Phase-boundary reduction and low-rank compensation for permanent derivatives

Scott Schubr
Luminary Oddities

25 October 2026

Abstract

We study a compensated homogeneous permanent derivative on centered Hermitian positive semidefinite matrices of order six. An exact Hermitian sum-of-squares certificate proves $F_4(K) > 8000(\operatorname{tr} K)^4$ for every nonzero centered complex positive semidefinite rank-one matrix, with no conjugate-disjointness assumption. We also prove a quantitative phase-boundary reduction: positivity of the actual derivative propagates from a saturated phase endpoint to its entire feasible ray, with a strictly positive reserve. Combining the two results gives $F_4(K) > 5040(\operatorname{tr} K)^4$ whenever $\operatorname{rank}(\operatorname{Re} K) \le 2$, including unequal real spectra and singular boundaries. A complementary common-state tensor dual certifies every feasible phase over an explicit noncirculant real base and a closed neighborhood outside the previously established spectral domain. The proofs use an opposite bound for the pure quartic phase term, exact rational certificates, and complete original-permanent coefficient reconstruction.

1 The functional and the new domains

All matrices have order six unless another size is specified. Put $J = \mathbf{1}\mathbf{1}^*$, $P = I - J/6$ and $W = 6I - J$. Centered means $K\mathbf{1} = 0$. A positive semidefinite matrix is Hermitian; a bar denotes entrywise conjugation. Entrywise products are denoted by $\circ$, and all unmarked matrix powers are ordinary products. Define

$$f_K(v) = \operatorname{per}(K + vJ), \qquad \mathcal{P}(K) = f_K(1) + f_K'(1), \qquad D(K) = \mathcal{P}(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2,$$

where the absolute square is entrywise. The actual additive derivative is

$$E_j(K) = [x^j]\, \partial_v D(xK + vW)|_{v=0}\,, \qquad F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{1}$$

In particular, vW is added *after* xK; it is not multiplied by another x. The quantities E_3, E_4 have degrees three and four, so F_4 is homogeneous of degree four under real scaling.

Our first main theorem removes the conjugate-disjointness condition from the previously established rank-one domain.

Theorem 1.1 (Unrestricted complex rank one)**.** *For every nonzero centered vector $z \in \mathbb{C}^6$, with $K = zz^*$ and $r = \operatorname{tr} K$, one has*

$$F_4(K) > 8000r^4.$$

The phases and zero coordinates of z are arbitrary. In particular, $z^T z = 0$ is not required.

The proof in Section 5 is computer-assisted: it reconstructs the complete polynomial from the original permanent, then verifies an exact positive-definite Hermitian coefficient matrix after a trace-cubed multiplier.

The receiving theorem for this endpoint result concerns the actual functional, rather than an independently estimated quadratic term. Write $K = R + iS$, with R real symmetric and S real antisymmetric, and put $r = \operatorname{tr} R$, $\beta = \operatorname{tr} R^2$. For the actual omission coefficients

$$\alpha_j(K) = [x^j] \operatorname{per}(J + xK), \qquad \delta_j(K) = \partial_v \alpha_j(K + vW)|_{v=0}, \quad j = 4, 5, \tag{2}$$

define

$$M(R) = (20160 - 464\sqrt{2})\delta_5(R) + 14800r\delta_4(R) + (24744 - 10368\sqrt{2})r^2\beta. \tag{3}$$

The degrees of δ_4 and δ_5 are three and four.

Theorem 1.2 (Paid phase-boundary propagation)**.** *Let $K = R + iS \succeq 0$ be centered and nonzero. If $S \neq 0$, there exist $0 < \kappa \leq 1$ and a centered matrix $K_* = R + iS_* \succeq 0$ such that $S = \kappa S_*$ and*

$$\operatorname{rank} K_* \leq \operatorname{rank} R - 1 \leq 4.$$

Writing $y = \kappa^2$, one has

$$F_4(K) \geq yF_4(K_*) + (1 - y)M(R), \qquad M(R) > 2016r^4. \tag{4}$$

The matrix K_* is saturated on its phase ray. Thus a negative value of F_4 forces a negative value at a saturated endpoint of residual rank at most four. If $S = 0$, the real-section theorem already gives strict positivity.

Here residual rank refers to the centered matrix K, not to a larger matrix obtained by adjoining a positive J anchor. A singular real part is allowed, and no lower bound on its positive eigenvalues is assumed.

Corollary 1.3 (Low real-part rank)**.** *If $K \succeq 0$ is centered and nonzero, and* $\operatorname{rank}(\operatorname{Re} K) \leq 2$*, then*

$$F_4(K) > 5040(\operatorname{tr} K)^4.$$

Both nonzero real eigenvalues may be unequal, and every imaginary part allowed by positive semidefiniteness is included.

The complementary full-support application in Section 7 uses a common-state tensor dual. It proves $F_4 > 16000r^4$ on an explicit closed neighborhood of real bases lying outside the spectral pinch of the predecessor [3]. It is a neighborhood theorem, not a uniform result over all real bases.

Relation to earlier work

The real-section bound [1], the conjugate-disjoint locus [2], and the spectral-pinch theorem [3] are the precise starting points. The locus $K\overline{K} = 0$ imposes $z^T z = 0$ in rank one. Theorem 1.1 treats the remaining rank-one vectors as well. Corollary 1.3 is a new consequence of that endpoint theorem and the paid scalar receiving identity; it does not follow from spectral decomposition alone.

