A square-integral threshold for subadditivity on shrinking corridors

Scott Schubr

Abstract. Füredi and Ruzsa asked whether the window $n \le m \le \mu n$ in their restricted Fekete lemma can shrink to $n \le m \le n + r(n)$ with $r(n) = o(n)$. We answer this question completely. Writing $r(n) = n\,w(\log n)$, the conclusion survives exactly when $\int w^2 = \infty$; thus it survives for the sublinear width $r(n) = n/\sqrt{\log n}$, and it fails, as Füredi and Ruzsa expected, for every slower width such as $r(n) = n^\alpha$ $(\alpha < 1)$ or $r(n) = n/(\log n)^{1/2+\varepsilon}$. Let w be a measurable nonincreasing relative width. For exact eventual integer subadditivity on $n \le m \le n + nw(\log n)$, we prove a sharp square-integral dichotomy in the nonnegative nondecreasing linearly bounded class. Divergence of $\int w^2$ forces normalized-slope convergence, including widths with jumps and missing adjacent pairs. With eventual adjacent-pair legality, convergence holds for arbitrary real sequences, with limit in $\mathbb{R} \cup \{-\infty\}$. Thus $n/\sqrt{\log n}$ is a sufficient sublinear corridor. We further replace adjacent legality by balanced legal predecessors of gap $O(N/[(\log N)^{3/2}(\log\log N)^\eta])$, $\eta > 1/2$. Growing local maxima preserve a smaller literal corridor, and dyadic propagation transfers their limit to the original sequence. A staircase example satisfies this condition while failing every bound $O(N/(\log N)^{3/2+\varepsilon})$, $\varepsilon > 0$, and every fixed sublinear power bound. When $\int w^2$ is finite, strictly increasing integer submultiplicative and supermultiplicative sequences can have two distinct exponential rates. Conversely, isolated outputs permit oscillation even when every finite positive-power integral of w diverges. These results distinguish integrated width from integer connectivity.

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1. The question and the result

Fekete's lemma concerns real sequences satisfying $a_{n+m} \le a_n + a_m$ for all positive integer pairs. Restricted-pair variants can still imply convergence of a_n/n. In particular, Füredi and Ruzsa [4] establish convergence when the inequality is required eventually on $n \le m \le \mu n$, for any fixed $\mu > 1$. Section 7 of their accessible preprint asks whether some width $r(n) = o(n)$ could replace the fixed relative width, and records the expectation that it cannot. The answer has a sharp threshold: writing $r(n) = n\,w(\log n)$, convergence is forced exactly when $\int^\infty w^2 = \infty$, for monotone counting sequences in every case (Corollary 1) and for arbitrary real sequences whenever $r(n) \ge 1$ eventually (Theorem 5 and the paragraph following Corollary 1), while genuinely slow widths such as $r(n) = n^\alpha$, $\alpha < 1$, or $r(n) = n/(\log n)^{1/2+\varepsilon}$ admit oscillating integer examples. In particular we give a sufficient sublinear width, $r(n) = n/\sqrt{\log n}$, for arbitrary real sequences, and the limit equals $\inf_{k \ge N_0} a_k/k$ exactly as in their Theorem 2 (Remark 4). More generally, a divergent square integral of the relative width forces convergence when adjacent integer pairs are eventually admissible. No monotonicity or normalized-slope bound is needed in that statement, and the limit may be $-\infty$.

For nondecreasing nonnegative linearly bounded sequences, including logarithms of monotone exponentially bounded counting sequences, the adjacent-pair condition can be omitted. In this class we obtain a complete square-integral dichotomy for every measurable nonincreasing relative width, including jumps and integer gaps. The finite-integral counterexamples are themselves strictly increasing integer counting sequences. For unrestricted real sequences, however, no criterion based only on divergence of finite positive-power integrals can replace integer connectivity: an explicit deep-jump width and isolated odd outputs give a counterexample.

We strengthen the balanced-predecessor method of [6]: it suffices that every sufficiently large output has a legal split of gap $O(N/[(\log N)^{3/2}(\log\log N)^\eta])$ for some $\eta > 1/2$. This includes all earlier bounds $O(N^\alpha)$, $\alpha < 1$, and $O(N/(\log N)^{3/2+\varepsilon})$, $\varepsilon > 0$. A growing local-maximum regularization repairs missing adjacent pairs, and a separate low-value argument transfers convergence back. A new staircase witness satisfies the iterated-logarithmic condition but no bound from either earlier family.

Vanishing normalized corrections that absorb an oscillatory defect have earlier uses. The error-correction arguments in [4] are relevant background. In a vector-valued setting, Kulikov and Shao [5, Section 3], following a MathOverflow discussion [7], compensate a squared angular discrepancy by a small radial correction. Their norm ratios converge; the scalar inequality studied here is a separate condition. Our construction uses exact dyadic homogeneity to make the scalar discrepancy quadratic in the relative corridor width, and a shifted square-width tail to pay it without assuming regular variation.

For the complementary convergence theorem, dyadic rescalings of a monotone sequence decrease pointwise. Integer rounding is handled before

any limiting argument. A divergent sum of admissible squared midpoint steps then forces the limit to be convex in the distributional sense. Classical mollification and monotonicity identify this limit at every point; dyadic homogeneity makes it linear. This handles jumps of w for which adjacent integer pairs need not be eventually admissible. When adjacent pairs are admissible, a monotone prefix envelope preserves the entire corridor, and its dense record indices transfer convergence back to an arbitrary real sequence by continuous interpolation. The latter step avoids any a.e.-to-pointwise inference without monotonicity.

All logarithms below are natural. We call a positive sequence exponentially bounded if $b_n \le \exp(Kn)$ for some finite K and all $n \ge 1$.

The convergence and oscillation classification.

Monotone counting class	Theorem 3 and Corollary 1 give the square-integral dichotomy without an adjacent-pair condition.
Arbitrary real sequences	Theorem 5 gives the divergent-integral conclusion with eventual adjacent legality; the limit lies in $\mathbb{R} \cup \{-\infty\}$.
Balanced predecessors	Theorem 7 replaces adjacent legality by the stated iterated-logarithmic legal-split bound.
Finite-integral examples	Theorem 2 constructs increasing integer examples with two distinct exponential rates.
Integer connectivity	Theorem 14 separates all finite-power integral budgets from legal access to individual integer outputs.

