Oscillating sequences with subadditivity on shrinking corridors

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Abstract

Let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing. For exact eventual subadditivity on the integer corridor $n \le m \le n + nw(\log n)$, we identify a square-integral threshold. If $\int_T^\infty w(s)^2\, ds$ diverges, normalized slopes converge for nondecreasing nonnegative linearly bounded sequences, without continuity or an adjacent-pair assumption. In this divergent-integral case, if adjacent integer pairs are eventually admissible, a prefix-envelope and dyadic-record argument removes all monotonicity, sign and growth hypotheses: every real sequence satisfying the corridor inequality has a normalized limit in $\mathbb{R} \cup \{-\infty\}$. In particular, $n/\sqrt{\log n}$ is a sufficient sublinear corridor for arbitrary real sequences. If the integral is finite, a shifted-tail correction gives strictly increasing integer submultiplicative and supermultiplicative counterexamples with logarithmic growth endpoints 1.99 and 2.01. Thus the criterion is an exact dichotomy in the monotone counting class, and also for arbitrary real sequences when the adjacent-pair condition holds. Without monotonicity, adjacent-pair legality cannot simply be omitted: we construct a positive nonincreasing width for which every finite positive-power integral diverges, yet an integer-valued linearly bounded subadditive sequence has normalized lower and upper limits zero and one.

1 The question and the result

Fekete's lemma concerns real sequences satisfying $a_{n+m} \le a_n + a_m$ for all positive integer pairs. Restricted-pair variants can still imply convergence of a_n/n. In particular, Füredi and Ruzsa [1] establish convergence when the inequality is required eventually on $n \le m \le \mu n$, for any fixed $\mu > 1$. Section 7 of their accessible preprint asks whether some width $r(n) = o(n)$ could replace the fixed relative width. We give a sufficient sublinear width, $r(n) = n/\sqrt{\log n}$, for arbitrary real sequences. More generally, a divergent square integral of the relative width forces convergence when adjacent integer pairs are eventually admissible. No monotonicity or normalized-slope bound is needed in that statement, and the limit may be $-\infty$.

For nondecreasing nonnegative linearly bounded sequences, including logarithms of monotone exponentially bounded counting sequences, the adjacent-pair condition can be omitted. In this class we obtain a complete square-integral dichotomy for every measurable nonincreasing relative width, including jumps and integer gaps. The finite-integral counterexamples are themselves strictly increasing integer counting sequences. For unrestricted real sequences, however, no criterion based only on divergence of finite positive-power integrals can replace integer connectivity: an explicit deep-jump width and isolated odd outputs give a counterexample.

Vanishing normalized corrections that absorb an oscillatory defect have earlier uses. The error-correction arguments in [1] are relevant background. In a vector-valued setting, Kulikov and Shao [2, Section 3], following a MathOverflow discussion [3], compensate a squared angular discrepancy by

a small radial correction. Their norm ratios converge; their vector inequality does not imply the scalar inequality studied here. Our construction uses exact dyadic homogeneity to make the scalar discrepancy quadratic in the relative corridor width, and a shifted square-width tail to pay it without assuming regular variation.

For the complementary convergence theorem, dyadic rescalings of a monotone sequence decrease pointwise. Integer rounding is handled before any limiting argument. A divergent sum of admissible squared midpoint steps then forces the limit to be convex in the distributional sense. Classical mollification and monotonicity identify this limit at every point; dyadic homogeneity makes it linear. This handles jumps of w for which adjacent integer pairs need not be eventually admissible. When adjacent pairs are admissible, a monotone prefix envelope preserves the entire corridor, and its dense record indices transfer convergence back to an arbitrary real sequence by continuous interpolation. The latter step avoids any a.e.-to-pointwise inference without monotonicity.

All logarithms below are natural. We call a positive sequence exponentially bounded if $b_n \le \exp(Kn)$ for some finite K and all $n \ge 1$.

