Optimal energy for Dirichlet repairs in divisor bands

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Luminary Oddities

26 October 2026

Abstract

We construct finite arithmetic corrections that impose an exact complex Dirichlet value using a prescribed band of divisor coordinates. For $q = 1/2 - a + iT$, $0 < a < 1/2$, and the critical Bernoulli penalty $\omega(r) = r\log^2(er)$, the minimum energy for value t on $N < r \le 2N$ is of order $|t|^2 N^{3-2a}\log^2(eN)$, with explicit constants and onset independent of T. The minimizing correction has complete native norm $O_a(|t|N^{-a})$ and first absolute coefficient moment $O_a(|t|N^{3/2-a})$. Applied to native approximants at a smaller scale, it gives actual finite anchored and q-null trials with energy $O_a(N^{3-a}\log^2(eN))$ and an explicit approximation bound relative to the best native trial at the smaller scale. At a specified height-dependent cutoff, their complete compensated mean, including its boundary term, is $O_a(N^{-3/2-5a/2})$. The approximation limit is the optimal native distance, uniformly in T.

1 The repair problem and its contribution

In the complete Nyman–Beurling approximation norm, finite fractional-part trials approximate the constant function. An additional exact Dirichlet null constrains their coefficients and can increase both the approximation error and the coefficient energy controlling arithmetic tails. We determine that cost for a prescribed band in divisor coordinates, and then bound the same correction's complete approximation norm and mean.

High-index corrections in native Nyman–Beurling spaces have explicit precedents: the neutral-density argument of [1, Section on constructive density] uses two small-norm generators to impose two arithmetic moment constraints. The coefficient estimate of [2, Lemma 3] gives support-dependent control. The Bernoulli divisor energy is supplied by [3, Theorem 1]; its general regularity criteria are developed in [4]. Finite Hilbert-space extremizers are standard. Our result concerns the specific Dirichlet row, its full-integer band capacity uniformly in height, and simultaneous bounds in the approximation norm, coefficient moment and complete mean.

Let $c = (c_d)$ be finite, with

$$A_c(s) = \sum_{d\ge1} c_d d^{-s}, \quad A_c(1) = 0, \quad v_r = \sum_{r|d} c_d/d, \quad c_d = d\sum_{k\ge1}\mu(k)v_{dk}.$$

Thus $v_1 = 0$ and every finite v on $r \ge 2$ realizes an actual finite anchored trial. Define

$$J_z(r) = \sum_{d|r}\mu(r/d)d^z, \qquad A_c(s) = \sum_{r\ge2} J_{1-s}(r)v_r,$$

and the fixed energy

$$W_\omega(c) = \sum_{r\ge2} p_r|v_r|^2, \quad p_r = r\log^2(er)w_r, \quad w_r = (J_4(r) + 10J_2(r))/180. \tag{1}$$

Jordan factorization gives

$$r^4/[180\zeta(4)] \le w_r \le 7r^4/360, \qquad r \ge 2. \tag{2}$$

Here and below $\zeta(x) = \sum_{n\ge1} n^{-x}$ for real $x > 1$. Fix

$$0 < a < \tfrac{1}{2}, \quad \sigma = \tfrac{1}{2} - a, \quad u = \tfrac{1}{2} + a, \quad q = \sigma + iT, \quad \lambda = u + iT, \quad T \in \mathbb{R}. \tag{3}$$

2 The exact band energy and its uniform order

For $I_N = \{N+1, \dots, 2N\}$ define

$$\kappa_N(q) = \sum_{r\in I_N} |J_{1-q}(r)|^2/p_r. \tag{4}$$

Put

$$Z_u = \zeta(u+1), \quad E_u = \frac{2^{-u}}{u} + \frac{2^{1-u}}{1-u}, \quad N_0(u) = \left\lceil (2Z_uE_u)^{1/u} \right\rceil,$$

$$\ell_u = \frac{45}{448Z_u^2}, \qquad h_u = 360\zeta(4)\zeta(2u+1)Z_u^2. \tag{5}$$

Theorem 2.1 (Optimal prescribed-band repair)**.** *For every integer $N \ge \max\{2, N_0(u)\}$ and every real T,*

$$\ell_u \frac{N^{-3+2a}}{\log^2(eN)} \le \kappa_N(q) \le h_u \frac{N^{-3+2a}}{\log^2(eN)}. \tag{6}$$

Among divisor vectors supported in I_N with $\sum J_{1-q}(r)v_r = t$, the exact minimum energy is

$$\mathcal{E}_N(t,q) = |t|^2/\kappa_N(q). \tag{7}$$

It is attained by

$$v_r = \frac{t\,\overline{J_{1-q}(r)}}{p_r\kappa_N(q)} \quad (r \in I_N), \qquad v_r = 0 \quad (r \notin I_N). \tag{8}$$

In particular, for nonzero t the minimum is $\Theta_a(|t|^2 N^{3-2a}\log^2(eN))$, uniformly in T.

