# Occupation bounds for coupled permanent cofactors

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Luminary Oddities

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**Abstract**

For a complex positive semidefinite Gram matrix $B = VV^*$, put $a = \operatorname{per} B$ and $c = \operatorname{per}|B|^2$. We couple the permanent-cofactor contractions of $B$ and $|B|^2$ and prove the sharp all-order bound $0 \le U - aT/n! \le (n-1)(c - a^2/n!)I$. A central-transposition statistic gives an order-six refinement. For the actual bordered permanent defect, we identify a nonnegative cofactor variance, characterize its zero set by parallel and orthogonal Gram lines, and retain every kernel mixture. A two-sided relative stability estimate in the unit Gram factors gives explicit occupied-vertex domains without spectral-gap assumptions. The transfer is conditional on the smaller complementary self-Chollet inequality; it is unconditional when that complement has order at most four. A general conditional occupation radius of order $n^{-3}$ gives ratio at most $2/3$. At order seven, larger explicit radii give ratio at most $7/8$. An unconditional entrywise neighborhood of repeated orthogonal blocks, of radius of order $n^{-4}$, certifies every admissible border. An explicit larger neighborhood of a rank-five base is given at order seven. We also give exact occupation-box matrix certificates and finite completeness at strictly positive fixed bases, with a conditional all-low extension through kernel zeros. Exact adaptive examples illustrate subdivision. The original residual occupation polynomial, thinning law and nonreal-root examples are retained.

## 1 Introduction and notation

For a square matrix $A$, $\operatorname{per} A = \sum_{\sigma\in\mathfrak{S}_n}\prod_i A_{i,\sigma(i)}$; $A(j \mid i)$ deletes row $j$ and column $i$. The symbol $\circ$ denotes entrywise multiplication, $*$ the conjugate transpose, and $|B|^2 = B \circ \overline{B}$, not the operator square.

Throughout, $V \in \mathbb{C}^{n\times d}$, $B = VV^*$, and $d$ is arbitrary. Define

$$a = \operatorname{per} B, \qquad c = \operatorname{per}|B|^2, \qquad R_{ij} = \operatorname{per} B(j \mid i), \qquad L_{ij} = \operatorname{per}|B|^2(j \mid i), \tag{1}$$

$$T = V^*RV, \qquad U = V^*(L \circ B)V, \qquad r = c - \frac{a^2}{n!}, \qquad S = U - \frac{a}{n!}T. \tag{2}$$

All operator inequalities below act on $\mathbb{C}^d$ unless specified otherwise. No rank, normalization of the diagonal, or invertibility hypothesis is imposed.

Permanent-cofactor Gram matrices and their Schur products are classical. Paksoy [4], in particular, uses omitted symmetric tensors and a weighted cofactor matrix with constant permanent row sum. Symmetrizers and bosonic permanent states are standard [5, 6]; Schur weights and content eigenvalues are standard representation theory [7, 8]. Our focus is the *joint residual* $S$, its sharp norm bound, and a computable refinement that preserves its common Gram origin. Li, Zhao and Luo [3] represent coefficients of ordinary and doubled bordered permanent polynomials by squared norms over complete occupation sectors. The polynomial studied below instead comes from the joint residual projection $P_{\mathrm{diag}} - P_{\mathrm{sep}}$; its coefficients measure one shared residual state. The separate positivity of $T$ and $U$, or a bound on either one alone, does not supply this residual estimate.

The motivation is the compensation in a bordered permanent inequality. Chollet asks whether $\operatorname{per}(A\circ D) \le \operatorname{per} A \operatorname{per} D$ for complex positive semidefinite $A, D$; see [1, 2]. The results here concern coupled cofactor operators and sufficient border certificates, rather than an unrestricted solution. They are distinct from the author's rank-restricted scalar results [11] and cofactor-comparison limitations [12]. We consolidate the sharp residual estimate and its refinements in one note.

## 2 A sharp bound in every order

**Theorem 2.1.** *With* (1)–(2)*, for every* $n \geq 2$*,*

$$0 \preceq S \preceq (n-1) r I_d. \tag{3}$$

*The coefficient* $n-1$ *cannot be reduced for actual Gram factors. The conclusion includes arbitrary complex phases and singular factors.*

### 2.1 The two invariant projections

We use the conjugate-linear-first inner product. Regard row $i$ of $V$ as the ket $v_i = \sum_j V_{ij} e_j$ in $\mathcal{H} = \mathbb{C}^d$. Its physical Gram is $\overline{B}$. Put

$$f = v_1 \otimes \cdots \otimes v_n, \qquad W = f \otimes \overline{f}.$$

On $\mathcal{H}^{\otimes n} \otimes \overline{\mathcal{H}}^{\otimes n}$, let $P_{\mathrm{sep}}$ average independent permutations of the two legs, and let $P_{\mathrm{diag}}$ average equal simultaneous permutations. Both are orthogonal projections and $P_{\mathrm{sep}} P_{\mathrm{diag}} = P_{\mathrm{sep}}$. In particular $a = n! \| \operatorname{Sym} f \|^2 \geq 0$. Direct expansion gives

$$\| P_{\mathrm{sep}} W \|^2 = \frac{a^2}{(n!)^2}, \qquad \| P_{\mathrm{diag}} W \|^2 = \frac{c}{n!}. \tag{4}$$

For unit $z \in \mathbb{C}^d$, let $p_z = |\overline{z}\rangle\langle\overline{z}|$ and let $N_z = \sum_{j=1}^n p_z^{(j)}$ act on the first leg. The conjugation in $p_z$ matches the row convention $B = VV^*$. This operator commutes with both projections.