Hermitian norm-power stabilization is classical [4, 5]. Those results assume a positive sphere minimum before concluding a sum-of-squares representation. Here positivity follows from an explicit finite certificate with complete coefficient reconstruction. Likewise, tensor positivity and semidefinite duality are established methods. Our dual supplies a receiving inequality for this signed permanent derivative and a specified all-phase domain.

Ordinary permanent comparisons, such as the border theorem in [7], concern endpoint inequalities. Nonnegativity of an endpoint polynomial does not imply positivity of a selected signed coefficient or derivative. The survey [8] also cautions against transferring unrelated real-part or singular-matrix reductions.

2 Exact coefficients and inherited positive reserves

We use the real-section theorem in its published normalization [1, Theorem1.1]:

$$F_4(R) \ge A\delta_5(R) + 14800r\delta_4(R) + C_0 r^2\beta, \quad A = 20160 - 464\sqrt{2}, \quad C_0 = 39744 - 10368\sqrt{2}, \quad (5)$$

for every centered real positive semidefinite R. These two constants are positive. This is an inherited theorem.

For completeness, the signs of the omission coefficients have a direct finite-dimensional interpretation. Represent K by centered Gram forms g_i, use the Fischer product $\|z^\gamma\|^2 = \gamma!$, and put $p_j = \sum_i g_i^j$. Set

$$\xi_i = g_i^4 - \tfrac{1}{2}g_i^2 p_2 - \tfrac{1}{3}g_i p_3, \qquad \psi_i = g_i^3 - \tfrac{1}{2}g_i p_2.$$

Expansion of the Gram permanent and the omitted elementary symmetric forms gives

$$\delta_5 = 6\sum_i \|\xi_i\|^2 - \left\|\sum_i \xi_i\right\|^2, \qquad \delta_4 = 2\left(6\sum_i \|\psi_i\|^2 - \left\|\sum_i \psi_i\right\|^2\right). \quad (6)$$

The factor two is the residual factor $(6-4)!$. One can verify these identities by expanding products of linear forms: their Fischer pairings sum all Gram pairings. Centering gives the displayed Newton identities; adding a common omitted symmetric form leaves the variance unchanged. Cauchy–Schwarz proves both signs. This explanation concerns positive semidefinite Gram matrices.

Lemma 2.1 (Intrinsic phase constraints)**.** *If $R + iS \succeq 0$, let $q = \|S\|_{\mathrm{F}}^2$ and $\beta = \operatorname{tr} R^2$. Then*

$$\ker R \subseteq \ker S, \qquad q \le \beta, \qquad \beta + q \le r^2, \qquad q^2 \le \tfrac{1}{2}r^2\beta.$$

Moreover $\beta \ge r^2/m$ when $m = \operatorname{rank} R > 0$.

Proof. For real $v \in \ker R$, positive semidefiniteness and $v^*Kv = 0$ give $Kv = 0$, hence $Sv = 0$. Complexification proves the kernel statement on the full support. Both K and $\overline{K}$ are PSD, so $\operatorname{tr}(K\overline{K}) = \beta - q \ge 0$. Also $\operatorname{tr} K^2 = \beta + q \le (\operatorname{tr} K)^2$. Thus $q \le r^2/2$ and $q^2 \le (r^2/2)\beta$. The last assertion is Cauchy–Schwarz on the m nonzero eigenvalues of R. □

Conjugation symmetry and degree give

$$F_4(R + ixS) = a + b(R,S)x^2 + c(S)x^4, \qquad a = F_4(R). \quad (7)$$

No sign is assumed for b, c, or curvature along a phase ray. The following original-coefficient recurrence specifies every polynomial used in our certificates, independently of the contraction formulas below.

Put

$$f_k = [x^k]\operatorname{per}(J + xK), \qquad d_k = [x^k]\,\partial_v \operatorname{per}(J + xK + vW)|_{v=0}\,.$$

Thus $d_{j-1} = \delta_j(K)$. The ordinary derivative coefficients are

$$O_j = 2\sum_{k=0}^{j}(7-k)(6-j+k)f_k d_{j-k}, \qquad j = 3, 4. \quad (8)$$

For $A_K = K + \overline{K}$, $B_K = K \circ \overline{K}$, set

$$N_{ij}(x) = 2 + \sqrt{2}x(A_K)_{ij} + x^2(B_K)_{ij},$$
$$T_j = [x^j]\sum_{\pi\in S_6}\sum_i W_{i,\pi(i)}\big(2\sqrt{2} + x(A_K)_{i,\pi(i)}\big)\prod_{\ell\ne i} N_{\ell,\pi(\ell)}(x). \quad (9)$$

Then

$$F_4(K) = O_4 - T_4 + \frac{r}{2}(O_3 - T_3). \tag{10}$$

Indeed $\mathcal{P}(K) = \sum_k (7-k)\alpha_k(K)$, which yields (8). Differentiating an entrywise absolute square gives the marked slot in (9).

For efficient exact reconstruction, extend a column set C of size i by an unused column j and use

$$P_{C\cup\{j\}} \mathrel{+}= P_C n_{ij}, \qquad \dot{P}_{C\cup\{j\}} \mathrel{+}= \dot{P}_C n_{ij} + P_C \dot{n}_{ij},$$

starting at $P_\varnothing = 1$, $\dot{P}_\varnothing = 0$. Ordinary slots are $n_{ij} = 1 + xK_{ij}$, $\dot{n}_{ij} = W_{ij}$; paired slots are (9). Truncate only above degree four for these derivative coefficients. This is the product rule on all permutations, with no interpolation or numerical fitting.

