Coupled permanent cofactors: sharp occupation bounds and compensation geometry

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Luminary Oddities

Abstract

Ordinary and entrywise-squared Gram matrices have cofactor operators linked by their common Gram vectors. This paper proves a sharp residual operator bound in every order, with optimal factor $n-1$, and a sharp inequality coupling ordinary occupation coefficients to a cofactor-weighted variance. The proofs include singular Gram matrices, complex phases and arbitrary ambient kernel components. They use nested orthogonal permutation projections and the number of tensor factors in a selected one-dimensional mode. For six Gram vectors, these identities give explicit sufficient regions for the self-conjugate permanent inequality at order seven. A sharp constraint on ordinary occupation coefficients has a complete equality classification and yields a region of small positive ordinary loss without a diagonal cutoff. A separate creation-operator estimate proves strict diagonal monotonicity and convexity of a transformed permanent deficit. Stronger auxiliary bounds based on weighted Carlen–Lieb–Loss estimates, coherent-state projections and matrix scaling hold on specified structured families and quantitative neighborhoods. Exact counterexamples distinguish these auxiliary tests from the actual defect. A rank-one three-coordinate example gives a positive but strictly decreasing transformed deficit along a range-preserving PSD direction. All restricted-family finite covers, their analytic tails and the finite operator calculation are specified in the article. The results retain their stated sufficient-domain hypotheses; they do not prove the unrestricted Chollet conjecture.

Keywords. Permanent; Gram matrix; coupled cofactors; occupation bounds; positive semidefinite matrices; Chollet inequality.
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Part I. Universal coupled operators and occupation bounds

1 Introduction and notation

Permanent-cofactor estimates enter Chollet's inequality through the expansion of a bordered Gram matrix. The ordinary Gram factors and their doubled counterparts produce different contractions, but those contractions are not independent: they arise from the same vectors. We study the residual operator and the variance that retain this common origin. Their estimates lead both to all-order inequalities and to explicit compensation regions for six-factor borders.

The distinction between the actual bordered defect and a stronger capacity comparison is essential. The sharp occupation constraint later in the paper gives a cutoff-free compensation strip, while the capacity regions require their own hypotheses. Likewise, diagonal monotonicity of the transformed deficit does not imply monotonicity in every PSD direction.

For a square matrix A, $\operatorname{per} A = \sum_{\sigma\in\mathfrak{S}_n}\prod_i A_{i,\sigma(i)}$; $A(j \mid i)$ deletes row j and column i. The symbol $\circ$ denotes entrywise multiplication, $*$ the conjugate transpose, and $|B|^2 = B\circ\overline{B}$, not the operator square.

Throughout, $V \in \mathbb{C}^{n\times d}$, $B = VV^*$, and d is arbitrary. Define

$$a = \operatorname{per} B, \qquad c = \operatorname{per}|B|^2, \qquad R_{ij} = \operatorname{per} B(j \mid i), \qquad L_{ij} = \operatorname{per}|B|^2(j \mid i), \tag{1}$$

$$T = V^*RV, \qquad U = V^*(L \circ B)V, \qquad r = c - \frac{a^2}{n!}, \qquad S = U - \frac{a}{n!}T. \tag{2}$$

All operator inequalities below act on $\mathbb{C}^d$ unless specified otherwise. No rank, normalization of the diagonal, or invertibility hypothesis is imposed.

Permanent-cofactor Gram matrices and their Schur products are classical. Paksoy [12], in particular, uses omitted symmetric tensors and a weighted cofactor matrix with constant permanent row sum. Symmetrizers and bosonic permanent states are standard [7, 21]; Schur weights and content eigenvalues are standard representation theory [6, 16]. Recent coordinate-difference inequalities for entry-weighted cofactors are proved by Aouichaoui and Zhang [1]. Our focus is the *joint residual* S, its sharp norm bound, and a computable refinement that preserves the common origin of ordinary and doubled Gram factors. Li, Zhao and Luo [8] represent coefficients of ordinary and doubled bordered permanent polynomials by squared norms over complete occupation sectors. The polynomial studied below instead comes from the joint residual projection $P_{\mathrm{diag}} - P_{\mathrm{sep}}$; its coefficients measure one shared residual state. The separate positivity of T and U, or a bound on either one alone, does not supply this residual estimate.

The motivation is the compensation in a bordered permanent inequality. Chollet asks whether $\operatorname{per}(A \circ D) \le \operatorname{per} A \operatorname{per} D$ for complex positive semidefinite A, D; see [5, 15]. The results here concern coupled cofactor operators and sufficient border certificates, rather than an unrestricted solution.

The bordered inequality and the role of the operators

Suppose now that B has diagonal one and that z is a unit vector. Write $x = Vz$, $p_i = |x_i|^2$ and $t = z^*Tz$. The Gram matrix obtained by adjoining z is

$$A_z = \begin{pmatrix} B & Vz \\ z^*V^* & 1 \end{pmatrix}.$$

Its scalar permanent defect is exactly

$$G(z) := (\operatorname{per} A_z)^2 - \operatorname{per}|A_z|^2 = (a+t)^2 - c - p^{\mathsf{T}}Lp. \tag{3}$$

Thus $G(z) \ge 0$ proves the self-conjugate permanent inequality for this bordered matrix. At six base rows, A_z has order seven. The auxiliary cofactor estimate replaces $p^{\mathsf{T}}Lp$ by z^*Uz, losing the nonnegative variance $E = z^*Uz - p^{\mathsf{T}}Lp$. The residual operator S and the later occupation estimates retain information about the same vectors and therefore about this discarded term.

Here *occupation* means the number of factors in a selected one-dimensional mode of a symmetric tensor. An occupation coefficient is the squared norm of that orthogonal component, with its displayed factorial normalization. This standard number-operator language will always refer to a specified tensor state; coefficients of different states are not independent variables that may be added arbitrarily.

Quantity or claim	Meaning and scope
a, c, T, U, r, S	Ordinary permanent, doubled permanent, their cofactor operators, and the joint residual defined in (1)–(2). The first theorem holds in all orders without diagonal normalization.
$G = \mathcal{D}$	Actual bordered scalar defect (3); a nonnegative value proves that particular self-conjugate inequality.
$Q = G - E$	A stronger cofactor sufficient test. Its failure does not imply $G < 0$.
$P, F(s), L$	On a specified six-row selected-mode pencil, $P = F(1) + F'(1)$ and $L = F(\sqrt{2}) - P$ is the ordinary loss. These symbols are introduced with that pencil.
U_{opt}	A weighted CLL upper envelope for the forward doubled permanent. A capacity test is a stronger sufficient condition, not an equivalent reformulation of $G \geq 0$.

The first part proves the universal operator statement. The second part proves variance and occupation constraints and their direct consequences for G. The third part, organized as supporting appendices, develops stronger conditional projection and capacity tests with explicit structured domains. The fourth part continues the appendices with sharp six-row occupation geometry and studies diagonal directions. Structured capacity results are statements about stronger certificates; when scalar low-rank coverage was already available, they are not claimed as new scalar permanent cases.

A reserve is a proved nonnegative lower bound for a specified term in an inequality. Each is used with its stated normalization and domain.

Results and dependencies. Every reserve below belongs to the same ordinary, doubled and omitted Gram factors. The principal results and their consequences are organized as follows.