Set $h = (P_{\mathrm{diag}} - P_{\mathrm{sep}}) W$. For $x = Vz$, differentiation of the two permanent norm formulas gives

$$\| h \|^2 = \frac{r}{n!}, \tag{5}$$

$$\langle h, N_z h \rangle = \frac{1}{n!} \left( z^* U z - \frac{a}{n!} z^* T z \right). \tag{6}$$

Here the conjugate physical Gram deformation is $B + t\, xx^*$, and

$$\frac{d}{dt} \operatorname{per}(B + t\, xx^*) \Big|_{t=0} = z^* T z.$$

The paired deformation is $|B|^2 + t(B \circ \overline{xx^*})$, whose permanent derivative is $z^* U z$. For example the latter derivative is $\sum_{ij} L_{ji} B_{ij} \overline{x_i} x_j$; $L$ is real symmetric, yielding the defined $U$. The independent-projection expectation is $a(z^* T z)/(n!)^2$ and the diagonal one is $(z^* U z)/n!$. Their difference is (6); the cross terms vanish because $N_z$ preserves the nested orthogonal components.

### 2.2 Excluding full occupation

*Proof of Theorem 2.1.* The spectrum of $N_z$ is contained in $\{0, \ldots, n\}$. Its $n$-eigenspace has first leg $\overline{z}^{\otimes n}$. A diagonal-invariant vector in that eigenspace has a symmetric first leg; diagonal invariance then forces its second leg to be symmetric too. Thus the $n$-occupation part of range $P_{\mathrm{diag}}$ lies in range $P_{\mathrm{sep}}$ and is absent from $h$. Consequently

$$0 \leq \langle h, N_z h \rangle \leq (n-1) \| h \|^2.$$

Use (5)–(6) and vary unit $z$.

For sharpness, take $n-1$ equal unit rows in one coordinate and one unit row in an orthogonal coordinate. Then $B = J_{n-1} \oplus [1]$, $a = c = (n-1)!$ and, in the repeated-row direction, $z^* T z = (n-1) a$, $z^* U z = (n-1) c$. The upper bound is an equality. □

In particular $r \geq 0$. If $r = 0$ then $S = 0$. Both $T$ and $U$ vanish on $\ker V$. Therefore, with $\mathcal{A} = (\ker V)^\perp$ and its orthogonal projection $P_{\mathcal{A}}$,

$$0 \preceq S \preceq (n-1) r P_{\mathcal{A}}. \tag{7}$$

This statement retains the kernel rather than replacing the ambient space.

## 3 The occupation polynomial and a cancellation

For unit $z$, write $x = Vz$, $X = xx^*$ and $H = B - X \succeq 0$.

**Proposition 3.1.** *The polynomial*

$$G_z(s) = \operatorname{per}\big(|B|^2 + (s-1)(B \circ \overline{X})\big) - \frac{a}{n!}\operatorname{per}\big(B + (s-1)X\big) = \sum_{k=0}^{n} q_k(z)s^k \tag{8}$$

*has $q_k(z) \geq 0$ and $q_n(z) = 0$. More precisely, if $E_k$ is the $k$-occupation spectral projector of $N_z$, then $q_k(z) = n!\|E_k h\|^2$. Thus*

$$G_z(1) = r, \qquad G_z'(1) = z^* S z, \tag{9}$$

$$q_0(z) = \operatorname{per}(H \circ \overline{B}) - \frac{a}{n!}\operatorname{per} H. \tag{10}$$

*Proof.* For real $s \geq 0$ multiply the selected first-leg mode by $\sqrt{s}$. The resulting operator is $s^{N_z/2}$ and commutes with both projections. The diagonal projected squared norm is the first permanent in (8) divided by $n!$; the independent projected squared norm is $a\ \operatorname{per}(B+(s-1)X)/(n!)^2$. Their difference is $\|s^{N_z/2}h\|^2$. Polynomial identification proves the coefficient formula. The full-occupation exclusion proves $q_n = 0$. The identities at $s = 1, 0$ follow directly. □

**Proposition 3.2** (Exact kernel thinning)**.** *Let $w \in \mathcal{A}$ and $\kappa \in \ker V$ be unit vectors. For $0 \leq p \leq 1$ and real $\theta$ put $z_p = \sqrt{p}\, w + e^{i\theta}\sqrt{1-p}\,\kappa$. Then*

$$G_{z_p}(s) = G_w(1 - p + ps), \tag{11}$$

$$q_k(z_p) = \sum_{j=k}^{n-1} \binom{j}{k} p^k (1-p)^{j-k} q_j(w). \tag{12}$$

*In particular $G'_{z_p}(1) = pG'_w(1)$ and $G''_{z_p}(1) = p^2 G''_w(1)$. The formulas include the endpoints and every relative complex phase.*

*Proof.* $Vz_p = \sqrt{p}\, Vw$, and substitution into (8) gives the first identity. Binomial expansion gives the rest. □

**Proposition 3.3** (Top occupation)**.** *Put $b_i = \prod_{j \neq i} x_j$, $y = \overline{b}$, $J = \mathbf{1}\mathbf{1}^*$ and*

$$F_0 = R \circ \overline{B} - \frac{a}{n} J.$$

*Then $F_0 \succeq 0$, $F_0\mathbf{1} = 0$, and*

$$q_{n-1} = y^*\left(R \circ |B|^2 - \frac{a}{n}B\right) y = y^*(B \circ F_0)y = y^*(H \circ F_0)y. \tag{13}$$

*These are polynomial identities, including when some $x_i = 0$.*

*Proof.* The usual omitted symmetric-tensor Gram construction gives $R \succeq 0$. The Schur product theorem gives $F = R \circ \overline{B} \succeq 0$. Column-$i$ Laplace expansion gives $\sum_j R_{ij}\overline{B_{ij}} = a$. Thus $F\mathbf{1} = a\mathbf{1}$ and removing that Hermitian eigencomponent gives $F_0 \succeq 0$. This construction is classical; compare [4].