3 An upper certificate for the pure quartic phase

For a centered real antisymmetric S, put

$$B = S \circ S, \quad b_S = B\mathbf{1}, \quad q = \|S\|_{\mathrm{F}}^2, \quad u = \|b_S\|^2, \quad v = \|B\|_{\mathrm{F}}^2, \quad w = \operatorname{tr} S^4, \quad X = \operatorname{tr}(S^2 B).$$

The matrix B is entrywise nonnegative; it need not be PSD.

Proposition 3.1 (Opposite quartic bound)**.** *For every such S,*

$$\begin{aligned} c(S) = {} & (604800 - 960\sqrt{2})u + (-483840 + 192\sqrt{2})v \\ & + (-46080 + 192\sqrt{2})q^2 - 80640w - 241920X \;\le\; 30000q^2. \end{aligned} \tag{11}$$

Proof. The evaluation $c = F_4(iS)$ is algebraic, at a generally indefinite Hermitian matrix, and no variance sign is used there. The original coefficient rules give

$$\delta_5(iS) = 30u - 24v - 2q^2 - 4w - 12X, \qquad \frac{T_4(iS)}{8\sqrt{2}} = 120u - 24v - 24q^2.$$

The ordinary quartic is $20160\delta_5(iS) - 5760q^2$. Subtraction proves the formula. These identities can also be reconstructed entirely in the ten free skew coordinates using (8)–(10).

Since $7/5 < \sqrt{2} < 10/7$, the rational polynomial

$$\begin{aligned} L = {} & (-604800 + 1344)u + (483840 - 1920/7)v \\ & + (76080 - 1920/7)q^2 + 80640w + 241920X \end{aligned} \tag{12}$$

satisfies

$$30000q^2 - c - L = 960(\sqrt{2} - 7/5)u + 192(10/7 - \sqrt{2})(v + q^2) \ge 0.$$

Parametrize the full centered skew space by $x_{ij} = S_{ij}$, $1 \le i < j \le 5$, ordered lexicographically, with $S_{ji} = -S_{ij}$, $S_{i6} = -\sum_{j\le 5} S_{ij}$ and $S_{6i} = -S_{i6}$. The final row sum vanishes automatically. Let m_2 list the 55 monomials $x_a x_b$, $a \le b$. The rational matrix in `quartic_upper_certificate.json` satisfies

$$L = m_2^T G_2 m_2, \qquad G_2 \succ 0.$$

The program `quartic_upper.py` reconstructs every contraction in (12), compares all quartic coefficients, and checks exact rational/integer positivity. There are 555 nonzero coefficients; full dictionary equality checks all vanishing coefficients as well. The 55 positive pivots, or equivalently the positive integer-cleared principal minors, prove the claimed sign. All data, coordinates and operations are specified by this identity and the supplied matrix; a floating solver is not an acceptance premise. Thus $L \ge 0$ and (11) follows. □

The spectral predecessor [3, Proposition3.1] supplies the separate bound $c(S) \geq -30000q^2$. We use the upper bound for actual boundary propagation, and the lower bound only for the sufficient quadratic reserve in Section 6.

4 The actual phase-boundary theorem

Lemma 4.1 (Positive chord reserve)**.** *If $R + iS \succeq 0$ is centered, nonzero, and $0 \leq y \leq 1$, then*

$$F_4(R + i\sqrt{y}S) = yF_4(R + iS) + (1 - y)(a - cy) \geq yF_4(R + iS) + (1 - y)M(R).$$

In particular $M(R) > 2016r^4$.

Proof. Expanding (7) proves the identity. If $c \geq 0$, $a - cy \geq a - 30000q^2$ by Proposition 3.1; if $c < 0$, $a - cy \geq a \geq a - 30000q^2$. Now apply (5) and Lemma 2.1 to get

$$a - 30000q^2 \geq A\delta_5(R) + 14800r\delta_4(R) + (C_0 - 15000)r^2\beta = M(R).$$

The real part has rank at most five. Since $C_1 := 24744 - 10368\sqrt{2} > 10080$, we have $M(R) \geq C_1 r^4/5 > 2016r^4$. The exact strict comparison follows from $14664^2 - 2 \cdot 10368^2 = 42048 > 0$. □

Proof of Theorem 1.2. Let $V = \operatorname{ran} R$, of dimension m. By kernel compatibility, S acts on this support. On V define the real skew matrix $T = R_V^{-1/2} S_V R_V^{-1/2}$. The Hermitian matrix iT has paired real eigenvalues $\pm\sigma_j$. Positive semidefiniteness of $I + iT$ gives $\kappa := \|T\|_{\mathrm{op}} \leq 1$. For $S \neq 0$, $\kappa > 0$. Put $S_* = S/\kappa$. Then $I + iT/\kappa \succeq 0$ is singular. Congruence on V, followed by adjoining the common kernel, proves $K_* = R + iS_* \succeq 0$ and $\operatorname{rank} K_* \leq m - 1$. No inverse is used outside V. Centering and trace are unchanged. Apply Lemma 4.1 with endpoint S_* and $y = \kappa^2$. If $y < 1$ and $F_4(K) \leq 0$, its endpoint is strictly negative; if $y = 1$, the point is already saturated. Real points are covered by (5). □

Corollary 4.2 (Quantitative conjugate mixing)**.** *For every centered $U \succeq 0$ and $\lambda, \mu \geq 0$,*

$$\begin{aligned} F_4(\lambda U + \mu\overline{U}) \geq (\lambda^2 - \mu^2)^2 F_4(U) \\ + 4\lambda\mu(\lambda + \mu)^2 M(\operatorname{Re} U). \end{aligned} \tag{13}$$

Thus nonnegative $F_4(U)$ implies strict positivity of every mixture with $\lambda\mu > 0$ and $U \neq 0$.