1. Theorem 2.1 gives the sharp all-order residual bound. The occupation identities and variance refinements retain the same residual state. Theorem E.1 identifies a sharp six-factor occupation constraint and its equality geometry; Theorem F.1 gives a uniform row reserve and strict diagonal monotonicity for the transformed deficit.

2. Corollary E.2 and Theorem 10.2 give representative compensation regions. Capacity certificates impose stronger comparisons, with their own stated geometric or occupation hypotheses. These consequences retain the distinction between the actual defect and an auxiliary capacity target.

3. The restricted-family theorems in Theorems B.2, D.2 and D.3 use complete rational covers with analytic tails. Other exact examples establish nonemptiness or delimit proposed extensions. The final reproducibility section identifies each checker's precise role.

The boundary theorem in Section 10 adds an explicit order-seven sufficient domain, including full-rank matrices. Its joint root-product proof uses common-factor coefficient constraints, weighted Cauchy–Schwarz and an elementary PSD permanent upper bound. The preceding boundary proof also uses the classical Carlen–Lieb–Loss permanent inequality [4]. The sharp retained-curvature floor and exact trace–occupation overlap retain information lost by independently summing scalar reserves. Section 10.2 gives further sufficient conditions by classical matrix scaling [18]. Earlier sufficient conditions, including those of Sa-nguansin and Rodtes [17], are credited.

2 A sharp bound in every order

Theorem 2.1. *With* (1)–(2)*, for every* $n \geq 2$*,*

$$0 \preceq S \preceq (n-1) r I_d. \tag{4}$$

The coefficient $n-1$ *cannot be reduced for actual Gram factors. The conclusion includes arbitrary complex phases and singular factors.*

A two-row example. Take $V = I_2$, so $B = I_2$ and $a = c = 1$. Deleting a row and column gives $R = L = I_2$, whence $T = U = I_2$, $r = 1 - 1/2 = 1/2$ and $S = I_2 - I_2/2 = I_2/2$. Thus the upper bound in Theorem 2.1 is equality. For any unit z, put $p_1 = |z_1|^2$, $p_2 = |z_2|^2$; then $t = 1$ and

$$E = 1 - p_1^2 - p_2^2 = 2p_1p_2, \qquad G = 4 - 1 - p_1^2 - p_2^2 = 2 + 2p_1p_2.$$

This explicitly separates the residual operator, its discarded variance, and the actual bordered defect.

2.1 The two invariant projections

We use the conjugate-linear-first inner product. Regard row i of V as the ket $v_i = \sum_j V_{ij} e_j$ in $\mathcal{H} = \mathbb{C}^d$. Its physical Gram is $\overline{B}$. Put

$$f = v_1 \otimes \cdots \otimes v_n, \qquad W = f \otimes \overline{f}.$$

On $\mathcal{H}^{\otimes n} \otimes \overline{\mathcal{H}}^{\otimes n}$, let P_{sep} average independent permutations of the two legs, and let P_{diag} average equal simultaneous permutations. Both are orthogonal projections and $P_{\mathrm{sep}} P_{\mathrm{diag}} = P_{\mathrm{sep}}$. In particular $a = n! \| \operatorname{Sym} f \|^2 \geq 0$. Direct expansion gives

$$\|P_{\mathrm{sep}} W\|^2 = \frac{a^2}{(n!)^2}, \qquad \|P_{\mathrm{diag}} W\|^2 = \frac{c}{n!}. \tag{5}$$

For unit $z \in \mathbb{C}^d$, let $p_z = |\overline{z}\rangle\langle\overline{z}|$ and let $N_z = \sum_{j=1}^n p_z^{(j)}$ act on the first leg. The conjugation in p_z matches the row convention $B = VV^*$. This operator commutes with both projections.

Set $h = (P_{\mathrm{diag}} - P_{\mathrm{sep}})W$. For $x = Vz$, differentiation of the two permanent norm formulas gives

$$\|h\|^2 = \frac{r}{n!}, \tag{6}$$

$$\langle h, N_z h \rangle = \frac{1}{n!} \left(z^* U z - \frac{a}{n!} z^* T z \right). \tag{7}$$

Here the conjugate physical Gram deformation is $B + t\, xx^*$, and

$$\frac{d}{dt} \operatorname{per}(B + t\, xx^*) \Big|_{t=0} = z^* T z.$$

The paired deformation is $|B|^2 + t(B \circ \overline{xx^*})$, whose permanent derivative is $z^* U z$. For example the latter derivative is $\sum_{ij} L_{ji} B_{ij} \overline{x_i} x_j$; L is real symmetric, yielding the defined U. The independent-projection expectation is $a(z^*Tz)/(n!)^2$ and the diagonal one is $(z^*Uz)/n!$. Their difference is (7); the cross terms vanish because N_z preserves the nested orthogonal components.

2.2 Excluding full occupation

Proof of Theorem 2.1. The spectrum of N_z is contained in $\{0,\dots,n\}$. Its n-eigenspace has first leg $\overline{z}^{\otimes n}$. A diagonal-invariant vector in that eigenspace has a symmetric first leg; diagonal invariance then forces its second leg to be symmetric too. Thus the n-occupation part of range P_{diag} lies in range P_{sep} and is absent from h. Consequently

$$0 \le \langle h, N_z h\rangle \le (n-1)\|h\|^2.$$

Use (6)–(7) and vary unit z.

For sharpness, take $n-1$ equal unit rows in one coordinate and one unit row in an orthogonal coordinate. Then $B = J_{n-1} \oplus [1]$, $a = c = (n-1)!$ and, in the repeated-row direction, $z^*Tz = (n-1)a$, $z^*Uz = (n-1)c$. The upper bound is an equality. □

In particular $r \ge 0$. If $r = 0$ then $S = 0$. Both T and U vanish on $\ker V$. Therefore, with $\mathcal{A} = (\ker V)^\perp$ and its orthogonal projection $P_{\mathcal{A}}$,

$$0 \preceq S \preceq (n-1) r P_{\mathcal{A}}. \tag{8}$$

This statement retains the kernel rather than replacing the ambient space.

3 The occupation polynomial and a cancellation

For unit z, write $x = Vz$, $X = xx^*$ and $H = B - X \succeq 0$.