In powers of $t = s - 1$, the coefficient of $t^{n-1}$ in the first permanent in (8) is $\sum_{ij} |B_{ij}|^2 \operatorname{per} B(i \mid j)\overline{b_i} b_j$. Exchanging $i, j$ gives $y^*(R \circ |B|^2)y$. The coefficient in the second permanent is $(n-1)!\, y^* B y$. Both leading $t^n$ terms in $G_z$ equal $a\prod_i |x_i|^2$ and cancel. Translating $t = s - 1$ therefore leaves the next coefficient unchanged, proving the first equality in (13). Finally $\operatorname{diag}(\overline{x})y = \overline{\prod_i x_i}\,\mathbf{1}$, so $y^*(X \circ F_0)y = 0$. This proves the last equality without division. □

For later directional tests at $n = 6$, set $r_+ = r - q_0$ and $m_2 = G_z''(1)$. The elementary moment bound is

$$z^* S z \le \min\left\{5r_+,\ \frac{m_2}{2j} + \frac{j+1}{2} r_+ :\ j = 1, \dots, 4\right\}. \tag{14}$$

Indeed $(k-j)(k-j-1) \ge 0$ at integer $1 \le k \le 5$. Multiplication by $q_k$ and summation proves each affine bound. For fixed feasible $r_+, m_2$, mixtures on consecutive occupations attain its segments, so this is the exact finite-support relaxation. It is an elementary instance of discrete moment linear programming [9]. Retaining $q_5$ also gives $z^* S z \le 4(r - q_0) + q_5$. Neither estimate alone couples its inputs to the compensation budget.

## 4 A central-transposition refinement at order six

In this section $n = 6$. On the first tensor leg let $C = \sum_{i<j} U_{(ij)}$ be the sum of transpositions and put

$$v = 720\,\langle h, Ch\rangle, \qquad M = \frac{v + 21r}{6}.$$

**Theorem 4.1.** *For every complex Gram factor of order six,*

$$0 \le S \le M I_d \le 5r I_d. \tag{15}$$

*For each transposition $\pi$ form $A^{\pi}_{ij} = B_{\pi(i),j}\overline{B_{ij}}$. Then*

$$v = \sum_{\pi \text{ transposition}} \operatorname{per} A^{\pi} - \frac{15a^2}{720}. \tag{16}$$

*The conclusion holds in every ambient dimension and at singular boundaries.*

*Proof.* On a first-leg Schur sector of shape $\lambda \vdash 6$, the largest one-mode occupation is $\lambda_1$: column strictness in semistandard tableaux permits a fixed letter at most once in each column. The central transposition sum acts by the content sum $\kappa_\lambda = \sum_{(i,j)\in\lambda}(j - i)$; see [8, Example 3.15]. The full comparison is

| $\lambda$ | $\lambda_1$ | $\kappa_\lambda$ | $(\kappa_\lambda + 21)/6$ |
|---|---|---|---|
| (6) | 6 | 15 | 6 |
| (5, 1) | 5 | 9 | 5 |
| (4, 2) | 4 | 5 | 13/3 |
| (4, 1, 1) | 4 | 3 | 4 |
| (3, 3) | 3 | 3 | 4 |
| (3, 2, 1) | 3 | 0 | 7/2 |
| (3, 1, 1, 1) | 3 | −3 | 3 |
| (2, 2, 2) | 2 | −3 | 3 |
| (2, 2, 1, 1) | 2 | −5 | 8/3 |
| (2, 1, 1, 1, 1) | 2 | −9 | 2 |
| $(1^6)$ | 1 | −15 | 1 |

so $N_z \le (C + 21I)/6$. The separately symmetric first-leg sector is absent from $h$: first-leg symmetry and diagonal invariance force second-leg symmetry. All remaining sectors have $\kappa_\lambda \le 9$. Hence

$$720\,\langle h, N_z h\rangle \le (v + 21r)/6 \le 5r.$$

Use (6). Nonnegativity follows from Theorem 2.1.

The class sum $C$ commutes with both projections, and acts as 15 on their separately symmetric range. Consequently

$$\langle h, Ch\rangle = \langle W, CP_{\mathrm{diag}} W\rangle - 15\|P_{\mathrm{sep}} W\|^2.$$

Expand the diagonal average and the fifteen swaps. Summing over its permutation index gives per $A^\pi$ for each $\pi$ (equivalently its complex conjugate under the opposite permutation convention). The sum is real since $CP_{\mathrm{diag}}$ is self-adjoint. Equations (4) give (16). □

More precisely, if $r_\lambda = 720\|E_\lambda h\|^2$, then $S \preceq (\sum_\lambda \lambda_1 r_\lambda) I$ and $v = \sum_\lambda \kappa_\lambda r_\lambda$. The single statistic $v$ compresses these masses. Since $S$ vanishes on $\ker V$, (15) also gives $S \preceq M P_{\mathcal{A}}$.