Theorem 1. *Let $T \in \mathbb{R}$ and let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing, with*

$$\int_T^\infty w(s)^2\, ds < \infty. \tag{1}$$

There exist positive strictly increasing exponentially bounded integer sequences (u_n) and (v_n), and an integer N_0, such that

$$u_{n+m} \le u_n u_m, \qquad v_{n+m} \ge v_n v_m \quad \text{if } N_0 \le n \le m \le n + nw(\log n), \tag{2}$$

while

$$\liminf_{n\to\infty} \frac{\log u_n}{n} = \liminf_{n\to\infty} \frac{\log v_n}{n} = 2 - \frac{1}{100},$$
$$\limsup_{n\to\infty} \frac{\log u_n}{n} = \limsup_{n\to\infty} \frac{\log v_n}{n} = 2 + \frac{1}{100}.$$

The sequences may depend on w. Jump discontinuities and zeros of w are allowed.

Equivalently, the permitted width $r(x) = xw(\log x)$ has nonincreasing relative size and satisfies

$$0 \le r(x) \le x, \qquad \int^\infty \frac{r(x)^2}{x^3}\, dx < \infty.$$

This follows by the substitution $x = e^s$. The condition restricts which pairs are tested; it allows no error in the tested inequalities.

2 Proof of Theorem 1

Put

$$\varepsilon = \frac{1}{100}, \qquad \omega = \frac{2\pi}{\log 2}, \qquad A_0(x) = x[2 + \varepsilon \sin(\omega \log x)], \qquad B = \varepsilon\omega(1 + \omega).$$

Then $A_0(2x) = 2A_0(x)$ and

$$A_0''(x) = \frac{\varepsilon\omega}{x}[\cos(\omega \log x) - \omega \sin(\omega \log x)], \qquad |A_0''(x)| \le \frac{B}{x}.$$

For $t \ge T + \log 3$ and $x \ge 3e^T$, define

$$Q(t) = \int_{t-\log 3}^\infty w(s)^2\, ds, \qquad g(x) = xQ(\log x), \qquad C = \frac{B}{\log(3/2)},$$

and set $A(x) = A_0(x) + Cg(x)$, $V(x) = 4x - A(x)$.

The normalized correction vanishes

Monotonicity and (1) imply $w(t) \to 0$. Also $Q(t) \to 0$, so $g(x)/x \to 0$. The tail integral is locally absolutely continuous, and hence so is g, with

$$g'(x) = Q(\log x) - w(\log x - \log 3)^2 \quad \text{for almost every sufficiently large } x.$$

Both terms tend to zero. Thus, for every $\eta > 0$, $|g'(x)| \le \eta$ for almost every sufficiently large x. No derivative of w is required.

The oscillatory defect is quadratic

Suppose $n \le m \le n + nw(\log n)$, and write $N = n + m$, $d = m - n$. Here $m \le 2n$ and $N \le 3n$. Taylor expansion about $N/2$, using dyadic homogeneity and $|A_0''| \le B/n$ on $[n, m]$, gives

$$\begin{aligned} |A_0(N) - A_0(n) - A_0(m)| &= |2A_0(N/2) - A_0(n) - A_0(m)| \\ &\le \frac{Bd^2}{4n} \le \frac{Bn}{4} w(\log n)^2 \le \frac{BN}{8} w(\log n)^2. \end{aligned} \tag{3}$$

These estimates hold for real inputs in the domain, and therefore for integer inputs.

A shifted tail pays the defect

For $x = n, m$,

$$Q(\log x) - Q(\log N) = \int_{\log x - \log 3}^{\log N - \log 3} w(s)^2 \, ds.$$

For sufficiently large n, the interval is contained in $[T, \infty)$. Its upper endpoint is at most $\log n$, since $N \le 3n$. Monotonicity therefore gives a lower bound $w(\log n)^2$ for the integrand throughout the interval. Also $n, m \le 2N/3$. Consequently

$$\begin{aligned} g(n) + g(m) - g(N) &= n[Q(\log n) - Q(\log N)] + m[Q(\log m) - Q(\log N)] \\ &\ge \left[n \log \frac{N}{n} + m \log \frac{N}{m} \right] w(\log n)^2 \\ &\ge N \log(3/2) w(\log n)^2. \end{aligned} \tag{4}$$

Combining (3) and (4) yields

$$A(n) + A(m) - A(N) \ge \frac{7BN}{8} w(\log n)^2 \ge 0.$$

Thus $A(N) \le A(n) + A(m)$ and $V(N) \ge V(n) + V(m)$ on the corridor. If $w(\log n) = 0$, only $m = n$ is allowed, and the same conclusion holds. The shifted tail makes no comparison assumption between nearby values of w.