Corollary 1. *Fix a measurable nonincreasing $w : [T, \infty) \to [0, 1]$. Every positive nondecreasing exponentially bounded integer sequence with eventual exact submultiplicativity on $n \le m \le n + nw(\log n)$ has a limiting nth-root rate if and only if*

$$\int_T^\infty w(s)^2 \, ds = \infty.$$

Proof. The divergent case is Theorem 3. In the finite case the sequence (u_n) of Theorem 2 belongs to the stated class and has distinct logarithmic endpoint rates. □

If $nw(\log n) \ge 1$ eventually, the same integral dichotomy holds for universal convergence of a_n/n in $\mathbb{R} \cup \{-\infty\}$ among arbitrary real sequences satisfying the corridor subadditivity condition. The divergent case is Theorem 5; the finite case follows by taking logarithms of the integer counterexample.

Theorem 2. *Let $T \in \mathbb{R}$ and let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing, with*

$$\int_T^\infty w(s)^2 \, ds < \infty. \tag{1}$$

There exist positive strictly increasing exponentially bounded integer sequences (u_n) and (v_n), and an integer N_0, such that

$$u_{n+m} \le u_n u_m, \qquad v_{n+m} \ge v_n v_m \quad \text{if } N_0 \le n \le m \le n + nw(\log n), \tag{2}$$

while

$$\liminf_{n\to\infty} \frac{\log u_n}{n} = \liminf_{n\to\infty} \frac{\log v_n}{n} = 2 - \frac{1}{100},$$
$$\limsup_{n\to\infty} \frac{\log u_n}{n} = \limsup_{n\to\infty} \frac{\log v_n}{n} = 2 + \frac{1}{100}.$$

The sequences may depend on w. Jump discontinuities and zeros of w are allowed.

Equivalently, the permitted width $r(x) = xw(\log x)$ has nonincreasing relative size and satisfies

$$0 \le r(x) \le x, \qquad \int^{\infty} \frac{r(x)^2}{x^3}\, dx < \infty.$$

This follows by the substitution $x = e^s$. The condition restricts which pairs are tested; it allows no error in the tested inequalities.

2. Proof of Theorem 2

Put

$$\varepsilon = \frac{1}{100}, \qquad \omega = \frac{2\pi}{\log 2}, \qquad A_0(x) = x[2+\varepsilon \sin(\omega \log x)], \qquad B = \varepsilon\omega(1+\omega).$$

Then $A_0(2x) = 2A_0(x)$ and

$$A_0''(x) = \frac{\varepsilon\omega}{x}[\cos(\omega \log x) - \omega \sin(\omega \log x)], \qquad |A_0''(x)| \le \frac{B}{x}.$$

For $t \ge T + \log 3$ and $x \ge 3e^T$, define

$$Q(t) = \int_{t-\log 3}^{\infty} w(s)^2\, ds, \qquad g(x) = xQ(\log x), \qquad C = \frac{B}{\log(3/2)},$$

and set $A(x) = A_0(x) + Cg(x)$, $V(x) = 4x - A(x)$.

The normalized correction vanishes

Monotonicity and (1) imply $w(t) \to 0$. Also $Q(t) \to 0$, so $g(x)/x \to 0$. The tail integral is locally absolutely continuous, and hence so is g, with

$$g'(x) = Q(\log x) - w(\log x - \log 3)^2 \quad \text{for almost every sufficiently large } x.$$

Both terms tend to zero. Thus, for every $\eta > 0$, $|g'(x)| \le \eta$ for almost every sufficiently large x. No derivative of w is required.

The oscillatory defect is quadratic

Suppose $n \le m \le n + nw(\log n)$, and write $N = n + m$, $d = m - n$. Here $m \le 2n$ and $N \le 3n$. Taylor expansion about $N/2$, using dyadic homogeneity and $|A_0''| \le B/n$ on $[n, m]$, gives

$$\begin{aligned} |A_0(N) - A_0(n) - A_0(m)| &= |2A_0(N/2) - A_0(n) - A_0(m)| \\ &\le \frac{Bd^2}{4n} \le \frac{Bn}{4} w(\log n)^2 \le \frac{BN}{8} w(\log n)^2. \end{aligned} \tag{3}$$

These estimates hold for real inputs in the domain, and therefore for integer inputs.

A shifted tail pays the defect

For $x = n, m$,

$$Q(\log x) - Q(\log N) = \int_{\log x - \log 3}^{\log N - \log 3} w(s)^2 \, ds.$$

For sufficiently large n, the interval is contained in $[T, \infty)$. Its upper endpoint is at most $\log n$, since $N \le 3n$. Monotonicity therefore gives a lower bound $w(\log n)^2$ for the integrand throughout the interval. Also $n, m \le 2N/3$. Consequently

$$\begin{aligned} g(n) + g(m) - g(N) &= n[Q(\log n) - Q(\log N)] + m[Q(\log m) - Q(\log N)] \\ &\ge \left[n \log \frac{N}{n} + m \log \frac{N}{m} \right] w(\log n)^2 \\ &\ge N \log(3/2) w(\log n)^2. \end{aligned} \tag{4}$$

Combining (3) and (4) yields

$$A(n) + A(m) - A(N) \ge \frac{7BN}{8} w(\log n)^2 \ge 0.$$

Thus $A(N) \le A(n) + A(m)$ and $V(N) \ge V(n) + V(m)$ on the corridor. If $w(\log n) = 0$, only $m = n$ is allowed, and the same conclusion holds. The shifted tail makes no comparison assumption between nearby values of w.

Monotonicity, oscillation and integer rounding

The derivative

$$A_0'(x) = 2 + \varepsilon[\sin(\omega \log x) + \omega \cos(\omega \log x)]$$

lies uniformly in $(1, 3)$. Since g' tends uniformly to zero off a null set, A' and V' are greater than one almost everywhere on a sufficiently late tail. Local absolute continuity gives

$$A(y) - A(x) \ge y - x, \qquad V(y) - V(x) \ge y - x$$

for all tail points $y > x$. Moreover

$$\frac{A(x)}{x} = 2 + \varepsilon \sin(\omega \log x) + o(1), \qquad \frac{V(x)}{x} = 2 - \varepsilon \sin(\omega \log x) + o(1).$$

Integers nearest $2^{j+1/4}$ and $2^{j+3/4}$ approach the two extreme sine phases. Both normalized slopes therefore have lower endpoint $2 - \varepsilon$ and upper endpoint $2 + \varepsilon$.