Lemma 2.2 (Divisor mean-square majorant)**.** *For $\beta > 0$, let $\tau_\beta(r) = \sum_{d|r} d^{-\beta}$. For $x \ge 1$,*

$$\sum_{r\le x} \tau_\beta(r)^2 \le x\,\zeta(2\beta+1)\zeta(\beta+1)^2. \tag{9}$$

Proof. Expand the square and count multiples of $\operatorname{lcm}(d,e)$. The result is at most

$$x \sum_{d,e\ge1} (d^\beta e^\beta \operatorname{lcm}(d,e))^{-1}.$$

Writing $d = gh, e = gk$, $\gcd(h,k) = 1$, makes the denominator $g^{2\beta+1}h^{\beta+1}k^{\beta+1}$. Dropping coprimality gives the convergent product in (9). □

Proof of Theorem 2.1. Set $g_u(r) = J_u(r)/r^u = \sum_{d|r} \mu(d) d^{-u} > 0$. Euler factorization and the reverse triangle inequality give $|J_{u-iT}(r)| \geq J_u(r) = r^u g_u(r)$, independently of T. The exact band sum satisfies

$$\left| \sum_{N<r\leq 2N} g_u(r) - N/Z_u \right| \leq E_u N^{1-u}.$$

Indeed, the error in $\lfloor 2N/d \rfloor - \lfloor N/d \rfloor - N/d$ is at most 1. The corresponding divisor sum is at most $(2N)^{1-u}/(1-u)$. The omitted tail of the absolutely convergent Möbius series for $1/Z_u$ is at most $2^{-u}N^{1-u}/u$ after multiplication by N. For $N \geq N_0(u)$, Cauchy–Schwarz therefore gives

$$\sum_{I_N} g_u(r)^2 \geq N/(4Z_u^2).$$

Use $r \geq N$ in the numerator and $p_r \leq (7/360)(2N)^5 \log^2(2eN)$ in the denominator. This yields

$$\kappa_N(q) \geq \frac{45}{112 Z_u^2} \frac{N^{2u-4}}{\log^2(2eN)} \geq \ell_u \frac{N^{2u-4}}{\log^2(eN)}.$$

For the upper bound, $|J_{u-iT}(r)| \leq r^u \tau_u(r)$. The lower weight in (2) and Lemma 2.2 at $x = 2N$ imply

$$\kappa_N(q) \leq \frac{180\zeta(4) N^{2u-5}}{\log^2(eN)} \sum_{r\leq 2N} \tau_u(r)^2 \leq h_u \frac{N^{2u-4}}{\log^2(eN)}.$$

Here $2u - 4 = -3 + 2a$. Finally weighted Cauchy–Schwarz gives $|t|^2 \leq \kappa_N W_\omega$, and (8) attains equality. The finite divisor inverse realizes its anchor and prescribed value. □

The optimality assertion concerns the specified divisor band. The actual coefficient correction may also change indices below N; the band requirement is on v, not on a preserved coefficient prefix.

3 Native and coefficient costs of the same correction

For integer $d \geq 2$, let $\gamma_d(n) = \{n/d\}$ and use the complete norm

$$\|f\|_{\mathrm{nat}}^2 = \sum_{n\geq 1} |f(n)|^2/[n(n+1)].$$

The finite inverse defines $f_c = \sum_{d\geq 2} c_d \gamma_d$. Splitting the sum at d gives $\|\gamma_d\|_{\mathrm{nat}}^2 \leq 2/d$. Write $S_1(c) = \sum_{d\geq 1} d|c_d|$, including c_1. Define

$$A_a = \sqrt{\frac{1440\zeta(4)\zeta(2)\zeta(3/2)^2}{\ell_u}}, \qquad B_a = \sqrt{\frac{180\zeta(4)\zeta(2)^2 \log 2}{\ell_u}}. \tag{10}$$

Theorem 3.1 (Native norm and coefficient-moment bounds)**.** *For the actual correction* (8)*, with* $N \geq \max\{2, N_0(u)\}$*,*

$$\|f_c\|_{\mathrm{nat}} \leq A_a |t| N^{-a}, \qquad S_1(c) \leq B_a |t| N^{3/2-a}. \tag{11}$$

