to the universal log-convex envelope dichotomy and the same-width application to the actual Riemann function; the logarithmic family provides an explicit threshold model.

2 Finite moment bounds for the complex graph

We first isolate the classical finite propagation input. All Hilbert inner products below are linear in the first argument.

Lemma 2.1 (Finite Bang propagation)**.** *Let $f \in C^N([0,T];\mathbb{C})$, $N \geq 1$, and suppose $|f^{(j)}| \leq L_j$ for a positive log-convex sequence $L_0, \dots, L_N$. If $f^{(j)}(0) = 0$ for $j < N$, then*

$$\sum_{j=s+1}^{N} L_{j-1}/L_j > eT, \quad 0 \leq s < N \quad \Longrightarrow \quad |f(t)| \leq L_0 e^{-s} (0 \leq t \leq T).$$

Proof. Put $b_j = |f^{(j)}|/(e^j L_j)$ and $H = \max_{0 \leq j \leq N} b_j$. Then $H(0) \leq e^{-N}$. On $e^{-l} < H < e^{-(l-1)}$ only indices $j \leq l-1$ can maximize, since $b_j \leq e^{-j}$. These entries are locally Lipschitz even though $f^{(N)}$ is only continuous. Almost everywhere on that level strip,

$$|(\log H)'| \leq eL_l/L_{l-1}.$$

Indeed $|b_j'| \leq e(L_{j+1}/L_j)H$, and log-convexity bounds the active ratios by L_l/L_{l-1}. Every crossing of the strip costs at least $L_{l-1}/(eL_l)$. Before the first upper-level hit choose the last lower-level hit; these crossing intervals are disjoint. Reaching e^{-s} would require time at least $e^{-1}\sum_{l=s+1}^{N} L_{l-1}/L_l > T$, a contradiction. The modulus derivative inequality applies to complex functions. No derivative of order $N+1$ is used. This is the classical argument of [9]. □

Assume for now that G, G' are bounded and holomorphic on S_y, and μ is any finite positive Borel measure on $\mathbb{R}$. Use (2) and suppose, through order N, that

$$M_n \leq B_0 C_0^n m_n \quad (0 \leq n \leq N), \qquad B_0, C_0 \geq 1, \quad m_0 = 1, \tag{4}$$

where m_n is positive, nondecreasing and log-convex. Define

$$\begin{aligned} L &= \max(1+y, |z_*|, 1), & a &= 1+y, \\ B_g &= (|G(z_*)|+3)B_0, & \Lambda_g &= a + 2LC_0, \\ K_N &= \operatorname{span}\{\mathbf{T}_G(z^j) : 0 \le j \le N\}, & A_{s,N} &= \sum_{j=s+1}^{N} m_{j-1}/m_j. \end{aligned} \tag{5}$$

Let Π_N be orthogonal projection onto K_N, which is closed even if its coefficient Gram is singular. For $T \ge 0$ put

$$e_N(T) = \min\left(1, \frac{(\Lambda_g T)^N m_N}{N!}, \{e^{-s} : 0 \le s < N, A_{s,N} > e\Lambda_g T\}\right), \tag{6}$$

omitting the last set when empty.

Theorem 2.2 (Finite graph approximation)**.** *Under (4), for $V(k) = \mathbf{T}_G(e^{ikz})$,*

$$\sup_{|k|\le T} \|(I - \Pi_N)V(k)\|_{\mathcal{H}} \le B_g e^{aT} e_N(T). \tag{7}$$

For a finite complex measure π supported in $[-T,T]$ and $f(z) = \int e^{ikz} d\pi(k)$, the corresponding distance is at most $B_g \operatorname{TV}(\pi) e^{aT} e_N(T)$.

Proof. The exact curve is

$$V(k) = (G(z_*)e^{ikz_*}, (G' + ikG)e^{ikz}, e^{ikx}).$$

For $n \ge 1$ its area derivative is

$$e^{ikz}\left[(iz)^n(G' + ikG) + ni(iz)^{n-1}G\right]. \tag{8}$$

All these terms are needed. Since $|z| \le L(1+|x|)$ and $|e^{ikz}| \le e^{y|k|}$, the moment budget gives

$$\|V^{(n)}(k)\| \le B_g(1+|k|)e^{y|k|}(2LC_0)^n m_n, \qquad 0 \le n \le N. \tag{9}$$

At $n = 0$ sum the three component norms to obtain $B_0(|G(z_*)| + 2 + |k|)e^{y|k|}$. For $n \ge 1$ the additional n term is bounded using $n+1 \le 2^n$ and $m_{n-1} \le m_n$. Finite moments through $2N$ justify C^N strong differentiability: difference quotients at order $n-1$ are bounded on compact k intervals by a constant times $(1+|x|)^n$ in the relevant L^2 components; continuity at order n follows by dominated convergence.

For $t \ge 0$ use two smooth curves $Z_\pm(t) = e^{-at}V(\pm t)$. Leibniz, $m_j \le m_n$ and $(1+t)e^{-t} \le 1$ imply

$$\|Z_\pm^{(n)}(t)\| \le B_g \Lambda_g^n m_n \quad (0 \le n \le N).$$

This controls the full half-lines without differentiating $e^{-a|t|}$ at zero. Fix k_0 and set $r = (I - \Pi_N)V(k_0)$. If $r \ne 0$, the unit linear functional $\ell(h) = \langle h, r\rangle / \|r\|$ annihilates K_N and has $\ell(V(k_0)) = \|r\|$. Every jet $V^{(j)}(0) = \mathbf{T}_G((iz)^j)$ for $j \le N$ lies in K_N. Apply Lemma 2.1 to $\ell(Z_\pm)/B_g$ with $L_j = \Lambda_g^j m_j$, choosing the sign of k_0. It gives $\|r\| \le B_g e^{a|k_0|} e^{-s}$ under the clock in (6). The order-zero bound and the integral Taylor remainder give the other two alternatives in that minimum. This proves (7).

The compact-band curve V is continuous in $\mathcal{H}$ and bounded there. Its Bochner integral against π equals $\mathbf{T}_G(f)$; finite-measure Fubini identifies both derivative and observation components. Apply $I - \Pi_N$ and its norm bound to that integral. □

For a holomorphic H^1 target we use the norm $\|f\|^2_{H^1(S_y)} = \int_{S_y}(|f|^2 + |f'|^2)dA$. The Cauchy–Riemann equation and Plancherel give

$$\widehat{f}_v(k) = e^{-kv}\alpha(k), \qquad \|f\|^2_{H^1(S_y)} = \frac{1}{2\pi}\int_{\mathbb{R}}(1+k^2)\frac{\sinh(2yk)}{k}|\alpha(k)|^2dk. \tag{10}$$

The quotient at zero is $2y$. One obtains the representation first distributionally from $\partial_v \widehat{f}_v = -k\widehat{f}_v$; L^2 uniqueness and integration in v then give the stated identity. Let f_T truncate α to $[-T,T]$, and define

$$R_T^2 = \frac{1}{2\pi}\int_{|k|>T}(1+k^2)\frac{\sinh(2yk)}{k}|\alpha(k)|^2dk, \qquad V_T = \frac{1}{2\pi}\int_{-T}^{T}|\alpha(k)|dk.$$

The second quantity is finite by Cauchy–Schwarz. With $r_* = (y - |\Im z_*|)/2$, a valid graph-map constant is

$$C_G^2 = 2(\|G\|_\infty^2 + \|G'\|_\infty^2) + \frac{|G(z_*)|^2}{\pi r_*^2} + \frac{4\mu(\mathbb{R})}{\pi y^2}. \tag{11}$$

The product rule bounds the area component; the area mean-value inequality on a disk about z_* bounds the anchor. Disks of radius $y/2$ centered on every real x give $\sup_{\mathbb{R}}|h|^2 \le 4\|h\|^2_{L^2(S_y)}/(\pi y^2)$. Thus $\mathbf{T}_G : H^1_{\mathrm{hol}}(S_y) \to \mathcal{H}$ is bounded even when μ is singular or has unbounded density. Theorem 2.2 therefore gives the full same-width bound

$$\operatorname{dist}(\mathbf{T}_G(f), K_N) \le C_G R_T + B_g V_T e^{aT} e_N(T). \tag{12}$$

Theorem 2.3 (Qualitative polynomial approximation)**.** *If every M_n in (2) is finite and $\sum_{n\ge1} M_n^{-1/n} = \infty$, the closed span K of $\mathbf{T}_G(\mathbb{C}[z])$ contains $\mathbf{T}_G(f)$ for every $f \in H^1_{\mathrm{hol}}(S_y)$.*

Proof. On each compact k interval, (8) gives $\|V^{(n)}(k)\| \le CR^nM_n$ after absorbing $n+1 \le 2^n$. For every continuous linear functional ℓ annihilating K, all jets of $\ell(V)$ vanish at zero. The classical Denjoy–Carleman sufficient moment criterion, applied to its real and imaginary parts, makes $\ell(V(k)) = 0$ on every connected compact real interval. Its criterion is unchanged by CR^n; see [3, 4]. The annihilator characterization of a closed subspace gives $V(k) \in K$ for all real k. Compact Fourier bands of f are Bochner integrals of these vectors and hence belong to K. Only after truncating do we pass to the full target, using (10) and the bounded graph map (11). No integrability of the untruncated mode curve against α is assumed. □

We also record the scalar estimate used later. This is the existing projection–Bang theorem of [13], included to identify the precise approximation norm.