For sufficiently large integers n, set

$$u_n = \lceil e^{A(n)} \rceil, \qquad v_n = \lfloor e^{V(n)} \rfloor.$$

The increment bounds and positivity of the exponents make both sequences strictly increasing on a late tail. The product $u_n u_m$ is an integer at least $e^{A(n)+A(m)} \ge e^{A(N)}$, so $u_N \le u_n u_m$. Likewise the integer $v_n v_m$ is at most $e^{V(n)+V(m)} \le e^{V(N)}$, so $v_N \ge v_n v_m$.

Rounding changes the logarithms by quantities tending to zero. Increase N_0 so that both first tail values exceed N_0, and assign the earlier terms the values $1, 2, \dots, N_0 - 1$. This supplies positive strictly increasing sequences on all of $\mathbb{N}$. The inequalities concern only tail inputs. Bounded normalized exponents, with a finite prefix absorbed into a constant, give exponential boundedness. The claimed endpoints follow. □

3. Convergence at divergent square integral

Theorem 3. *Let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing, and suppose*

$$\int_T^\infty w(t)^2 \, dt = \infty.$$

Let (a_n) be nondecreasing, with $0 \le a_n \le Kn$ for all sufficiently large integers n and some finite K. If, for some N_0,

$$a_{n+m} \le a_n + a_m \quad \text{whenever } N_0 \le n \le m \le n + nw(\log n),$$

then a_n/n converges to a finite limit. In particular, every positive nondecreasing exponentially bounded integer sequence satisfying the submultiplicative inequality on this corridor has a limiting nth-root rate.

Proof. For sufficiently large real x put

$$A(x) = a_{\lceil x \rceil}, \qquad F_k(x) = 2^{-k} A(2^k x).$$

On every compact subinterval of $(0, \infty)$ these expressions are defined for all sufficiently large k. The allowed diagonal pair and monotonicity give exactly

$$A(2x) = a_{\lceil 2x \rceil} \le a_{2\lceil x \rceil} \le 2a_{\lceil x \rceil} = 2A(x).$$

Thus $F_{k+1} \le F_k$, and F_k is nondecreasing in x, with $0 \le F_k(x) \le K(x + 2^{-k})$. Consequently F_k decreases pointwise to a finite nondecreasing function f, converges to it locally in L^1, and satisfies

$$0 \le f(x) \le Kx, \qquad f(2x) = 2f(x).$$

The functions F_k are step functions; no continuity is assumed.

Admissible midpoint steps and exact integer rounding

Fix $0 < a < b$ and a nonnegative smooth test function ϕ with compact support in (a, b). Set

$$q_k = w(k \log 2 + \log(2b)).$$

For large k, suppose $0 < h \le aq_k/16$ and $2^k a q_k \ge 16$. For $x \in [a, b]$ put

$$n = \lceil 2^k (x - h) \rceil, \qquad m = \lceil 2^k (x + h) \rceil.$$

Since $q_k \le 1$, we have $h \le a/16$, and

$$2^k a/2 \le n \le 2^k b + 1 \le 2^{k+1} b,$$

$$m - n \le 2^{k+1} h + 1 \le 2^k a q_k/8 + 1 \le 2^k a q_k/2 \le nw(\log n).$$

Here $w(\log n) \geq q_k$ follows from $\log n \leq k \log 2 + \log(2b)$. Both indices exceed the threshold for large k. Also $n+m \geq \lceil 2^{k+1}x \rceil$. Monotonicity and the literal integer hypothesis therefore imply

$$F_{k+1}(x) \leq \frac{F_k(x-h) + F_k(x+h)}{2}. \tag{5}$$

No enlargement of the permitted integer corridor has been made.

If $w(t) \to 0$, the integral test for the nonincreasing function w^2 gives $\sum_k q_k^2 = \infty$. Call a scale good if $2^k a q_k \geq 16$. At bad scales $q_k^2 < (16/a)^2 4^{-k}$, so their squared sum is finite. On the good scales choose $h_k = a q_k/16$. Then $h_k \to 0$, $\sum h_k^2 = \infty$, and (5) holds. If instead $\lim w(t) = L > 0$, choose $h_k = c/\sqrt{k}$ for any fixed $c > 0$; all sufficiently large scales are good and the same properties hold. In either case there is an unbounded set G of admissible scales such that

$$h_k \longrightarrow 0 \quad (k \in G), \qquad \sum_{k \in G} h_k^2 = \infty.$$

Bad scales are omitted only from the midpoint comparison, not from the decreasing dyadic sequence.

A telescoping argument forces convexity

Write $D_k = F_k - F_{k+1} \geq 0$ and $\Delta_h \phi(x) = \phi(x-h) + \phi(x+h) - 2\phi(x)$. Multiplying (5) by ϕ and translating variables gives, for $k \in G$,

$$\int D_k \phi \geq -\frac{1}{2} \int F_k \Delta_{h_k} \phi.$$

All integrands have support in one fixed compact subinterval of $(0, \infty)$ for large k. Since $\Delta_h \phi / h^2 \to \phi''$ uniformly and $F_k \to f$ locally in L^1 with uniform local bounds,

$$\frac{1}{h_k^2} \int F_k \Delta_{h_k} \phi \longrightarrow \int f \phi''.$$

If the last integral were negative, then for some $\eta > 0$ and every sufficiently late good scale, $\int D_k \phi \geq \eta h_k^2$. This contradicts the finite nonnegative telescoping sum

$$\sum_{k \geq k_0} \int D_k \phi = \int (F_{k_0} - f)\phi < \infty.$$

Hence $\int f \phi'' \geq 0$ for every nonnegative smooth compactly supported ϕ: the limit is convex in the distributional sense.

We include the classical representative argument to make the pointwise conclusion explicit. On an interior compact interval convolve f with a nonnegative smooth approximate identity. The resulting smooth functions have nonnegative second derivative and are convex. Local boundedness and the elementary secant bounds for convex functions make them uniformly Lipschitz on a smaller interval. A subsequence converges uniformly there to a continuous convex function g, while approximate identities converge to f locally in L^1. Thus $g = f$ almost everywhere. These representatives agree on overlapping intervals. At any point, take agreement points approaching from

both sides; monotonicity of f and continuity of g squeeze f to g. Therefore f itself is continuous and convex everywhere. This step uses monotonicity, not only local boundedness.