Proposition 3.1. *The polynomial*

$$G_z(s) = \operatorname{per}\big(|B|^2 + (s-1)(B \circ \overline{X})\big) - \frac{a}{n!}\operatorname{per}\big(B + (s-1)X\big) = \sum_{k=0}^{n} q_k(z) s^k \tag{9}$$

has $q_k(z) \ge 0$ and $q_n(z) = 0$. More precisely, if E_k is the k-occupation spectral projector of N_z, then $q_k(z) = n!\|E_k h\|^2$. Thus

$$G_z(1) = r, \qquad G_z'(1) = z^* S z, \tag{10}$$

$$q_0(z) = \operatorname{per}(H \circ \overline{B}) - \frac{a}{n!}\operatorname{per} H. \tag{11}$$

Proof. For real $s \ge 0$ multiply the selected first-leg mode by $\sqrt{s}$. The resulting operator is $s^{N_z/2}$ and commutes with both projections. The diagonal projected squared norm is the first permanent in (9) divided by $n!$; the independent projected squared norm is $a\ \operatorname{per}(B+(s-1)X)/(n!)^2$. Their difference is $\|s^{N_z/2}h\|^2$. Polynomial identification proves the coefficient formula. The full-occupation exclusion proves $q_n = 0$. The identities at $s = 1, 0$ follow directly. □

Proposition 3.2 (Exact kernel thinning)**.** *Let $w \in \mathcal{A}$ and $\kappa \in \ker V$ be unit vectors. For $0 \le p \le 1$ and real θ put $z_p = \sqrt{p}\, w + e^{i\theta}\sqrt{1-p}\,\kappa$. Then*

$$G_{z_p}(s) = G_w(1 - p + ps), \tag{12}$$

$$q_k(z_p) = \sum_{j=k}^{n-1} \binom{j}{k} p^k (1-p)^{j-k} q_j(w). \tag{13}$$

In particular $G'_{z_p}(1) = pG'_w(1)$ and $G''_{z_p}(1) = p^2 G''_w(1)$. The formulas include the endpoints and every relative complex phase.

Proof. $Vz_p = \sqrt{p}\,Vw$, and substitution into (9) gives the first identity. Binomial expansion gives the rest. □

Proposition 3.3 (Top occupation)**.** *Put* $b_i = \prod_{j\neq i} x_j$, $y = \overline{b}$, $J = \mathbf{1}\mathbf{1}^*$ *and*

$$F_0 = R \circ \overline{B} - \frac{a}{n} J.$$

Then $F_0 \succeq 0$, $F_0\mathbf{1} = 0$, *and*

$$q_{n-1} = y^* \left(R \circ |B|^2 - \frac{a}{n} B \right) y = y^*(B \circ F_0)y = y^*(H \circ F_0)y. \tag{14}$$

These are polynomial identities, including when some $x_i = 0$.

Proof. The usual omitted symmetric-tensor Gram construction gives $R \succeq 0$. The Schur product theorem gives $F = R \circ \overline{B} \succeq 0$. Column-$i$ Laplace expansion gives $\sum_j R_{ij}\overline{B_{ij}} = a$. Thus $F\mathbf{1} = a\mathbf{1}$ and removing that Hermitian eigencomponent gives $F_0 \succeq 0$. This construction is classical; compare [12].

In powers of $t = s - 1$, the coefficient of t^{n-1} in the first permanent in (9) is $\sum_{ij} |B_{ij}|^2 \operatorname{per} B(i \mid j)\overline{b_i}b_j$. Exchanging i, j gives $y^*(R \circ |B|^2)y$. The coefficient in the second permanent is $(n-1)!\, y^*By$. Both leading t^n terms in G_z equal $a \prod_i |x_i|^2$ and cancel. Translating $t = s - 1$ therefore leaves the next coefficient unchanged, proving the first equality in (14). Finally $\operatorname{diag}(\overline{x})y = \overline{\prod_i x_i}\,\mathbf{1}$, so $y^*(X \circ F_0)y = 0$. This proves the last equality without division. □

For later directional tests at $n = 6$, set $r_+ = r - q_0$ and $m_2 = G_z''(1)$. The elementary moment bound is

$$z^*Sz \leq \min\left\{ 5r_+,\ \frac{m_2}{2j} + \frac{j+1}{2} r_+ :\ j = 1, \ldots, 4 \right\}. \tag{15}$$

Indeed $(k-j)(k-j-1) \geq 0$ at integer $1 \leq k \leq 5$. Multiplication by q_k and summation proves each affine bound. For fixed feasible r_+, m_2, mixtures on consecutive occupations attain its segments, so this is the exact finite-support relaxation. It is an elementary instance of discrete moment linear programming [14]. Retaining q_5 also gives $z^*Sz \leq 4(r - q_0) + q_5$. Neither estimate alone couples its inputs to the compensation budget.

3.1 A central-transposition refinement at order six

In this section $n = 6$. On the first tensor leg let $\mathcal{C} = \sum_{i<j} U_{(ij)}$ be the sum of transpositions and put

$$v = 720\,\langle h, \mathcal{C}h \rangle, \qquad M = \frac{v + 21r}{6}.$$

Theorem 3.4. *For every complex Gram factor of order six,*

$$0 \preceq S \preceq MI_d \preceq 5rI_d. \tag{16}$$

For each transposition π *form* $A^\pi_{ij} = B_{\pi(i),j}\overline{B_{ij}}$. *Then*

$$v = \sum_{\pi\ \textit{transposition}} \operatorname{per} A^\pi - \frac{15a^2}{720}. \tag{17}$$

The conclusion holds in every ambient dimension and at singular boundaries.

Proof. On a first-leg Schur sector of shape $\lambda \vdash 6$, the largest one-mode occupation is λ_1: column strictness in semistandard tableaux permits a fixed letter at most once in each column. The central transposition sum acts by the content sum $\kappa_\lambda = \sum_{(i,j)\in\lambda}(j-i)$; see [16, Example 3.15]. The full comparison is

λ	λ_1	κ_λ	$(\kappa_\lambda+21)/6$
(6)	6	15	6
$(5,1)$	5	9	5
$(4,2)$	4	5	13/3
$(4,1,1)$	4	3	4
$(3,3)$	3	3	4
$(3,2,1)$	3	0	7/2
$(3,1,1,1)$	3	-3	3
$(2,2,2)$	2	-3	3
$(2,2,1,1)$	2	-5	8/3
$(2,1,1,1,1)$	2	-9	2
(1^6)	1	-15	1

so $N_z \preceq (\mathcal{C}+21I)/6$. The separately symmetric first-leg sector is absent from h: first-leg symmetry and diagonal invariance force second-leg symmetry. All remaining sectors have $\kappa_\lambda \le 9$. Hence

$$720\,\langle h, N_z h\rangle \le (v+21r)/6 \le 5r.$$

Use (7). Nonnegativity follows from Theorem 2.1.

The class sum $\mathcal{C}$ commutes with both projections, and acts as 15 on their separately symmetric range. Consequently

$$\langle h, \mathcal{C}h\rangle = \langle W, \mathcal{C}P_{\mathrm{diag}}W\rangle - 15\|P_{\mathrm{sep}}W\|^2.$$

Expand the diagonal average and the fifteen swaps. Summing over its permutation index gives $\operatorname{per} A^\pi$ for each π (equivalently its complex conjugate under the opposite permutation convention). The sum is real since $\mathcal{C}P_{\mathrm{diag}}$ is self-adjoint. Equations (5) give (17). □

More precisely, if $r_\lambda = 720\|E_\lambda h\|^2$, then $S \preceq (\sum_\lambda \lambda_1 r_\lambda)I$ and $v = \sum_\lambda \kappa_\lambda r_\lambda$. The single statistic v compresses these masses. Since S vanishes on $\ker V$, (16) also gives $S \preceq MP_{\mathcal{A}}$.