Monotonicity, oscillation and integer rounding

The derivative

$$A_0'(x) = 2 + \varepsilon[\sin(\omega \log x) + \omega \cos(\omega \log x)]$$

lies uniformly in $(1, 3)$. Since g' tends uniformly to zero off a null set, A' and V' are greater than one almost everywhere on a sufficiently late tail. Local absolute continuity gives

$$A(y) - A(x) \ge y - x, \qquad V(y) - V(x) \ge y - x$$

for all tail points $y > x$. Moreover

$$\frac{A(x)}{x} = 2 + \varepsilon \sin(\omega \log x) + o(1), \qquad \frac{V(x)}{x} = 2 - \varepsilon \sin(\omega \log x) + o(1).$$

Integers nearest $2^{j+1/4}$ and $2^{j+3/4}$ approach the two extreme sine phases. Both normalized slopes therefore have lower endpoint $2 - \varepsilon$ and upper endpoint $2 + \varepsilon$.

For sufficiently large integers n, set

$$u_n = \lceil e^{A(n)} \rceil, \qquad v_n = \lfloor e^{V(n)} \rfloor.$$

The increment bounds and positivity of the exponents make both sequences strictly increasing on a late tail. The product $u_n u_m$ is an integer at least $e^{A(n)+A(m)} \geq e^{A(N)}$, so $u_N \leq u_n u_m$. Likewise the integer $v_n v_m$ is at most $e^{V(n)+V(m)} \leq e^{V(N)}$, so $v_N \geq v_n v_m$.

Rounding changes the logarithms by quantities tending to zero. Increase N_0 so that both first tail values exceed N_0, and assign the earlier terms the values $1, 2, \ldots, N_0 - 1$. This supplies positive strictly increasing sequences on all of $\mathbb{N}$. The inequalities concern only tail inputs. Bounded normalized exponents, with a finite prefix absorbed into a constant, give exponential boundedness. The claimed endpoints follow. □

3 Convergence at divergent square integral

Theorem 2. *Let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing, and suppose*

$$\int_T^\infty w(t)^2 \, dt = \infty.$$

Let (a_n) be nondecreasing, with $0 \leq a_n \leq Kn$ for all sufficiently large integers n and some finite K. If, for some N_0,

$$a_{n+m} \leq a_n + a_m \quad \textit{whenever } N_0 \leq n \leq m \leq n + nw(\log n),$$

then a_n/n converges to a finite limit. In particular, every positive nondecreasing exponentially bounded integer sequence satisfying the submultiplicative inequality on this corridor has a limiting nth-root rate.

Proof. For sufficiently large real x put

$$A(x) = a_{\lceil x \rceil}, \qquad F_k(x) = 2^{-k} A(2^k x).$$

On every compact subinterval of $(0, \infty)$ these expressions are defined for all sufficiently large k. The allowed diagonal pair and monotonicity give exactly

$$A(2x) = a_{\lceil 2x \rceil} \leq a_{2\lceil x \rceil} \leq 2a_{\lceil x \rceil} = 2A(x).$$

Thus $F_{k+1} \leq F_k$, and F_k is nondecreasing in x, with $0 \leq F_k(x) \leq K(x + 2^{-k})$. Consequently F_k decreases pointwise to a finite nondecreasing function f, converges to it locally in L^1, and satisfies

$$0 \leq f(x) \leq Kx, \qquad f(2x) = 2f(x).$$

The functions F_k are step functions; no continuity is assumed.

Admissible midpoint steps and exact integer rounding

Fix $0 < a < b$ and a nonnegative smooth test function ϕ with compact support in (a, b). Set

$$q_k = w(k \log 2 + \log(2b)).$$

For large k, suppose $0 < h \le aq_k/16$ and $2^k a q_k \ge 16$. For $x \in [a, b]$ put

$$n = \lceil 2^k (x - h) \rceil, \qquad m = \lceil 2^k (x + h) \rceil.$$

Since $q_k \le 1$, we have $h \le a/16$, and

$$2^k a/2 \le n \le 2^k b + 1 \le 2^{k+1} b,$$

$$m - n \le 2^{k+1} h + 1 \le 2^k a q_k/8 + 1 \le 2^k a q_k/2 \le n w(\log n).$$

Here $w(\log n) \ge q_k$ follows from $\log n \le k \log 2 + \log(2b)$. Both indices exceed the threshold for large k. Also $n + m \ge \lceil 2^{k+1} x \rceil$. Monotonicity and the literal integer hypothesis therefore imply

$$F_{k+1}(x) \le \frac{F_k(x - h) + F_k(x + h)}{2}. \tag{5}$$

No enlargement of the permitted integer corridor has been made.