## 5 Compensation and exact examples

For any $n$ define $\delta = a^2 - c$ and $D_0 = \delta I_d + 2aT - U$. Let $P_i$ be the rank-one matrix $(P_i)_{\alpha\beta} = \overline{V_{i\alpha}} V_{i\beta}$, and set

$$\mathcal{K} = \delta I_{d^2} + a(T \otimes I + I \otimes T) + T \otimes T - \sum_{ij} L_{ij} P_i \otimes P_j. \tag{17}$$

**Lemma 5.1.** *$D_0 \succeq 0$ implies $\mathcal{K} \succeq 0$. For the bordered Gram matrix $\widetilde{B}_z = \begin{pmatrix} B & Vz \\ z^* V^* & \|z\|^2 \end{pmatrix}$,*

$$\operatorname{per}(\widetilde{B}_z)^2 - \operatorname{per}|\widetilde{B}_z|^2 = \langle z \otimes z, \mathcal{K}(z \otimes z)\rangle.$$

*Proof.* $L$ is real symmetric PSD by the omitted-tensor Gram construction. Factor $L = CC^T$ over the reals and put $A_k = \sum_i C_{ik} P_i$. Then $A_k$ are Hermitian and $\sum_k A_k^2 = U$, since $P_i P_j$ has entries $\overline{V_{i\alpha}} B_{ij} V_{j\beta}$. Expansion gives the identity

$$\mathcal{K} = \frac{D_0 \otimes I + I \otimes D_0}{2} + T \otimes T + \frac{1}{2} \sum_k (A_k \otimes I - I \otimes A_k)^2.$$

Every displayed term is PSD when $D_0 \succeq 0$. Finally border expansion gives $\operatorname{per} \widetilde{B}_z = a\|z\|^2 + z^* T z$ and $\operatorname{per}|\widetilde{B}_z|^2 = c\|z\|^4 + \sum_{ij} L_{ij} |(Vz)_i|^2 |(Vz)_j|^2$. Subtracting proves the formula. □

Theorem 2.1 gives, in every order,

$$D_0 \geq \left[ \left(1 + \frac{n-1}{n!}\right) a^2 - nc \right] I_d + \left(2 - \frac{1}{n!}\right) aT.$$

At order six this certifies $D_0, \mathcal{K} \succeq 0$ whenever $a > 0$ and $c/a^2 \leq 145/864$. Theorem 4.1 improves the sufficient test to

$$D_0 \succeq (\delta - M) I_d + \frac{1439a}{720} T, \qquad M \leq \delta. \tag{18}$$

This is a condition on the actual mixed-permanent statistic.

Here are exact factors witnessing strict improvement of the scalar test. Let $W \in \mathbb{C}^{6\times 6}$ have first three rows $(1, \frac{1}{2}, 0, 0, 0, 0)$, $(1, \frac{i}{2}, 0, 0, 0, 0)$, $(1, -\frac{1}{2}, 0, 0, 0, 0)$, and last three rows $e_2^T, e_3^T, e_4^T$, using coordinates $0, \ldots, 5$. For $0 \leq i \leq 5$, $0 \leq j \leq 4$ set

$$E_{ij} = \frac{((i+2j) \bmod 3) - 1 + i_c(((2i+j) \bmod 3) - 1)}{10000}, \qquad E_{i5} = 0,$$

where $i_c = \sqrt{-1}$ is distinguished from the row index. Put $V_5 = W + E$ and $V_6 = V_5 + 10^{-4} e_2 e_5^T$. The accompanying exact-rational verifier proves, for $B_j = V_j V_j^*$,

$$\operatorname{rank} V_j = j, \quad B_j \text{ has no zero entries}, \qquad \frac{c}{a^2} > \frac{145}{864}, \quad \frac{\delta - M}{a^2} > \frac{17}{1000} \quad (j = 5, 6).$$

Both have a triangle $B_{ij} B_{jk} B_{ki}$ with nonzero imaginary part, which cannot be removed by diagonal unitary row gauges. Numerically $c/a^2$ is about 0.20174. Strict margins persist on relative open neighborhoods of the corresponding rank strata. These examples distinguish two sufficient scalar certificates.

For unit $z$, writing $t = z^*Tz$, the remaining directional target is

$$z^*Sz \le \delta + \frac{1439a}{720}t + t^2.$$

Moment bounds such as (14) can certify individual directions. For a rank-five factor in six ambient coordinates, Proposition 3.2 retains every active/kernel mixture.

## 6 Actual residual polynomials need not be real-rooted

The positivity in Proposition 3.1 does not imply real stability. This limitation matters when importing probability-law inequalities. Consider the Gaussian-integer factor

$$V_6 = \begin{pmatrix} -1 & -i & 1-i & 1-i & -1+i & 0 \\ -1-i & 1 & i & -1 & -1 & -1 \\ i & -1-i & -1-i & 1+i & -1-i & 1-i \\ -1-i & 1+i & i & -1+i & -1 & -1-i \\ -1-i & 1-i & 1-i & -i & 1+i & -1-i \\ -1+i & -1-i & -i & 1+i & 1 & -1+i \end{pmatrix}.$$

Let $V_5$ be the same matrix with its last column zero, and take $z = e_0$. Here the names $V_5, V_6$ denote these obstruction factors only. Exact elimination gives ranks five and six; both Gram matrices have full support and nonzero imaginary triangle products. The coefficient lists $(q_0, \dots, q_5)$ are

$$\begin{aligned} \operatorname{rank} V = 6: \quad & (20542128507952/45,\ 5821468652200/9,\ 18083778219968/45, \\ & \ 2049107101804/15,\ 127236053248/5,\ 6211082816/3), \\ \operatorname{rank} V = 5: \quad & (403462265536/5,\ 127255137024,\ 458165154544/5, \\ & \ 187047753024/5,\ 8625123712,\ 878481472). \end{aligned}$$

All entries are positive and $q_6 = 0$. Nevertheless

$$4q_1^2 - 10q_0q_2 = \begin{cases} -651686468368265612895710 72/405, & \operatorname{rank} V = 6, \\ -458273045040072984596 48/5, & \operatorname{rank} V = 5. \end{cases}$$

If a degree-five polynomial with positive coefficients had only real roots, they would be negative and it could be written $q_0 \prod_{\ell=1}^{5}(1 + \rho_\ell s)$ with $\rho_\ell > 0$. Cauchy–Schwarz then gives the necessary Newton inequality $4q_1^2 \ge 10q_0q_2$. The exact reverse inequalities prove nonreal roots. This elementary condition belongs to the classical theory of real-rooted probability generating polynomials; see [10].