3.2 Compensation and exact examples

For any n define $\delta = a^2 - c$ and $D_0 = \delta I_d + 2aT - U$. Let P_i be the rank-one matrix $(P_i)_{\alpha\beta} = \overline{V_{i\alpha}}V_{i\beta}$, and set

$$\mathcal{K} = \delta I_{d^2} + a(T\otimes I + I\otimes T) + T\otimes T - \sum_{ij} L_{ij} P_i \otimes P_j. \tag{18}$$

Lemma 3.5. *$D_0 \succeq 0$ implies $\mathcal{K} \succeq 0$. For the bordered Gram matrix* $\widetilde{B}_z = \begin{pmatrix} B & Vz \\ z^*V^* & \|z\|^2 \end{pmatrix}$,

$$\operatorname{per}(\widetilde{B}_z)^2 - \operatorname{per}|\widetilde{B}_z|^2 = \langle z\otimes z, \mathcal{K}(z\otimes z)\rangle.$$

Proof. L is real symmetric PSD by the omitted-tensor Gram construction. Factor $L = CC^T$ over the reals and put $A_k = \sum_i C_{ik}P_i$. Then A_k are Hermitian and $\sum_k A_k^2 = U$, since P_iP_j has entries $\overline{V_{i\alpha}}B_{ij}V_{j\beta}$. Expansion gives the identity

$$\mathcal{K} = \frac{D_0\otimes I + I\otimes D_0}{2} + T\otimes T + \frac{1}{2}\sum_k (A_k\otimes I - I\otimes A_k)^2.$$

Every displayed term is PSD when $D_0 \succeq 0$. Finally border expansion gives $\operatorname{per} \widetilde{B}_z = a\|z\|^2 + z^*Tz$ and $\operatorname{per} |\widetilde{B}_z|^2 = c\|z\|^4 + \sum_{ij} L_{ij}|(Vz)_i|^2|(Vz)_j|^2$. Subtracting proves the formula. □

Theorem 2.1 gives, in every order,

$$D_0 \succeq \left[\left(1 + \frac{n-1}{n!}\right) a^2 - nc\right] I_d + \left(2 - \frac{1}{n!}\right) aT.$$

At order six this certifies $D_0, \mathcal{K} \succeq 0$ whenever $a > 0$ and $c/a^2 \le 145/864$. Theorem 3.4 improves the sufficient test to

$$D_0 \succeq (\delta - M) I_d + \frac{1439a}{720} T, \qquad M \le \delta. \tag{19}$$

This is a condition on the actual mixed-permanent statistic.

Here are exact factors witnessing strict improvement of the scalar test. Let $W \in \mathbb{C}^{6\times 6}$ have first three rows $(1, \frac{1}{2}, 0, 0, 0, 0)$, $(1, \frac{i}{2}, 0, 0, 0, 0)$, $(1, -\frac{1}{2}, 0, 0, 0, 0)$, and last three rows e_2^T, e_3^T, e_4^T, using coordinates $0, \dots, 5$. For $0 \le i \le 5$, $0 \le j \le 4$ set

$$E_{ij} = \frac{((i+2j) \bmod 3) - 1 + i_c(((2i+j) \bmod 3) - 1)}{10000}, \qquad E_{i5} = 0,$$

where $i_c = \sqrt{-1}$ is distinguished from the row index. Put $V_5 = W + E$ and $V_6 = V_5 + 10^{-4} e_2 e_5^T$. Exact rational arithmetic proves, for $B_j = V_j V_j^*$,

$$\operatorname{rank} V_j = j, \quad B_j \text{ has no zero entries}, \qquad \frac{c}{a^2} > \frac{145}{864}, \quad \frac{\delta - M}{a^2} > \frac{17}{1000} \quad (j = 5, 6).$$

Both have a triangle $B_{ij}B_{jk}B_{ki}$ with nonzero imaginary part, which cannot be removed by diagonal unitary row gauges. Numerically c/a^2 is about 0.20174. Strict margins persist on relative open neighborhoods of the corresponding rank strata. These examples distinguish two sufficient scalar certificates.

For unit z, writing $t = z^*Tz$, the remaining directional target is

$$z^*Sz \le \delta + \frac{1439a}{720} t + t^2.$$

Moment bounds such as (15) can certify individual directions. For a rank-five factor in six ambient coordinates, Proposition 3.2 retains every active/kernel mixture.

3.3 Actual residual polynomials need not be real-rooted

The positivity in Proposition 3.1 does not imply real stability. This limitation matters when importing probability-law inequalities. Consider the Gaussian-integer factor

$$V_6 = \begin{pmatrix} -1 & -i & 1-i & 1-i & -1+i & 0 \\ -1-i & 1 & i & -1 & -1 & -1 \\ i & -1-i & -1-i & 1+i & -1-i & 1-i \\ -1-i & 1+i & i & -1+i & -1 & -1-i \\ -1-i & 1-i & 1-i & -i & 1+i & -1-i \\ -1+i & -1-i & -i & 1+i & 1 & -1+i \end{pmatrix}.$$

Let V_5 be the same matrix with its last column zero, and take $z = e_0$. Here the names V_5, V_6 denote these obstruction factors only. Exact elimination gives ranks five and six; both Gram matrices have

full support and nonzero imaginary triangle products. The coefficient lists $(q_0, \dots, q_5)$ are

$$\begin{aligned}\operatorname{rank} V = 6: &\quad (20542128507952/45,\ 5821468652200/9,\ 18083778219968/45,\\ &\qquad 2049107101804/15,\ 127236053248/5,\ 6211082816/3),\\ \operatorname{rank} V = 5: &\quad (403462265536/5,\ 127255137024,\ 458165154544/5,\\ &\qquad 187047753024/5,\ 8625123712,\ 878481472).\end{aligned}$$

All entries are positive and $q_6 = 0$. Nevertheless

$$4q_1^2 - 10q_0q_2 = \begin{cases} -651686468368265612895710 72/405, & \operatorname{rank} V = 6,\\ -45827304504007298459648/5, & \operatorname{rank} V = 5.\end{cases}$$

If a degree-five polynomial with positive coefficients had only real roots, they would be negative and it could be written $q_0 \prod_{\ell=1}^{5}(1+\rho_\ell s)$ with $\rho_\ell > 0$. Cauchy–Schwarz then gives the necessary Newton inequality $4q_1^2 \ge 10q_0q_2$. The exact reverse inequalities prove nonreal roots. This elementary condition belongs to the classical theory of real-rooted probability generating polynomials; see [19].

In the rank-five example $e_5 \in \ker V_5$. For every $p > 0$, a nonreal root ρ is carried by Proposition 3.2 to $(\rho - 1 + p)/p$, still nonreal. Thus every nontrivial active/kernel mixture along this ray retains the obstruction, independently of relative complex phase. At $p = 0$ the polynomial is constant. These are not compensation counterexamples: the exact directional budgets $(a+t)^2 - c - z^*Uz$ are respectively 906092392444480 and 337221858201088.

