Odd Weil-form bounds by positive exterior elimination

Scott Schubr
Luminary Oddities

October 26, 2026

Abstract

We give a complete lower certificate for the odd part of the original minimal theta-weighted Weil form. A positive infinite-rank exterior is absorbed through its directional covariance with the signed low-frequency features. An endpoint chord for the receiving resolvent reduces the required information to two finite matrices, without assuming a positive lower bound for the true feature Gram. At arithmetic cutoff eight this yields

$$q_H[h] \geq -1.88 \times 10^{-9} \|h\|_2^2 \qquad (h \in \mathcal{V} \text{ odd}).$$

The proof retains the original polar constraint, four odd exact derivative-null rows, the complete arithmetic remainder and all low/exterior mixed terms. Directed ball arithmetic verifies the resulting matrix of order 4085. The method also has a concrete boundary: at arithmetic cutoff eleven, with the same frequency band and multiplication parameter, the actual positive exterior has odd ambient norm greater than 1.7. We prove the first-moment receiver's optimality in its finite information class and a convergent higher-moment refinement.

1 The question and the two arithmetic regimes

Quantitative Weil-form certificates seek a lower bound on the complete explicit form in a fixed norm and on its original constrained domain. Signed low-frequency reduction is useful only when the omitted exterior and its coupling to the retained features are controlled. A pointwise exterior floor is one way to do this. Increasing the arithmetic cutoff can destroy that floor even while improving the arithmetic tail.

Here we retain the band $[-2400, 2400]$ and increase the arithmetic cutoff from seven to eight. The exterior is no longer nonnegative. Its positive defect, however, has a whole-operator upper bound below $3/4$. Rather than charge that number as a scalar loss against the low cap, we enclose the defect's covariance with the low and constraint columns and eliminate it through a positive resolvent. This gives a new odd certificate with the same 4080 phase coordinates and five global rows. No additional exterior carrier columns are needed in this implementation.

The odd sector retains the Weil positivity endpoint: complete lower certificates on this same original odd domain, with allowances tending to zero, would establish that endpoint. This makes a quantitatively sharper odd certificate a natural target even when the available certificate is at one fixed allowance.

The original realization H and its form domain $\mathcal{V}$ are defined in Section 2. Our main application is the following.

Theorem 1.1 (Original odd-form certificate)**.** *Every odd $h \in \mathcal{V}$ satisfies*

$$q_H[h] \geq -\nu_8 \|h\|_2^2, \qquad \nu_8 = \frac{47}{25000000000}. \tag{1}$$

Equivalently, every odd compact smooth f with $\mathcal{M}f(1/2) = \mathcal{M}f(-1/2) = 0$ satisfies

$$Q[f] \geq -\nu_8 \int_{\mathbb{R}} |f(u)|^2 \frac{\cosh(u/2)}{\psi(u)}\, du.$$

The same statement holds on the complex odd subspace.

The predecessor [1] proves the both-parity estimate with allowance $\nu_7 = \frac{4279999016263}{2\cdot 10^{20}}$ and a retained positive potential on an exact-null complement. Restricting its scalar conclusion to odd vectors gives a matched comparison: $\nu_7/\nu_8 \simeq 11.382976107$. The new bound strengthens that odd scalar conclusion; the predecessor's even conclusion and its more detailed positive-potential statement retain their own scope.

The exterior elimination has an arithmetic boundary that is visible in the actual operator, rather than only in a comparison majorant.

Theorem 1.2 (Supercritical exterior at cutoff eleven)**.** *Put $\mu = 1/5$, $t_\alpha = s/(\alpha+\mu s^2)^{1/2}$ and*

$$v_{11}(\xi) = 1_{|\xi|>2400}(\mu - m_{11}(\xi))_+, \qquad O_{11,\alpha} = M_{t_\alpha} v_{11}(D) M_{t_\alpha},$$

where m_{11} is the full retained symbol of Section 2. For every $0 < \alpha \leq 10^{-9}$,

$$\|O_{11,\alpha}|_{L^2_{\mathrm{odd}}}\| > \frac{17}{10}. \tag{2}$$

This is the ambient odd restriction; no polar or derivative-null constraints are imposed on this auxiliary operator.

The two theorems answer a common method question. Cutoff eight admits a subcritical positive exterior and a successful signed receiving cap. At cutoff eleven the same ambient positive-exterior inverse cannot serve as a positive reference at the stated parameters. The obstruction is specific to that relaxation: a joint signed treatment or a genuinely constrained exterior is a different recipient.

The realization, complete arithmetic estimates, exact-null deflation and signed-Gram reduction are inherited from [2, 1, 3]. Positive functional calculus, Schur complements and endpoint chords are classical. In particular the inequality behind our first-moment receiver is the endpoint chord appearing in operator Kantorovich arguments; see [4, Remark 2.3]. We use those tools with the actual theta exterior, preserve its directional mixed covariance, and certify the resulting original-domain bound. Compact-window Weil certificates such as [7] have a related finite-certification purpose but a different recipient. The new source-specific ingredients are the complete positive-exterior enclosure, its executed covariance and the contrasting cutoff-eleven lower bound. Desogus [5] proposes an all-support positivity argument based on physical-support induction, complete arithmetic routing and a rooted common-source Schur form. The certificate here uses a reciprocal-theta weighted norm, a fixed arithmetic cutoff and a complete positive-exterior enclosure.

2 The original form, domain and exact null rows

Inner products are conjugate-linear first. Our transforms are

$$\mathcal{M}f(z) = \int_{\mathbb{R}} f(u) e^{zu}\, du, \qquad \widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-iu\xi}\, du,$$

with Plancherel measure $d\xi/(2\pi)$. Let $\Psi_0 = \Gamma'/\Gamma$, let Λ be von Mangoldt's function, and put $c_n = \Lambda(n)/\sqrt{n}$ and $S_b f(u) = f(u+b)$. The physical Weil form is

$$\begin{aligned} Q[f] = &\int_{\mathbb{R}} [\operatorname{Re}\Psi_0(1/4 + i\xi/2) - \log\pi] |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi} \\ &- \sum_{n\geq 2} c_n \{\langle f, S_{\log n} f\rangle + \langle f, S_{-\log n} f\rangle\}. \end{aligned} \tag{3}$$

Use the normalized positive even theta kernel

$$\psi(u) = 2\sum_{k\geq 1}(2\pi^2 k^4 e^{9u/2} - 3\pi k^2 e^{5u/2})e^{-\pi k^2 e^{2u}}, \tag{4}$$

$$c(u) = \cosh(u/2), \quad w(u) = \psi(u)c(u), \quad s(u) = \sqrt{\psi(u)/c(u)}.$$