Proof. When $\lambda + \mu > 0$, use (7) at $t = (\lambda - \mu)/(\lambda + \mu)$ and Lemma 4.1 with $y = t^2$, then multiply by $(\lambda + \mu)^4$. Conjugation symmetry permits either sign of t. Here $M(\operatorname{Re} U)$ uses the *original* trace $\operatorname{tr} U$, not the trace of the mixture. At $t = 0$ the scalar identity is valid directly. If $U = 0$ or $\lambda = \mu = 0$, every term vanishes. □

There is also a valid normalized monotonicity. For an endpoint ray $f(y) = a + by + cy^2$ and $0 < y \leq 1$,

$$\frac{d}{dy}\frac{f(y)}{y} = c - \frac{a}{y^2} \leq c - a \leq -M(R) < -2016r^4. \tag{14}$$

Consequently its nonpositive region, if present, extends outward along the ray. The argument uses the stated sufficient estimates. Positive curvature and interior minima are compatible with the theorem.

5 An exact certificate for every complex rank-one endpoint

Here we give a fully specified generative certificate for Theorem 1.1. Let $z_1, \dots, z_5$ and $w_1, \dots, w_5$ be independent formal variables, and put $z_6 = -\sum_{i\le 5} z_i$, $w_6 = -\sum_{i\le 5} w_i$. Eventually $w = \overline{z}$. Set

$$K_{ij} = z_i w_j, \qquad r = \sum_{i=1}^{6} z_i w_i = \sum_{i\le 5} z_i w_i + (\sum_{i\le 5} z_i)(\sum_{i\le 5} w_i).$$

Let $e_k(z)$ be the elementary symmetric polynomial in all six coordinates, with $e_0 = 1$, and write $z^{(i)}$ for omission of coordinate i. The original ordinary coefficients simplify to

$$\begin{aligned} f_k &= (6-k)!k!\, e_k(z)e_k(w), \\ d_k &= 6(5-k)!k! \sum_i e_k(z^{(i)})e_k(w^{(i)}) - (6-k)f_k, \qquad 0 \le k \le 5. \end{aligned} \tag{15}$$

The first identity follows by choosing the k retained rows and columns of the rank-one matrix and summing their $k!$ bijections. For the second, the $6I$ part of W gives six times the sum of the five-row cofactors; the $-J$ part gives $-(6-k)f_k$. Thus (15), (8) and the paired rule specify the entire actual rank-one polynomial without an assumed sign.

In (9), the corresponding entries are particularly simple:

$$(A_K)_{ij} = z_i w_j + w_i z_j, \qquad (B_K)_{ij} = z_i w_i z_j w_j.$$

Define $p(z, w)$ by (10) with these entries and (15). It is a real symmetric Hermitian polynomial of bidegree $(4, 4)$, uniquely written $p = p_0 + \sqrt{2}p_1$ over $\mathbb{Q}$. The supplied `rank_one_polynomial.json` contains these complete coefficient dictionaries. Direct original subset/product-rule reconstruction agrees with all 4900 rational and 4700 nonzero radical coefficients, including all omitted zero terms. This is a complete polynomial identity check, not a collection of sample rank-one evaluations.

Let $\mathcal{I}_7 = \{\alpha \in \mathbb{N}^5 : |\alpha| = 7\}$, ordered lexicographically, and $m_7(z) = (z^\alpha)_{\alpha\in\mathcal{I}_7}$. There are $\binom{11}{4} = 330$ entries. Define the symmetric matrices H_0, H_1 by the exact coefficient rule

$$r^3[p(z, w) - 8000r^4] = \sum_{\alpha,\beta\in\mathcal{I}_7} (H_0 + \sqrt{2}H_1)_{\alpha\beta} z^\alpha w^\beta. \tag{16}$$

This rule specifies every matrix entry; symmetry makes it the usual Hermitian quadratic form $m_7(z)^*(H_0 + \sqrt{2}H_1)m_7(z)$ when $w = \overline{z}$. The data file `rank_one_trace_cubed.json`, together with the exact r^7 subtraction, supplies the same complete identity. The multiplier and norm subtraction are independently reconstructed, rather than accepted as unexplained matrix data.

Lemma 5.1 (Exact positive-definite certificate)**.** *The matrix $H_0 + \sqrt{2}H_1$ in* (16) *is positive definite.*

Proof. Use the rational endpoints

$$a_- = \frac{1414213562373095}{10^{15}}, \qquad a_+ = \frac{1414213562373096}{10^{15}}.$$

Integer squaring verifies $a_- < \sqrt{2} < a_+$. For each endpoint, clear its positive denominator from $H_0 + a_\pm H_1$ to obtain a symmetric integer 330-by-330 matrix $G_\pm$. The supplied exact checker computes

$$\chi_\pm(t) = \det(tI + G_\pm).$$

Every one of the 331 coefficients is strictly positive, for each of the two matrices. These are exact integer characteristic-polynomial computations; the procedure uses FLINT arithmetic through `python-flint`, not approximate eigenvalues. For $t \geq 0$, such a polynomial is strictly positive. Symmetry of $G_\pm$ makes all its eigenvalues real, and a nonpositive eigenvalue would give a root $t \geq 0$. Hence $G_- \succ 0$ and $G_+ \succ 0$. The exact-radical matrix is a convex combination of the two rational endpoint matrices before denominator clearing, and is therefore positive definite.