Part II. Variance identities and sufficient scalar inequalities

4 Variance geometry of the actual border gap

In this section B is a correlation matrix. Set $w_i = V^*e_i$, so $w_i^*w_j = B_{ij}$ and $\|w_i\| = 1$. For unit z put $p_i = |w_i^*z|^2$, $P_i = w_iw_i^*$, $t = z^*Tz$ and $\delta = a^2 - c$. The normalized bordered matrix and its exact defect are

$$A_z = \begin{pmatrix} B & Vz \\ z^*V^* & 1 \end{pmatrix}, \tag{20}$$

$$\mathcal{D}(z) = (\operatorname{per} A_z)^2 - \operatorname{per}|A_z|^2 = \delta + 2at + t^2 - p^{\mathsf{T}}Lp. \tag{21}$$

Indeed, border expansion gives $\operatorname{per} A_z = a + t$ and $\operatorname{per}|A_z|^2 = c + p^{\mathsf{T}}Lp$. Thus the sufficient cofactor surrogate $Q(z) = \delta + 2at + t^2 - z^*Uz$ discards the nonnegative term

$$E(z) = z^*Uz - p^{\mathsf{T}}Lp = \sum_{i,j} L_{ij}\langle r_i, r_j\rangle, \qquad r_i = (I - zz^*)P_iz, \qquad \mathcal{D}(z) = Q(z) + E(z). \tag{22}$$

Here $U = \sum_{i,j} L_{ij}P_iP_j$. Equivalently, writing $L = CC^{\mathsf{T}}$ with real C and $H_j = \sum_i C_{ij}P_i$, $E = \sum_j [z^*H_j^2z - (z^*H_jz)^2]$. These are ordinary Hermitian variances.

Proposition 4.1 (Cofactor rank and variance rigidity)**.** *Partition the unit Gram vectors into m classes of parallel complex lines. Then* $\operatorname{rank} L = m$*, its kernel consists of the vectors whose entries sum to zero within each class, and its range consists of vectors constant within each class. If $\mu = \lambda_{\min}^{+}(L)$, then, for every unit z,*

$$E(z) \ge \mu \sum_i p_i(1 - p_i).$$

Consequently $E(z) = 0$ precisely when every w_i is parallel or orthogonal to z.

Proof. The doubled vectors $d_i = w_i \otimes \overline{w_i}$ have Gram matrix $|B|^2$. They coincide within a parallel class and are nonproportional between classes. Omitted symmetric products of the d_i have Gram matrix $L/(n-1)!$. Identify the distinct d_i with nonassociate linear forms $\ell_1, \dots, \ell_m$, of multiplicities n_j. After cancelling $\prod_j \ell_j^{n_j-1}$, their distinct omitted products are $\prod_{h\neq j} \ell_h$. A linear relation, reduced modulo ℓ_j, has only its jth term left. Unique factorization forces that coefficient to vanish. This is the elementary reciprocal linear-form independence also underlying [20]. It proves the asserted Gram kernel and range.

Within each class P_i, and hence r_i, are identical. Each coordinate column of $(r_i)_i$ therefore lies in range L. The positive-spectrum bound for L in (22) yields $E \geq \mu \sum_i \|r_i\|^2 = \mu \sum_i p_i(1-p_i)$. It vanishes exactly when every p_i is zero or one. □

If exactly $k \geq 1$ vectors are parallel to z in this zero-variance case, row phases and ordering split $B = J_k \oplus B_\perp$. Direct block permanents give

$$t = ka, \qquad p^{\mathsf{T}} L p = kc, \qquad \mathcal{D}(z) = (k+1)\delta + k(k+1)a^2.$$

In particular $\delta \geq 0$ makes these active directions strictly positive. If all vectors are orthogonal to z, then $\mathcal{D} = \delta$. An irreducible B of rank at least two has no zero-variance active direction; the all-parallel rank-one case is an exception.

For active unit h and kernel unit κ, write $z_p = \sqrt{p}\, h + e^{i\theta}\sqrt{1-p}\,\kappa$. Direct homogeneity of the two terms in (21) gives

$$\mathcal{D}(z_p) = (1-p^2)\delta + p^2 \mathcal{D}(h) + 2a(h^* T h)p(1-p). \tag{23}$$

Thus $\delta \geq 0$ and the active-endpoint inequality suffice for every kernel weight and relative phase. Enlarging the Gram dimension represents positive Schur slack in the same way.

4.1 Relative stability and occupied-vertex domains

Matrix-norm derivative and perturbation bounds for permanents are developed in [3]. The following elementary comparison instead measures perturbations directly in unit Gram factors; we use its multiplicative form for the quantitative transfer below.

Lemma 4.2 (Relative Gram-permanent stability)**.** *Replace at most r of N unit vectors by unit vectors, each at distance at most ε. If P_0, P are their Gram permanents, then*

$$(1+\sqrt{N}\varepsilon)^{-2r} P_0 \leq P \leq (1+\sqrt{N}\varepsilon)^{2r} P_0.$$

Gram ranks and ambient dimension are unrestricted.

Proof. For one replacement let g be the symmetric product of the other $N-1$ vectors. Orthogonal projection gives $\|\operatorname{Sym}((v-w)\otimes g)\| \leq \|v-w\|\|g\|$. On each homogeneous degree define $c_w^* g = \sqrt{N}\operatorname{Sym}(w \otimes g)$, with c_w its graded adjoint. For unit w the classical identity is $\|c_w^* g\|^2 = \|g\|^2 + \|c_w g\|^2 \geq \|g\|^2$; see the bosonic conventions in [7, 21]. The permanent norm formula gives

$$|\sqrt{P(v)} - \sqrt{P(w)}| \leq \sqrt{N}\|v-w\|\sqrt{P(w)}.$$

Interchange v, w and apply the two ratio bounds successively. The products are nonzero, for example by the product-of-linear-forms representation. The vacuum case $N = 1$ is included. □

Theorem 4.3 (Transfer from an orthogonal complement)**.** *Let A be the Gram matrix of unit vectors $w_1, \dots, w_n, z$, and set $p_i = |w_i^* z|^2$. Suppose $0 \le \tau < 1/2$, each p_i belongs to $[0, \tau] \cup [1-\tau, 1]$, and exactly $k \ge 1$ belong to the second interval. Replace those k vectors by their closest phase multiples of z, and the others by their normalized projections onto $z^\perp$. Let C be the Gram matrix of the latter $n-k$ vectors. If $\operatorname{per}|C|^2 \le (\operatorname{per} C)^2$, then*

$$\frac{\operatorname{per}|A|^2}{(\operatorname{per} A)^2} \le \frac{(1+\sqrt{2(n+1)\tau})^{6n}}{(k+1)!}. \tag{24}$$

The empty complement has permanent one.

Proof. Each ordinary replacement has distance at most $\sqrt{2\tau}$: its squared distance is $2 - 2\sqrt{p_i}$ or $2 - 2\sqrt{1-p_i}$. For doubled unit vectors $d(v) = v \otimes \bar{v}$,

$$\|d(v) - d(w)\|^2 = 2(1 - |v^* w|^2).$$

For these particular projections the doubled squared distances are $2(1-p_i)$ or $2p_i$, so the same bound holds. Put $\rho = \sqrt{2(n+1)\tau}$. With $P = \operatorname{per} A$ and $Q = \operatorname{per}|A|^2$, Lemma 4.2 gives $P \ge (1+\rho)^{-2n} P_0$ and $Q \le (1+\rho)^{2n} Q_0$. The reference is $J_{k+1} \oplus C$, hence $Q_0/P_0^2 \le 1/(k+1)!$. Combine the inequalities. □

Corollary 4.4 (A polynomial occupation radius)**.** *Under the complementary hypothesis in Theorem 4.3, if $\tau \le [1152 n^2 (n+1)]^{-1}$, then $\operatorname{per}|A|^2 \le \frac{2}{3}(\operatorname{per} A)^2$.*