If $w(t) \to 0$, the integral test for the nonincreasing function w^2 gives $\sum_k q_k^2 = \infty$. Call a scale good if $2^k a q_k \ge 16$. At bad scales $q_k^2 < (16/a)^2 4^{-k}$, so their squared sum is finite. On the good scales choose $h_k = aq_k/16$. Then $h_k \to 0$, $\sum h_k^2 = \infty$, and (5) holds. If instead $\lim w(t) = L > 0$, choose $h_k = c/\sqrt{k}$ for any fixed $c > 0$; all sufficiently large scales are good and the same properties hold. In either case there is an unbounded set G of admissible scales such that

$$h_k \longrightarrow 0 \quad (k \in G), \qquad \sum_{k \in G} h_k^2 = \infty.$$

Bad scales are omitted only from the midpoint comparison, not from the decreasing dyadic sequence.

A telescoping argument forces convexity

Write $D_k = F_k - F_{k+1} \ge 0$ and $\Delta_h \phi(x) = \phi(x - h) + \phi(x + h) - 2\phi(x)$. Multiplying (5) by ϕ and translating variables gives, for $k \in G$,

$$\int D_k \phi \ge -\frac{1}{2} \int F_k \Delta_{h_k} \phi.$$

All integrands have support in one fixed compact subinterval of $(0, \infty)$ for large k. Since $\Delta_h \phi / h^2 \to \phi''$ uniformly and $F_k \to f$ locally in L^1 with uniform local bounds,

$$\frac{1}{h_k^2} \int F_k \Delta_{h_k} \phi \longrightarrow \int f \phi''.$$

If the last integral were negative, then for some $\eta > 0$ and every sufficiently late good scale, $\int D_k \phi \ge \eta h_k^2$. This contradicts the finite nonnegative telescoping sum

$$\sum_{k \ge k_0} \int D_k \phi = \int (F_{k_0} - f) \phi < \infty.$$

Hence $\int f\phi'' \geq 0$ for every nonnegative smooth compactly supported ϕ: the limit is convex in the distributional sense.

We include the classical representative argument to make the pointwise conclusion explicit. On an interior compact interval convolve f with a nonnegative smooth approximate identity. The resulting smooth functions have nonnegative second derivative and are convex. Local boundedness and the elementary secant bounds for convex functions make them uniformly Lipschitz on a smaller interval. A subsequence converges uniformly there to a continuous convex function g, while approximate identities converge to f locally in L^1. Thus $g = f$ almost everywhere. These representatives agree on overlapping intervals. At any point, take agreement points approaching from both sides; monotonicity of f and continuity of g squeeze f to g. Therefore f itself is continuous and convex everywhere. This step uses monotonicity, not only local boundedness.

Linearity and convergence at every index

Extend f to zero by $f(0) = 0$. The bounds $0 \leq f(x) \leq Kx$ give a continuous convex extension, by taking positive endpoints tending to zero in the convex inequality. Thus $f(x)/x$ is nondecreasing. Its equal values at x and $2x$ make it constant on every $[x, 2x]$, and hence everywhere: $f(x) = \ell x$.

Finally $F_k \to \ell x$ uniformly on $[1, 2]$. Indeed, use pointwise convergence at finitely many grid points and squeeze between adjacent endpoints using monotonicity of F_k in x and continuity of ℓx. This avoids applying Dini's theorem to discontinuous functions. For an integer n, take $k = \lfloor \log_2 n \rfloor$ and $x = n/2^k \in [1, 2)$. Then

$$\frac{a_n}{n} = \frac{F_k(x)}{x} \longrightarrow \ell.$$

For integer counting sequences set $a_n = \log u_n$. The hypotheses give a nondecreasing nonnegative linearly bounded sequence, and exponentiating its limiting slope proves the final assertion. □

4 Removing monotonicity when adjacent pairs are admissible

Theorem 3. *Let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing, with*

$$\int_T^\infty w(t)^2\, dt = \infty, \qquad nw(\log n) \geq 1 \quad \textit{for all sufficiently large integers } n.$$

If an arbitrary real sequence (a_n) satisfies

$$a_{n+m} \leq a_n + a_m \quad \textit{eventually whenever } n \leq m \leq n + nw(\log n),$$

then a_n/n converges in $\mathbb{R} \cup \{-\infty\}$. No monotonicity, sign or growth bound is assumed. Consequently every positive real sequence satisfying the corresponding submultiplicative inequality has a finite nth-root limit in $[0, \infty)$; for positive integer sequences this limit is at least one.