Lemma 2.4 (Scalar finite projection)**.** *Let ω be finite positive and $\|x^j\|_{L^2(\omega)} \le BC^jm_j$ for $0 \le j \le N$, with $m_0 = 1$ log-convex. If $b(x) = \int_{-T}^{T} e^{itx}d\pi(t)$, $\mathrm{TV}(\pi) \le \Lambda$, and $A_{s,N} > eCT$, then*

$$\operatorname{dist}_{L^2(\omega)}(b, \mathcal{P}_N) \le B\Lambda e^{-s}.$$

Proof. Project onto the finite-dimensional polynomial image, which is closed without an invertible Gram. For residual $r \ne 0$ put $F(t) = \int \overline{r}e^{itx}d\omega$. Its derivatives through N are bounded by $B\|r\|C^jm_j$ and its jets through N vanish. Lemma 2.1 gives $|F(t)| \le B\|r\|e^{-s}$ on $[-T,T]$, using $F(-t)$ on the negative half. Fubini and projection give $\|r\|^2 = \int F\,d\pi$; divide the bound by $\|r\|$. □

3 Full-strip zero-jet cores

The passage from an arbitrary zero-compatible target to the H^1 approximation theorem requires a bounded quotient first. The following disk lemma is a joint-topology consequence of classical Dirichlet clipping and outer cyclicity [10, 5]. We give the construction because the extra observation norm must be controlled at the same time.

Write $\mathcal{D}$ for the disk Dirichlet space with norm $\|u\|_{H^2}^2 + \pi^{-1}\int_{\mathbb{D}} |u'|^2$, and $\mathcal{D}_\infty = \mathcal{D} \cap H^\infty$. An anchored derivative norm is equivalent to this norm; at zero this follows from the power-series inequalities $\sum |a_n|^2 \le |a_0|^2 + \sum n|a_n|^2$.

Lemma 3.1 (Compatible bounded quotient cutoffs)**.** *Let $g = BO \in \mathcal{D}_\infty$ have inner factor B and outer factor O, with O cyclic in $\mathcal{D}$. Let λ be any positive Borel measure on the open disk. If $u \in \mathcal{D} \cap BH^2$ and $f = u/g \in L^2(\lambda)$, there are analytic contractions $\theta_j \to 1$ locally uniformly such that*

$$u_j = u\theta_j \in \mathcal{D}_\infty, \qquad q_j = f\theta_j \in \mathcal{D}_\infty, \qquad \|u_j - u\|_{\mathcal{D}} + \|q_j - f\|_{L^2(\lambda)} \to 0.$$

In particular $|q_j| \le |f|$. No Carleson or trace inequality for λ is assumed.

Proof. For an H^2 inner–outer factorization $v = Io$, let $C_M v = Io_M$ be analytic magnitude clipping, with normalized outer factor having boundary modulus $\min(|o^*|, M)$. The classical local Dirichlet formulas give

$$\|C_M v\|_{\mathcal{D}} \le \|v\|_{\mathcal{D}}, \qquad \|C_M v\|_\infty \le M. \tag{13}$$

To recall why, the outer energy is the nonnegative boundary double integral with integrand $(e^{2a} - e^{2b})(a-b)$, $a = \log|o^*(\zeta)|$, $b = \log|o^*(\eta)|$. Truncating both at $\log M$ reduces both ordered differences. The positive inner contribution, weighted by the outer modulus squared, also decreases, as does the H^2 norm. The same formula puts the outer maximum $o \vee M$ in $\mathcal{D}$. These are the cutoff facts in [10, Sections2 and5]. The quotient $(C_M v)/v$ is an analytic contraction. Its logarithm is the negative outer tail $(\log|o^*| - \log M)_+$, so it tends locally uniformly to one. Local convergence, weak compactness and (13) imply strong $\mathcal{D}$ convergence to v. Inner division decreases the Dirichlet integral and preserves the H^2 norm whenever the quotient is in H^2.

First clip $u_M = C_M u = u\chi_M$. Its inner factor still contains B, so $h_M = u_M/B \in \mathcal{D}_\infty$. We have $u_M \to u$ in $\mathcal{D}$ and $f_M = f\chi_M \to f$ in the given $L^2(\lambda)$ by dominated convergence. For the outer divisor choose polynomials p_n with $v_n = Op_n \to 1$ in $\mathcal{D}$. If o_n is the normalized outer factor of v_n, set

$$a_n = o_n \vee 1, \qquad \varphi_n = C_1 v_n = v_n/a_n.$$

Here $a_n \in \mathcal{D}_\infty$, $|a_n| \ge 1$ in the disk and $1/a_n \in \mathcal{D}_\infty$ by differentiation of the inverse. Consequently $\varphi_n/O = p_n/a_n \in \mathcal{D}_\infty$. Moreover $\|\varphi_n\|_\infty \le 1$ and $D(\varphi_n) \le D(v_n) \to 0$. To see its local limit, use $\log^+|v_n^*| \le |v_n^* - 1|$ on the circle. Its L^1 norm tends to zero because $v_n \to 1$ in H^2. Thus $a_n \to 1$ and $\varphi_n \to 1$ locally uniformly, and then strongly in $\mathcal{D}$.

For fixed M put

$$u_{M,n} = u_M\varphi_n, \qquad q_{M,n} = h_M p_n/a_n = f_M\varphi_n.$$

Both are in the algebra $\mathcal{D}_\infty$. In $(u_M(\varphi_n - 1))' = u_M'(\varphi_n - 1) + u_M\varphi_n'$, the first term tends to zero by dominated convergence and the second by $\|u_M\|_\infty\|\varphi_n'\|_2 \to 0$. The anchor converges locally. The observation follows by dominated convergence against $|f_M|^2 d\lambda$. Choose M first and then n; the products $\theta_j = \chi_M\varphi_n$ are contractions, yielding the claim. □

Lemma 3.2 (Finite order excludes strip-end singular factors)**.** *Let G be nonzero entire of finite order, with G, G' bounded on $\overline{S}_y$ and $G, G' \in L^2(S_y)$. The bounded disk Dirichlet function $g = G \circ \tau^{-1}$, $\tau(z) = \tanh(\pi z/(4y))$, has pure Blaschke inner factor and cyclic outer factor.*

Proof. Bounded G' and finite strip L^2 norm force uniform decay of G at both horizontal ends. Otherwise points with $|G| \geq \epsilon$ give disjoint disks of a fixed radius, on whose intersections with the strip the Lipschitz bound makes $|G| \geq \epsilon/2$; their areas have a fixed positive lower bound, contradicting integrability. The same reasoning applies at boundary-adjacent points. Thus g is continuous on the closed disk, with value zero at its two ends. At every other boundary point G continues analytically, so its boundary zero set E is countable compact, accumulating only at the two ends.

We establish a finite-order lower bound on each real end. Choose integer $p > \operatorname{ord} G$ and σ with $\operatorname{ord} G < \sigma < p + 1$. Hadamard factorization gives

$$G(z) = z^m e^{Q(z)} \prod_j E_p(z/a_j), \quad E_p(w) = (1-w)\exp\left(\sum_{l=1}^{p} w^l/l\right),$$

with $\deg Q \leq p$ and zero count $n(R) \leq C(1+R)^\sigma$. For large R remove from $[R, R+1]$ the disks of radius R^{-H} about zeros $|a_j| \leq 4R$, where $H > \sigma + 1$. Their total real length is $O(R^{\sigma - H}) < 1$. Choose an unremoved x_R. For near zeros, with $a_* = \inf_j |a_j| > 0$,

$$\log|1 - x_R/a_j| \geq -(H+1)\log R - C, \qquad \Re \sum_{l=1}^{p} (x_R/a_j)^l/l \geq -C_p(1 + R/a_*)^p.$$

Summing bounds the near product by a finite-power loss $O(R^{p+\sigma} + R^\sigma \log R)$. For the far zeros, $|x_R/a_j| < 1/2$ and

$$\sum_{|a_j|>4R} \log|E_p(x_R/a_j)| \geq -C_p R^{p+1} \sum_{|a_j|>4R} |a_j|^{-p-1} \geq -CR^\sigma.$$

Stieltjes integration of the zero-count bound proves the last inequality. The exponential prefactor costs at most CR^p. Hence $|G(x_R)| \geq e^{-CR^{H_0}}$ for some finite H_0; using $[-R-1, -R]$ gives the negative-end bound. If there are no nonzero zeros, the exponential polynomial factor alone proves it.