The program `rank_one.py` checks the complete original polynomial, the multiplier identity, the full 330-monomial basis, symmetry, the radical enclosure, the norm subtraction and both characteristic polynomials. These finite exact operations and the coefficient definition (16) are the certificate. No spectral estimate or stabilization existence theorem is used as a positivity premise. □

Proof of Theorem 1.1. For $z \neq 0$ centered, at least one of its first five coordinates is nonzero, so $m_7(z) \neq 0$. Also $r = \sum |z_i|^2 > 0$. Lemma 5.1 and (16) give $r^3[F_4(zz^*) - 8000r^4] > 0$. Divide by r^3. □

Proof of Corollary 1.3. If $S = 0$, (5) and $\beta \geq r^2/2$ give the claim. If $\operatorname{rank} R = 1$, kernel compatibility forces $S = 0$ on its one-dimensional real support. Otherwise $\operatorname{rank} R = 2$ and saturation produces a nonzero endpoint K_* of complex rank at most one, with trace r. Theorem 1.1 gives $F_4(K_*) > 8000r^4$. Meanwhile $M(R) \geq C_1 r^2 \beta \geq C_1 r^4/2 > 5040r^4$. The chord (4) is a convex combination of two terms strictly larger than $5040r^4$. This proves the assertion, including unequal eigenvalues, zero rows and singular endpoints. □

6 A common-state all-phase tensor dual

Normalize $r = 1$ throughout Sections 6–7. General trace is restored by division of R, S by r and multiplication of the functional by r^4. The lower quartic bound in [3], (5) and Lemma 2.1 give the sufficient bound

$$F_4(R + iS) \geq Q(R, S) := M(R) + b(R, S). \tag{17}$$

Unlike Theorem 1.2, this is a sufficient reserve test; its failure would not refute actual F_4 positivity.

Let A_R be the top-left 5-by-5 block of R. Let $B_1, \ldots, B_{10}$ be the elementary skew basis in that block, ordered by (i, j), $1 \leq i < j \leq 5$, and $E = [I_5; -\mathbf{1}^T]$. Write $S_5 = \sum_a s_a B_a$, $S = E S_5 E^T$, and define

$$G_{ab} = \langle E B_a E^T, E B_b E^T \rangle_{\mathrm{F}}, \qquad X = ss^T.$$

The matrix $\mathcal{H}(R)$ of the actual form $b(R, S)$ is defined by full polarization, as specified in Appendix A. Thus $b = \langle \mathcal{H}, X \rangle$ and $q = \langle G, X \rangle$. Define

$$\mathcal{T}(X) = \sum_{a,b} X_{ab} B_a \otimes B_b, \qquad \mathcal{T}^*(Y)_{ab} = \left\langle Y, \frac{B_a \otimes B_b + B_b \otimes B_a}{2} \right\rangle.$$

All pairings here are real Frobenius pairings.

Proposition 6.1 (Common-state dual)**.** *Fix real centered PSD R of trace one. Suppose $Y_-, Y_+ \succeq 0$ are real symmetric 25-by-25 matrices, $\mu \geq 0$, and*

$$Z = \mathcal{H}(R) + \mathcal{T}^*(Y_- - Y_+) + \mu G \succeq 0.$$

Then every S with $R + iS \succeq 0$ satisfies

$$F_4(R + iS) \geq Q(R, S) \geq M(R) - \langle Y_- + Y_+, A_R \otimes A_R \rangle - \mu \min(\operatorname{tr} R^2, 1/2). \tag{18}$$

The matrices may be singular.

Proof. Centering gives $K = E(A_R + iS_5)E^T$. Its principal block and its conjugate are PSD. Averaging their same-state tensor products gives

$$A_R \otimes A_R - S_5 \otimes S_5 \succeq 0, \qquad A_R \otimes A_R + S_5 \otimes S_5 \succeq 0. \tag{19}$$

For example the first is the average of $(A_R + iS_5) \otimes (A_R + iS_5)$ and its conjugate; the second uses conjugate partners in the two slots. Also $X \succeq 0$ and $\langle G, X\rangle \le \min(\operatorname{tr} R^2, 1/2)$ by Lemma 2.1. Since $\mathcal{T}(X) = S_5 \otimes S_5$,

$$\langle \mathcal{H}, X\rangle = \langle Z, X\rangle - \langle Y_- - Y_+, \mathcal{T}(X)\rangle - \mu\langle G, X\rangle.$$

Pair the first constraint of (19) with Y_- and the second with Y_+. Positivity of trace pairings gives (18). No factors are chosen independently, and no commutation assumption is used. □

7 An explicit nonpinched full-support domain

Let Y have rows

$$(1, 2, -1),\ (2, -1, 0),\ (-1, 0, 2),\ (0, -2, -1),\ (-2, 1, 1),\ (0, 0, -1)$$

and set

$$R_0 = \frac{1}{10}P + \frac{1}{56}YY^T. \tag{20}$$

Its columns are centered and $\|Y\|_{\mathrm{F}}^2 = 28$, so R_0 is centered PSD of trace one and is at least $P/10$ on the centered space. Its first and sixth diagonal entries differ by $5/56$; hence no row permutation makes it circulant. For the first column v of Y,

$$\frac{v^T R_0 v}{v^T v} = \frac{37}{112} > \frac{9}{40}.$$

Two centered directions orthogonal to the columns of Y have eigenvalue $1/10 < 7/40$. This base lies outside both endpoints of the closed pinch $(7/40)P \preceq R \preceq (9/40)P$ in [3].