In the rank-five example $e_5 \in \ker V_5$. For every $p > 0$, a nonreal root $\rho$ is carried by Proposition 3.2 to $(\rho - 1 + p)/p$, still nonreal. Thus every nontrivial active/kernel mixture along this ray retains the obstruction, independently of relative complex phase. At $p = 0$ the polynomial is constant. These are not compensation counterexamples: the exact directional budgets $(a + t)^2 - c - z^*Uz$ are respectively 906092392444480 and 337221858201088.

## 7 Variance geometry of the actual border gap

In this section $B$ is a correlation matrix. Set $w_i = V^*e_i$, so $w_i^*w_j = B_{ij}$ and $\|w_i\| = 1$. For unit $z$ put $p_i = |w_i^*z|^2$, $P_i = w_iw_i^*$, $t = z^*Tz$ and $\delta = a^2 - c$. The normalized bordered matrix and its exact defect are

$$A_z = \begin{pmatrix} B & Vz \\ z^*V^* & 1 \end{pmatrix}, \tag{19}$$

$$\mathcal{D}(z) = (\operatorname{per} A_z)^2 - \operatorname{per}|A_z|^2 = \delta + 2at + t^2 - p^{\mathsf{T}}Lp. \tag{20}$$

Indeed, border expansion gives $\operatorname{per} A_z = a + t$ and $\operatorname{per} |A_z|^2 = c + p^{\mathsf{T}} L p$. Thus the sufficient cofactor surrogate $Q(z) = \delta + 2at + t^2 - z^* U z$ discards the nonnegative term

$$E(z) = z^* U z - p^{\mathsf{T}} L p = \sum_{i,j} L_{ij} \langle r_i, r_j \rangle, \qquad r_i = (I - z z^*) P_i z, \qquad \mathcal{D}(z) = Q(z) + E(z). \tag{21}$$

Here $U = \sum_{i,j} L_{ij} P_i P_j$. Equivalently, writing $L = CC^{\mathsf{T}}$ with real $C$ and $H_j = \sum_i C_{ij} P_i$, $E = \sum_j [z^* H_j^2 z - (z^* H_j z)^2]$. These are ordinary Hermitian variances.

**Proposition 7.1** (Cofactor rank and variance rigidity)**.** *Partition the unit Gram vectors into $m$ classes of parallel complex lines. Then* $\operatorname{rank} L = m$*, its kernel consists of the vectors whose entries sum to zero within each class, and its range consists of vectors constant within each class. If $\mu = \lambda_{\min}^{+}(L)$, then, for every unit $z$,*

$$E(z) \geq \mu \sum_i p_i (1 - p_i).$$

*Consequently $E(z) = 0$ precisely when every $w_i$ is parallel or orthogonal to $z$.*

*Proof.* The doubled vectors $d_i = w_i \otimes \overline{w_i}$ have Gram matrix $|B|^2$. They coincide within a parallel class and are nonproportional between classes. Omitted symmetric products of the $d_i$ have Gram matrix $L/(n-1)!$. Identify the distinct $d_i$ with nonassociate linear forms $\ell_1, \dots, \ell_m$, of multiplicities $n_j$. After cancelling $\prod_j \ell_j^{n_j - 1}$, their distinct omitted products are $\prod_{h \neq j} \ell_h$. A linear relation, reduced modulo $\ell_j$, has only its $j$th term left. Unique factorization forces that coefficient to vanish. This is the elementary reciprocal linear-form independence also underlying [14]. It proves the asserted Gram kernel and range.

Within each class $P_i$, and hence $r_i$, are identical. Each coordinate column of $(r_i)_i$ therefore lies in range $L$. The positive-spectrum bound for $L$ in (21) yields $E \geq \mu \sum_i \|r_i\|^2 = \mu \sum_i p_i (1 - p_i)$. It vanishes exactly when every $p_i$ is zero or one. □

If exactly $k \geq 1$ vectors are parallel to $z$ in this zero-variance case, row phases and ordering split $B = J_k \oplus B_\perp$. Direct block permanents give

$$t = ka, \qquad p^{\mathsf{T}} L p = kc, \qquad \mathcal{D}(z) = (k+1)\delta + k(k+1)a^2.$$

In particular $\delta \geq 0$ makes these active directions strictly positive. If all vectors are orthogonal to $z$, then $\mathcal{D} = \delta$. An irreducible $B$ of rank at least two has no zero-variance active direction; the all-parallel rank-one case is an exception.

For active unit $h$ and kernel unit $\kappa$, write $z_p = \sqrt{p}\, h + e^{i\theta} \sqrt{1 - p}\, \kappa$. Direct homogeneity of the two terms in (20) gives

$$\mathcal{D}(z_p) = (1 - p^2)\delta + p^2 \mathcal{D}(h) + 2a(h^* T h) p (1 - p). \tag{22}$$

Thus $\delta \geq 0$ and the active-endpoint inequality suffice for every kernel weight and relative phase. Enlarging the Gram dimension represents positive Schur slack in the same way.