Theta inversion makes ψ even and $\mathcal{M}\psi(z) = \xi(1/2+z)$ for the completed xi-function. Define

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle\sqrt{w}, h\rangle = \langle\sqrt{w}\tanh(u/2), h\rangle = 0\},$$
$$a(\xi) = \operatorname{Re}\Psi_0(1/4 + i\xi/2) - \Psi_0(1/4), \qquad m_0 = \Psi_0(1/4) - \log\pi.$$

The positive series

$$a(\xi) = \sum_{j\geq 0}\frac{(\xi/2)^2}{(j+1/4)((j+1/4)^2 + (\xi/2)^2)} \tag{5}$$

shows that $a \geq 0$ and is increasing for $\xi \geq 0$. On $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ take

$$q_A[h] = \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2\frac{d\xi}{2\pi}.$$

The specified minimal domain $\mathcal{V}$ is the closure of this core in its graph form norm. The graph map $h \mapsto \sqrt{a}\,\widehat{sh}$ is closable. Two fixed compact bump corrections give density in $\mathcal{K}$, since the ratio of the two row profiles is the nonconstant function $\tanh(u/2)$.

With $P_{\mathcal{K}}$ the orthogonal projection onto $\mathcal{K}$, let

$$B = P_{\mathcal{K}}\left[m_0 M_s^2 - \sum_{n\geq 2} c_n M_s(S_{\log n} + S_{-\log n})M_s\right]\Big|_{\mathcal{K}}, \qquad H = A + B. \tag{6}$$

The complete series converges in operator norm by the tail estimate below. On the core, $f = sh$ has both polar moments zero and

$$q_H[h] = Q[f], \qquad \|h\|_2^2 = \int_{\mathbb{R}} |f|^2 c/\psi. \tag{7}$$

Conversely every such compact f gives $h = f/s \in \mathcal{C}$. This fixes the original realization throughout.

The restricted real-odd Weil criterion [5, Theorem 1.2], together with the polar repair in [2, Section 2.1], gives

$$\text{RH} \iff q_H[h] \geq 0 \quad \text{for every real odd } h \in \mathcal{V}. \tag{8}$$

To see the domain bridge, take an arbitrary real odd compact smooth g, and set $m = \mathcal{M}g(1/2)$. The odd function $F = g + 4m\psi'$ has both polar moments zero, since $\mathcal{M}\psi'(1/2) = -1/4$ and $\mathcal{M}\psi'(-1/2) = 1/4$. Its zero-side Weil pairing equals that of g, because $\mathcal{M}\psi'$ vanishes at all centered zeros. Even smooth cutoffs, followed by one fixed odd compact bump repair, approximate F with both moments exact. Continuity of the explicit formula passes positivity on the original compact core to the zero-side pairing of g. The restricted odd criterion gives the reverse implication in (8); under RH, Weil positivity on the compact core extends to its specified form closure.

Write $D_u = d/du$ and

$$q_j = (D_u^2 - 1/4)D_u^j\psi, \qquad z_j = q_j/s, \qquad 0 \leq j \leq 7. \tag{9}$$

The exact-null statement [1, Section on exact deflation] is $z_j \in \operatorname{Dom} H \cap \mathcal{K}$, $Hz_j = 0$, with parity $(-1)^j$. Here is the mechanism. The transform of q_j is $(z^2 - 1/4)(-z)^j\xi(1/2+z)$: it

vanishes at the polar arguments and at every centered zero entering the Weil pairing. Thus its complete bilinear pairing with every compact test vanishes. The theta derivative envelopes give $z_j \in L^2$, form-domain cutoff approximation and $q_A(z_j, g) = -\langle Bz_j, g\rangle$; the representation theorem gives the operator identity. These domain and cutoff details are proved in [1, 3] and use the same $\mathcal{V}$ defined above.

Let Z_8 be their span, Π_8 its orthogonal projection and $P_8 = I - \Pi_8$. A finite projection onto vectors in $\operatorname{Dom} H$ preserves $\mathcal{V}$. Consequently

$$q_H[h] = q_H[P_8 h], \qquad h \in \mathcal{V}. \tag{10}$$

For odd vectors the relevant null rows are exactly $j = 1, 3, 5, 7$. This is nullity for the complete H, rather than for any truncated arithmetic form.

For $M \geq 2$, let B_M retain every $n \leq M$ in (6) and the same m_0 term, and define

$$m_M(\xi) = \operatorname{Re} \Psi_0(1/4 + i\xi/2) - \log \pi - 2 \sum_{2 \leq n \leq M} c_n \cos(\xi \log n). \tag{11}$$

All prime powers in this expression have their literal von Mangoldt coefficients.

3 Eliminating a positive infinite-rank exterior

The next result is an operator statement, independent of theta arithmetic.

Theorem 3.1 (Directional first-moment receiver)**.** *Let $0 \leq O \leq \tau I$ on a Hilbert space, let $a > \tau$, and let $W : \mathbb{C}^d \to \mathcal{H}$ be bounded. Put $G = W^*W$ and $T = W^*OW$. Then*

$$W^*(aI - O)^{-1}W \leq \frac{1}{a}\left(G + \frac{T}{a - \tau}\right) =: G_{\mathrm{ch}}. \tag{12}$$

This is the least Loewner upper bound valid from only G, T and the interval $[0, \tau]$, including singular and noncommuting G, T. Suppose $C = C^*$ and finite Hermitian matrices $\overline{C}, \overline{G}$ satisfy

$$C \leq \overline{C}, \quad G_{\mathrm{ch}} \leq \overline{G}, \quad \overline{G} > 0, \quad \beta\overline{G} - \overline{G}\,\overline{C}\,\overline{G} \geq 0 \tag{13}$$

for $\beta \geq 0$. Then

$$WCW^* \leq \beta(aI - O). \tag{14}$$

Proof. For $0 \leq x \leq \tau < a$, clearing denominators gives

$$\frac{1}{a - x} \leq \frac{1}{a} + \frac{x}{a(a - \tau)}.$$

Functional calculus and congruence give (12). The equivalent inequality $a(a-\tau)(aI-O)^{-1} + (aI - O) \leq (2a - \tau)I$ is the endpoint chord in the operator Kantorovich literature [4]. For sharpness, any admissible $0 \leq T \leq \tau G$ has the realization

$$O_* = \operatorname{diag}(0, \tau I), \qquad W_* = \begin{bmatrix} (G - T/\tau)^{1/2} \\ (T/\tau)^{1/2} \end{bmatrix}.$$

It has the specified two moments and gives equality in (12). No commutation is involved. The case $\tau = 0$ is immediate.