Proof. Now $\rho \le 1/(24n)$ and $(1+\rho)^{6n} \le \sum_{j \ge 0} (6n\rho)^j \le 4/3$. Use $(k+1)! \ge 2$. □

The complementary hypothesis is automatic when $n - k \le 4$ by the complex small-order result [15]. For $n = 6, k = 1$ it instead requires the order-five statement, supplied by the self-contained reconstruction in Lemma 5.3 below of the coefficient argument in [8]. The general transfer retains its stated hypothesis. Exact rational comparisons in (24) give the ratio ceiling $7/8$ at these larger order-seven radii:

k	1	2	3	4	5	6
τ	1/57344	1/8064	1/2016	1/896	1/504	1/224

Choose respectively $\rho = 1/64, 1/24, 1/12, 1/8, 1/6, 1/4$. At the uniform radius $\tau_0 = 1/57344$,

$$(65/64)^{-24} - \tfrac{1}{2}(65/64)^{12} > 1/12, \qquad (65/64)^{36}/2 < 7/8.$$

The reference permanent $P_0 = (k+1)! \operatorname{per} C \ge 2$, by the classical PSD permanent diagonal bound [10]. Thus $\mathcal{D} \ge P_0^2/12 \ge 1/3$ on this uniform domain. These estimates are independent of positive Gram/cofactor eigenvalues. The all-low vertex is excluded. The conclusions have their stated finite sufficient domains.

For an active unit direction at order six, let $s = \lambda_{\min}^+(B)$ and $\tau = \min\{s/12, 1/57344\}$. If $E < \mu\tau(1-\tau)$, Proposition 4.1 forces every occupation outside $[\tau, 1-\tau]$. The active frame bound $\sum_i p_i \ge s$ excludes the all-low alternative. Thus the occupied vertex theorem applies, with its complementary hypothesis, and gives $\mathcal{D} \ge 1/3$. Only this conversion from variance uses s, μ; the direct geometric domain does not. In this small domain the surrogate is positive too: $E \le \|L\| \sum_i p_i(1-p_i) \le 4320/57344 < 1/3$, since $\operatorname{Tr} L \le 6 \cdot 5! = 720$.

Theorem 4.5 (Neighborhoods of repeated orthogonal blocks)**.** *Let $B_0 = \bigoplus_j J_{m_j}$ have order n, with some $m_j \geq 2$. For any complex PSD correlation matrix A of order $n+1$ with leading block B, if*

$$\max_{i,j} |B_{ij} - (B_0)_{ij}| \leq \frac{1}{2304n^3(n+1)},$$

then $\operatorname{per}|A|^2 \leq \frac{2}{3}(\operatorname{per} A)^2$*. Gram ranks and the normalized border are unrestricted.*

Proof. At B_0 one has $a_0 = c_0 = \prod_j m_j! \geq 2$ and $U_0 = T_0$: $L_0 = R_0$ and $L_0 \circ B_0 = R_0$ by block cofactors. For every unit border vector, $p^{\mathsf{T}} L_0 p \leq z^* U_0 z = t$, so $Q_0 = a_0 + p^{\mathsf{T}} L_0 p \leq a_0 + t = P_0$. Thus $Q_0/P_0^2 \leq 1/a_0 \leq 1/2$. If the leading-block entrywise distance is at most d, then $\|B - B_0\| \leq nd$. Square-root factors differ rowwise by at most $\varepsilon = \sqrt{nd}$, by the PSD square-root bound below. Pad the factors to represent any admissible border by the same unit vector in both Gram systems. Their doubled-row distances are at most 2ε. Lemma 4.2 bounds the new ratio by

$$\frac{(1 + 2\sqrt{n+1}\varepsilon)^{6n}}{a_0}.$$

At the stated radius $2\sqrt{n+1}\varepsilon \leq 1/(24n)$; the same geometric-series estimate as Corollary 4.4 gives at most $4/(3a_0) \leq 2/3$. □

Corollary 4.6 (A neighborhood of a rank-five base)**.** *Let A be any complex PSD correlation matrix of order seven, with leading block B of order six. If*

$$\max_{i,j} |B_{ij} - (J_2 \oplus I_4)_{ij}| \leq 1/884736,$$

then $\operatorname{per}|A|^2 \leq \frac{7}{8}(\operatorname{per} A)^2$*. There is no rank assumption on A or B.*

Proof. At the reference base $B_0 = J_2 \oplus I_4$, one has $a = c = 2$ and $U = T$ by direct block cofactors. For every unit border vector in any ambient dimension, (22) gives

$$\tfrac{1}{2}(a+t)^2 - c - p^{\mathsf{T}} L p = t + \tfrac{1}{2}t^2 + E \geq 0.$$

Thus every reference border has ratio at most $1/2$. If its six unit base vectors are each perturbed by at most ε, keep the same unit border vector. The doubled-vector distances are at most 2ε. Lemma 4.2, with $N = 7, r = 6$, bounds the new ratio by

$$\tfrac{1}{2}(1 + 2\sqrt{7}\varepsilon)^{12}(1 + \sqrt{7}\varepsilon)^{24} \leq \tfrac{1}{2}(1 + 6\varepsilon)^{36} < 7/8 \quad (\varepsilon \leq 1/384).$$

For the stated entrywise radius, $\|B - B_0\| \leq 6/884736$ and the PSD square-root inequality gives $\|\sqrt{B} - \sqrt{B_0}\| \leq \sqrt{\|B - B_0\|} \leq 1/384$. Its rowwise bound compares unit Gram factors. Pad them, if necessary, by an unused coordinate to represent arbitrary normalized border slack. Isometric equivalence of Gram realizations lets the same border vector be used in both factors. Hence the comparison covers every admissible A, including singular bases and full-rank borders. □

The square-root bound used here follows directly from operator monotonicity: if $\|B - B_0\| \leq d$, then $\sqrt{B} \preceq \sqrt{B_0 + dI} \preceq \sqrt{B_0} + \sqrt{d}\, I$, and similarly in reverse. The neighborhood is a quantitative application.

4.2 Occupation-box certificates for the actual quartic

Return to $\mathcal{D}$ in (21). This section also holds for unnormalized B, with $0 \le p_i \le B_{ii}$. For real y set

$$D(y) = \delta I + 2aT - 2V^* \operatorname{diag}(Ly)V, \qquad F_y(z) = y^{\mathsf{T}} L y + z^* D(y) z + (z^* T z)^2.$$

Then $F_y(z) = \mathcal{D}(z) + (p-y)^{\mathsf{T}} L (p-y)$. Consequently

$$\min_{\|z\|=1} \mathcal{D}(z) = \min_{0 \le y_i \le B_{ii}} \left\{ y^{\mathsf{T}} L y + \max_{\xi \in [\lambda_{\min} T, \lambda_{\max} T]} \lambda_{\min}(D(y) + 2\xi T - \xi^2 I) \right\}. \tag{25}$$

The compact product minimum is exact because $y = p$ is allowed. At each fixed y, the inner scalar dual follows from convexity of the complex numerical range of $D(y) + iT$ and minimax for the convex function $d + t^2$; cf. [9]. The elementary scalar identity is $t^2 = \max_\xi(2\xi t - \xi^2)$. There is no interchange of the outer minimum and inner maximum in (25).