## 8 Relative stability and occupied-vertex domains

Matrix-norm derivative and perturbation bounds for permanents are developed in [13]. The following elementary comparison instead measures perturbations directly in unit Gram factors; we use its multiplicative form for the quantitative transfer below.

**Lemma 8.1** (Relative Gram-permanent stability)**.** *Replace at most $r$ of $N$ unit vectors by unit vectors, each at distance at most $\varepsilon$. If $P_0, P$ are their Gram permanents, then*

$$(1 + \sqrt{N}\varepsilon)^{-2r} P_0 \leq P \leq (1 + \sqrt{N}\varepsilon)^{2r} P_0.$$

*Gram ranks and ambient dimension are unrestricted.*

*Proof.* For one replacement let $g$ be the symmetric product of the other $N-1$ vectors. Orthogonal projection gives $\|\operatorname{Sym}((v-w)\otimes g)\| \le \|v-w\|\|g\|$. On each homogeneous degree define $c_w^* g = \sqrt{N}\operatorname{Sym}(w\otimes g)$, with $c_w$ its graded adjoint. For unit $w$ the classical identity is $\|c_w^* g\|^2 = \|g\|^2+\|c_w g\|^2 \ge \|g\|^2$; see the bosonic conventions in [5, 6]. The permanent norm formula gives

$$|\sqrt{P(v)}-\sqrt{P(w)}| \le \sqrt{N}\|v-w\|\sqrt{P(w)}.$$

Interchange $v, w$ and apply the two ratio bounds successively. The products are nonzero, for example by the product-of-linear-forms representation. The vacuum case $N=1$ is included. □

**Theorem 8.2** (Transfer from an orthogonal complement)**.** *Let $A$ be the Gram matrix of unit vectors $w_1,\dots,w_n,z$, and set $p_i = |w_i^* z|^2$. Suppose $0 \le \tau < 1/2$, each $p_i$ belongs to $[0,\tau]\cup[1-\tau,1]$, and exactly $k \ge 1$ belong to the second interval. Replace those $k$ vectors by their closest phase multiples of $z$, and the others by their normalized projections onto $z^\perp$. Let $C$ be the Gram matrix of the latter $n-k$ vectors. If $\operatorname{per}|C|^2 \le (\operatorname{per} C)^2$, then*

$$\frac{\operatorname{per}|A|^2}{(\operatorname{per} A)^2} \le \frac{(1+\sqrt{2(n+1)\tau})^{6n}}{(k+1)!}. \tag{23}$$

*The empty complement has permanent one.*

*Proof.* Each ordinary replacement has distance at most $\sqrt{2\tau}$: its squared distance is $2-2\sqrt{p_i}$ or $2-2\sqrt{1-p_i}$. For doubled unit vectors $d(v) = v\otimes\bar v$,

$$\|d(v)-d(w)\|^2 = 2(1-|v^*w|^2).$$

For these particular projections the doubled squared distances are $2(1-p_i)$ or $2p_i$, so the same bound holds. Put $\rho = \sqrt{2(n+1)\tau}$. With $P = \operatorname{per} A$ and $Q = \operatorname{per}|A|^2$, Lemma 8.1 gives $P \ge (1+\rho)^{-2n}P_0$ and $Q \le (1+\rho)^{2n}Q_0$. The reference is $J_{k+1}\oplus C$, hence $Q_0/P_0^2 \le 1/(k+1)!$. Combine the inequalities. □

**Corollary 8.3** (A polynomial occupation radius)**.** *Under the complementary hypothesis in Theorem 8.2, if $\tau \le [1152n^2(n+1)]^{-1}$, then $\operatorname{per}|A|^2 \le \frac{2}{3}(\operatorname{per} A)^2$.*

*Proof.* Now $\rho \le 1/(24n)$ and $(1+\rho)^{6n} \le \sum_{j\ge0}(6n\rho)^j \le 4/3$. Use $(k+1)! \ge 2$. □

The complementary hypothesis is automatic when $n-k \le 4$ by the complex small-order result [2]. For $n=6, k=1$ it instead requires the order-five statement; the through-six preprint [3] reports it. Our transfer theorem explicitly retains that hypothesis. Exact rational comparisons in (23) give the ratio ceiling $7/8$ at these larger order-seven radii:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\tau$ | 1/57344 | 1/8064 | 1/2016 | 1/896 | 1/504 | 1/224 |

Choose respectively $\rho = 1/64, 1/24, 1/12, 1/8, 1/6, 1/4$. At the uniform radius $\tau_0 = 1/57344$,

$$(65/64)^{-24} - \tfrac{1}{2}(65/64)^{12} > 1/12, \qquad (65/64)^{36}/2 < 7/8.$$

The reference permanent $P_0 = (k+1)!\operatorname{per} C \ge 2$, by the classical PSD permanent diagonal bound [15]. Thus $\mathcal{D} \ge P_0^2/12 \ge 1/3$ on this uniform domain. These estimates are independent of positive Gram/cofactor eigenvalues. The all-low vertex is excluded. The conclusions have their stated finite sufficient domains.