Proof. Inversion and the generator estimate give $\|f_c\|_{\mathrm{nat}} \leq \sqrt{2} \sum_{I_N} |v_r| \sigma_{1/2}(r)$, where $\sigma_\beta(r) = \sum_{d|r} d^\beta$. Also $S_1(c) \leq \sum_{I_N} |v_r| \sigma_2(r)$. Since $\sigma_{1/2}(r) = \sqrt{r}\, \tau_{1/2}(r)$, Lemma 2.2 gives

$$\sum_{I_N} \sigma_{1/2}(r)^2 \leq 4N^2 \zeta(2)\zeta(3/2)^2.$$

Consequently its dual sum with weight p_r^{-1} is at most $720\zeta(4)\zeta(2)\zeta(3/2)^2N^{-3}/\log^2(eN)$. For the other moment, $\sigma_2(r) \le \zeta(2)r^2$ and $\sum_{I_N} 1/r \le \log 2$, so its dual sum is at most $180\zeta(4)\zeta(2)^2\log 2/\log^2(eN)$. Apply weighted Cauchy–Schwarz, (7), and the lower capacity bound (6). All native sums and their cross terms are included in these complete-norm inequalities. □

4 Simultaneous approximation, energy and complete mean

For a finite anchored trial set $\mathcal{F}(c) = \|1 + f_c\|_{\mathrm{nat}}^2$, and let d_n be its minimum square-root error over coefficient indices at most n, with no additional q-null. These are complete native norms. The finite native Gram is positive definite by integer-jump Möbius inversion, so its optimizer $b^{(n)}$ exists uniquely.

We use the coefficient estimate

$$\sum_{d=2}^{n} |b_d| \le P_n := n\sqrt{4n+1}. \tag{12}$$

For completeness, set $f(0) = 0$. The anchored floor formula gives $c_k = -\sum_{d|k} \mu(k/d)[f(d) - f(d-1)]$. Thus

$$\sum_{k=2}^{n} |c_k| \le n \sum_{j=1}^{n} |f(j)| \left(\frac{1}{j} + \frac{1}{j+1}\right) \le n\sqrt{4n+1}\|f\|_{\mathrm{nat}}.$$

The squared dual weight is exactly $4n + 1 - 1/(n+1)$. Orthogonal projection gives $\|f_{b^{(n)}}\|_{\mathrm{nat}} \le 1$, proving (12). This is the estimate of [2, Lemma 3] in the present sign convention.

Theorem 4.1 (A simultaneous finite family)**.** *Let $N \ge \max\{2, N_0(u)\}$ be an integer such that $n = \lfloor N^{a/3} \rfloor \ge 2$. Starting with $b^{(n)}$, apply (8) with $t = -A_{b^{(n)}}(q)$, and call the resulting sum $c^{(N,T)}$. It is a finite actual coefficient vector supported in indices at most $2N$, with $A_c(1) = A_c(q) = 0$. Uniformly in T,*

$$\sqrt{\mathcal{F}(c)} \le d_n + \frac{3\sqrt{5}}{2} A_a N^{-a/2}, \tag{13}$$

$$W_\omega(c) \le \left(\frac{7}{72} + \frac{45}{4\ell_u}\right) N^{3-a} \log^2(eN), \tag{14}$$

$$S_1(c) \le \sqrt{5}\left(\frac{5}{4} + \frac{3}{2}B_a\right) N^{3/2-a/2}. \tag{15}$$

Proof. The anchor coefficient obeys $|b_1| \le P_n/2$, and hence $|A_b(q)| \le (3/2)P_n \le (3\sqrt{5}/2)N^{a/2}$. Theorem 3.1 and the triangle inequality prove (13). For $r \ge 2$, $|v_r(b)| \le P_n/r$. From (2),

$$W_\omega(b) \le (7/360)P_n^2[n(n+1)/2]^2 \log^2(en) \le (7/72)n^7 \log^2(en).$$

The base divisor support lies below $n < N$, while the repair lies in I_N. Their energies add exactly, even though their coefficient supports can overlap. By (6), the repair costs at most $45N^{3-a}\log^2(eN)/(4\ell_u)$. Since $7a/3 \le 3 - a$ for $0 < a < 1/2$, this proves (14). Finally $S_1(b) \le (n+1/2)P_n \le (5\sqrt{5}/4)n^{5/2}$. Add the repair bound (11); the base exponent $5a/6$ is at most $3/2 - a/2$. This gives (15). □

The bound (13) compares the constrained trial with the best native approximation at the smaller support scale n; it is an oracle inequality in this precise sense.