Set $A = aI - O > 0$ and $U = A^{-1/2}W$. From (12), $U^*U \leq \overline{G}$. Hence $J = U\overline{G}^{-1/2}$ is a contraction. The last inequality of (13) is equivalent to $\overline{G}^{1/2}\overline{C}\overline{G}^{1/2} \leq \beta I$. Thus

$$UCU^* \leq U\overline{C}U^* \leq \beta JJ^* \leq \beta I.$$

The last step uses the nonnegative threshold β. Congruence with $A^{1/2}$ proves (14). The actual G need not have a positive spectral floor. □

Higher moments give a systematic refinement.

Proposition 3.2. *Let $T_j = W^*O^jW$, $T_0 = G$. For every $k \geq 1$,*

$$W^*(aI - O)^{-1}W \leq H_k := \sum_{j=0}^{k-1} \frac{T_j}{a^{j+1}} + \frac{T_k}{a^k(a-\tau)}. \tag{15}$$

These uppers decrease in Loewner order to the exact receiving Gram. Their residual norm is at most $\|G\|\tau^k/[a^k(a-\tau)]$.

Proof. The finite Neumann expansion has positive remainder $O^k/[a^k(aI-O)] \leq O^k/[a^k(a-\tau)]$. Moreover $H_k - H_{k+1} = (\tau T_k - T_{k+1})/[a^{k+1}(a-\tau)] \geq 0$. The displayed geometric bound proves norm convergence at fixed $a > \tau$. □

For application, suppose a signed target K has a finite approximation and positive exterior with

$$K \leq WCW^* + \eta I + O, \qquad 0 \leq O \leq \tau I, \quad a = 1 - \eta > \tau.$$

If Theorem 3.1 gives (14) with $0 \leq \beta < 1$, then

$$K \leq [\beta a + \eta + (1-\beta)\tau]I = [1 - (1-\beta)(a-\tau)]I \leq I. \tag{16}$$

When some columns are exact global constraint rows, a negative row penalty can be retained in C and applied on the ambient space. It vanishes on the exact row kernel. In that implementation $G = W^*W$ and $T = W^*OW$ are the raw augmented moments. A hard-projected implementation would instead require the transported covariance F^*POPF for $W = PF$. These are distinct finite reductions.

4 The complete cutoff-eight exterior

In the remainder of the certificate proof fix

$$M = 8, \quad \alpha = 10^{-9}, \quad \mu = 1/5, \quad L = 2400, \quad t = s/(\alpha + \mu s^2)^{1/2}. \tag{17}$$

The nonzero retained channels are $2, 3, 4, 5, 7, 8$.

4.1 All omitted arithmetic

For $u \geq 0$, the positive theta series gives

$$\psi(u) \leq 8\pi^2 e^{9u/2 - \pi e^{2u}}, \qquad s(u)^2 \leq 16\pi^2 e^{4u - \pi e^{2u}}. \tag{18}$$

For example, after extracting the first exponential, use $k^2 - 1 \geq 3(k-1)$ and $\sum k^4 q^{k-1} = (1 + 11q + 11q^2 + q^3)/(1-q)^5 < 2$ for $q = e^{-3\pi} < 1/100$. The known log-concavity of ψ [6, Theorems 4.2 and 4.5] makes s even and log-concave. For $U = M_s S_b M_s$, let $S = s(0)^2$, $P = s(b/2)^2$ and $Q = s(b)^2$. Midpoint concavity gives $\|U\| = P$ and $\|U^2\| = SQ$. The multiplication operator $UU^* + U^*U$ has norm at most $P^2 + SQ$: on each half-line one summand is at most P^2 and the other at most SQ. Therefore

$$\|U + U^*\| \leq \sqrt{P^2 + 3SQ}. \tag{19}$$

The coarse paired triangle estimate from (18) is $32\pi^2 n^2 e^{-\pi n}$, since $c_n \leq 1$. Exact geometric summation yields $\sum_{n \geq k} n^2 e^{-\pi n} < 2k^2 20^{-k}$ for $k \geq 2$. Hence

$$\|B - B_8\| \leq \frac{640 \cdot 17^2}{20^{17}} + \sum_{(n,p) \in E} \frac{\log p}{\sqrt{n}} \sqrt{s(\tfrac{1}{2}\log n)^4 + 3s(0)^2 s(\log n)^2} < \delta := \frac{88}{10^{11}}, \tag{20}$$

where $E = \{(9,3), (11,11), (13,13), (16,2)\}$. The finite expression is enclosed with the first eight theta atoms and a complete absolute remainder 2^{-250}; Appendix B gives the scaled evaluation bound.

4.2 A whole-line exterior cover

Let $b(\xi) = \mu - m_8(\xi)$. The positive series (5) supplies an all-ray floor once $a(4800) - d_8 > \mu$, where $d_8 = -m_0 + 2\sum_{n\le 8} c_n < 12$. On $[2400, 4800]$ bisect whole closed intervals. Accept a box when interval evaluation gives $m_8 > \mu$ throughout. Otherwise bisect down to width at most $1/32$ and retain it as unresolved. The supplied exact rational leaf list is a gap-free cover; it has 58 unresolved leaves in three connected components. Its last unresolved endpoint is $6348825/2048$. Thus every exterior frequency outside these leaves and their reflections has $b \le 0$.

For each unresolved positive leaf $I_l = [a_l, b_l]$, choose an exact outward dyadic constant $\gamma_l > \max\{0, \sup_{I_l} b\}$, retaining only positive constants. Put

$$\gamma(\xi) = \sum_l \gamma_l 1_{I_l \cup (-I_l)}(\xi), \qquad \widehat{O} = M_t \gamma(D) M_t. \tag{21}$$

The enclosure checks $0 < \gamma_l < 1$ and

$$A_\gamma := \sum_l \gamma_l |I_l| < 0.198. \tag{22}$$

It follows pointwise that $1_{|\xi|>L} b(\xi) \le \gamma(\xi)$. The dyadic envelope in (21), rather than a pointwise sampled exterior, is the operator used in the finite covariance.