Theorem 7.1 (Nonpinched all-phase neighborhood)**.** *Let R be real symmetric, centered, of trace one, and $\|R - R_0\|_{\mathrm{F}} \le 10^{-8}$. Then every centered $K = R + iS \succeq 0$ satisfies*

$$F_4(K) > 16000.$$

At $R = R_0$ the stronger bound $F_4(K) > 16863$ holds. Every feasible phase is included, including singular and arbitrarily ill-conditioned matrices. For general trace $r > 0$, replace the condition by $\|R/r - R_0\|_{\mathrm{F}} \le 10^{-8}$ and the bounds by $16000r^4$ and $16863r^4$.

Proof. The file `nonpinched_dual_certificate.json` specifies exact rational matrices Y_-, Y_+, $\mu = 1$, and a rational Loewner lower bound Z_{low} for the dual matrix at R_0. Write $\mathcal{H} = \mathcal{H}_0 + \sqrt{2}\mathcal{H}_1$ and use the same radical endpoints a_-, a_+ as in Lemma 5.1. With $L = \sum_{a,b} |(\mathcal{H}_1)_{ab}|$, the lower matrix is

$$Z_{\mathrm{low}} = \mathcal{H}_0 + \mathcal{T}^*(Y_- - Y_+) + G + a_-\mathcal{H}_1 - (a_+ - a_-)LI.$$

It is a valid bound because $\|\mathcal{H}_1\|_{\mathrm{op}} \le L$. The exact checker reconstructs all 100 entries and proves

$$Y_- \succ 0, \quad Y_+ \succ 0, \quad Z_{\mathrm{low}} - \tfrac{1}{2}G \succ 0$$

by rational LDL or integer-cleared principal minors, with full reconstruction. In particular all $25 + 25 + 10$ positivity conditions are checked. Proposition 6.1 then gives at R_0

$$Q \geq \frac{202449575232388474393831}{120050000000000000000} > 16863. \tag{21}$$

This is a certificate for all phases, not a finite screen of phases. The remaining argument proves uniformity in the stated ball.

Put $\varepsilon = 10^{-8}$. Appendix B proves

$$\mathcal{H}(R) \succeq \mathcal{H}(R_0) - L_\varepsilon G, \qquad L_\varepsilon = 23000000(2\varepsilon + \varepsilon^2) + 1513728\varepsilon = \frac{4751372823}{10^{10}} < \tfrac{1}{2}. \tag{22}$$

Thus the same fixed dual matrices have $Z(R) \succ 0$ over the whole ball. Since $R_0 \succeq P/10$ and $\|R - R_0\|_{\mathrm{op}} \leq \varepsilon < 1/10$, every permitted real base is itself PSD. Their connecting segment is PSD of trace one, with all entries of absolute value at most one.

The absolute original coefficient budgets for δ_5, δ_4 are $36000, 72000$, respectively. Indeed,

$$5! \sum_{ij} |W_{ij}| = 7200.$$

The remaining marked-slot counts are $\binom{5}{4} = 5$ and $\binom{5}{3} = 10$. Telescoping degree-four and degree-three monomials gives

$$|\Delta\delta_5| \leq 144000\varepsilon, \qquad |\Delta\delta_4| \leq 216000\varepsilon.$$

Also $|\Delta\beta| \leq 2\varepsilon$. Consequently

$$|\Delta M| \leq (20160 \cdot 144000 + 14800 \cdot 216000 + 24744 \cdot 2)\varepsilon.$$

Let $L_Y = \sum_{ij} |(Y_- + Y_+)_{ij}|$. Principal compression is Frobenius-contracting, so the tensor cost changes by at most $L_Y(2 + \varepsilon)\varepsilon$. The final norm cost changes by at most 2ε, since $\mu = 1$ and $\min(\beta, 1/2)$ is one-Lipschitz in β. Subtracting these rational losses from (21), the exact checker gives

$$Q \geq \frac{4034047683499583514296594 49}{24010000000000000000000} > 16801 > 16000.$$

This proves the theorem. The diagonal and spectral separations just computed are much larger than ε; the entire ball remains noncirculant and outside the earlier pinch. Homogeneity proves the unnormalized statement. □

8 The saturated boundary reduction

Theorem 1.2 reduces actual centered positivity to a saturated boundary with complex residual rank at most four. Theorem 1.1 settles all its rank-one endpoints and hence all real supports of dimension at most two. The common-state dual settles a different full-support neighborhood.

A natural next question is a complete exact certificate for two-column Gram factors, with the right-unitary column symmetry retained. A rank-one Hermitian certificate alone sees only the symmetric one-column sector; it cannot settle the other two-column sectors. Numerical positivity or an exact receiving basis without its positive Gram matrix is not such a certificate. Symmetry reduction is a classical way to organize this task [6]; it does not supply the missing sign. Higher residual ranks, the other homogeneous blocks and the full permanent comparison remain separate obligations.