Proposition 4.7 (Finite box certificate)**.** *Cover $\prod_i [0, B_{ii}]$ by finitely many boxes of center h and halfwidth $r \ge 0$. A box is certified if some real shift ξ gives*

$$[\delta + h^{\mathsf{T}} L h - r^{\mathsf{T}} L r - \xi^2] I + 2(a + \xi) T - 2V^* \operatorname{diag}(Lh) V \succeq 0. \tag{26}$$

If every box is certified, then $\mathcal{D}(z) \ge 0$ for all unit complex z. For a strictly positive fixed-base instance, a sufficiently fine finite cover admits these certificates. With rational real and imaginary input coordinates, the cover and shifts can be rational.

Proof. All entries of L are nonnegative. If $|p-h| \le r$, then $(p-h)^{\mathsf{T}} L (p-h) \le r^{\mathsf{T}} L r$. Use this with $F_h - \mathcal{D} = (p-h)^{\mathsf{T}} L (p-h)$ and $(t-\xi)^2 \ge 0$ to obtain (26) as a lower bound. If $\min \mathcal{D} = m > 0$, every $F_h \ge m$. The exact fixed-center dual supplies a shift attaining that margin before the box error. A fine mesh with $r^{\mathsf{T}} L r < m/2$ leaves a strict PSD margin. Continuity permits rational nearby shifts. This is finite existence at one strictly positive base; no uniform complexity bound or termination at zero margin follows. □

Proposition 4.8 (The all-low box)**.** *For a correlation base, put $J = V^* \operatorname{diag}(L\mathbf{1})V$. Then T and J have kernel exactly $\ker V$. Every fixed base therefore admits $\eta > 0$ with $2aT - \eta J \succeq 0$. If also $\delta \ge 0$, this test certifies the entire box $0 \le p_i \le \eta$, with $\mathcal{D} \ge \delta + t^2$.*

Proof. The omitted-product argument for the original unit Gram vectors gives the same line-class range description for R, after phases make parallel rows equal. The image of V lies in that range, so R is positive on it and $\ker T = \ker V$. Under a row gauge $B = DB_0D^*$, $R(B) = DR(B_0)D^*$ and $V = DV_0$, preserving this argument. Every diagonal entry of L is at least one by the PSD permanent diagonal bound; hence J has the same kernel. Positivity on the active subspace gives an admissible η. Finally, entrywise nonnegativity of L gives $p^{\mathsf{T}} L p \le \eta p^{\mathsf{T}} L \mathbf{1} = \eta z^* J z$. Substitute into (21). □

There is a conditional finite-completeness extension through kernel zeros. Suppose $\delta \ge 0$ and $\mathcal{D}(z) > 0$ whenever $Vz \ne 0$. Use Proposition 4.8 on an all-low box. On the remaining nonnegative auxiliary centers with $\max_i h_i \ge \eta/2$, each F_h is strictly positive on the whole unit sphere: off the kernel this follows from $F_h \ge \mathcal{D} > 0$, and on it $F_h = \delta + h^{\mathsf{T}} L h > 0$ because $L_{ii} \ge 1$. Compactness over those centers gives a common positive F_h margin. Fine subdivision and the fixed-center dual then give finitely many spectral leaves. This assumes the active-endpoint sign.

An adaptive cover may also accept a box by Theorem 4.3 if every coordinate is wholly low or high at a valid radius and at least one is high. Its complementary hypothesis must still be supplied.

The all-low box cannot be accepted by that theorem. Standard rigorous partition methods [11] motivate this organization; the quadratic conjugacy and scalar numerical-range dual are classical. The reusable point here is a certificate of the actual quartic, without first proving the stronger cofactor surrogate. The following exact adaptive example illustrates subdivision. In particular, the rational full-support, phase-bearing rank-six base specified in Appendix I has a two-leaf proof of $\mathcal{D} \geq 13/20$. An exact Rayleigh witness rules out that margin for every single-full-box shift, while the exact shifted cofactor test also certifies 13/20 on the same base. This is a subdivision example, rather than evidence that retained variance is necessary there.

5 Uniform cofactor-weighted variance

The coefficient $\lambda^{+}_{\min}(L)$ in Proposition 4.1 can degenerate when Gram lines merge. The following trace-contraction estimate removes that coefficient. We first state it for trace-one operators; its variance must then be the operator variance, rather than a Bernoulli expression.

Theorem 5.1 (Trace-one operator variance)**.** *Let $P_1, \dots, P_n$ be positive semidefinite trace-one operators, $n \geq 2$, and let H_0 be the span of their ranges, of dimension r. For any unit z, set*

$$Q_z = zz^*, \quad \pi = \|P_{H_0} z\|^2, \quad h_{ij} = \operatorname{Tr}(P_i P_j), \quad L_{ij} = \operatorname{per} h(j \mid i), \quad \rho_i = (I - Q_z) P_i z,$$

and put

$$v = \sum_i \|\rho_i\|^2 = \sum_i [z^* P_i^2 z - (z^* P_i z)^2], \qquad E = \sum_{i,j} L_{ij} \langle \rho_i, \rho_j \rangle.$$

Writing $y = \sqrt{E/(n-1)!}$ and $k = (n-1) r^{(n-2)/2} \sqrt{\pi(r-\pi)}$, one has

$$v \leq r^{n-1} y^2 + k y. \tag{27}$$

In particular, for $v > 0$,

$$E \geq (n-1)! \left[\frac{2v}{k + \sqrt{k^2 + 4 r^{n-1} v}} \right]^2. \tag{28}$$

For $v = 0$ the conclusion is $E \geq 0$. The support dimension r need not be bounded by n.

Proof. Work in the Hilbert–Schmidt space $K = \operatorname{End}(H_0)$. Let q_i be the averaged symmetric tensor product of the $n-1$ operators other than P_i. The permanent norm identity gives $\langle q_i, q_j \rangle = L_{ij}/(n-1)!$; here h and L are real symmetric. Hence, for $Y = \sum_i \rho_i \otimes q_i$,

$$E = (n-1)! \|Y\|^2.$$

The trace functional ℓ on K has norm $\sqrt{r}$ and $\ell(P_i) = 1$. Full contraction and one uncontracted slot give

$$S_1 = \sum_i \rho_i = (I \otimes \ell^{\otimes(n-1)}) Y,$$
$$Z = \sum_i \rho_i \otimes \sum_{j \neq i} P_j = (n-1)(I \otimes I_K \otimes \ell^{\otimes(n-2)}) Y.$$

The factor $n-1$ compensates for averaged symmetrization. Therefore

$$\|S_1\| \leq r^{(n-1)/2} \|Y\|, \qquad \|Z\| \leq (n-1) r^{(n-2)/2} \|Y\|.$$

On $z^{\perp} \otimes K$ the complex-linear functional $F(s \otimes P) = z^* P s$ has squared norm

$$\|F\|^2 = \|P_{H_0} z\|^2 \operatorname{Tr}[P_{H_0}(I - zz^*)] = \pi(r - \pi).$$