Lemma 4.1 (Complete exterior contraction)**.** *Uniformly for $\alpha \ge 10^{-11}$ at $\mu = 1/5$,*

$$\int_{\mathbb{R}} t_\alpha(u)^2 \, du < 11.704, \qquad 0 \le \widehat{O}_\alpha \le \frac{3}{4} I. \tag{23}$$

Proof. The function $t_\alpha^2 = \psi/(\alpha c + \psi/5)$ decreases with α. At 10^{-11}, 2500 whole cells of width $1/1000$ cover $[0, 5/2]$. Evaluate the eight-atom theta enclosure and its complete remainder on each cell, divide by the positive denominator, clip the upper by the independently valid bound 5, and sum with reflection. This gives a complete bounded-part upper below 11.703168183340002. For the two unbounded tails use $s^2 \le 240 e^{4u - 3e^{2u}}$. The substitution $v = e^{2u}$ and $e^5 > 148$ give

$$\int_{|u|>5/2} t_\alpha^2 \, du \le 240 \cdot 10^{11} \int_{148}^{\infty} v e^{-3v} \, dv < 2^{-520}.$$

This proves the first bound. The Fourier factorization of (21) is Hilbert–Schmidt and gives exactly

$$\operatorname{tr} \widehat{O}_\alpha = \frac{A_\gamma}{\pi} \int_{\mathbb{R}} t_\alpha^2 < \frac{(198/1000)(11704/1000)}{157/50} < \frac{3}{4},$$

using $\pi > 157/50$. Positivity bounds its norm by its trace. □

For $x = (\alpha + \mu s^2)^{1/2} h$, the full symbol identity and (20) now imply, first on the core and then by closure,

$$q_H[h] + (\alpha + \delta)\|h\|^2 \ge \|x\|^2 - \langle x, K_8 x \rangle, \qquad K_8 \le K_{\mathrm{low}} + \widehat{O}, \tag{24}$$

where $K_{\mathrm{low}} = M_t[1_{[-L,L]} b](D) M_t$. The notation K_8 in this inequality denotes the complete signed transformed defect; its negative high-frequency part can equivalently be retained as a lower form. All bounds above are valid on its form domain, and the upper relaxation on the right is bounded.

5 The signed low block and the original odd return

Partition $[-17/10, 17/10]$ into 136 cells of half-width $r = 1/80$. With centers $c_a = -17/10 + (2a+1)r$, use

$$\phi_{aj}(u) = 1_{\text{cell } a}(u)\sqrt{\frac{2j+1}{2r}}P_j((u-c_a)/r), \qquad 0 \le j < 60. \tag{25}$$

Odd reflection folds the 68 negative cells into 4080 raw phase columns. A raw folded column has squared norm 2 before multiplication by t. Append the five complete odd profiles

$$t\sinh(u/2), \qquad tc(u)q_j(u)/\psi(u), \quad j = 1, 3, 5, 7, \tag{26}$$

after a fixed invertible triangular dyadic change of coordinates. These are precisely the transformed original polar and derivative-null rows, because $tcq_j/\psi = z_j/(\alpha + \mu s^2)^{1/2}$. The exact dyadic mixture is specified in the reproduction program. Its entries have modulus below 2 and squared Frobenius norm below 100; no orthonormality is assumed.

Let W consist of these 4085 columns, and set

$$D = \operatorname{diag}(2I_{4080}, I_5), \qquad C_{\text{aug}} = D^{-1}\operatorname{diag}(C_{\text{fold}}, -1000I_5)D^{-1}. \tag{27}$$

The phase coefficient is the exact signed low-band Fourier compression. The factor D^{-1} on each side is essential: normalized odd folding divides the integer folded matrix by 2, and converting normalized columns to raw columns divides by a further 2. The penalty in (27) vanishes on every exact row kernel.

The complete phase and spatial estimate of Appendix A gives

$$\|K_{\text{low}} - F_{\text{phase}}\| < \eta := 9 \cdot 10^{-9}. \tag{28}$$

For $h \in \mathcal{V} \cap Z_8^{\perp}$ odd, the transformed x is in the five exact row kernel. Thus its low quadratic form equals the augmented one, and

$$\langle x, K_8 x\rangle \le \langle x, WC_{\text{aug}}W^*x\rangle + \eta\|x\|^2 + \langle x, \widehat{O}x\rangle.$$

All raw directional moments are retained:

$$G = W^*W, \qquad T = W^*\widehat{O}W. \tag{29}$$

With $a = 1 - \eta$ and $\tau = 3/4$, Theorem 3.1 gives

$$W^*(aI - \widehat{O})^{-1}W \le (1 + 18 \cdot 10^{-9})G + 5T. \tag{30}$$

The two coefficient enlargements follow from $1/a \le 1 + 2\eta$ and $1/[a(a - 3/4)] \le 5$.

5.1 Complete directed enclosures

Let $e = 10^{-9}$, and let C_0, G_0 be the exact saved dyadic centers in the folded metric. The full source-sum and analytic enclosures prove

$$C_{\text{aug}} \le \overline{C} := D^{-1}(C_0 + eD)D^{-1}, \qquad G \le G_0 + eD. \tag{31}$$

Here C_0 includes the literal row penalty $-1000I_5$. The physical and frequency error bounds include all global-row tails and all finite-source conversion errors.

There are 24 positive-frequency Gauss nodes on each of the 58 exterior leaves, hence 1392 nodes. Let F_d be the exact dyadic matrix of weighted odd Fourier features. Appendix A proves

$$T \le F_d^*F_d + e_T D, \qquad e_T < 2.365655088730 \cdot 10^{-12}. \tag{32}$$

In particular every low/exterior, row/exterior and row/row entry is included. Define

$$\overline{G} = (1 + 18 \cdot 10^{-9})(G_0 + 2eD) + 5(F_d^* F_d + e_T D), \qquad (33)$$
$$\beta = 99999999/100000000.$$

Since the actual source error is strictly below e, $G_0 + 2eD \geq G + eD > 0$. The other term in (33) is positive semidefinite, so $\overline{G} > 0$ without a lower floor for the true Gram.

The exact finite acceptance matrix is

$$J = \beta\overline{G} - \overline{G}\,\overline{C}\,\overline{G}. \qquad (34)$$

Floating-point Cholesky proposes only a square dyadic congruence witness V_d. Arb recomputes the complete target and $V_d^* J V_d$ at 192 bits. All 4085 rows satisfy strict positive diagonal dominance. The minimum diagonal lower minus the entire off-diagonal absolute row sum exceeds 0.9988015268483315; the maximum off-diagonal row sum is below 0.000794130062920. Hence $V_d^* J V_d > 0$, which also proves V_d invertible and $J > 0$.

The source normalized coefficient error is below $1.558 \cdot 10^{-10}$ before its analytic 10^{-12} allowance; the normalized Gram source error is below $1.055 \cdot 10^{-10}$ before its analytic 10^{-20} allowance. These are strictly inside the padding in (31). A separate reconstruction of the physical coefficient, receiving Gram and full target gives the same strict row bounds from the saved dyadic witness. Primitive generation, finite-target reconstruction and the analytic finite-to-infinite return have distinct reproducibility roles, described in the package instructions.