For an active unit direction at order six, let $s = \lambda_{\min}^+(B)$ and $\tau = \min\{s/12, 1/57344\}$. If $E < \mu\tau(1-\tau)$, Proposition 7.1 forces every occupation outside $[\tau, 1-\tau]$. The active frame bound $\sum_i p_i \ge s$ excludes the all-low alternative. Thus the occupied vertex theorem applies, with its complementary hypothesis, and gives $\mathcal{D} \ge 1/3$. Only this conversion from variance uses $s, \mu$; the direct geometric domain does not. In this small domain the surrogate is positive too: $E \le \|L\|\sum_i p_i(1-p_i) \le 4320/57344 < 1/3$, since $\operatorname{Tr} L \le 6\cdot 5! = 720$.

**Theorem 8.4** (Neighborhoods of repeated orthogonal blocks)**.** *Let $B_0 = \bigoplus_j J_{m_j}$ have order $n$, with some $m_j \geq 2$. For any complex PSD correlation matrix $A$ of order $n+1$ with leading block $B$, if*

$$\max_{i,j} |B_{ij} - (B_0)_{ij}| \leq \frac{1}{2304n^3(n+1)},$$

*then* $\operatorname{per}|A|^2 \leq \frac{2}{3}(\operatorname{per} A)^2$. *Gram ranks and the normalized border are unrestricted.*

*Proof.* At $B_0$ one has $a_0 = c_0 = \prod_j m_j! \geq 2$ and $U_0 = T_0$: $L_0 = R_0$ and $L_0 \circ B_0 = R_0$ by block cofactors. For every unit border vector, $p^{\mathsf{T}} L_0 p \leq z^* U_0 z = t$, so $Q_0 = a_0 + p^{\mathsf{T}} L_0 p \leq a_0 + t = P_0$. Thus $Q_0/P_0^2 \leq 1/a_0 \leq 1/2$. If the leading-block entrywise distance is at most $d$, then $\|B - B_0\| \leq nd$. Square-root factors differ rowwise by at most $\varepsilon = \sqrt{nd}$, by the PSD square-root bound below. Pad the factors to represent any admissible border by the same unit vector in both Gram systems. Their doubled-row distances are at most $2\varepsilon$. Lemma 8.1 bounds the new ratio by

$$\frac{(1 + 2\sqrt{n+1}\varepsilon)^{6n}}{a_0}.$$

At the stated radius $2\sqrt{n+1}\varepsilon \leq 1/(24n)$; the same geometric-series estimate as Corollary 8.3 gives at most $4/(3a_0) \leq 2/3$. □

**Corollary 8.5** (A neighborhood of a rank-five base)**.** *Let $A$ be any complex PSD correlation matrix of order seven, with leading block $B$ of order six. If*

$$\max_{i,j} |B_{ij} - (J_2 \oplus I_4)_{ij}| \leq 1/884736,$$

*then* $\operatorname{per}|A|^2 \leq \frac{7}{8}(\operatorname{per} A)^2$. *There is no rank assumption on $A$ or $B$.*

*Proof.* At the reference base $B_0 = J_2 \oplus I_4$, one has $a = c = 2$ and $U = T$ by direct block cofactors. For every unit border vector in any ambient dimension, (21) gives

$$\tfrac{1}{2}(a+t)^2 - c - p^{\mathsf{T}} L p = t + \tfrac{1}{2}t^2 + E \geq 0.$$

Thus every reference border has ratio at most $1/2$. If its six unit base vectors are each perturbed by at most $\varepsilon$, keep the same unit border vector. The doubled-vector distances are at most $2\varepsilon$. Lemma 8.1, with $N = 7, r = 6$, bounds the new ratio by

$$\tfrac{1}{2}(1 + 2\sqrt{7}\varepsilon)^{12}(1 + \sqrt{7}\varepsilon)^{24} \leq \tfrac{1}{2}(1 + 6\varepsilon)^{36} < 7/8 \quad (\varepsilon \leq 1/384).$$

For the stated entrywise radius, $\|B - B_0\| \leq 6/884736$ and the PSD square-root inequality gives $\|\sqrt{B} - \sqrt{B_0}\| \leq \sqrt{\|B - B_0\|} \leq 1/384$. Its rowwise bound compares unit Gram factors. Pad them, if necessary, by an unused coordinate to represent arbitrary normalized border slack. Isometric equivalence of Gram realizations lets the same border vector be used in both factors. Hence the comparison covers every admissible $A$, including singular bases and full-rank borders. □

The square-root bound used here follows directly from operator monotonicity: if $\|B - B_0\| \leq d$, then $\sqrt{B} \leq \sqrt{B_0 + dI} \leq \sqrt{B_0} + \sqrt{d}\, I$, and similarly in reverse. The neighborhood is a quantitative application.

# 9 Occupation-box certificates for the actual quartic

Return to $\mathcal{D}$ in (20). This section also holds for unnormalized $B$, with $0 \leq p_i \leq B_{ii}$. For real $y$ set

$$D(y) = \delta I + 2aT - 2V^* \operatorname{diag}(Ly) V, \qquad F_y(z) = y^{\mathsf{T}} L y + z^* D(y) z + (z^* T z)^2.$$

Then $F_y(z) = \mathcal{D}(z) + (p-y)^{\mathsf{T}} L(p-y)$. Consequently

$$\min_{\|z\|=1} \mathcal{D}(z) = \min_{0 \le y_i \le B_{ii}} \left\{ y^{\mathsf{T}} L y + \max_{\xi \in [\lambda_{\min} T, \lambda_{\max} T]} \lambda_{\min}(D(y) + 2\xi T - \xi^2 I) \right\}. \tag{24}$$

The compact product minimum is exact because $y = p$ is allowed. At each fixed $y$, the inner scalar dual follows from convexity of the complex numerical range of $D(y) + iT$ and minimax for the convex function $d + t^2$; cf. [16]. The elementary scalar identity is $t^2 = \max_\xi (2\xi t - \xi^2)$. There is no interchange of the outer minimum and inner maximum in (24).