To state the complete tail in the same normalization as [5], put $F_K(x) = x^{-\lambda} - K^{-2a}x^{-q}$ and

$$\mathcal{M}_K(c) = \sum_d c_d \int_K^\infty B_1(\{x/d\})F_K'(x)\,dx, \qquad B_1(x) = x - \tfrac{1}{2}. \tag{16}$$

One integration by parts using the continuous periodic primitive $dB_2(\{x/d\})/2$ gives

$$|\mathcal{M}_K(c)| \le H(a,T)S_1(c)K^{-u-1},$$

where

$$H(a,T) = \frac{2a+\mathcal{A}_1}{12}, \quad \mathcal{A}_1 = \frac{|\lambda(\lambda+1) - q(q+1)|}{u+1} + \frac{2a|q(q+1)|}{(\sigma+1)(u+1)}.$$

Indeed the boundary term is $(\lambda - q)K^{-\lambda-1}\sum_d c_d dB_2(\{K/d\})/2$. Its absolute bound contributes $2aS_1K^{-u-1}/12$. The remaining derivative integral contributes $\mathcal{A}_1S_1K^{-u-1}/12$: scale $x = Ky$ and split the derivative difference into $[\lambda(\lambda+1) - q(q+1)]y^{-2a} + q(q+1)(y^{-2a} - 1)$. Their integrals are $1/(u+1)$ and $2a/[(\sigma+1)(u+1)]$. This is the complete mean, including the boundary term.

Since $\lambda(\lambda+1) - q(q+1) = 4a(1+iT)$, $|q| \le 1 + |T|$ and $|q+1| \le (\sigma+1)(1+|T|)$ imply $H(a,T) \le 5a(1+|T|)^2/[6(u+1)]$.

Corollary 4.2 (A uniform complete tail)**.** *For the same family and hypotheses as Theorem 4.1, let*

$$K_N(T) = \left\lceil N^2(1+|T|)^{2/(u+1)} \right\rceil.$$

Then

$$|\mathcal{M}_{K_N(T)}(c^{(N,T)})| \le \frac{5a\sqrt{5}}{6(u+1)}\left(\frac{5}{4} + \frac{3}{2}B_a\right)N^{-3/2-5a/2}. \tag{17}$$

Proof. Substitute (15). The cutoff absorbs the entire $(1+|T|)^2$ factor and contributes N^{-2u-2}. □

The height-dependent cutoff in this corollary is an arithmetic mean-tail cutoff. The approximation error in (13) was already measured in the complete native norm.

5 The approximation endpoint and concrete use

Let $d_\infty = \inf_n d_n$. The lower bound $d_\infty \le \sqrt{\mathcal{F}(c^{(N,T)})}$ and (13) show that this family's native error converges to d_∞ uniformly in real T, for every fixed a. Its zero case is the classical strong Nyman–Beurling criterion [6].

Here is the exact normalization bridge. For unanchored generators $e_d(y) = \{y/d\}$ in $L^2((0,\infty), y^{-2}dy)$, write $f = \sum c_d e_d$ and $S = \sum c_d/d$. On $(0,1)$, $f(y) = Sy$, so $|S| \le \|\mathbf{1}_{[1,\infty)} + f\|$. The anchored correction $f - Se_1$ vanishes there and equals $\sum_{d\ge2} c_d\gamma_d(n)$ on $n \le y < n+1$. Since $\|e_1\| \le \sqrt{2}$, its approximation cost is at most $1+\sqrt{2}$ times the unanchored error. The reverse closure implication is inclusion. Thus the native zero endpoint is precisely the cited criterion. An approximation upper bound can therefore be transferred through (13) with its original hypotheses.

For example, the RH-conditional theorem of Balazard and de Roton [7, stated theorem, Section 1] and this anchoring bound imply, under RH, for every $\varepsilon > 0$,

$$\sup_T \mathcal{F}(c^{(N,T)}) \ll_{a,\varepsilon} \frac{(\log\log N)^{5/2+\varepsilon}}{\sqrt{\log N}}.$$

The energy and complete-tail estimates above apply independently of that conditional input.

For a fixed known native trial b of support at most $n < N$, the same repair gives

$$\sqrt{\mathcal{F}(b + \text{repair})} \le \sqrt{\mathcal{F}(b)} + A_a |A_b(q)| N^{-a}.$$

This permits a certified complete trial to be reused with the exact q-null while retaining an explicit error payment. The band method's energy order is N^{3-2a} per fixed prescribed Dirichlet value. A correction only at coefficient index N, with its anchor adjustment, gives the sufficient order N^{4-2a} in this metric. The comparison concerns these proved constructions; the prescribed-band optimum is the precise optimality statement.

6 Finite controls and reproducibility

The accompanying interval program reconstructs the actual finite divisor rows, band capacities, extremizers, inverse coefficients and full energy for $a = 1/4$ at representative heights. It compares them with both capacity bounds and the complete coefficient/native majorants. Exact null identities follow from divisor inversion and (8); interval residual checks are consistency controls. The infinite, all-height conclusions follow from the analytic proofs. Reader instructions specify dependencies and the reference output.

Acknowledgment of assistance. AI assistance supported derivation, literature comparison, independent source review and computational checks. The author is responsible for the mathematical statements and presentation.

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