5.2 Proof of the original-domain theorem

Apply Theorem 3.1 with (30)–(34). It gives $WC_{\mathrm{aug}}W^* \leq \beta(aI - \widehat{O})$. On the exact row kernel, (16) and (24) imply $q_H[h] \geq -(\alpha + \delta)\|h\|^2$. For arbitrary odd $h \in \mathcal{V}$, use (10) and $\|P_8 h\| \leq \|h\|$. Finally

$$\alpha + \delta = 10^{-9} + 88 \cdot 10^{-11} = \frac{47}{25000000000}.$$

This proves Theorem 1.1.

The strict finite reserve also retains positive terms. Set $\kappa = (1 - \beta)(a - 3/4) = 249999991/10^{17}$, $\lambda = \mu\kappa$ and $\nu = \alpha + \delta - \alpha\kappa$. The same proof gives

$$q_H[h] \geq -\nu\|h\|^2 + \nu\|\Pi_8 h\|^2 + \lambda\|sP_8 h\|^2 \qquad (h \in \mathcal{V} \text{ odd}). \qquad (35)$$

The simpler scalar endpoint (1) avoids making this small reserve central to the statement.

6 Why the cutoff-eleven exterior is supercritical

The nonzero channels in m_{11} are $2, 3, 4, 5, 7, 8, 9, 11$. In particular the new terms at 9 and 11 have coefficients $\log 3/3$ and $\log 11/\sqrt{11}$. We prove Theorem 1.2 by a whole-interval frequency test.

Let $I = [2447, 2448]$ and $A = \int_I v_{11}(\xi)\, d\xi$. Direct interval evaluation on 2048 equal whole cells gives

$$A > 1.35214062295504 > 27/20. \qquad (36)$$

On each cell, its width times the positive part of the interval lower for $\mu - m_{11}$ is a lower integral bound.

Every theta atom is positive for $u \geq 0$, since its parameter $r_k = \pi k^2 e^{2u}$ is at least π. Thus $\psi \geq \psi_1 > 0$. At $\alpha_0 = 10^{-9}$,

$$t_{\alpha_0}^2 \geq \frac{5}{1 + 5\alpha_0 \cosh(u/2)/\psi_1(u)}.$$

Interval evaluation of this stable quotient on 1000 equal whole cells covering $[0,1]$ gives a lower above 4.8992668, hence

$$t_\alpha(u)^2 > 24/5 \quad (|u| \le 1,\ 0 < \alpha \le \alpha_0). \tag{37}$$

Evenness and monotonicity in α justify the full stated range.

Use frequency measure $d\xi/(2\pi)$ and define

$$h(\xi) = \operatorname{sgn}(\xi)\sqrt{v_{11}(\xi)}1_{I\cup(-I)}(\xi), \qquad Tg(u) = t_\alpha(u)\int \sqrt{v_{11}(\xi)}g(\xi)e^{i\xi u}\frac{d\xi}{2\pi}.$$

The multiplier v_{11} has compact support, because (5) eventually dominates the fixed finite trigonometric sum. Thus this is a bounded factorization $O_{11,\alpha} = TT^*$. Odd frequency functions map to odd physical functions up to a common imaginary scalar. The test satisfies

$$\|h\|^2 = A/\pi, \qquad Th(u) = \frac{i}{\pi}t_\alpha(u)S(u), \qquad S(u) = \int_I v_{11}(\xi)\sin(\xi u)\,d\xi.$$

For $\xi, \eta \in I$,

$$\begin{aligned}\int_{-1}^{1}\sin(\xi u)\sin(\eta u)\,du &= \operatorname{sinc}(\xi-\eta) - \operatorname{sinc}(\xi+\eta)\\ &\ge \frac{5}{6} - \frac{1}{4800}.\end{aligned}$$

Indeed $|\xi-\eta| \le 1$ and $\operatorname{sinc} x \ge 1 - x^2/6$ there, while $\xi+\eta \ge 4894 > 4800$. The oscillation at the large carrier is retained in this identity. Consequently

$$\begin{aligned}\|O_{11,\alpha}|_{\mathrm{odd}}\| \ge \frac{\|Th\|^2}{\|h\|^2} &\ge \frac{24/5}{\pi A}\int_{-1}^{1} S(u)^2\,du\\ &\ge \frac{3999A}{1000\pi} > \frac{3999}{1000}\frac{27}{20}\frac{7}{22} = \frac{755811}{440000} > \frac{17}{10}.\end{aligned}$$

The equality of the relevant norms follows from $O = TT^*$ on the odd subspaces; alternatively apply Cauchy–Schwarz to $f = Th$ to obtain the same lower bound for TT^*. Multiplying Th by $-i$ gives a real odd physical test. This proves Theorem 1.2.

Reproducibility

The supplement contains the source manuscript, the exact rational exterior cover, the frozen full certificate generator, saved dyadic data, a separately written finite-target reconstruction and the scalar analytic verifiers. Its reader instructions specify a fresh generation path and a saved-data reconstruction path. All acceptance inequalities use directed arithmetic or exact rational comparisons. Floating-point centers and witnesses are proposals whose validity is decided by the enclosing inequalities. The domain and analytic arguments are supplied above.

Acknowledgment

AI assistance was used in mathematical development, literature comparison, manuscript preparation and computational verification.

A Analytic approximation bounds

This appendix gives the complete analytic errors used in the directed matrices. Classical Legendre and Gaussian estimates are recorded explicitly so that the finite source enclosures can be checked against the same continuum recipient.

A.1 All-degree phase approximation and spatial error

The Fourier transform of (25) is

$$\widehat{\phi}_{aj}(\xi) = e^{-ic_a\xi}(-i)^j\sqrt{2r(2j+1)}\,\mathrm{j}_j(r\xi),$$

where j_j is the spherical Bessel function. Rodrigues' formula, j integrations by parts and the beta integral give $|\mathrm{j}_j(x)| \le |x|^j/(2j+1)!!$ on the real axis. Let F_L be Fourier restriction, and Π the orthogonal projection onto all local columns. Summing the squared tails over the cells and all $j \ge p = 60$ gives

$$e_N^2 := \|F_L(1_{[-R,R]} - \Pi)\|_{\mathrm{HS}}^2 \le \frac{2RL}{\pi}\frac{(rL)^{2p}}{((2p+1)!!)^2(1-q)}, \quad q = \frac{(rL)^2}{(2p+1)(2p+3)} = \frac{300}{4961} < 1. \tag{38}$$

Here $R = 17/10$, $rL = 30$, and exact rational evaluation with $\pi > 3$ gives $e_N < 3\cdot 10^{-11}$. The factor $2j+1$ in the column normalization cancels the integrated power ξ^{2j}; each subsequent term has ratio at most q.