Proof. The hypotheses give, for every sufficiently large integer n,

$$a_{2n} \leq 2a_n, \qquad a_{2n+1} \leq a_n + a_{n+1}. \tag{6}$$

These inequalities imply an automatic upper linear bound. Choose n_0 large enough for (6), and choose a finite C bounding a_j/j for $n_0 \leq j \leq 2n_0 + 1$. Strong induction using the even and odd alternatives in (6) gives $a_j \leq Cj$ for all $j \geq n_0$. Enlarge C to be positive and absorb the finite prefix if needed.

Thus $S = \limsup a_n/n$ is finite or $-\infty$. In the latter case the required extended-real convergence already holds. Otherwise add a linear term Dn so that the shifted limsup is positive. This preserves every pair inequality. We use a_n for the shifted sequence until the last step; no lower bound is imposed.

The prefix envelope preserves the corridor

Put

$$B_n = \max\{0, a_1, \dots, a_n\}.$$

It is nondecreasing and $0 \le B_n \le C'n$ for some finite C'. For a sufficiently large allowed pair $n \le m$, consider any $j \le n+m$. If $j \le n$, then $a_j \le B_n \le B_n + B_m$. If $n < j \le 2n$, split $j = p + q$ with $p = \lfloor j/2 \rfloor$ and $q = \lceil j/2 \rceil$. Both indices are eventually in the original tail, their gap is at most one, and $q \le n$. Hence

$$a_j \le a_p + a_q \le 2B_n \le B_n + B_m.$$

If $2n < j \le n+m$, use $p = n$, $q = j-n$. Then $n < q \le m$, so this split lies in the original permitted corridor and again $a_j \le B_n + B_m$. The same upper bound holds for zero. Taking the maximum proves

$$B_{n+m} \le B_n + B_m.$$

Theorem 2 therefore gives $B_n/n \to L \ge 0$, with $L \ge \limsup a_n/n > 0$.

Continuous interpolation and dense records

Let A be the piecewise-linear interpolant of a_n. It satisfies $A(2x) \le 2A(x)$ for all sufficiently large real x. Indeed, write $x = n + \alpha$. If $0 \le \alpha \le 1/2$, then

$$\begin{aligned} A(2x) &= (1-2\alpha)a_{2n} + 2\alpha a_{2n+1} \\ &\le (1-2\alpha)2a_n + 2\alpha(a_n + a_{n+1}) = 2A(x). \end{aligned}$$

For $1/2 \le \alpha \le 1$,

$$\begin{aligned} A(2x) &= (2-2\alpha)a_{2n+1} + (2\alpha-1)a_{2n+2} \\ &\le (2-2\alpha)(a_n + a_{n+1}) + (2\alpha-1)2a_{n+1} = 2A(x). \end{aligned}$$

Only the three diagonal or adjacent-diagonal integer pairs are used; A need not be monotone.

For each sufficiently large n, choose $j_n \le n$ with $B_n = a_{j_n}$. Such records exist because $B_n \to \infty$. They satisfy $j_n \to \infty$, since finitely many fixed terms have bounded values. For every $0 < \eta < L$, eventually

$$(L-\eta)n \le B_n = a_{j_n} \le B_{j_n} \le (L+\eta)j_n.$$

It follows that

$$j_n/n \longrightarrow 1, \qquad a_{j_n}/n = B_n/n \longrightarrow L.$$

The high records are asymptotically next to every large index, regardless of arbitrarily negative values between records.