A The complete quadratic phase form

This appendix supplies the actual form used in the dual. All its formulas are at trace one; the general trace restoration is stated below. For real symmetric centered R and centered skew S, set

$$\begin{gathered}
t = \operatorname{diag} R, \quad b_R = \operatorname{diag}(R^2), \quad B = S \circ S, \quad b_S = B\mathbf{1}, \quad D_1 = R \circ S, \quad d = D_1\mathbf{1}, \\
\beta = \operatorname{tr} R^2, \quad d_2 = t^T t, \quad q = \|S\|_{\mathrm{F}}^2, \\
H = \sum R_{ij} S_{ij}^2, \quad V = t^T b_S, \quad W_1 = \operatorname{tr}(R(-S^2)), \quad C = \sum t_i R_{ij} S_{ij}^2, \\
H_2 = \sum (R^2)_{ij} S_{ij}^2, \quad J_2 = \sum R_{ij}^2 S_{ij}^2, \quad N_2 = \sum (S^2)_{ij} R_{ij}^2, \\
Z_2 = \sum (RS + SR)_{ij} R_{ij} S_{ij}.
\end{gathered}$$

All entry sums range over ordered pairs. Put $\delta_2 = [x^2]\delta_5(R+ixS)$ and $P_2 = [x^2]T_4(R+ixS)/(8\sqrt{2})$. Their inherited exact contractions [3] are

$$\begin{aligned}
\delta_2 = {} & 144(t^{\circ 2})^T b_S - 60 b_R^T b_S - 72\|d\|^2 - 24 t^T R b_S - 48(St)^T d \\
& - 36H_2 + 12N_2 - 24Z_2 + 288C + 144J_2 - 12H - 6d_2 q + 4\beta q \\
& + 48 \sum_i t_i (RS^2)_{ii} - 24 \sum_i t_i (SRS)_{ii} - 16 \operatorname{tr}(R^2 S^2) + 8 \operatorname{tr}(RSRS),
\end{aligned} \tag{23}$$

$$\begin{aligned}
P_2 = {} & -2016(t^{\circ 2})^T b_S + 816 b_R^T b_S + 408 q d_2 - 1728C \\
& + 672 t^T R b_S - 144\beta q + 240H + 288H_2 - 432J_2,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
b(R,S) = {} & 20160\delta_2 + 476352H + 1097856V - 137376q - 181440W_1 \\
& + 5760q(2\beta - 6d_2) - 8\sqrt{2}P_2.
\end{aligned} \tag{25}$$

For general trace, replace $-12H$ by $-12rH$ in (23), $240H$ by $240rH$ in (24), multiply the H, V, W_1 terms in (25) by r, and replace its standalone q term by $r^2 q$. The result has bidegree $(2,2)$ in R, S.

To check the expansion, use

$$\begin{gathered}
K \circ K = R \circ R - x^2 B + 2ixD_1, \qquad K^2 = R^2 - x^2 S^2 + ix(RS + SR), \\
\operatorname{tr}(K\overline{K}) = \beta - x^2 q, \qquad K = R + ixS.
\end{gathered}$$

Collecting the original coefficients by (8)–(10) gives exactly (23)–(25). In particular the ordinary quartic change is $20160\delta_2 - 69120H + 5760q(2\beta - 6d_2)$; half the ordinary-minus-paired cubic change is $545472H + 1097856V - 137376q - 181440W_1$. The accompanying `quadratic_coefficients.py` reconstructs the complete identities in all $15 + 10$ free centered real symmetric/skew coordinates, checking 4530 nonzero omission coefficients and 3630 nonzero paired coefficients. Its equality checks include all zero coefficients; it is an exact finite polynomial verification.

For full centered skew basis $S_a = EB_aE^T$, the matrix used in the dual is

$$\mathcal{H}_{aa} = b(R, S_a), \qquad \mathcal{H}_{ab} = \tfrac{1}{2}\big[b(R, S_a + S_b) - b(R, S_a) - b(R, S_b)\big].$$

Thus no selected phases, occupations or independent Gram factors are substituted for the actual ten-coordinate quadratic form.

B Uniform variation budgets for the nonpinched domain

Remove the linear trace-one terms from (23), (24): let $\widehat{\delta}_2 = \delta_2 + 12H$ and $\widehat{P}_2 = P_2 - 240H$, interpreted as the displayed quadratic polynomials in R, without assuming $\operatorname{tr} R = 1$. Define

$$A_2(R,S) = 20160\widehat{\delta}_2 + 5760q(2\operatorname{tr} R^2 - 6\|\operatorname{diag} R\|^2) - 8\sqrt{2}\widehat{P}_2.$$

At trace one,

$$b = A_2 + (234432 - 1920\sqrt{2})H + 1097856V - 181440W_1 - 137376q. \tag{26}$$

The following estimates hold for *every* centered real symmetric matrix E, not only PSD or trace-zero matrices.

Put $\epsilon = \|E\|_{\mathrm{op}}$. Centering gives $\|E\|_{\mathrm{F}} \le \sqrt{5}\epsilon$; $|E_{ij}| \le \epsilon$ and $(E^2)_{ii} \le \epsilon^2$. Also $\|S\|_{\mathrm{op}} \le \sqrt{q/2}$ and $\|(E \circ S)\mathbf{1}\|^2 \le \epsilon^2 q$ by rowwise Cauchy–Schwarz. The complete coefficient-weighted bounds for $\widehat{\delta}_2$, divided by $\epsilon^2 q$, are

$(\operatorname{diag} E)^{\circ 2} \cdot b_S$	144	$\operatorname{diag}(E^2) \cdot b_S$	60
$\|(E \circ S)\mathbf{1}\|^2$	72	$(\operatorname{diag} E)^T E b_S$	$24\sqrt{5}$
$(S \operatorname{diag} E)^T (E \circ S)\mathbf{1}$	$48\sqrt{5/2}$	H_2	36
N_2	12	Z_2	48
C	288	J_2	144
$\|\operatorname{diag} E\|^2 q$	30	$\operatorname{tr}(E^2)q$	20
$\sum E_{ii}(ES^2)_{ii}$	48	$\sum E_{ii}(SES)_{ii}$	24
$\operatorname{tr}(E^2S^2)$	16	$\operatorname{tr}(ESES)$	8.