**Proposition 9.1** (Finite box certificate)**.** *Cover $\prod_i [0, B_{ii}]$ by finitely many boxes of center $h$ and halfwidth $r \ge 0$. A box is certified if some real shift $\xi$ gives*

$$[\delta + h^{\mathsf{T}} L h - r^{\mathsf{T}} L r - \xi^2] I + 2(a + \xi) T - 2V^* \operatorname{diag}(Lh) V \succeq 0. \tag{25}$$

*If every box is certified, then $\mathcal{D}(z) \ge 0$ for all unit complex $z$. For a strictly positive fixed-base instance, a sufficiently fine finite cover admits these certificates. With rational real and imaginary input coordinates, the cover and shifts can be rational.*

*Proof.* All entries of $L$ are nonnegative. If $|p - h| \le r$, then $(p-h)^{\mathsf{T}} L (p-h) \le r^{\mathsf{T}} L r$. Use this with $F_h - \mathcal{D} = (p-h)^{\mathsf{T}} L (p-h)$ and $(t-\xi)^2 \ge 0$ to obtain (25) as a lower bound. If $\min \mathcal{D} = m > 0$, every $F_h \ge m$. The exact fixed-center dual supplies a shift attaining that margin before the box error. A fine mesh with $r^{\mathsf{T}} L r < m/2$ leaves a strict PSD margin. Continuity permits rational nearby shifts. This is finite existence at one strictly positive base; no uniform complexity bound or termination at zero margin follows. □

**Proposition 9.2** (The all-low box)**.** *For a correlation base, put $J = V^* \operatorname{diag}(L\mathbf{1}) V$. Then $T$ and $J$ have kernel exactly* $\ker V$. *Every fixed base therefore admits $\eta > 0$ with $2aT - \eta J \succeq 0$. If also $\delta \ge 0$, this test certifies the entire box $0 \le p_i \le \eta$, with $\mathcal{D} \ge \delta + t^2$.*

*Proof.* The omitted-product argument for the original unit Gram vectors gives the same line-class range description for $R$, after phases make parallel rows equal. The image of $V$ lies in that range, so $R$ is positive on it and $\ker T = \ker V$. Under a row gauge $B = DB_0D^*$, $R(B) = DR(B_0)D^*$ and $V = DV_0$, preserving this argument. Every diagonal entry of $L$ is at least one by the PSD permanent diagonal bound; hence $J$ has the same kernel. Positivity on the active subspace gives an admissible $\eta$. Finally, entrywise nonnegativity of $L$ gives $p^{\mathsf{T}} L p \le \eta p^{\mathsf{T}} L \mathbf{1} = \eta z^* J z$. Substitute into (20). □

There is a conditional finite-completeness extension through kernel zeros. Suppose $\delta \ge 0$ and $\mathcal{D}(z) > 0$ whenever $Vz \ne 0$. Use Proposition 9.2 on an all-low box. On the remaining nonnegative auxiliary centers with $\max_i h_i \ge \eta/2$, each $F_h$ is strictly positive on the whole unit sphere: off the kernel this follows from $F_h \ge \mathcal{D} > 0$, and on it $F_h = \delta + h^{\mathsf{T}} L h > 0$ because $L_{ii} \ge 1$. Compactness over those centers gives a common positive $F_h$ margin. Fine subdivision and the fixed-center dual then give finitely many spectral leaves. This assumes the active-endpoint sign.

An adaptive cover may also accept a box by Theorem 8.2 if every coordinate is wholly low or high at a valid radius and at least one is high. Its complementary hypothesis must still be supplied. The all-low box cannot be accepted by that theorem. Standard rigorous partition methods [17] motivate this organization; the quadratic conjugacy and scalar numerical-range dual are classical. The reusable point here is a certificate of the actual quartic, without first proving the stronger cofactor surrogate. The supplied adaptive examples illustrate subdivision and exact checking. In particular, the first rational full-support, phase-bearing rank-six base in `adaptive_certificates.json` has a two-leaf proof of $\mathcal{D} \ge 13/20$. An exact Rayleigh witness rules out that margin for every single-full-box shift, while the exact shifted cofactor test also certifies $13/20$ on the same base. This is a subdivision example, rather than evidence that retained variance is necessary there.

## 10 Scope and reproducibility

The sharp theorem is analytic. The sector refinement uses classical Schur weights and content eigenvalues; its specialization and mixed-permanent evaluation produce the stated coupled bound. The moment relaxation and thinning algebra are classical tools. The cofactor PSD construction in Proposition 3.3 is credited to the existing omitted-tensor method; its role here is the residual coefficient cancellation.

The standalone standard-library program `verify.py` reconstructs the displayed factors, exact ranks, phase witnesses, mixed permanents, occupation coefficients, Newton defects and strict scalar margins. It uses rational arithmetic and explicit checks that remain active under `python -O`. No private computational files are needed.

The extensions use classical reciprocal-linear-form independence, bosonic creation calculus, numerical-range convexity and rigorous partition methods. Their role is the explicit variance geometry, relative-factor transfer and actual-quartic certificate. The finite adaptive example improves a two-leaf bound over the corresponding single full-box bound, while the exact shifted cofactor surrogate also passes that example. It is not a separation from that surrogate. The standalone `verify_adaptive.py` checks the supplied rational tree certificates, scalar constants and rejection tests. The conclusions in [7] concern Haar averages.

## AI assistance

AI tools assisted mathematical exploration, drafting and checking. The author is responsible for the statements and presentation.

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