Records force a lower bound at every late point

Fix any sufficiently large real x. Put $n_k = \lfloor 2^k x \rfloor$ and $y_k = j_{n_k}/2^k$. Then

$$y_k \longrightarrow x, \qquad 2^{-k}A(2^k y_k) = \frac{B_{n_k}}{2^k} \longrightarrow Lx.$$

For large k, the points y_k lie in one fixed late neighborhood of x on which $A(2y) \le 2A(y)$ holds, as it does on every successive doubling. Iterating gives

$$A(y_k) \ge 2^{-k}A(2^k y_k).$$

Continuity of the fixed interpolant A at x yields $A(x) \geq Lx$. Since x was arbitrary on a sufficiently late tail, $a_n/n \geq L$ for every sufficiently large integer n. Conversely $a_n \leq B_n$ and $B_n/n \to L$ give the matching upper bound. Hence $a_n/n \to L$ for the shifted sequence. Subtract D to obtain the finite limit for the original sequence. Together with the earlier $S = -\infty$ case, this proves the real-sequence assertion.

For a positive real sequence u_n take $a_n = \log u_n$. Exponentiating its extended-real limit gives a finite root limit, with $e^{-\infty} = 0$. If the terms are positive integers, $a_n \geq 0$, so the limit is at least one. □

Remark 4. *Under the width hypotheses of Theorem 3, sign reversal also gives convergence of a_n/n in $\mathbb{R} \cup \{+\infty\}$ for every eventually corridor-superadditive real sequence. Thus a positive real corridor-supermultiplicative sequence has a root limit in $(0, \infty]$: applying the automatic upper bound to the negative logarithms supplies a positive lower exponential bound. For positive integer sequences the submultiplicative limit lies in $[1, \infty)$ and the supermultiplicative limit in $[1, \infty]$.*

5 Divergent integral budgets without integer connectivity

The adjacent-pair hypothesis in Theorem 3 cannot be removed for all measurable nonincreasing widths. The obstruction is already visible for nonnegative integer-valued sequences with a linear bound.

Lemma 5 (Isolated outputs)**.** *Let $\mathcal{P} \subseteq \mathbb{N} \times \mathbb{N}$ be a set of permitted pairs. If infinitely many positive integers have no representation $n + m$ with $(n, m) \in \mathcal{P}$, there is a sequence $a_n \in \{0, n\}$ satisfying $a_{n+m} \leq a_n + a_m$ for every $(n, m) \in \mathcal{P}$, but*

$$\liminf_{n\to\infty} a_n/n = 0, \qquad \limsup_{n\to\infty} a_n/n = 1.$$

Proof. Choose an infinite subset U of the unrepresented outputs with infinite complement, and put $a_n = n\mathbf{1}_{\{n\in U\}}$. Every permitted sum lies outside U, so its left side is zero and the right side is nonnegative. The two infinite sets give the asserted normalized limits. □

Theorem 6. *There is a positive measurable nonincreasing function $w : [0, \infty) \to (0, 1]$, tending to zero, such that*

$$\int_0^\infty w(s)^p \, ds = \infty \quad \text{for every real } p > 0,$$

yet there is a nonnegative integer-valued sequence (a_n) with $a_n \leq n$ for which

$$a_{n+m} \leq a_n + a_m \quad \text{whenever } 1 \leq n \leq m \leq n + nw(\log n),$$

and $\liminf a_n/n = 0$, $\limsup a_n/n = 1$. Consequently there is a positive integer sequence (u_n) with $1 \leq u_n \leq 2^n$, satisfying $u_{n+m} \leq u_n u_m$ on this entire corridor, but with root lower and upper limits 1 and 2.

Proof. Define integers and plateau endpoints by

$$b_1 = 4, \qquad b_{j+1} = b_j + 2^{(j+2)b_j+6}, \qquad A_j = 2^{b_j}, \qquad s_j = b_j \log 2 \quad (j \geq 1).$$

Set $w(s) = 1$ for $0 \leq s < s_1$, and

$$w(s) = \frac{1}{8A_j} \quad \text{for } s_j \leq s < s_{j+1}.$$

This function has the claimed positivity, measurability, monotonicity and zero limit. For each fixed real $p > 0$ its jth plateau contributes

$$\int_{s_j}^{s_{j+1}} w(s)^p \, ds = 2^{(j+2-p)b_j+6-3p} \log 2.$$

For all sufficiently large j one has $j + 2 - p \geq 1$; as $b_j \to \infty$, these contributions tend to infinity. Thus each of the stated integrals diverges for this same width.