Both $F_L M_t$ and $F_L \Pi M_t$ have norm at most $j = 9/4$, since $\|t\|_\infty \le \sqrt{5} < 9/4$. On $|u| > R$, the theta envelope and $\pi > 3$ give

$$j_R^2 := \|1_{|u|>R}t\|_\infty^2 < 169\alpha^{-1}e^{4R-3e^{2R}} < 169\alpha^{-1}(3/8)^{80} < (7\cdot 10^{-12})^2.$$

For the middle inequality use $e^{17/5} > 29$, proved by a finite positive Taylor sum. Hence the two maps differ in norm by less than $j_R + je_N < 7.45\cdot 10^{-11}$. The series (5) gives $a(2400) \le 4 + \frac{1}{2}\log(1 + 4\cdot 2400^2) < 13$. With $d_8 < 12$, we have $\|1_{[-L,L]}b\|_\infty < 26$. Expanding the difference of the two signed products yields

$$\|K_{\mathrm{low}} - F_{\mathrm{phase}}\| \le 2\cdot 26\cdot\frac{9}{4}\left(7\cdot 10^{-12} + \frac{9}{4}3\cdot 10^{-11}\right) = 8.7165\cdot 10^{-9} < \eta.$$

The common norm bound for both maps includes the projection; thus the expansion needs no additional unbounded approximate-map factor.

A.2 Gaussian quadrature and low-frequency coefficients

If a function is holomorphic and bounded by M on the Bernstein ellipse of parameter $\rho > 1$ over a cell of half-width h, its n-point Gauss–Legendre error is at most

$$E(M,n,h,\rho) = \frac{8hM\rho^{-2n}}{1-\rho^{-1}}. \tag{39}$$

Indeed the Chebyshev tail after degree $2n-1$ has uniform bound $2M\rho^{-2n}/(1-\rho^{-1})$. Exactness of the rule on that polynomial and positivity of weights, whose sum on the reference interval is 2, bound the integral and rule errors together by four times that tail; rescale by h.

Use 24 Gauss nodes on each of the 16 panels of width $1/4$ covering $[0,4]$ and 1198 panels of width 2 covering $[4,2400]$. There are 29136 positive-frequency nodes. The holomorphic continuation is

$$b(z) = \mu + \log\pi - \tfrac{1}{2}\{\Psi_0(1/4 + iz/2) + \Psi_0(1/4 - iz/2)\} + 2\sum_{n\le 8} c_n\cos(z\log n).$$

Neither a real-part operation nor the band indicator is analytically continued. On the parameter-four small-panel ellipses $|\Im z| \le 15/64$, both digamma arguments have positive real part. On the large-panel ellipses, $\Re z \ge 23/8$, $|\Im z| \le 15/8$ and $|z| < 2403$. One

digamma recurrence moves each argument to positive real part. The elementary series estimate $|\Psi_0(w)| \le 1 + 1/\Re w + 2|w|$ there, with the recurrence term away from zero, gives the same coarse bound 2500 for the symmetric digamma part as in [1, computation appendix]. For the six channels, $c_n < 1$ and $|\cos(z \log n)| \le e^{(15/8)\log 8} < 64$; thus $|b(z)| < 4000$.

Put $A_j(z) = \sqrt{2r(2j+1)}\,\mathrm{j}_j(rz)$. Its Legendre integral representation and $|P_j| \le 1$ on $[-1,1]$ give $|A_j(z)| < 3$ on these ellipses. The lag phases for $|d| \le 135$ have modulus at most $e^{2r|d||\Im z|} < e^7 < 2187$. Each scalar coefficient integrand is therefore bounded by 10^8. Summing (39) over all panels gives the per-entry bound

$$E_f = \frac{16}{3}\, 2400 \cdot 10^8\, 4^{-48}.$$

The normalized odd fold has at most two lag contributions per column, so its operator error is at most $2 \cdot 4080 E_f < 10^{-12}$. Source conversion and finite summation errors are enclosed separately.

A.3 Complex theta profiles at the changed multiplication reference

For $z = x + iy$, $x \ge 0$, $|y| \le 1/100$, set $r_1 = \pi e^{2z}$ and $\psi_1(z) = e^{z/2}(4r_1^2 - 6r_1)e^{-r_1}$. We have $\Re r_1 > 2.99$. Direct atom comparison gives

$$|\psi_k(z)/\psi_1(z)| \le 3k^4 e^{-\Re r_1 (k^2-1)}, \qquad \sum_{k\ge 2} |\psi_k/\psi_1| < 1/10.$$

For the last inequality separate $k = 2$ and sum the geometrically decreasing remainder, uniformly for $\Re r_1 \ge 2.99$. Hence ψ is zero-free on the strip, and

$$1.8\pi^2 e^{(9/2)x - \pi e^{2x}\cos(2y)} \le |\psi(z)| \le 120 e^{(9/2)x - \pi e^{2x}\cos(2y)}. \tag{40}$$

Evenness handles $x < 0$.

On $|y| \le 1/200$, the denominator $d(z) = \alpha \cosh(z/2) + \psi(z)/5$ remains in a right half-plane at the actual $\alpha = 10^{-9}$. For $0 \le x \le 1.2$, the unwrapped first-atom phase is at most $(13/2)/200 + 2\pi e^{2.4}/200 < 1$. Thus $\Re\psi_1 > \frac{1}{2}|\psi_1|$, and the relative tail above gives $\Re\psi > 0$. For $x \ge 1.2$, the upper envelope in (40) is decreasing and is bounded by

$$120 \cdot 729 (3/8)^{32} < 2.047 \cdot 10^{-9}.$$

Use $e^{2.4} > 11$, $e^{5.4} < 729$, and $\pi e^{2.4}\cos(1/100) > 32$. Since the theta term is multiplied by $1/5$ and $\Re\cosh(z/2) \ge \cos(1/400)$, both regions give

$$\Re d(z) > \alpha/2, \qquad |t^2(z)| = |\psi(z)/d(z)| < 4 \cdot 10^{17}. \tag{41}$$

The deliberately loose second bound follows already from the envelope maximum $|\psi| < 17$. This proof uses the changed α, rather than transferring a numerical Gram from another reference.