Linearity and convergence at every index

Extend f to zero by $f(0) = 0$. The bounds $0 \le f(x) \le Kx$ give a continuous convex extension, by taking positive endpoints tending to zero in the convex inequality. Thus $f(x)/x$ is nondecreasing. Its equal values at x and $2x$ make it constant on every $[x, 2x]$, and hence everywhere: $f(x) = \ell x$.

Finally $F_k \to \ell x$ uniformly on $[1, 2]$. Indeed, use pointwise convergence at finitely many grid points and squeeze between adjacent endpoints using monotonicity of F_k in x and continuity of ℓx. This avoids applying Dini's theorem to discontinuous functions. For an integer n, take $k = \lfloor \log_2 n \rfloor$ and $x = n/2^k \in [1, 2)$. Then

$$\frac{a_n}{n} = \frac{F_k(x)}{x} \longrightarrow \ell.$$

For integer counting sequences set $a_n = \log u_n$. The hypotheses give a nondecreasing nonnegative linearly bounded sequence, and exponentiating its limiting slope proves the final assertion. □

Remark 4. *In Theorems 3 and 5 the limit equals* $\inf_{k \ge N_0} a_k/k$*, where* N_0 *is the threshold of the corridor hypothesis, matching the form of the conclusion in* [4, Theorem 2]*. Indeed the diagonal pair* (k, k) *is admissible for every width, so* $a_{2^j k} \le 2^j a_k$ *for* $k \ge N_0$ *and* $j \ge 0$*. Dividing by* $2^j k$ *and letting* $j \to \infty$ *along the convergent sequence* a_n/n *gives* $\lim a_n/n \le a_k/k$*, while the limit is trivially at least the infimum.*

4. Removing monotonicity when adjacent pairs are admissible

Theorem 5. *Let* $w : [T, \infty) \to [0, 1]$ *be measurable and nonincreasing, with*

$$\int_T^\infty w(t)^2\, dt = \infty, \qquad nw(\log n) \ge 1 \quad \textit{for all sufficiently large integers } n.$$

If an arbitrary real sequence (a_n) *satisfies*

$$a_{n+m} \le a_n + a_m \quad \textit{eventually whenever } n \le m \le n + nw(\log n),$$

then a_n/n *converges in* $\mathbb{R} \cup \{-\infty\}$*. No monotonicity, sign or growth bound is assumed. Consequently every positive real sequence satisfying the corresponding submultiplicative inequality has a finite* n*th-root limit in* $[0, \infty)$*; for positive integer sequences this limit is at least one.*

Proof. The hypotheses give, for every sufficiently large integer n,

$$a_{2n} \le 2a_n, \qquad a_{2n+1} \le a_n + a_{n+1}. \tag{6}$$

These inequalities imply an automatic upper linear bound. Choose n_0 large enough for (6), and choose a finite C bounding a_j/j for $n_0 \le j \le 2n_0 + 1$.

Strong induction using the even and odd alternatives in (6) gives $a_j \le Cj$ for all $j \ge n_0$. Enlarge C to be positive and absorb the finite prefix if needed.

Thus $S = \limsup a_n/n$ is finite or $-\infty$. In the latter case the required extended-real convergence already holds. Otherwise add a linear term Dn so that the shifted limsup is positive. This preserves every pair inequality. We use a_n for the shifted sequence until the last step; no lower bound is imposed.

The prefix envelope preserves the corridor

Put

$$B_n = \max\{0, a_1, \dots, a_n\}.$$

It is nondecreasing and $0 \le B_n \le C'n$ for some finite C'. For a sufficiently large allowed pair $n \le m$, consider any $j \le n + m$. If $j \le n$, then $a_j \le B_n \le B_n + B_m$. If $n < j \le 2n$, split $j = p + q$ with $p = \lfloor j/2 \rfloor$ and $q = \lceil j/2 \rceil$. Both indices are eventually in the original tail, their gap is at most one, and $q \le n$. Hence

$$a_j \le a_p + a_q \le 2B_n \le B_n + B_m.$$

If $2n < j \le n + m$, use $p = n$, $q = j - n$. Then $n < q \le m$, so this split lies in the original permitted corridor and again $a_j \le B_n + B_m$. The same upper bound holds for zero. Taking the maximum proves

$$B_{n+m} \le B_n + B_m.$$

Theorem 3 therefore gives $B_n/n \to L \ge 0$, with $L \ge \limsup a_n/n > 0$.

Continuous interpolation and dense records

Let A be the piecewise-linear interpolant of a_n. It satisfies $A(2x) \le 2A(x)$ for all sufficiently large real x. Indeed, write $x = n + \alpha$. If $0 \le \alpha \le 1/2$, then

$$\begin{aligned} A(2x) &= (1 - 2\alpha)a_{2n} + 2\alpha a_{2n+1} \\ &\le (1 - 2\alpha)2a_n + 2\alpha(a_n + a_{n+1}) = 2A(x). \end{aligned}$$

For $1/2 \le \alpha \le 1$,

$$\begin{aligned} A(2x) &= (2 - 2\alpha)a_{2n+1} + (2\alpha - 1)a_{2n+2} \\ &\le (2 - 2\alpha)(a_n + a_{n+1}) + (2\alpha - 1)2a_{n+1} = 2A(x). \end{aligned}$$

Only the three diagonal or adjacent-diagonal integer pairs are used; A need not be monotone.

For each sufficiently large n, choose $j_n \le n$ with $B_n = a_{j_n}$. Such records exist because $B_n \to \infty$. They satisfy $j_n \to \infty$, since finitely many fixed terms have bounded values. For every $0 < \eta < L$, eventually

$$(L - \eta)n \le B_n = a_{j_n} \le B_{j_n} \le (L + \eta)j_n.$$

It follows that

$$j_n/n \longrightarrow 1, \qquad a_{j_n}/n = B_n/n \longrightarrow L.$$

The high records are asymptotically next to every large index, regardless of arbitrarily negative values between records.