We claim that every odd integer $N_j = 3A_j + 1$ is an unrepresented output of the permitted corridor. If $N_j = n + m$ were permitted, then $w \leq 1$ would give $m \leq 2n$, hence $n \geq N_j/3 > A_j$. On the other hand, since A_j is even and $n \leq m$,

$$n \leq \lfloor N_j/2 \rfloor = 3A_j/2 < A_{j+1}.$$

Thus $\log n$ lies in the jth plateau and

$$0 \leq m - n \leq nw(\log n) = \frac{n}{8A_j} \leq \frac{3}{16} < 1.$$

Integer inputs would force $m = n$, contradicting oddness of N_j. Now set $a_n = n$ at the outputs N_j and $a_n = 0$ elsewhere. No permitted left side is nonzero, so every required inequality holds. The spikes give normalized value one, while every even index gives zero. Finally take $u_n = 2^{a_n}$. □

The sequence in Theorem 6 is not nondecreasing. It therefore does not contradict Theorem 2 or the monotone counting classification below. The example separates an integrated width budget from legal decompositions at individual integer outputs; it does not say that every width with infinitely many missing adjacent pairs permits oscillation. Sign reversal also gives a bounded-slope superadditive example. The elementary lemma states the mechanism, rather than a characterization of every permitted-pair set.

6 The sharp classification and its scope

Corollary 7. *Fix a measurable nonincreasing $w : [T, \infty) \to [0, 1]$. Every positive nondecreasing exponentially bounded integer sequence with eventual exact submultiplicativity on $n \leq m \leq n + nw(\log n)$ has a limiting nth-root rate if and only if*

$$\int_T^\infty w(s)^2 \, ds = \infty.$$

Proof. The divergent case is Theorem 2. In the finite case the sequence (u_n) of Theorem 1 belongs to the stated class and has distinct logarithmic endpoint rates. □

If $nw(\log n) \geq 1$ eventually, the same integral dichotomy holds for universal convergence of a_n/n in $\mathbb{R} \cup \{-\infty\}$ among arbitrary real sequences satisfying the corridor subadditivity condition. The divergent case is Theorem 3; the finite case follows by taking logarithms of the integer counterexample.

Corollary 8. *The conclusion of Theorem 1 holds for each of the following eventual corridor widths:*

$$r(n) = \frac{n}{(\log n)^p} \quad (p > 1/2), \qquad r(n) = \frac{n}{\sqrt{\log n}(\log\log n)^a} \quad (a > 1/2),$$

and

$$r(n) = \frac{n}{\sqrt{\log n}\sqrt{\log\log n}(\log\log\log n)^a} \quad (a > 1/2).$$

Proof. On a sufficiently late tail use, respectively,

$$w(s) = s^{-p}, \qquad w(s) = s^{-1/2}(\log s)^{-a}, \qquad w(s) = \frac{1}{\sqrt{s}\sqrt{\log s}(\log\log s)^{a}}.$$

Each is nonincreasing, bounded by one and square integrable, by the integral test. □

The second width eventually exceeds every width in the first family. The same sequences also satisfy the inequalities on any narrower corridor.

Conversely, for $r(n) = n/(\log n)^p$ with $0 < p \le 1/2$, Theorem 3 forces normalized-slope convergence for arbitrary real sequences: the square integral diverges and the width eventually exceeds one. In particular $r(n) = n/\sqrt{\log n} = o(n)$ answers positively the existence of an adequate sublinear corridor asked in [1, Section 7], as phrased in the cited preprint. For the second and third logarithmic families, the same unrestricted convergence conclusion holds when $a \le 1/2$. Their width ratios are eventually nonincreasing and bounded by one, the square integrals diverge, and the widths tend to infinity. These statements and the counterexamples give exact thresholds within those families.

The unrestricted theorem retains eventual admissibility of adjacent integer pairs. Theorem 6 shows that even divergence of every finite positive-power integral cannot replace integer connectivity in the unrestricted class. Theorem 2, with monotonicity, still covers every divergent-square-integral width, including these deep-jump widths. The record argument is essential: an a.e. linear limiting profile of continuous rescalings alone would not justify all-index convergence without monotonicity. The constructions and convergence criteria concern scalar sequences.

Use of AI tools. AI tools assisted in developing and checking the arguments and preparing the manuscript. The proofs are given in full, and no computational result is needed.

References

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