For $|\Re z| \le 13/5$, $|\Im z| \le 1/200$ and $|\zeta| \le 1/200$, the upper and lower envelopes give

$$\left|\frac{\psi(z+\zeta)}{\psi(z)}\right| \le \frac{120}{1.8\pi^2} e^{9/400} \exp\{\pi e^{26/5}[1 - e^{-1/100}\cos(1/50)]\} < 10^4. \tag{42}$$

When the small circle crosses $\Re z = 0$, evenness gives the same bound. Cauchy's estimate now yields, for $0 \le j \le 7$,

$$|q_j(z)/\psi(z)| \le 2 \cdot 10^4 (j+2)! 200^{j+2} < 10^{60}. \tag{43}$$

The factor two bounds the sum of the $(j+2)$nd derivative term and the quarter of the jth derivative term. All endpoints in (41)–(43) admit direct rational or directed-ball evaluation in the supplied scalar controls.

A.4 The complete physical Gram

The physical cells have half-width $r = 1/80$ and use 800 Gauss nodes. On their parameter-7/5 ellipses the imaginary height is $3/700 < 1/200$ and the real extension stays inside $|\Re z| < 13/5$. The Legendre bound $|P_j| \le (7/5)^j$ gives $|\phi_{aj}| < 10^{12}$, while $|c|, |\sinh(z/2)| < 3$. Simplify each product before bounding: phase–null entries are $t^2 c\phi q_j/\psi$, and null–null entries are $t^2c^2(q_i/\psi)(q_j/\psi)$. Equations (41)–(43) and the dyadic row-change bound give modulus below 10^{160} for every Gram integrand. Thus (39), all physical cells, all row sums and all folds give an error bounded conservatively by

$$100 \cdot 10^6 \frac{8r\, 10^{160}(5/7)^{1600}}{1 - 5/7}. \tag{44}$$

The spare factor 100 allows the row change a second time and is harmless.

The 68 negative phase cells and 32 negative row-only cells cover $[-5/2, 0]$ exactly; reflection gives the full bounded region. Global rows must still be integrated to infinity. To pay their tails define

$$P_0(v) = 4v^2 - 6v, \qquad \mathcal{L}P = (1/2 - 2v)P + 2vP', \qquad Q_j = \mathcal{L}^{j+2}P_0 - \tfrac{1}{4}\mathcal{L}^j P_0.$$

Then $\deg Q_j = j + 4$ and $q_j(u) = e^{u/2}\sum_{k\ge1} Q_j(k^2 r_1)e^{-k^2 r_1}$ for real $u \ge 0$, $r_1 = \pi e^{2u}$. If $Q_j(v) = \sum_l d_{jl} v^l$, put $A_j = \sum_l |d_{jl}| 400^{l-j-4}$. For $u \ge 5/2$, $r_1 > 400$; $\sum_{k\ge1} k^{2j+8} e^{-r_1(k^2-1)} < 2$ and $\psi(u) \ge 2e^{u/2} r_1^2 e^{-r_1}$. Therefore

$$\frac{|q_j(u)|^2}{\psi(u)} \le 2A_j^2 e^{u/2} r_1^{2j+6} e^{-r_1}.$$

Using $c(u) \le e^{u/2}$, changing to r_1 and adding both tails gives

$$T_j := \int_{|u|>5/2} \left| \frac{q_j}{s\sqrt{\alpha + \mu s^2}} \right|^2 du \le \frac{3A_j^2}{\alpha} \Gamma(2j + 7, 400). \tag{45}$$

Each polar tail is at most $T_{\mathrm{pol}} = (8/\alpha)\Gamma(3, 400)$. Integer incomplete gamma values are evaluated from

$$\Gamma(n + 1, a) = n! e^{-a} \sum_{k=0}^{n} a^k/k!.$$

The omitted row Gram is positive semidefinite; its norm after mixing is at most $100(\sum_{j=0}^{7} T_j + 2T_{\mathrm{pol}})$. Positivity pays every mixed tail entry. Combining this with (44) gives a total physical Gram analytic error below 10^{-20}. This bound applies to the actual full global rows in (26).

A.5 Complete outer Fourier features and the moment error

Let W_j be any column of W and $H_j(\xi) = \widehat{tW_j}(\xi)$. Since these functions are real odd, reflected frequency pairing gives exactly

$$T_{ij} = \sum_l \gamma_l \int_{I_l} H_i(-\xi) H_j(\xi) \frac{d\xi}{\pi}. \tag{46}$$

The holomorphic continuation of the product is $H_i(-z)H_j(z)$, not a conjugate evaluated at a complex argument.

The negative-half-line physical integration uses the same 100 cells and 800 Gauss nodes. Its analytic integrand is t^2 times a phase or row profile times $e^{i\xi z}$; no complex square-root choice is used. Every carrier has $|\xi| < 3101$, so on the physical ellipses $|e^{i\xi z}| < e^{16} < 10^7$. The

profile bounds above absorb this into 10^{160}. Summing the quadrature and reflection factors conservatively gives

$$E_{\text{phys}} \le \frac{1000 \cdot 10^{160}(5/7)^{1600}}{1 - 5/7}.$$

The complete t tail satisfies $\|1_{|u|>5/2}t\|_2 < 2^{-260}$. Every full W_j has norm below 10: phase norms squared are at most 10, and the complete enclosed global-row Gram diagonal is below 100. Cauchy–Schwarz pays each global Fourier tail by $10 \cdot 2^{-260}$; phase tails vanish. Including the node weight, which is below 1, the total weighted Fourier-entry analytic error is

$$E_{\text{phys}} + 10 \cdot 2^{-260} < 10^{-25}. \tag{47}$$

For frequency quadrature, each exterior leaf has half-width at most $1/64$. Its parameter-four ellipse has imaginary height at most $15/512 < 1/32$. For any $0 \le b \le 1/32$,

$$\int_{\mathbb{R}} e^{2b|u|} t(u)^2 \, du < 25. \tag{48}$$

Indeed the bounded region contributes at most $2\int t^2 < 24$ by Lemma 4.1. On the exterior the theta envelope and $e^5 > 148$ give

$$\int_{|u|>5/2} e^{|u|/16} t^2 \, du \le \frac{240}{\alpha} \int_{148}^{\infty} v^{1+1/32} e^{-3v} \, dv \le \frac{240}{\alpha} \left(\frac{148^2}{3} + \frac{296}{9} + \frac{2}{27} \right) e^{-444} < 1.$$

Hence $|H_j(z)| \le 5\|W_j\| < 50$ on all frequency ellipses. Since $\gamma_l < 1$, the integrand in (46) has modulus below 2500. Apply the 24-point rule separately on each constant-density leaf; it never crosses a density discontinuity. The sum of positive leaf half-widths is $2175/4096$, so the complete per-entry frequency error is at most

$$E_{\text{entry}} \le \frac{32}{3} \frac{2175}{4096} 2500 \cdot 4^{-48} < 1.8 \cdot 10^{-25}.$$

The 4085-dimensional operator error is below 10^{-20}. This proof uses the full Fourier profiles through (48), including their infinite physical tails.