All named contractions in this table are evaluated at E. For N_2, the symmetric entrywise-nonnegative matrix $E \circ E$ has maximal row sum at most ϵ^2; its operator norm is therefore at most ϵ^2, without presuming it PSD. Pair it with $-S^2 \succeq 0$. For Z_2, use $\|ES + SE\|_{\mathrm{F}} \le 2\epsilon\sqrt{q}$ and $\|E \circ S\|_{\mathrm{F}} \le \epsilon\sqrt{q}$. Weighted traces follow from operator/nuclear-norm duality, and $|\operatorname{tr}(ESES)| \le \|ES\|_{\mathrm{F}}^2$. These observations justify every row, including the indefinite E cases. With $\sqrt{5} < 9/4$ and $\sqrt{5/2} < 8/5$, their sum is less than 1100. The eight analogous bounds for $\widehat{P}_2$ are

$$2016,\ 816,\ 2040,\ 1728,\ 672\sqrt{5},\ 720,\ 288,\ 432,$$

whose sum is less than 10000. Hence

$$|A_2(E,S)| \le [20160 \cdot 1100 + 5760 \cdot 40 + 12 \cdot 10000]\epsilon^2 q \le 23000000\|E\|_{\mathrm{F}}^2 q.$$

For fixed S, this homogeneous quadratic form therefore has selfadjoint operator norm at most $23000000q$ on the Frobenius space of centered symmetric matrices. If $\|R - R_0\|_{\mathrm{F}} \le \varepsilon$ and $\|R_0\|_{\mathrm{F}} \le 1$, polarization gives

$$|A_2(R,S) - A_2(R_0,S)| \le 23000000(2\varepsilon + \varepsilon^2)q.$$

The changes of H, V, W_1 are each at most εq: use entry bounds, diagonal bounds and pairing with $-S^2 \succeq 0$, respectively. The sum of their absolute coefficients in (26) is less than 1513728. Thus

$$|b(R,S) - b(R_0,S)| \le L_\varepsilon q,$$

which is exactly the matrix inequality (22) by full polarization. This proves the uniform variation premise used in Theorem 7.1.

C Exact verification and reproducibility

The editable manuscript and its mathematical certificates are accompanied by one command, `python verify.py`; `python -O verify.py` has the same acceptance conditions. The pinned dependencies and commands are stated in the supplied `README.md` and `requirements.txt`. The exact characteristic-polynomial stage uses `python-flint`; rational coefficient reconstruction uses the stated symbolic/rational routines. An installation without the required exact-arithmetic dependency has not verified that stage.

The proof computations have four distinct roles:

1. `exact_permanents.py` implements the original subset product rule. The rank-one program reconstructs the complete bidegree $(4, 4)$ polynomial and its trace-cubed identity, rather than accepting finite controls as a universal proof.

2. `rank_one.py` reconstructs the $8000r^7$ subtraction, checks the complete 330-monomial Hermitian matrix and both exact integer characteristic polynomials in Lemma 5.1.

3. `quartic_upper.py` reconstructs the opposite quartic polynomial and its rational SOS certificate on all 55 monomials.

4. `phase_contractions.py`, `quadratic_coefficients.py` and `dual.py` reconstruct the actual phase form, complete dual identity, rational positivity and nonpinched neighborhood budget.

The mathematical data files specify the coefficient dictionaries and matrices used in these identities. Floating optimization, approximate eigenvalues and optimizer status are not acceptance premises. Additional exact feasible examples in `supporting_checks.py` corroborate normalization and singular cases.

The classical positivity in Lemma 5.1 is a test of a real symmetric coefficient matrix, not a numerical minimum search on complex vectors. Similarly the common-state dual proves every phase from explicit PSD inequalities; it does not extrapolate from sampled phases. These distinctions are necessary for the domains claimed in the main theorems.

Computational assistance

AI assistance was used in research, drafting and checking. The author is responsible for the mathematical statements and presentation.

References

[1] S. Schubr, *Real-section permanent-derivative compensation and quadratic phase control*, author-published preprint, 2026. Luminary Research Archive, edition1.

[2] S. Schubr, *Uniform permanent-derivative compensation on the conjugate-disjoint Gram locus*, author-published preprint, 2026. Luminary Research Archive, edition1.

[3] S. Schubr, *Full-phase permanent-derivative compensation under real-part spectral pinching*, author-published preprint, 2026. Luminary Research Archive, edition1.

[4] D. W. Catlin and J. P. D'Angelo, A stabilization theorem for Hermitian forms and applications to holomorphic mappings, *Math. Res. Lett.* **3** (1996), no. 2, 149–166. doi:10.4310/MRL.1996.v3.n2.a2.

[5] A. Drouot and M. Zworski, A quantitative version of Catlin–D'Angelo–Quillen theorem, *Anal. Math. Phys.* **3** (2013), no. 1, 1–19. doi:10.1007/s13324-012-0035-4.

[6] K. Gatermann and P. A. Parrilo, Symmetry groups, semidefinite programs, and sums of squares, *J. Pure Appl. Algebra* **192** (2004), 95–128. https://arxiv.org/abs/math/0211450.

[7] Q. Li, Q. Zhao and W. Luo, *Chollet's permanent conjecture through order six via border induction*, preprint, 31 August 2026. Preprints.org, version1.

[8] F. Zhang, *An update on a few permanent conjectures*, preprint, 2016. https://arxiv.org/abs/1608.02844.