Records force a lower bound at every late point

Fix any sufficiently large real x. Put $n_k = \lfloor 2^k x \rfloor$ and $y_k = j_{n_k}/2^k$. Then

$$y_k \longrightarrow x, \qquad 2^{-k} A(2^k y_k) = \frac{B_{n_k}}{2^k} \longrightarrow Lx.$$

For large k, the points y_k lie in one fixed late neighborhood of x on which $A(2y) \le 2A(y)$ holds, as it does on every successive doubling. Iterating gives

$$A(y_k) \ge 2^{-k} A(2^k y_k).$$

Continuity of the fixed interpolant A at x yields $A(x) \ge Lx$. Since x was arbitrary on a sufficiently late tail, $a_n/n \ge L$ for every sufficiently large integer n. Conversely $a_n \le B_n$ and $B_n/n \to L$ give the matching upper bound. Hence $a_n/n \to L$ for the shifted sequence. Subtract D to obtain the finite limit for the original sequence. Together with the earlier $S = -\infty$ case, this proves the real-sequence assertion.

For a positive real sequence u_n take $a_n = \log u_n$. Exponentiating its extended-real limit gives a finite root limit, with $e^{-\infty} = 0$. If the terms are positive integers, $a_n \ge 0$, so the limit is at least one. □

Remark 6. *Under the width hypotheses of Theorem 5, sign reversal also gives convergence of a_n/n in $\mathbb{R} \cup \{+\infty\}$ for every eventually corridor-superadditive real sequence. Thus a positive real corridor-supermultiplicative sequence has a root limit in $(0, \infty]$: applying the automatic upper bound to the negative logarithms supplies a positive lower exponential bound. For positive integer sequences the submultiplicative limit lies in $[1, \infty)$ and the supermultiplicative limit in $[1, \infty]$.*

5. Balanced integer predecessors

The following theorem strengthens the balanced-predecessor criterion of [6]. This section supersedes that record. We give its full argument using Theorem 5 as the adjacent-corridor input. Classical balanced recursions [3, 1, 2] and low-slope propagation provide background. Wallace's author summary [8] concerns particular extremal recurrences in sequential search; its displayed output $n + 2p + 1$ differs from the child-index sum $2n + 2p$.

All logarithms are natural. Fix $T \in \mathbb{R}$ and a measurable nonincreasing function $w : [T, \infty) \to (0, 1]$. A pair of positive integers (p, q) is *legal* if

$$p \le q \le p + pw(\log p). \tag{7}$$

Only indices large enough to lie in the domain will be used. A real sequence $a = (a_n)_{n \ge 1}$ is eventually corridor-subadditive if there is n_0 such that

$$a_{p+q} \le a_p + a_q \quad \text{for every legal pair with } p \ge n_0. \tag{8}$$

The endpoint in (7) is literal: there is no ceiling or replacement of a width less than one by one.

Theorem 7. *Assume*

$$\int_T^\infty w(s)^2\,ds = \infty. \tag{9}$$

Suppose that every sufficiently large integer N has a legal split $N = p + q$ satisfying

$$q - p \le \frac{CN}{(\log N)^{3/2}(\log\log N)^\eta}, \qquad \eta > \tfrac{1}{2}, \tag{10}$$

for fixed finite $C > 0$ and fixed η. Then every real sequence satisfying (8) *has*

$$\lim_{n\to\infty} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}.$$

No sign, monotonicity or growth bound on a is required.

The constant C and the onset of the splits may depend on w. Its predecessor condition includes every gap bound $O(N^\alpha)$, $\alpha < 1$, and every bound $O(N/(\log N)^{3/2+\varepsilon})$, $\varepsilon > 0$: their ratios to the right side of (10) tend to zero for any fixed $\eta > 1/2$. Eventual adjacent legality also implies (10). Proposition 11 shows that the converses to these implications fail.

Local-maximum regularization

We apply the adjacent-corridor conclusion of Theorem 5 to a new majorant, proving its integer hypothesis rather than imposing it on the original sequence.

On a sufficiently late tail set

$$h(x) = \frac{Hx}{\sqrt{\log x}(\log\log x)^\eta}, \qquad H\log 2 > 4C. \tag{11}$$

This function is eventually increasing and concave, with $h \to \infty$, $h = o(x)$ and $h' \to 0$. To check these claims and the margins below, put

$$\rho(s) = s^{-1/2}(\log s)^{-\eta}, \qquad \lambda(s) = \frac{1}{2s} + \frac{\eta}{s\log s}.$$

Then $\rho' = -\lambda\rho$, $\lambda' = O(s^{-2})$, and

$$h'(x) = H\rho(\log x)[1 - \lambda(\log x)],$$

$$h''(x) = \frac{H\rho(\log x)}{x}[\lambda(\log x)^2 - \lambda(\log x) - \lambda'(\log x)] < 0$$

eventually. In particular, $h(n+1) - h(n) = o(1)$. Since $\rho''(s) = O(\rho(s)s^{-2})$, Taylor's formula gives, uniformly for c in a fixed bounded interval,

$$\rho(s + c) = \rho(s)\left[1 - \frac{c}{2s} + o(s^{-1})\right]. \tag{12}$$

Define

$$k_n = \lfloor h(n) \rfloor, \qquad b_n = \max_{|j-n|\le k_n} a_j \tag{13}$$

for all sufficiently large n, with integer j in the maximum. Early values of b may be chosen arbitrarily. Each maximum is finite.

Lemma 8. *Eventually*

$$b_{2n} \le 2b_n, \qquad b_{2n+1} \le b_n + b_{n+1}. \tag{14}$$

Proof. For every j in the b_{2n} or b_{2n+1} window, choose a legal predecessor split promised by the appropriate hypothesis. Both children tend to infinity, since their difference is $o(j)$. Their distance from the relevant half-center is bounded by half the output-window radius, half the predecessor gap, and an absolute rounding constant.

This bound is at most

$$\frac{h(2n)}{2} + (C + o(1))\frac{n}{(\log n)^{3/2}(\log\log n)^{\eta}} + O(1).$$

Indeed $j = 2n + O(h(2n) + 1) = 2n(1 + o(1))$, so the predecessor gap at j, divided by two, has precisely the second bound. Writing $\ell = \log n$, equation (12) gives

$$h(n) - \frac{h(2n)}{2} = \left(\frac{H\log 2}{2} + o(1)\right)\frac{n}{\ell^{3/2}(\log \ell)^{\eta}}. \tag{15}$$

The choice in (11) leaves an unbounded spare margin, exceeding the predecessor payment and every absolute rounding error. Also $h(n+1) - h(n) = o(1)$, so floors and the center shift by one are paid. Thus both children lie in $[n - k_n, n + k_n]$ for an even center. For an odd center, both lie in the intersection

$$[n - k_n, n + k_n] \cap [n + 1 - k_{n+1}, n + 1 + k_{n+1}].$$

Apply the actual inequality (8) to each selected split. In the odd case its right side is at most $2\min(b_n, b_{n+1}) \le b_n + b_{n+1}$, even if these values are negative. Taking maxima proves (14). □

Lemma 9. *Put $v(s) = w(s+\log 2)/8$. Then b is eventually corridor-subadditive for the literal width v.*

Proof. The diagonal case follows from Lemma 8. For an off-diagonal pair write $m = n + d$, where

$$1 \le d \le \frac{nw(\log(2n))}{8}, \qquad 1 \le \frac{m}{n} \le \frac{9}{8}. \tag{16}$$

Let $j = n + m + e$ be any integer in the b_{n+m} window. Set

$$\theta = \frac{h(n)}{h(n) + h(m)}, \qquad p = n + \lfloor \theta e \rfloor, \qquad q = j - p. \tag{17}$$

We check the window inclusions and original legality separately.