In the canonical factorization $g = BSO$, a singular inner measure is supported on E, since off E the function extends and stays nonzero. A measure on countable E is atomic. An atom at a finite boundary point would force decay $e^{-c/(1-r)}$, contradicting the finite analytic order of vanishing there. An end atom would force $|G(x)| \leq C\exp[-c\exp(\pi|x|/(2y))]$ on an entire real ray, contradicting the good-point lower bound. Thus S is constant. Classical inner division puts O in $\mathcal{D}_\infty$. Since $|O| \geq |g|$ in the disk, its closed liminf-zero set is contained in E. The countable BS-exceptional-set cyclicity theorem [5, Section7], with the Dirichlet factorization facts of [10], makes O cyclic. These classical inputs concern exactly the bounded outer factor just obtained. □

Theorem 3.3 (Full same-width joint core)**.** *Let G be a nonzero entire function of finite order, with G, G' bounded on $\overline{S}_y$. Let μ be finite positive on $\mathbb{R}$ and suppose the moments (2) are all finite with $\sum M_n^{-1/n} = \infty$. Fix z_* with $G(z_*) \neq 0$. For every holomorphic U on S_y such that $f = U/G$ is holomorphic and*

$$|U(z_*)|^2 + \int_{S_y} |U'|^2 dA + \int_{\mathbb{R}} |f|^2 d\mu < \infty,$$

there are ordinary polynomials P_n satisfying

$$|G(z_*)P_n(z_*) - U(z_*)|^2 + \int_{S_y} |(GP_n - U)'|^2 + \int_{\mathbb{R}} |P_n - U/G|^2 d\mu \longrightarrow 0. \tag{14}$$

Conversely every such graph limit retains all interior zero jets of G. The assertion is for the original full strip, including widths whose boundary contains zeros.

Proof. The zeroth moment and Lemma 3.2 supply a pure Blaschke inner factor B and cyclic outer factor after conformal mapping. The disk image of U is Dirichlet and hence H^2; cancellation of every interior zero with its full multiplicity puts it in BH^2. Apply Lemma 3.1, with the pushforward of μ on the open real diameter. Conformal invariance of derivative energy gives $U_j = Gf_j \to U$ in anchored strip Dirichlet norm and $f_j \to f$ in $L^2(\mu)$, with both U_j and f_j bounded Dirichlet functions.

For fixed j put $f_{j,\epsilon} = e^{-\epsilon z^2} f_j$. Boundedness of f_j and its finite derivative energy put this quotient in full $H^1_{\mathrm{hol}}(S_y)$. Product differentiation gives

$$(Gf_{j,\epsilon} - U_j)' = (e^{-\epsilon z^2} - 1)U_j' - 2\epsilon z e^{-\epsilon z^2} U_j.$$

The first term tends to zero in L^2 by dominated convergence. The squared norm of the second is at most $C_{j,y}\epsilon^2 \int_{\mathbb{R}} (x^2+y^2)e^{-2\epsilon x^2}dx = O_{j,y}(\sqrt{\epsilon})$. The anchor converges. On the real line the Gaussian has modulus at most one, so dominated convergence controls its original quotient observation error. Theorem 2.3 now approximates each $f_{j,\epsilon}$ by ordinary polynomials in the joint graph. Choose successively j, ϵ, a Fourier band and a polynomial degree. Triangle inequalities prove (14).

Anchored derivative convergence of holomorphic functions implies local uniform convergence, and then convergence of every fixed local derivative. Every GP_n has each interior zero jet of G, so the limit does too. This proves the converse. □

Corollary 3.4 (Logarithmic core guarantee)**.** *Under the remaining hypotheses of Theorem 3.3, if*

$$M_n \le BC^n n! \log(e+n)^{\alpha n} \quad (n \ge 0), \qquad 0 \le \alpha \le 1, \tag{15}$$

then its conclusion holds.

Proof. Since $n! \le n^n$, the inverse roots dominate a positive constant times $1/[n\log(e+n)^\alpha]$. This series diverges through $\alpha = 1$. The statement is the classical Carleman implication in this joint approximation norm. □

4 Exact failures beyond the logarithmic threshold

We now show that exponent one is sharp for a universal core guarantee from (15). Infinite convolution and Fourier gaps are classical nonquasianalytic mechanisms [6, 2]; the point here is to control the full joint graph and the actual observation.

Theorem 4.1 (Own-weight core failure for every $\alpha > 1$)**.** *For every $\alpha > 1$ there is a real-even entire function G of order one, with only real zeros and with G, G' bounded on each closed strip, such that the following hold. Its physical functions*

$$\gamma_v(x) = \frac{1}{8v}\partial_v |G(x+iv)|^2 \quad (v > 0), \qquad W_y(x) = \sup_{0 \le v \le y} |\gamma_v(x)| \tag{16}$$

have continuous removable values at zero, $\gamma_v(x) > 0$ for every real x and $v > 0$, and the complete measure (2) with $d\mu = W_y(x)dx$ satisfies (15) at this α, for each fixed $y > 0$. Nevertheless its ordinary polynomial joint core fails at every positive width, even for entire zero-compatible targets of either parity.

Proof. Choose $c_0 > 0$ and $a_j = c_0/[j\log(e+j)^\alpha]$, so $A = \sum a_j < \infty$. Let X_j be independent uniform variables on $[-a_j, a_j]$. Their sum converges absolutely for every outcome and has an even

nonnegative smooth density ϕ, supported in $[-A, A]$, of integral one. To justify smoothness and its budget, write $u_a = (2a)^{-1}1_{[-a,a]}$. Its distributional derivative has total variation $1/a$. Differentiate the first n convolution factors; leaving a further box in the probability tail gives

$$\|\phi^{(n)}\|_2 \le \left(\prod_{j=1}^{n} a_j^{-1}\right)(2a_{n+1})^{-1/2} \le B_0 C_0^n m_n(\alpha). \tag{17}$$

Leaving two undifferentiated boxes instead makes the derivative a continuous function, since their convolution is continuous and convolution with a finite measure preserves continuity. This works for every n. The additional factor in the first bound grows only polynomially and logarithmically in n, and is absorbed in $B_0 C_0^n$.

Set $G(z) = \int \phi(t) e^{itz} dt$. Compact support gives entire exponential type at most A. Since the even smooth probability has positive mass on a negative interval away from zero, $G(iY)$ grows at least $ce^{c'Y}$, so the order is exactly one. Integration by parts gives rapid decay of every fixed derivative on each closed strip. Independence and normal convergence give the exact product

$$G(z) = \prod_{j\ge 1} \operatorname{sinc}(a_j z) = \prod_{j,m\ge 1}\left(1 - \frac{a_j^2 z^2}{\pi^2 m^2}\right). \tag{18}$$

Here $\sum a_j^2 < \infty$ ensures local convergence of both products. Off the real axis all factors are nonzero and the normal product is nonzero; all zeros are real and no quadratic exponential gauge is inserted.

Uniformly on $|v| \le y$, Plancherel and the product rule give

$$\|x^n G(x+iv)\|_2 = \sqrt{2\pi}\|\partial_t^n(e^{-vt}\phi(t))\|_2 \le B_y C_y^n m_n(\alpha).$$

The same holds for each fixed derivative $G^{(j)}$: replace ϕ by $t^j\phi$. Each Leibniz term is bounded using compact support and $n(n-1)\cdots(n-l+1)m_{n-l} \le m_n$, with $l \le j$. Binomial expansion gives the same bound with $(1+|x|)^n$.

For a paired real-zero factor,

$$\partial_v |1 - (x+iv)^2/r^2|^2 = 4v(r^2 + x^2 + v^2)/r^4 > 0.$$

Logarithmic differentiation of the locally normal product (18) proves $\gamma_v(x) > 0$ for $v > 0$. For the supremum weight put $A_x(v) = |G(x+iv)|^2$. Reality gives $A_x'(0) = 0$ and

$$\gamma_v(x) = \tfrac{1}{8}\int_0^1 A_x''(tv)dt, \quad A_x''(v) = 2|G'(x+iv)|^2 - 2\Re(G''(x+iv)\overline{G(x+iv)}). \tag{19}$$

For $h \in H^1([-y,y])$, the fundamental theorem and $2ab \le a^2 + b^2$ give

$$\sup|h|^2 \le (1 + 1/(2y))\int |h|^2 + \int |h'|^2. \tag{20}$$

Apply this to G, G', G'' and use (19) to obtain $W_y(x) \le C_y \sum_{j=0}^3 \int_{-y}^{y} |G^{(j)}(x+iv)|^2 dv$. The preceding moment estimates control this actual weight, the strip terms and the origin atom in (2). It is bounded, integrable, even and continuous; compactness of the vertical parameter and the removable formula give continuity, and $\gamma_y > 0$ gives positivity.