At the actual nodes the program encloses each weighted physical Fourier sum before conversion. The saved dyadic feature has per-entry radius $R_F = 2^{-55}$ after all source arithmetic; the observed maximum source radius is below $4.337 \cdot 10^{-19}$. Put

$$\epsilon_F = \sqrt{1392 \cdot 4085}(2^{-55} + 10^{-25}).$$

Then $\|F_{\text{exact}} - F_d\| \le \epsilon_F$ in the quadrature feature map. With the full dyadic Frobenius upper $\|F_d\| < 17.871275068587$,

$$\|T - F_d^* F_d\| \le \epsilon_F(2\|F_d\| + \epsilon_F) + 10^{-20} < 2.365655088730 \cdot 10^{-12}. \tag{49}$$

Its outward dyadic upper is e_T in (32). The use of $e_T D$ is conservative since $D \ge I$. Every cross covariance is retained in this norm estimate and in the exact product $F_d^* F_d$.

B Scaled sources, parity and finite acceptance

This appendix specifies the numerical formulas and how their complete enclosures support the final finite gate.

B.1 Scaled theta sources

At a real node $u \geq 0$, let $r_1 = \pi e^{2u}$ and use the exact polynomials of Appendix A. Evaluate

$$\mathcal{T}(u) = \sum_{k\geq 1} P_0(k^2 r_1) e^{-(k^2-1)r_1}, \qquad \mathcal{Q}_j(u) = \sum_{k\geq 1} Q_j(k^2 r_1) e^{-(k^2-1)r_1}.$$

Then $\psi = e^{u/2-r_1}\mathcal{T}$ and $q_j/\psi = \mathcal{Q}_j/\mathcal{T}$. The program checks positivity of the denominator before dividing and reflects odd quotients with $(-1)^j$. For a polynomial of degree $d \leq 11$ with coefficient absolute sum A, all terms after $k = 8$ have total absolute value at most

$$2A\,243^d (3/8)^{240}. \tag{50}$$

Indeed $r_1^d e^{-(k^2-1)r_1}$ decreases for $r_1 \geq 3$, $k \geq 9$. The successive majorants from $k = 9$ have ratio at most $(10/9)^{22} e^{-57} < 1/2$. For P_0 use $A = 10, d = 2$; multiplying back gives the absolute theta remainder 2^{-250} used earlier.

Each physical feature is multiplied by $\sqrt{r w_k t(u_k)^2}$ and by its exact dyadic row mixture before conversion. This square root occurs only at real positive nodes. Its analytic quadrature proof concerns the product profiles in Appendix A.3. Each conversion is individually enclosed at 192 bits. The source radii are 2^{-52} for frequency amplitudes, weighted symbols, cosines and sines, and 2^{-51} for weighted physical Gram features. For outer Fourier sums the weighted integrands are additionally stored as two dyadic doubles with residual below 2^{-90} before the directed matrix sums are formed.

B.2 Exact low block and odd metric

Let $\omega_j = (-i)^j$ and $A_j(\xi) = \sqrt{2r(2j+1)}\mathrm{j}_j(r\xi)$. The full lag block is

$$B_d(j,k) = \int_0^L \frac{b(\xi)}{\pi} A_j(\xi) A_k(\xi) \{ \operatorname{Re}(\overline{\omega}_j \omega_k) \cos(2rd\xi) - \operatorname{Im}(\overline{\omega}_j \omega_k) \sin(2rd\xi) \}\, d\xi.$$

Set $B_{-d} = B_d^T$. The integer odd fold has entries

$$C_{\mathrm{fold}}(aj, bk) = 2\{B_{a-b}(j,k) - (-1)^k B_{a+b-135}(j,k)\}, \quad 0 \leq a, b < 68. \tag{51}$$

The saved dyadic blocks obey $J B_d^T J = B_d$, $J_{jj} = (-1)^j$, exactly. Thus (51) gives a symmetric exact dyadic center. A rounded full matrix is only a witness proposal. The physical raw coefficient is the phase block of $D^{-1} C_0 D^{-1}$, not C_0 itself.

All frequency quadrature sums are independently re-enclosed from primitive balls. The per-block absolute row and column errors are combined according to the two lag terms in (51). Its leading factor two cancels the phase metric normalization. The physical Gram is reconstructed from all 68 phase cells and all 32 row-only cells before any source comparison. Its phase–phase errors divide by two, its phase–row errors by $\sqrt{2}$, and its row–row errors have unit normalization. For the resulting symmetric normalized errors, the maximum absolute row sum bounds the operator norm. Adding the complete analytic errors 10^{-12} and 10^{-20} respectively proves (31).

B.3 The outer feature matrix

For every positive exterior Gauss node ξ_k in a leaf with density γ_l, its real odd feature is

$$F_{kj} = 2\sqrt{\gamma_l h_l w_k/\pi} \int_{-\infty}^{0} t(u) W_j(u) \sin(\xi_k u)\, du.$$

The common imaginary Fourier phase cancels in F^*F; the factor two is physical odd reflection and the measure $1/\pi$ is the reflected frequency pairing. Source matrix products include every physical cell and all five complete rows. Source balls for roots, weights, theta quotients, trigonometric terms and finite sums are enclosed before the final dyadic conversion. Equations (47) and (49) pay the remaining physical and frequency approximation.

B.4 Accepted computation and reproduction paths

The full generator proposes centers and a congruence witness with NumPy. It then rebuilds the low coefficients, physical Grams, exterior Fourier matrix and complete finite target with Arb. It accepts only after every source-error gate and every strict row of the congruence passes. A saved favorable floating eigenvalue is never an acceptance premise. The final matrix has order 4085 and uses the exact rational $\beta = 99999999/10^8$, padding $e = 10^{-9}$ and the outward dyadic e_T.

The saved-data reconstruction separately rebuilds the physical coefficient, full receiving Gram and target in (34) and applies the exact dyadic witness. It reuses the accepted primitive/source-sum enclosures and the same Arb engine. The fresh-generation path regenerates the primitive arrays and all source enclosures. These two paths verify different portions of the proof; the reader instructions expose that distinction. Scalar programs separately verify the complete arithmetic tail, whole exterior cover, complete mass/trace bound, analytic budgets and cutoff-eleven lower test.

For a whole-interval lower check such as (36), the programs use width times the positive part of the interval lower endpoint. For upper covers such as (22), they use width times a complete outward positive upper. This distinction is needed at leaves that straddle a sign change. Gaussian nodes are used only after the corresponding holomorphic remainder has been paid.

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