The spare allocation margin $h(n) + h(m) - h(n+m)$ tends to infinity uniformly for $1 \le m/n \le 9/8$. With $r = m/n$ and $\ell = \log n$, equation (12) yields

$$\begin{aligned} h(n) + h(m) - h(n+m) = \left(\frac{H}{2} + o(1)\right)\frac{n}{\ell^{3/2}(\log \ell)^{\eta}} \\ \cdot [(1+r)\log(1+r) - r\log r]. \end{aligned} \tag{18}$$

The bracket is at least $2\log 2$. The remainder is uniform because $\log r$ and $\log(1+r)$ range over fixed bounded intervals. Both θ and $1-\theta$ stay bounded away from zero. Consequently

$$\theta h(n+m) < h(n) - 2, \qquad (1-\theta)h(n+m) < h(m) - 2$$

eventually. The floor error in (17) is at most one. Thus $|p-n| \le k_n$ and $|q-m| \le k_m$.

To check legality, exact subtraction gives

$$q - p = d + (1-2\theta)e + 2\{\theta e\}.$$

The mean-value theorem, eventual concavity and the positive allocation margin imply

$$|(1-2\theta)e| \le \frac{[h(m)-h(n)]h(n+m)}{h(n)+h(m)} \le h'(n)d = o(d)$$

uniformly in (16). Therefore, for sufficiently large n,

$$q - p \ge d/2 > 0, \qquad q - p \le 2d + 2 \le 4d \le \frac{nw(\log(2n))}{2}.$$

Since $h = o(n)$, we have $n/2 \le p \le 2n$ eventually. Nonincrease of w gives

$$pw(\log p) \ge \frac{n}{2} w(\log(2n)) \ge q - p.$$

Thus (p, q) is an original legal pair with $p + q = j$. Its true subadditivity inequality bounds a_j by $b_n + b_m$. Take the maximum over the entire output window. No continuity of w has been used. □

6. Convergence and transfer of low values

The function

$$V(s) = \max\{v(s), e^{-s}\}$$

is positive, nonincreasing and at most one on a late tail. Its square integral diverges. An integer pair allowed by V is either v-allowed or diagonal/adjacent, since $ne^{-\log n} = 1$. Lemmas 8 and 9 therefore give every V-corridor inequality for b. Theorem 5 yields

$$b_n/n \longrightarrow L \in \mathbb{R} \cup \{-\infty\}. \tag{19}$$

As $a_n \le b_n$, the case $L = -\infty$ already proves the desired conclusion. The finite case needs an additional argument.

Lemma 10 (Dyadic escape). *For every fixed positive integer n, the values*

$$\frac{xw(\log x)}{h(x)}, \qquad x = 2^k n,$$

are unbounded as $k \to \infty$.

Proof. If these ratios were bounded eventually, then on the logarithmic lattice $s = \log n + k \log 2$ we would have $w(s) \le C'\rho(s)$ for some finite C'. For a point between consecutive lattice points, monotonicity bounds w by its value at the left endpoint, and (12) shows that ρ at that endpoint is at most twice its value at the point, eventually. Hence $w(s) \le 2C'\rho(s)$ on a full tail. But

$$\int^{\infty} \rho(s)^2 \, ds = \int^{\infty} \frac{ds}{s(\log s)^{2\eta}} < \infty \qquad (2\eta > 1).$$

This contradicts (9). □

Completion of Theorem 7. Suppose L in (19) is finite, but a_n/n does not tend to L. As $a \le b$, there are $\delta > 0$ and arbitrarily large n with $a_n/n \le L - \delta$. Fix such an n in the subadditivity tail. Diagonal legality gives

$$a_x/x \le a_n/n \le L - \delta \quad (x = 2^k n).$$

By Lemma 10 there are arbitrarily large such x with

$$xw(\log x) \ge 4h(2x), \qquad h(2x) \le x/2. \tag{20}$$

Here $h(2x)/h(x)$ is bounded on a late tail for the choice (11). For every integer t with $|t| \le k_{2x}$ the pair consisting of x and $x + t$ is legal. For $t \ge 0$, use the first inequality in (20). For $t < 0$, the smaller child is $x + t \ge x/2$ and

$$(x + t)w(\log(x + t)) \ge (x/2)w(\log x) \ge 2h(2x) \ge |t|.$$

For any $\varepsilon > 0$, (19) gives $a_y \le b_y \le (L+\varepsilon)y$ for all sufficiently large y. Hence, uniformly throughout the output window,

$$a_{2x+t} \le a_x + a_{x+t} \le (L - \delta)x + (L + \varepsilon)(x + t).$$

Taking the maximum and dividing by $2x$ gives

$$\frac{b_{2x}}{2x} \le L - \frac{\delta}{2} + \frac{\varepsilon}{2} + o(1),$$

since $h(2x) = o(x)$. Choose $\varepsilon < \delta/2$ and let these dyadic x tend to infinity. This contradicts (19). Thus the limit of a_n/n is L as required. □

7. A width beyond adjacent and faster predecessors

Proposition 11. *There is a positive nonincreasing measurable width $w \to 0$ satisfying* (9) *and* (10) *with $\eta = 1$, such that adjacent pairs are illegal in infinitely many windows of lengths tending to infinity. For this width no legal predecessor bound $O(N^\alpha)$, with fixed $\alpha < 1$, or $O(N/(\log N)^{3/2+\varepsilon})$, with fixed $\varepsilon > 0$, holds eventually. The example can also be chosen so that $\int_0^\infty w(s)^p\, ds = \infty$ for every finite $p > 0$.*