Choose $k > 2A$ and $U(z) = e^{ikz}G(z)$. Its quotient is entire, its strip derivative energy is finite and its actual observation integral is $\int W_y < \infty$. On each horizontal line the Fourier support of

$(GP)'$ is contained in $[-A, A]$ for every polynomial P, while that of U' lies in $[k-A, k+A]$. These supports are disjoint. Consequently

$$\|(U - GP)'\|^2_{L^2(S_y)} = \|U'\|^2 + \|(GP)'\|^2 \geq \|U'\|^2 > 0. \tag{21}$$

With our Fourier convention the positive constant is $2\pi \int (t+k)\sinh(2y(t+k))\phi(t)^2 dt$. The target $\cos(kz)G$ or $\sin(kz)G$ gives the same obstruction in the even or odd sector. Extra anchor and observation summands cannot cancel the derivative lower bound. This proves failure at every degree and even with complex polynomial coefficients. □

Thus Corollary 3.4 has a sharp universal envelope boundary, although individual functions beyond it may still have a core. Even positive physical γ_v and entirely real zeros do not repair this approximation failure.

Proposition 4.2 (A negative band invisible to every square)**.** *For every $\alpha > 1$ there are real-even bounded continuous g, W, with $|g| \leq W$, $W\,dx$ finite with moment budget $BC^n m_n(\alpha)$, such that g has a negative interval but $\int P^2 g\,dx > 0$ for every nonzero real polynomial.*

Proof. Use G from Theorem 4.1 and set

$$w(x) = |G(x)|^2, \qquad g(x) = (1 + 2\cos(kx))w(x), \quad W(x) = 3w(x), \qquad k > 2A.$$

The real Fourier transform of the Schwartz function w is smooth and supported in $[-2A, 2A]$. All its jets at $\pm k$ vanish; hence $\int x^j \cos(kx)w(x)dx = 0$ for every j. Therefore $\int P^2 g = \int P^2 w > 0$. Strictness holds because nonzero entire G has a discrete real zero set. On $(2\pi/(3k), 4\pi/(3k))$ the cosine factor is negative, and a compact subinterval avoiding those zeros gives a strict negative band. The moments follow from the proof above. This g is a separate signed-weight model, not the physical γ_v of G, which is positive. It shows why Theorem 7.2 cannot extend universally to any $\alpha > 1$ solely from that moment budget. □

Remark 4.3 (Why finite order is substantive). Fix $y > 0$ and $k = \pi/(2y)$. Let

$$U_*(z) = z^2 e^{-z^2}, \quad G_*(z) = U_*(z)e^{-\cosh(kz)}, \quad W_*(x) = e^{-4\cosh(kx)}.$$

On the closed strip $\cos(kv) \geq 0$, so G_*, G_*' have Gaussian times exponential bounds and satisfy strong Carleman moment conditions. They have the same interior zero as U_*, and $W_*|U_*/G_*|^2 = e^{-2\cosh(kx)}$ is integrable. Nevertheless, under $w = \tanh(\pi z/(4y))$ the factor

$$e^{-\cosh(kz)} = \exp\left[-\tfrac{1}{2}\left(\frac{1+w}{1-w} + \frac{1-w}{1+w}\right)\right]$$

is singular inner with atoms at both ends. Every polynomial multiple of G_* retains it in H^2 after mapping, and so does any anchored-Dirichlet limit. The finite-order U_* has neither end atom, by its Gaussian real decay or Lemma 3.2's good-point argument. It cannot be such a limit. The chosen W_* is a declared positive observation. This example separates the finite-order divisor condition from the moment condition.

5 A universal log-convex envelope threshold

Let $m_0 = 1$, let $m_n > 0$ be log-convex, and put $r_n = m_n/m_{n-1}$ and $a_n = r_n^{-1}$. The following statement gives a universal guarantee for the actual joint strip graph measure across all admissible functions.

Theorem 5.1 (Universal log-convex envelope dichotomy)**.** *Retain the structural hypotheses of Theorem 3.3, replacing its Carleman sum by the envelope for its actual graph measure ν: $M_0 < \infty$ and $M_n \le BC^n m_n$ for all $n \ge 1$. If $\sum_{n\ge1} a_n = \infty$, all conclusions of that theorem hold. For the signed assertion, let σ be a finite even signed measure, let ω be a finite even positive measure with $\omega \ge |\sigma|$, and suppose $\|x^n\|_{L^2(\omega)} \le BC^n m_n$. If $\sum a_n = \infty$ and a bounded open interval $I \Subset (0,\infty)$ satisfies the whole-measure inequality $\sigma|_I \le -\eta\, dx|_I$ for some $\eta > 0$, the polynomial negative index is unbounded in each parity. Conversely, if $A = \sum_{n\ge1} a_n < \infty$, there exists a real even entire function G of order one with only real zeros whose actual physical weight W_y and full joint graph measure satisfy this envelope on every fixed strip, but whose polynomial graph core fails. There is also an even signed weight with a negative interval and this moment envelope that is positive on every nonzero polynomial square, in either parity.*

Proof. Log-convexity makes a_n decreasing and $m_n^{-1/n} = (a_1 \cdots a_n)^{1/n} \ge a_n$. Thus in the divergent case $\sum M_n^{-1/n} = \infty$ and Theorem 3.3 applies. For the signed assertion, if necessary replace m_n by $c^n m_n$, where $c = \max(1, 1/r_1)$. This makes the sequence nondecreasing, changes a_n to a_n/c, preserves divergence, and is absorbed in the moment constant C. Theorem 7.1 and its finite-clock construction then apply to every fixed dimension; the clock has unbounded range under precisely the displayed Carleman divergence. This asserts unbounded index. The assertion is qualitative in degree for an arbitrary envelope.

For the convergent case let $u_a = (2a)^{-1}\mathbf{1}_{[-a,a]}$ and set $\psi = u_1 * u_1$, $L = A + 2$, and

$$\phi = \psi * \mathop{*}_{j\ge1} u_{a_j}, \qquad G(z) = \int_{\mathbb{R}} e^{izt}\phi(t)\, dt.$$

The probability convolutions converge, are even and supported in $[-L, L]$. The distributional derivative of u_a has total variation $1/a$. Differentiating distinct first n tail factors and retaining the fixed ψ gives, for $q = 1, 2$,

$$\|\phi^{(n)}\|_q \le c_q m_n, \qquad c_1 = 1, \quad c_2 = \|\psi\|_2 = 1/\sqrt{3}.$$

These estimates pass from finite convolutions to the limit distributionally (and weakly in L^2); all Sobolev orders imply $\phi \in C_c^\infty$. For $q = 1$ the same derivative is the convolution of an L^1 function with finite measures, so the L^1 estimate also passes directly. Since $na_n \le A$, for $0 \le \ell \le n$,

$$\frac{m_{n-\ell}}{m_n} \le \frac{A^\ell}{n(n-1)\cdots(n-\ell+1)}.$$

Leibniz' rule, first for the tilt and then for t^j, therefore yields

$$\|D^n(t^j e^{-vt}\phi)\|_q \le c_q e^{y(L+A)}(L+A)^j m_n \quad (|v| \le y).$$

Indeed the tilt sum is bounded by $e^{A|v|}$, while the remaining finite sum is $\sum_{\ell\le j}\binom{j}{\ell}L^{j-\ell}A^\ell = (L+A)^j$. Fourier inversion and Plancherel now bound all fixed derivative strip norms with horizontal powers $|x|^n$. The passage to $(1+|x|)^n$ costs only $\sum_{k\le n}\binom{n}{k}m_k \le e^A m_n$ by the same ratio estimate. The physical-weight estimate in the preceding construction uses only vertical Sobolev, derivatives through three, and Cauchy–Schwarz of these weighted norms. It therefore gives the actual W_y, not a substitute observation measure, with moments bounded by a constant times m_n^2. The two area terms have the same bound. Finally $m_n \ge n!/A^n$ implies $1 \le e^A m_n$, paying the origin atom.

The locally uniformly convergent product is

$$G(z) = \operatorname{sinc}(z)^2 \prod_{j\ge1} \operatorname{sinc}(a_j z).$$

Here $\sum a_j^2 < \infty$. Its zeros are real, its order is one, and its physical γ_v is strictly positive for $v > 0$, by the paired-real-zero formula already proved. The order assertion also follows from the nonzero compact support and the imaginary-axis Laplace lower bound. Choose $k > 2L$. The derivative Fourier support of every polynomial multiple of G remains in $[-L, L]$, while that of $e^{ikz}G$ is in the disjoint translate. Plancherel on each horizontal line proves a strictly positive area distance, so the joint graph core fails for this admissible target. Sine and cosine combinations give the separate parities. Finally the signed weight $(1 + 2\cos kx)|G(x)|^2$ has a negative interval. Fourier support shows its cosine term integrates to zero against every polynomial square, leaving $\int |p|^2|G|^2 > 0$. Its absolute measure is dominated by $3|G|^2dx$, so the same envelope holds. This is the asserted countermodel. □

6 Same-width synthesis for the actual Riemann function

For the real-even function $F(z) = \xi((1 + iz)/2)/4$, write

$$\gamma_v(x) = \frac{\partial_v |F(x + iv)|^2}{8v}, \qquad \gamma_0(x) = \frac{F'(x)^2 - F(x)F''(x)}{4}, \qquad W_y(x) = \sup_{0 \le v \le y} |\gamma_v(x)|.$$

For a holomorphic function V, the normalized derivative energy is

$$\mathcal{E}_v[V] = \frac{1}{8\pi v} \int_{S_v} |V'|^2 \, dA \quad (v > 0), \qquad \mathcal{E}_0[V] = \frac{1}{4\pi} \int_{\mathbb{R}} |V'(x)|^2 \, dx.$$

These are the physical-weight and energy normalizations of [14].

Corollary 6.1 (Every positive strip width)**.** *Let $F(z) = \xi((1+iz)/2)/4$, as in [14]. Fix any $y > 0$. If U is holomorphic on S_y, $U' \in L^2(S_y)$, and cancels every interior zero of F to its full multiplicity, then, for every finite positive Borel observation measure μ whose actual joint graph moments satisfy the Carleman criterion and for which $U/F \in L^2(\mu)$, ordinary polynomials P_n satisfy*

$$|F(0)P_n(0) - U(0)|^2 + \int_{S_y} |(FP_n - U)'|^2 \, dA + \int_{\mathbb{R}} |P_n - U/F|^2 \, d\mu \longrightarrow 0.$$

In particular this holds for $\mu = W_y \, dx$, or any finite sum of the original absolute physical-potential measures at widths at most y. For $\mu = W_y \, dx$ all these absolute-potential errors tend to zero uniformly in width. The normalized substrip derivative energies also converge uniformly for $0 \le v \le y$, with the width-zero value understood by its middle-line limit. No boundary zero is required to be absent and no larger-strip extension of U is assumed.

Proof. The actual fixed-strip estimates proved in the lemma *Fixed-strip inputs* of [14] give, for every fixed j and enlarged finite strip,

$$|F^{(j)}(x + iv)| \le C(1 + |x|)^p e^{-\pi|x|/8}, \qquad W_y(x) \le C(1 + |x|)^p e^{-\pi|x|/4}.$$

Moreover F is entire of order one. Consequently F, F' are bounded on the closed strip and square-integrable over it. The full graph measure for each displayed physical observation has an exponential horizontal envelope apart from its origin atom. Direct gamma-integral estimates give $M_n \le BC^n n!$ (a fixed polynomial factor is absorbed in C^n), so its Carleman sum diverges. Theorem 3.3 therefore applies at this very y, including when F has zeros on its boundary: only interior divisors enter that theorem.

For the uniform potential assertion use $|\gamma_v| \le W_y$ for $0 \le v \le y$. For uniform energies write $V_n = FP_n - U$. For $y/2 \le v \le y$, the normalized substrip integral of $|V_n'|^2$ is bounded by a fixed multiple of its full-strip integral. For $0 < v \le y/2$, analytic submean on fixed-radius disks lying inside S_y, integrated horizontally, bounds every derivative-line integral in the central half-strip by a fixed multiple of the same full-strip integral. Averaging those line integrals over $[-v, v]$ proves the uniform assertion, including the middle-line limit. In this notation, $\sup_{0\le v\le y} \mathcal{E}_v[V_n] \to 0$. The same estimates bound $\sup_{0\le v\le y} \mathcal{E}_v[U]$; Cauchy–Schwarz then gives uniform convergence of $\mathcal{E}_v[FP_n]$ to $\mathcal{E}_v[U]$. This uses no horizontal boundary trace bound. If the data are real, conjugation and averaging yield real polynomials. For a prescribed parity, replace the observation measure by its sum with its reflection; the target norm remains finite for a target of that parity, and projection to that parity preserves convergence. The fixed strip and F are reflection symmetric. Finally each selected polynomial has finite degree, and all graph moments are finite, so sufficiently small rational perturbations of its real coefficients preserve its graph error. These operations give reality, parity and finite rational rounding without changing the observation norm. □

This removes the restriction $y > 1$ in the preceding actual-function same-boundary theorem of [14]; the predecessor explicitly left $0 < y \le 1$ open. The conclusion is a synthesis statement with the actual interior divisor prescribed.

Part II. Negative polynomial spaces and matching orders

The second branch concerns the original real signed form. Its common negative-span construction and matching Jacobi models have their own measure and moment hypotheses.

7 Whole negative spans under finite moment clocks

Let σ be an even finite real signed measure and ω an even positive finite measure with $|\sigma| \le \omega$. Assume that for a bounded open interval $I \Subset (0, \infty)$,

$$\sigma|_I \le -\eta\, dx, \qquad \eta > 0, \tag{22}$$

in measure order. This condition concerns the *entire* restriction, not just a negative density with additional positive singular mass. Negative singular mass is allowed. Choose a closed $J \Subset I$ of length $l > 0$ and center x_J, and a real $\chi \in C_c^3(I)$ with $0 \le \chi \le 1$ and $\chi = 1$ on J. For the Fourier convention $\widehat{\chi}(t) = (2\pi)^{-1} \int \chi(x) e^{-itx} dx$, put

$$\begin{aligned} \lambda_0 &= \eta l, & M_0 &= \sqrt{\omega(\mathbb{R})}, \\ \eta_0 &= \min\{1, \lambda_0/[8(2M_0+1)]\}, & \Lambda &= 2\int |\widehat{\chi}(t)|dt, \\ C_t &= \|\chi'''\|_1/\pi, & U_0 &= \max\{1, (4\sqrt{\omega(\mathbb{R})}C_t/\eta_0)^{1/2}\}. \end{aligned} \tag{23}$$

All are fixed independently of the number of targets. The Fourier integral is finite, for instance bounded by $(\|\chi\|_1 + \|\chi''\|_1)/\pi$.

Theorem 7.1 (Finite-clock negative spans)**.** *Suppose $\|x^j\|_{L^2(\omega)} \le BC^j m_j$ through N, where $B, C \ge 1$ and $m_0 = 1, m_1, \dots, m_N$ is positive nondecreasing and log-convex. For K with $d = \lfloor Kl/(2\pi) \rfloor \ge 1$, set*

$$D = 2d, \quad T = K + U_0\sqrt{D}, \quad s = \lceil \log \max\{1, 4B\Lambda D/\eta_0\} \rceil .$$

If $N > s$ and $A_{s,N} > eCT$, then in each parity there is a D-dimensional real subspace of $\mathcal{P}_N$, with rational polynomial basis, on whose entire nonzero span $q_\sigma < 0$. The same N works for both parities.

Proof. Take the $2d$ positive-side targets

$$h_{j,c} = \chi(x)\cos(2\pi j(x - x_J)/l), \qquad h_{j,s} = \chi(x)\sin(2\pi j(x - x_J)/l), \quad 1 \le j \le d.$$

For parity $\varepsilon \in \{0, 1\}$ fold each to $f_j(x) = h_j(x) + (-1)^\varepsilon h_j(-x)$, with D targets. On the complete period J, sine/cosine orthogonality gives $l/2$ times the coefficient norm squared. All remaining supported square mass has the same negative measure sign. Reflection doubles the reserve, not the dimension. Thus

$$\int \left(\sum_{j=1}^{D} v_j f_j\right)^2 d\sigma \le -\lambda_0 \|v\|_2^2 \qquad (v \in \mathbb{R}^D). \tag{24}$$

Each $|f_j| \le 1$, so $\|f_j\|_{L^2(\omega)} \le M_0$ uniformly. This is the full-period construction of [12].

Three integrations by parts give

$$\int_{|t|>R} |\widehat{\chi}(t)|dt \le \|\chi'''\|_1/(2\pi R^2).$$

Carrier modulation only shifts frequencies by at most K; parity reflection at most doubles variation. Each f_j has a Fourier density of variation at most Λ, whose tail beyond $T = K + R$ has variation at most C_t/R^2. Let b_j retain that density in $[-T, T]$. For $R = U_0\sqrt{D}$,

$$\|b_j - f_j\|_{L^2(\omega)} \le \sqrt{\omega(\mathbb{R})}C_t/(U_0^2 D) \le \eta_0/(4D).$$

These are real-axis errors of compact C^3 targets; no holomorphic H^1 property of the untruncated targets is assumed. Lemma 2.4 gives polynomial p_j of degree at most N with $\|p_j - b_j\|_{L^2(\omega)} \le B\Lambda e^{-s} \le \eta_0/(4D)$. Taking real parts and parity projection are contractions because ω is even. Round the finitely many permitted coefficients rationally with further error at most $\eta_0/(2D)$; finite moments justify this since

$$\left\|\sum_l \delta_l x^l\right\|_{L^2(\omega)} \le \sum_l |\delta_l| \|x^l\|_{L^2(\omega)}.$$

Hence $\|p_j - f_j\| \le \eta_D := \eta_0/D$.