Proof. Set $b_1 = 16$ and recursively define

$$\begin{aligned} A_j &= 2^{b_j}, \qquad d_j = b_j \lceil \sqrt{b_j} \rceil \lceil \log_2 b_j \rceil, \\ L_j &= \left\lfloor \frac{A_j}{16 d_j} \right\rfloor, \qquad M_j = A_j + L_j, \qquad b_{j+1} = M_j^3. \end{aligned} \tag{21}$$

Let $w(s) = 1$ for $0 \le s < \log A_1$, and

$$w(s) = M_j^{-1} \quad (\log A_j \le s < \log A_{j+1}).$$

It has the asserted regularity. Its jth contribution to the square integral is

$$\frac{(M_j^3 - b_j)\log 2}{M_j^2} = \left(M_j - \frac{b_j}{M_j^2} \right) \log 2 \longrightarrow \infty.$$

At integers $A_j \le n \le A_j + L_j - 1$ we have $nw(\log n) < 1$; therefore $(n, n+1)$ is illegal. For $M_j \le n < A_{j+1}$ it is legal.

Every even output has a diagonal split. For an odd output $N = 2n + 1$ outside a forbidden window use $(n, n+1)$. Inside the jth forbidden window use

$$p = A_j - 1, \qquad q = 2n - A_j + 2, \qquad q - p = 2(n - A_j) + 3 \le 2L_j + 1. \tag{22}$$

At $p = A_j - 1$, the width is at least $1/b_j$: before the first drop it is one, and subsequently $b_j = M_{j-1}^3 \ge M_{j-1}$. For $b_j \ge 16$,

$$\frac{A_j - 1}{b_j} \ge \frac{A_j}{8d_j} + 1 \ge 2L_j + 1.$$

Thus (22) is actually legal. Moreover $A_j/(16d_j) \ge 1$, so

$$2L_j + 1 \le \frac{3A_j}{16d_j} < \frac{A_j}{4d_j}.$$

In this window $2A_j < N < 3A_j$ and $\log N \le b_j$ for $b_j \ge 16$. Also $\log\log N \le \log b_j \le \lceil \log_2 b_j \rceil$. Consequently

$$q - p \le \frac{A_j}{4d_j} \le \frac{N}{(\log N)^{3/2} \log\log N}.$$

The adjacent gap one satisfies this bound eventually. This proves (10) with $\eta = 1$. Here $d_j \le b_j^3$ and $2^{b_j} \ge 16b_j^3$ for integers $b_j \ge 16$, so the lower bound $A_j/(16d_j) \ge 1$ used above holds at every stage.

At the last forbidden output $N_j = 2(A_j + L_j - 1) + 1$, any smaller child $p \in [A_j, \lfloor N_j/2 \rfloor]$ has $pw(\log p) < 1$ and cannot produce a legal odd sum. Thus every legal split has $p \le A_j - 1$ and gap at least $2L_j + 1$. The split in (22) attains this bound. Since

$$L_j \sim \frac{A_j}{16b_j^{3/2} \log_2 b_j}, \qquad b_j = \log_2 A_j,$$

the minimum gap divided by N_j^α tends to infinity for each fixed $\alpha < 1$. Its ratio to $N_j/(\log N_j)^{3/2+\varepsilon}$ also tends to infinity for every fixed $\varepsilon > 0$, since $b_j^\varepsilon / \log b_j \to \infty$. Thus neither earlier family of predecessor bounds applies.

For the last assertion replace only the last recurrence in (21) by $b_{j+1} = M_j^{j+3}$. All legality arguments used only $b_j \ge M_{j-1}$ and remain valid. For each fixed $p > 0$ the jth plateau contributes

$$\left(M_j^{j+3-p} - \frac{b_j}{M_j^p} \right) \log 2 \longrightarrow \infty$$

to $\int w^p$. Here eventually $j + 3 - p \ge 1$, and $b_j/M_j^p \to 0$. This proves simultaneous divergence for all finite positive powers. □

Remark 12. *Replacing d_j by b_j^2 in* (21)*, so that $L_j = \lfloor A_j/(16b_j^2) \rfloor$, the same argument gives a width satisfying* (9) *for which every sufficiently large N has a legal split with $q - p \le N/(\log N)^2$, while no bound $O(N^\alpha)$, $\alpha < 1$, holds eventually; this is the example of* [6, Proposition 4.1]*. The needed inequalities*

are $(A_j-1)/b_j \ge A_j/(8b_j^2)+1$ *and* $A_j/(4b_j^2) \le N/(\log N)^2$ *for* $b_j \ge 16$ *and* $2A_j < N < 3A_j$.

8. Divergent integral budgets without integer connectivity

The adjacent-pair hypothesis in Theorem 5 cannot be removed for all measurable nonincreasing widths. The obstruction is already visible for nonnegative integer-valued sequences with a linear bound.

Lemma 13 (Isolated outputs). *Let* $\mathcal{P} \subseteq \mathbb{N} \times \mathbb{N}$ *be a set of permitted pairs. If infinitely many positive integers have no representation* $n+m$ *with* $(n,m) \in \mathcal{P}$*, there is a sequence* $a_n \in \{0,n\}$ *satisfying* $a_{n+m} \le a_n + a_m$ *for every* $(n,m) \in \mathcal{P}$*, but*

$$\liminf_{n\to\infty} a_n/n = 0, \qquad \limsup_{n\to\infty} a_n/n = 1.$$

Proof. Choose an infinite subset U of the unrepresented outputs with infinite complement, and put $a_n = n\mathbf{1}_{\{n\in U\}}$. Every permitted sum lies outside U, so its left side is zero and the right side is nonnegative. The two infinite sets give the asserted normalized limits. □

Theorem 14. *There is a positive measurable nonincreasing function* $w : [0,\infty) \to (0,1]$*, tending to zero, such that*

$$\int_0^\infty w(s)^p\, ds = \infty \quad \textit{for every real } p > 0,$$

yet there is a nonnegative integer-valued sequence (a_n) *with* $a_n \le n$ *for which*

$$a_{n+m} \le a_n + a_m \quad \textit{whenever } 1 \le n \le m \le n + nw(\log n),$$

and $\liminf a_n/n = 0$, $\limsup a_n/n = 1$. *Consequently there is a positive integer sequence* (u_n) *with* $1 \le u_n \le 2^n$*, satisfying* $u_{n+m} \le u_n u_m$ *on this entire corridor, but with root lower and upper limits* 1 *and* 2.