For any real coefficient vector the target and error synthesis maps have norms at most $M_0\sqrt{D}$ and $\eta_D\sqrt{D}$. The bound $|\sigma| \le \omega$ controls *every* mixed term:

$$\left|q_\sigma\left(\sum v_j p_j\right) - q_\sigma\left(\sum v_j f_j\right)\right| \le D\eta_D(2M_0 + \eta_D)\|v\|_2^2 \le \lambda_0\|v\|_2^2/8. \tag{25}$$

The whole rounded Gram is at most $-7\lambda_0 I/8$. Strict negativity proves independence of the D polynomials. The construction and constants are common to both parities. □

Theorem 7.2 (Logarithmic inertia hierarchy)**.** *Under (22), assume for every $n \ge 0$*

$$\|x^n\|_{L^2(\omega)} \le BC^n m_n(\alpha), \qquad 0 \le \alpha \le 1,$$

with α fixed. There are constants $c_\alpha > 0$ and N_0 depending on the supplied band and moment data such that for $N \ge N_0$ and either parity,

$$i_-(\sigma, N, \varepsilon) \ge \begin{cases} c_\alpha(\log N)^{1-\alpha}, & \alpha < 1, \\ c_1 \log\log N, & \alpha = 1. \end{cases} \tag{26}$$

The negative spaces have rational polynomial bases. An envelope $B_1C_1^n n!\log(e+n)^{\alpha n}$ is also sufficient.

Proof. The ratios $m_n/m_{n-1} = n\log(e+n)^\alpha$ increase. Concavity gives $\log(e+j) \ge (j/n)\log(e+n)$, so

$$e^{-\alpha n} n!\log(e+n)^{\alpha n} \le m_n(\alpha) \le n!\log(e+n)^{\alpha n}. \tag{27}$$

This proves the last assertion after enlarging C. The terminal clock is $\sum_{j=s+1}^{N} 1/[j\log(e+j)^\alpha]$. Write $c = e+1$. Monotonicity and $\log(e+x) \le \log(cx)$ for $x \ge 1$ give

$$A_{s,N} \ge \int_{s+1}^{N+1} \frac{dx}{x\log(cx)^\alpha}.$$

For $\alpha < 1$, $r = 1-\alpha$, the degree

$$N_K = \left\lceil \exp\left([\log(c(s+1))^r + reCT + 1]^{1/r} \right) \right\rceil \tag{28}$$

ensures $A_{s,N_K} > eCT$. At $\alpha = 1$ take instead

$$N_K = \left\lceil \exp\left(\log(c(s+1)) e^{eCT+1} \right) \right\rceil. \tag{29}$$

These integers exceed s. With all constants fixed, $D = (l/\pi)K + O(1)$, $T = K + O(\sqrt{K})$ and $s = O(\log(K+2))$. Thus $\log N_K = O(K^{1/(1-\alpha)})$ below the endpoint, and $\log\log N_K = O(K)$ at it. Choose integer K a sufficiently small fixed multiple of $(\log N)^{1-\alpha}$, respectively $\log\log N$. Then $N_K \le N$ for every sufficiently large N, and Theorem 7.1 plus degree-space containment proves (26). □

At $\alpha = 0$ the order is the published exponential-moment benchmark; [12] gives a sharper coefficient in that stronger hypothesis. The scalar clocks and their logarithmic inversions already occur in [13]. The present receiving statement preserves a growing *whole* negative span under the weaker moment budget. No globally bounded density is needed.

8 Matching orders in the finite-measure class

The following construction matches the orders of Theorem 7.2. Its kernel asymptotic is the classical period-one theorem of Świderski–Trojan [15, Theorems1.2,4.2,4.6]. We check all its receiving hypotheses and retain the boundary term of the original polynomial form. The quadrature and parity mechanism is the one in [12].

Theorem 8.1 (Slow Jacobi models)**.** *For every fixed $0 \le \alpha \le 1$ and $a > 0$ there is an even probability measure ν_α with positive continuous density on $\mathbb{R}$ such that $d\sigma_{\alpha,a} = (x^2 - a^2)d\nu_\alpha$ has a negative interval, and*

$$i_-(\sigma_{\alpha,a}, N) \sim \frac{a}{\pi}\rho_N, \qquad i_-(\sigma_{\alpha,a}, N, \varepsilon) \sim \frac{a}{2\pi}\rho_N, \tag{30}$$

$$\rho_N = \sum_{j=0}^{N} \frac{1}{(j+1)\log(e+j+1)^\alpha},$$

$$\rho_N \sim \begin{cases} (\log N)^{1-\alpha}/(1-\alpha), & \alpha < 1, \\ \log\log N, & \alpha = 1. \end{cases}$$

The positive dominating measure $d\omega = (x^2+a^2)d\nu_\alpha$ satisfies $\|x^n\|_{L^2(\omega)} \le BC^n m_n(\alpha)$ for fixed B, C. Thus the lower orders cannot be increased universally in the finite signed-measure class.

Proof. Define on finitely supported sequences in $\ell^2(\mathbb{N}_0)$

$$Je_n = A_n e_{n+1} + A_{n-1} e_{n-1}, \qquad A_n = (n+1)\log(e+n+1)^\alpha, \quad A_{-1} = 0.$$

The coefficient Carleman sum $\sum 1/A_n$ diverges, so J is essentially selfadjoint and its e_0 spectral measure is moment-determinate. Every power applied to e_0 has finite support, so all moments exist. The sign change $e_n \mapsto (-1)^n e_n$ conjugates J to $-J$ and fixes e_0, making this measure even.

The function $L(t) = \log t + \alpha \log\log(e+t)$ is increasing and strictly concave for $t > 0$. Hence A_n increases, $1/A_n$ decreases to zero and $A_{n-1}/A_n = \exp(L(n) - L(n+1))$ increases to one. These bounded monotone sequences have summable first differences; $b_n/A_n = 0$ does also. They therefore meet the Stolz D_1 conditions in [15]. The period-one reference coefficients are $1, 0$, the limiting transfer at zero is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, of trace zero and discriminant -4, and $A_{n+1}/A_n \to 1$. The cited local-density and kernel theorems consequently give, on every compact real interval, a continuous positive density w and

$$\frac{K_N(x,x)}{\rho_N} \longrightarrow \frac{1}{2\pi w(x)}, \qquad K_N(x,x) = \sum_{j=0}^{N} p_j(x)^2, \tag{31}$$

locally uniformly. The reference equilibrium density at zero is $1/(2\pi)$, fixing this normalization. Moment determinacy identifies the local orthogonality measures with the same spectral measure; their continuous densities agree on overlaps. Exhaustion therefore gives one positive continuous density on all of $\mathbb{R}$. The reciprocal-sum asymptotics follow by integral comparison.

Here are explicit moment budgets, without inferring a global density estimate. Every path of $J^n e_0$ lies in indices $0, \dots, n$. The relevant finite compression has row sums at most $2A_n$, so

$$\left(\int x^{2n} d\nu_\alpha \right)^{1/2} = \|J^n e_0\| \le (2A_n)^n \le (8e)^n n! \log(e+n)^{\alpha n} \le C_0^n m_n(\alpha), \quad C_0 = 8e^2.$$

We used $n+1 \le 2n$, $\log(e+n+1) \le 2\log(e+n)$ and (27); $n = 0$ is immediate. Since $m_{n+1}/m_n \le 2 \cdot 4^n$ for every $n \ge 0$, Minkowski gives

$$\left(\int |x|^{2n} (x^2 + a^2) d\nu_\alpha \right)^{1/2} \le C_0^{n+1} m_{n+1} + a C_0^n m_n \le (2C_0 + a)(4C_0)^n m_n.$$

The same estimate dominates the total variation of σ.

For completeness we return the kernel asymptotic to zero counts. The Gaussian quadrature probability measure on the $N+1$ zeros $x_{N,j}$ of p_{N+1} is $\nu_N = \sum_j K_N(x_{N,j}, x_{N,j})^{-1} \delta_{x_{N,j}}$. It matches moments through $2N+1$. Its fixed second moment gives tightness, and each eventually exact higher even moment supplies uniform integrability of lower monomials. Every subsequential weak limit has the moments of ν_α, and determinacy gives $\nu_N \Rightarrow \nu_\alpha$. For a fixed bounded interval I,

$$\frac{\#\{j : x_{N,j} \in I\}}{\rho_N} = \int_I \frac{K_N(x,x)}{\rho_N} d\nu_N(x) \longrightarrow \int_I \frac{1}{2\pi w(x)} d\nu_\alpha(x) = \frac{|I|}{2\pi}.$$

Local uniformity controls the kernel error. The remaining integrand is bounded and continuous except at endpoints, which have zero limiting mass. Open or closed endpoints change the zero count by at most two.

In the orthonormal basis $p_0, \dots, p_N$ the *original* signed Gram is exactly

$$Q_N = J_N^2 - a^2 I + A_N^2 e_N e_N^*. \tag{32}$$

Multiplication by x has one missing p_{N+1} component; its squared norm is precisely the last term. Positive rank-one interlacing gives

$$\#\operatorname{spec}(J_N)\cap(-a,a)-1\le \operatorname{ind}_- Q_N\le \#\operatorname{spec}(J_N)\cap(-a,a).$$

Together with the zero count this proves the full asymptotic. Zero diagonal makes J_N bipartite. The two square blocks of J_N^2 have the same nonzero squared singular values and dimensions differing by at most one. After subtracting a^2I their negative indices differ by at most one; the boundary term affects one parity and changes its index by at most one. Each parity is thus half the full index up to $O(1)$, proving (30).

Finally w has a positive minimum on every compact subinterval of $(0,a)$, giving a supplied negative band for σ. Theorem 7.2 applies with the just-proved finite measure ω. Invertible real changes to monomial coordinates preserve inertia, and strict negativity on a finite subspace is open under coefficient perturbation; rational bases give the same maximal indices. □

The local absolute-continuity input above is the classical theorem [15, Theorem4.2]. It supplies local continuity and positivity, not a global bound on w or $(x^2+a^2)w$. This is why the matching claim is made in the finite-measure class of Theorem 7.2. The bounded-density physical application below is a separate member of that class.

Part III. One fixed kernel realizing both phenomena

The final application binds both branches to the actual physical weight.

9 The all-depth kernel and its actual physical weight

Fix $A>0$, $B\in\mathbb{R}$, and define

$$\phi(s)=e^{-Ae^{s^2}+Bs^2},\qquad F(z)=\int_{\mathbb{R}}\phi(s)e^{izs}ds. \tag{33}$$

For each fixed strip width $y>0$ use the actual

$$\gamma_v(x)=\frac{\partial_v|F(x+iv)|^2}{8v},\qquad \gamma_0(x)=\frac{F'(x)^2-F(x)F''(x)}{4},\qquad W_y=\sup_{0\le v\le y}|\gamma_v|. \tag{34}$$

The zero value is removable by reality and evenness. No sign is assigned when using absolute estimates for W_y.

Theorem 9.1 (Fourier moments with the original supremum)**.** *For fixed $y>0$ and finite integer $J\ge 0$, there are finite $C_0,C_1>0$ depending on A,B,y,J such that, for $q=1,2$,*

$$\sup_{|v|\le y,\,0\le j\le J}\|\partial_s^n(s^je^{-vs}\phi(s))\|_{L^q(\mathbb{R})}\le C_0n!C_1^n\log(n+e)^{n/2}\qquad(n\ge 0). \tag{35}$$

Consequently, uniformly on that strip,

$$\|x^nF^{(j)}(x+iv)\|_2\le\sqrt{2\pi}C_0n!C_1^n\log(n+e)^{n/2}. \tag{36}$$

For $J\ge 3$, W_y is bounded, continuous, even and integrable, and

$$\left(\int|x|^{2n}W_y(x)dx\right)^{1/2}\le C_yD_y^nn!\log(n+e)^{n/2}. \tag{37}$$

The complete measure (2) for $G=F$, $\mu=W_ydx$ has the same moment class and satisfies the Carleman condition.

Proof. For real s take the complex disk $|\zeta| \le c/(1+|s|)$, $c = 1/16$, and put $w = s+\zeta$, $\eta = 2c + c^2$. Then

$$|w^2 - s^2| \le \eta, \qquad \Re e^{w^2} \ge e^{-\eta}\cos(\eta)e^{s^2} =: a_0 e^{s^2} > 0.$$

The radius prevents the oscillatory cosine from changing sign. For $j \le J$ and $|v| \le y$, the fixed polynomial, quadratic and linear factors are dominated by half this negative double exponential, so for a fixed C_* and $d = Aa_0/2$,

$$\sup_{|\zeta| \le c/(1+|s|)} |w^j e^{-vw}\phi(w)| \le C_* e^{-de^{s^2}}.$$

Cauchy's estimate gives

$$|\partial_s^n(s^j e^{-vs}\phi(s))| \le C_* n! c^{-n}(1+|s|)^n e^{-de^{s^2}}. \tag{38}$$

To bound the integral over the whole real line, maximize $\frac{n}{2}\log(1+t) - \frac{d}{2}e^t$ for $t \ge 0$. At an interior maximizer $d(1+t)e^t = n$, so $t \le \max(0, \log(n/d))$; the endpoint is covered as well. Choose K_d with $1 + \max(0, \log(n/d)) \le K_d \log(n+e)$. Since $(1+|s|)^n \le 2^{n/2}(1+s^2)^{n/2}$,

$$\sup_s (1+|s|)^n e^{-(d/2)e^{s^2}} \le [2K_d \log(n+e)]^{n/2}.$$

The unused factor $e^{-(d/2)e^{s^2}}$ belongs to both L^1 and L^2 on the complete line. Multiplying its norm proves (35), including $n = 0$ after enlarging C_0. All fixed derivatives are Schwartz, so integration by parts and Plancherel give exactly

$$\int |x|^{2n}|F^{(j)}(x+iv)|^2 dx = 2\pi \int |\partial_s^n(s^j e^{-vs}\phi(s))|^2 ds.$$

This proves (36) with no omitted tail.

The identity (19) applies with $G = F$. Use (20) for F, F', F'' to obtain

$$W_y(x) \le C_y \sum_{j=0}^{3} \int_{-y}^{y} |F^{(j)}(x+iv)|^2 dv. \tag{39}$$

This controls the entire mixed term $F''\overline{F}$ by a valid absolute bound; it is not a sign argument. Tonelli and (36) prove (37). The same bound controls the strip terms for F, F'. Continuity follows from the continuous removable potential on the compact vertical interval; boundedness follows either from the Fourier L^1 bounds or fixed derivative decay. Evenness is inherited from real-even F. Finally $(1+|x|)^{2n} \le 2^{2n}(1+|x|^{2n})$ controls the atom and all horizontal moments. Their inverse roots dominate $c/[n\sqrt{\log(n+e)}]$, a divergent series. □

The L^1 half also gives, by n integrations by parts and the choice $n = \lfloor |x|/(4C_1\sqrt{\log(e+|x|)}) \rfloor$ for large $|x|$,

$$|F^{(j)}(x+iv)| \le C\exp[-c|x|/\sqrt{\log(e+|x|)}] \quad (|v| \le y, j \le J). \tag{40}$$

Indeed $n! \le n^n$, and the resulting integration-by-parts factor is at most 4^{-n}. This is consistent with the following exact obstruction; it explains why a positive exponential Fourier envelope cannot simply be assumed from the spatial decay.

Proposition 9.2 (No positive exponential Fourier envelope)**.** *For (33) there are no $a, C > 0$ and finite real b with $|F(x)| \le C(1+|x|)^b e^{-a|x|}$ for every real x.*

Proof. Such a bound would make the inverse Fourier integral holomorphic for $|\Im z| < a$, uniformly bounded in $\Re z$ on each smaller horizontal line. On the real axis it is ϕ, so the identity theorem identifies it with the entire ϕ throughout that strip. Fix $0 < t < a$ and $s_n = (2n+1)\pi/(2t)$. Then $e^{(s_n+it)^2} = -e^{s_n^2-t^2}$ and

$$\log|\phi(s_n+it)| = Ae^{s_n^2-t^2} + B(s_n^2-t^2) \to \infty,$$

contradicting the uniform bound on that horizontal line. □

The transform F is nonzero entire of exact order one. For clarity, every exponential spatial moment is finite, and $\phi(s) \le C_p e^{-2|s|^p}$ for every fixed $p > 1$. Young's inequality yields order at most $p/(p-1)$ for each such p, hence at most one. Positivity on any fixed negative interval gives $F(iY) \ge c_a e^{aY}$ for arbitrary fixed $a > 0$, excluding order less than one and finite exponential type. Closed-strip F, F' bounds follow from (35). Theorem 3.3 therefore applies to this *same* F and its actual W_y at every fixed $y > 0$.