Proof. Define integers and plateau endpoints by

$$b_1 = 4, \qquad b_{j+1} = b_j + 2^{(j+2)b_j+6}, \qquad A_j = 2^{b_j}, \qquad s_j = b_j \log 2 \quad (j \ge 1).$$

Set $w(s) = 1$ for $0 \le s < s_1$, and

$$w(s) = \frac{1}{8A_j} \quad \text{for } s_j \le s < s_{j+1}.$$

This function has the claimed positivity, measurability, monotonicity and zero limit. For each fixed real $p > 0$ its jth plateau contributes

$$\int_{s_j}^{s_{j+1}} w(s)^p\, ds = 2^{(j+2-p)b_j+6-3p} \log 2.$$

For all sufficiently large j one has $j+2-p \ge 1$; as $b_j \to \infty$, these contributions tend to infinity. Thus each of the stated integrals diverges for this same width.

We claim that every odd integer $N_j = 3A_j + 1$ is an unrepresented output of the permitted corridor. If $N_j = n+m$ were permitted, then $w \le 1$

would give $m \le 2n$, hence $n \ge N_j/3 > A_j$. On the other hand, since A_j is even and $n \le m$,

$$n \le \lfloor N_j/2 \rfloor = 3A_j/2 < A_{j+1}.$$

Thus $\log n$ lies in the jth plateau and

$$0 \le m - n \le nw(\log n) = \frac{n}{8A_j} \le \frac{3}{16} < 1.$$

Integer inputs would force $m = n$, contradicting oddness of N_j. Now set $a_n = n$ at the outputs N_j and $a_n = 0$ elsewhere. No permitted left side is nonzero, so every required inequality holds. The spikes give normalized value one, while every even index gives zero. Finally take $u_n = 2^{a_n}$. □

The sequence in Theorem 14 is not monotone, so it lies outside the class of Theorem 3. Its mechanism is the isolated-output principle of Lemma 13, which separates an integrated width budget from legal decompositions at individual integer outputs. Sign reversal gives a bounded-slope superadditive example.

9. The sharp classification and its scope

Corollary 15. *The conclusion of Theorem 2 holds for each of the following eventual corridor widths:*

$$r(n) = \frac{n}{(\log n)^p} \quad (p > 1/2), \qquad r(n) = \frac{n}{\sqrt{\log n}(\log\log n)^a} \quad (a > 1/2),$$

and

$$r(n) = \frac{n}{\sqrt{\log n}\sqrt{\log\log n}(\log\log\log n)^a} \quad (a > 1/2).$$

Proof. On a sufficiently late tail use, respectively,

$$w(s) = s^{-p}, \qquad w(s) = s^{-1/2}(\log s)^{-a}, \qquad w(s) = \frac{1}{\sqrt{s}\sqrt{\log s}(\log\log s)^a}.$$

Each is nonincreasing, bounded by one and square integrable, by the integral test. □

The second width eventually exceeds every width in the first family. The same sequences also satisfy the inequalities on any narrower corridor.

Conversely, for $r(n) = n/(\log n)^p$ with $0 < p \le 1/2$, Theorem 5 forces normalized-slope convergence for arbitrary real sequences: the square integral diverges and the width eventually exceeds one. In particular $r(n) = n/\sqrt{\log n} = o(n)$ answers positively the existence of an adequate sublinear corridor asked in [4, Section 7], as phrased in the cited preprint. For the second and third logarithmic families, the same unrestricted convergence conclusion holds when $a \le 1/2$. Their width ratios are eventually nonincreasing and bounded by one, the square integrals diverge, and the widths tend to infinity. These statements and the counterexamples give exact thresholds within those families.

The unrestricted theorem retains eventual admissibility of adjacent integer pairs. Theorem 14 shows that even divergence of every finite positive-power integral cannot replace integer connectivity in the unrestricted class. Theorem 3, with monotonicity, still covers every divergent-square-integral width, including these deep-jump widths. The record argument is essential: an a.e. linear limiting profile of continuous rescalings alone would not justify all-index convergence without monotonicity. The constructions and convergence criteria concern scalar sequences.

Theorem 7 replaces eventual adjacent-pair legality by balanced legal predecessors while retaining the divergent square-integral budget. Its staircase example separates the iterated-logarithmic predecessor condition from both earlier fixed-power and logarithmic families. The convergence conclusions use the quantitative gap bound (10) on legal splits.

Finite controls. The accompanying `verify.py` and `verify_balanced.py` give exact finite controls for isolated outputs, integer windows, parent allocations and staircase endpoints. They corroborate finite algebra; the universal and asymptotic results follow from the analytic proofs.

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References

[1] C. J. K. Batty, M. J. Pelling and D. G. Rogers, *Some recurrence relations of recursive minimization*, SIAM J. Algebraic Discrete Methods **3** (1982), 13–29. doi:10.1137/0603002.

[2] C. J. K. Batty and D. G. Rogers, *Some maximal solutions of the generalized subadditive inequality*, SIAM J. Algebraic Discrete Methods **3** (1982), 369–378. doi:10.1137/0603038.

[3] N. G. de Bruijn and P. Erdős, *Some linear and some quadratic recursion formulas. II*, Proc. Koninkl. Nederl. Akad. Wetensch. A **55** (1952), 152–163. Author archive.
[4] Z. Füredi and I. Z. Ruzsa, *Nearly subadditive sequences*, Acta Math. Hungar. **161** (2020), 401–411. doi:10.1007/s10474-020-01071-0. Question attribution and mathematical comparisons use arXiv:1810.11723v1, Section 7.
[5] A. Kulikov and F. Shao, *Fekete's lemma in Banach spaces*, arXiv:2411.17380v1 (2024).
[6] S. Schubr, *Fekete convergence from balanced integer predecessors in shrinking corridors*, author-published edition 2 (2026). Luminary Research Archive.
[7] F. Shao and contributors, *High dimensional Fekete's subadditive lemma*, MathOverflow discussion (2023), MathOverflow discussion.
[8] R. J. Wallace, *Optimal solutions of restricted subadditive inequalities*, Bull. Austral. Math. Soc. **38** (1988), 475–477 (author's thesis summary). doi:10.1017/S0004972700027878.

Scott Schubr
Luminary Oddities