Corollary 9.3 (Core and physical inertia on one fixed kernel)**.** *For every fixed real $M \ge 999$, put*

$$\phi_M(s) = e^{-e^{s^2}-Ms^2}, \qquad F_M(z) = \int \phi_M(s)e^{izs}ds, \qquad Q_y(P) = \frac{1}{2\pi}\int P(x)^2\gamma_y(x)dx.$$

The following conclusions hold for this same kernel.

1. *At every fixed $y > 0$, the ordinary polynomial multiples of F_M have the full zero-jet core (14), with the actual observation $W_y dx$. Boundary zeros are allowed.*

2. *For any nonreal zero $a+ib$ with $a, b > 0$, there is $\delta_0 > 0$ such that for every fixed $b-\delta_0 < y < b$, each parity satisfies*

$$i_-(Q_y, N, \varepsilon) \ge c_{M,y}\sqrt{\log N}$$

for all sufficiently large N, with rational polynomial bases strictly negative on their entire nonzero real spans. At least one such interval of widths exists, and it contains a rational width.

Meanwhile the continuous root-Laguerre hierarchy of ϕ_M is positive and uniformly strictly log-concave on the root half-line $u \ge 0$ at every depth; their entire real extensions have strictly totally positive translation kernels, as publicly proved in [11].

Proof. The first assertion follows from the moment and finite-order estimates above. For the nonreal zero we use the public [11, Theorem1.1, Corollary4.3]. Its fixed $M = 999$ case has two distinct whole-line rational cumulant enclosures, the sharper giving $\kappa_6 < -1.8\cdot 10^{-18}$; its classical Gaussian-multiplier argument extends the conclusion to every fixed $M \ge 999$. These public certificates supply the inherited nonreal-zero input. Positivity of ϕ_M gives $F_M(it) > 0$ for real t, and reality/evenness then supply a zero representative with both coordinates positive.

At a zero $a + ib$ of multiplicity m, write $F_M(z) = (z-a-ib)^m H(z)$ on a disk where H is nonzero. Bound $K = \sup|\partial_y \log|H|^2|$ on a smaller closed disk. At $x = a$, $y = b - \delta$,

$$\partial_y|F_M(a+iy)|^2 = \delta^{2m-1}|H(a+iy)|^2\left[-2m + \delta\partial_y \log|H(a+iy)|^2\right].$$

Choose $\delta_0 < b/2$ with the segment in that disk and $\delta_0 K \le m$. This derivative, and hence $\gamma_y(a)$, is strictly negative for every $0 < \delta < \delta_0$. Continuity supplies a bounded positive-side interval with a uniform negative margin; reflection supplies the other side. The entire residual H is retained,

so simplicity or a truncated zero product is not required. Apply Theorem 7.2 at $\alpha = 1/2$ with $\sigma = \gamma_y dx$, $\omega = W_y dx$, using Theorem 9.1. The positive factor $1/(2\pi)$ changes no index.

The inherited root properties concern $S_r(u) = \mathcal{T}^r(e^{-e^u - Mu})$, $\mathcal{T}f = (f')^2 - ff''$, with the explicit formula

$$S_r(u) = 2^{2^r - r - 1} \exp\big([2^r(1-M) - 1]u - 2^r e^u\big).$$

Its logarithmic curvature is $-2^r e^u$; the strict translation minors are the classical exponential-kernel structure verified in [11]. They are different forms from Q_y. □

The core statement holds at every width; the negative-band assertion holds at the specified later widths below a nonreal zero. No numerical zero location, width, negative polynomial vector or uniform onset is obtained from the existence-only cumulant input. The finite degree receivers become effective when their stated band or Fourier-tail data are supplied. The example is an analytic kernel family, not the arithmetic Riemann theta density.

A Explicit all-depth constants and degree substitution

This appendix evaluates the input budgets used in the application; it does not quantify the target-dependent bounded quotient cutoffs. Fix $A > 0$, $B \in \mathbb{R}$, $y > 0$ and integer $J \geq 3$. Set

$$\begin{aligned}
&c = 1/16, && \eta = 33/256, && d = A/4,\\
&K = \max(B,0) + (y+J)/2, && C_c = |B|\eta + yc + (y+J)/2, &&\\
&C_* = \exp[C_c + K\log_+(4K/A)], && K_d = 2 + \log_+(1/d), && C_1 = 16\sqrt{2K_d},\\
&C_0 = C_* \max\{1, e^{-d/2}\sqrt{2\pi/d}, e^{-d/2}(\pi/d)^{1/4}\}. &&&&
\end{aligned}$$

Here $\log_+ t = \max(0, \log t)$ and the product $K\log_+(4K/A)$ is defined as zero if $K = 0$. These constants work for both $q = 1, 2$ in (35). Indeed $e^{-\eta}\cos\eta > 1/2$. On the shrinking disk, $B\Re(w^2) \leq \max(B,0)s^2 + |B|\eta$ and $y|s| + J\log(1+|s|) \leq (y+J)(s^2+1)/2$. Thus the integrand is at most $\exp[-(A/2)e^{s^2} + Ks^2 + C_c] \leq C_* e^{-de^{s^2}}$, using $\sup_{u\geq 0}(Ku - (A/4)e^u) \leq K\log_+(4K/A)$. The maximum estimate in the proof of Theorem 9.1 uses $1 + \max(0, \log(n/d)) \leq K_d \log(n+e)$. Finally $e^{s^2} \geq 1 + s^2$ gives

$$\|e^{-(d/2)e^{s^2}}\|_q \leq e^{-d/2}(2\pi/(qd))^{1/(2q)}, \qquad q = 1, 2,$$

the two factors in C_0. These bounds include $n = 0$.

Write $C_s = 1 + 1/(2y)$ and $a_n = n! C_1^n \log(n+e)^{n/2}$. Equations (19)–(20) give

$$W_y \leq \tfrac{1}{4}\sup|F'|^2 + \tfrac{1}{8}\sup|F|^2 + \tfrac{1}{8}\sup|F''|^2 \leq \frac{C_s}{2}\sum_{j=0}^{3}\int_{-y}^{y}|F^{(j)}|^2 dv.$$

Therefore

$$\int |x|^{2n} W_y(x)dx \leq 8\pi y C_s C_0^2 a_n^2, \qquad \int_{\mathbb{R}}\int_{-y}^{y}|x|^{2n}(|F|^2 + |F'|^2)dv\,dx \leq 8\pi y C_0^2 a_n^2.$$

Put $H = 8\pi y(C_s+1)C_0^2$. For the complete measure (2) one has $M_0 \leq \sqrt{1+H}$, and for $n \geq 1$,

$$M_n \leq \sqrt{1+2H}\, n!(2C_1)^n \log(e+n)^{n/2}. \tag{41}$$

This follows from $(1+|x|)^{2n} \leq 2^{2n}(1+|x|^{2n})$, $a_n \geq 1$ and the separate origin atom.

Take anchor zero, $J = 3$, and use the product sequence $m_n = m_n(1/2)$. By (27), valid input and graph constants are

$$B_0 = \sqrt{1+2H}, \qquad C_\nu = \max(1, 2\sqrt{e}C_1),$$
$$L = a = 1 + y, \qquad B_g = (C_0 + 3)B_0, \qquad \Lambda_g = a + 2LC_\nu.$$

The L^1 bound also gives $|F(0)|, \|F\|_\infty, \|F'\|_\infty \le C_0$ on the same strip, so these constants have no unevaluated analytic supremum. The exact clock is $A_{s,N} = \sum_{j=s+1}^{N} 1/[j\sqrt{\log(e+j)}]$, and

$$A_{s,N} \ge 2\left\{\sqrt{\log((e+1)(N+1))} - \sqrt{\log((e+1)(s+1))}\right\}.$$

Consequently

$$N = \left\lceil (s+1)\exp\left\{e\Lambda_g T\sqrt{\log((e+1)(s+1))} + (e\Lambda_g T)^2/4 + 1\right\}\right\rceil \tag{42}$$

strictly satisfies $N > s$ and $A_{s,N} > e\Lambda_g T$. Indeed, after adding $\log(e+1)$, its logarithm exceeds

$$(\sqrt{\log((e+1)(s+1))} + e\Lambda_g T/2)^2.$$

Theorem 2.2 gives mode error at most $B_g e^{aT-s}$ and finite-band error at most $B_g\,\mathrm{TV}(\pi)e^{aT-s}$. For a full H^1 target, (12) holds with the explicit safe choice

$$C_G = \left[4C_0^2 + \frac{4C_0^2}{\pi y^2} + \frac{4H}{\pi y^2}\right]^{1/2}.$$

Choosing T to bound the stated H^1 tail and then s to bound the band term makes (42) a finite degree for that supplied target. Its size is the sufficient bound stated above. The bounded quotient and Gaussian choices preceding a general zero-jet target are target-dependent and are not assigned a uniform rate here.

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