For example, this follows by summing its squared coefficients over matrix units of K and an orthonormal basis of $z^{\perp}$ in $\operatorname{span}(H_0, z)$. Extra ambient directions contribute zero. Hermiticity, with $P_\Sigma = \sum_i P_i$, gives

$$F(S_1 \otimes P_\Sigma) = \|S_1\|^2, \qquad F\left(\sum_i \rho_i \otimes P_i\right) = v.$$

Since $S_1 \otimes P_\Sigma - Z = \sum_i \rho_i \otimes P_i$, the exact cancellation identity is $v = \|S_1\|^2 - F(Z)$. The norm bounds prove (27); its positive quadratic root gives (28). The proof applies to the stated mixed-state class. □

Corollary 5.2 (A first six-factor specialization)**.** *For the variance in* (22)*, let* $v = \sum_i p_i(1-p_i)$*. For every complex correlation base of order six and every unit border vector in any ambient dimension,*

$$E \geq \frac{v^2}{1600}.$$

If $\operatorname{rank} B \leq 5$*, then* $E \geq v^2/600$*. Both statements include every kernel mixture and all rank limits. The same constants hold for six trace-one operators with joint support dimension at most six or five, respectively, with their actual operator variance* v*.*

Proof. Trace-one positivity implies $0 \preceq P_i \preceq I$ and $\operatorname{Var}_z(P_i) \leq 1/4$, so $v \leq 3/2$. For $r \geq 2$, $\pi(r - \pi) \leq r - 1$. Put $k_r = 5r^2\sqrt{r-1}$. Theorem 5.1 gives

$$E \geq \frac{480 v^2}{[k_r + \sqrt{k_r^2 + 6r^5}]^2}.$$

At $r = 6$, $k_r^2 = 162000$ and $k_r^2 + 6r^5 = 208656$; their square roots are less than 403 and 457. Since $860^2 < 480 \cdot 1600$, the first bound follows. For $2 \leq r \leq 5$ the denominator increases with r; at $r = 5$ it is less than $250 + 286 = 536$, and $536^2 < 480 \cdot 600$. For $r = 1$ every operator is the same rank-one projector, and $E = n!v = 720v$, which covers both constants. □

The exponent two cannot be decreased in a uniform positive power-law bound. Indeed, for $s > 0$ take the first two unit Gram vectors to be $(e_0 \pm s e_1)/\sqrt{1+s^2}$, the other four mutually orthogonal and orthogonal to e_0, e_1, and take $z = e_1$. Directly,

$$v = \frac{2s^2}{(1+s^2)^2}, \qquad E = \frac{8s^4}{(1+s^2)^4} = 2v^2.$$

The bases have rank six for $s > 0$ and rank five at $s = 0$. This proves optimality of the exponent, not of our numerical constants.

5.1 A quantitative base-defect reserve

We include the small-order input explicitly. The coefficient induction below is the method of Li–Zhao–Luo [8]; its inclusion identifies the dependency of the quantitative extraction that follows.

Lemma 5.3 (Small-order coefficient induction)**.** *For every complex positive semidefinite matrix* A *of order at most six,* $\operatorname{per}|A|^2 \leq (\operatorname{per} A)^2$*.*

Proof. For PSD matrices M_j, every coefficient of $\operatorname{per}(\sum_j t_j M_j)$ is nonnegative: realize their Gram vectors in orthogonal labelled spaces, expand the symmetric product, and group by label occupations. The coefficients are factorial multiples of squared norms of the orthogonal occupation components.

A zero diagonal of $A \succeq 0$ gives a zero row and trivial permanents. Otherwise diagonal congruence reduces to diagonal one; both sides scale by the same factor. Write $A = \left(\begin{smallmatrix} B & x \\ x^* & 1 \end{smallmatrix}\right)$, $H = B - xx^* \succeq 0$, $X = xx^*$, and, if B has order m, set

$$F(s) = \operatorname{per}(H + sX) = \sum_{k=0}^{m} \alpha_k s^k,$$
$$\Gamma(s,t) = \operatorname{per}(H \circ \bar{H} + s(H \circ \bar{X} + \bar{H} \circ X) + tX \circ \bar{X}) = \sum_{j,k} \gamma_{jk} s^j t^k.$$

Every coefficient is nonnegative by the preceding argument and the Schur product theorem. Border expansion and rank-one differentiation give

$$\operatorname{per} A = \sum_k (k+1)\alpha_k, \qquad \operatorname{per}|A|^2 = \sum_{j,k} (k+1)\gamma_{jk}, \qquad \Gamma(s, s^2) = \operatorname{per}|H + sX|^2.$$

If $m \le 5$, then $(\sqrt{2})^k \le k+1$ for $0 \le k \le m$, whereas $k+1 \le 2^k \le 2^{j/2+k}$ for all $j, k \ge 0$. Assuming the assertion in order m, these comparisons yield

$$\operatorname{per}|A|^2 \le \Gamma(\sqrt{2}, 2) \le F(\sqrt{2})^2 \le (\operatorname{per} A)^2.$$

Start from scalar equality and induct to order six. At $k = 6$ the last ordinary coefficient comparison fails ($8 > 7$); this argument gives no unrestricted order-seven theorem. □

Lemma 5.4 (Sharp nonzero occupation mass)**.** *Let $f = \operatorname{Sym}(v_1 \otimes \cdots \otimes v_m)$ for unit vectors v_i, and let e be unit. If $q = \max_i |v_i^* e|^2$ and P_0 projects onto $\operatorname{Sym}^m(e^\perp)$, then*

$$\|(I - P_0)f\|^2 \ge q^m \|f\|^2.$$

The constant q^m is sharp among such products.

Proof. Choose a factor v attaining q, let $M = v \operatorname{Sym}^{m-1} H$, and let $N = \operatorname{Sym}^m(e^\perp)$. On $e^\perp$, projection onto $v^\perp$ has least singular value at least $\sqrt{q}$. Its symmetric tensor power is projection onto $M^\perp$ restricted to N, with least singular value at least $q^{m/2}$. Thus $P_N P_M P_N \preceq (1 - q^m) P_N$. Since $f \in M$, $\|P_N f\|^2 \le (1 - q^m)\|f\|^2$.

For sharpness take unit $u \perp e$, $0 < q < 1$, and $v = \sqrt{1-q}\,u + \sqrt{q}e$. In the binary polynomial model, $P_M u^m = u^m - [qu - \sqrt{q(1-q)}e]^m$ factors as

$$\prod_{\omega^m = 1} [(1 - \omega q)u + \omega\sqrt{q(1-q)}e].$$

The squared e-overlap of each normalized factor is $q(1-q)/(1 + q - 2q \operatorname{Re}\omega) \le q$, with equality at $\omega = 1$. Moreover $\|P_M u^m\|^2 = 1 - q^m$ and its zero-mode projection is $(1 - q^m)u^m$, giving nonzero-mode fraction exactly q^m. The endpoints follow directly. □

Theorem 5.5 (Coherence reserve)**.** *For a complex correlation matrix B of order six, put $\eta = \max_{i \ne j} |B_{ij}|^2$, $r_0 = \eta^5$, and*

$$d_0 = \frac{(6 - 4\sqrt{2})r_0}{1 + 5r_0}, \qquad \bar{d}_0 = \frac{r_0}{3(1 + 5r_0)}.$$

Then

$$\frac{\delta}{a^2} \geq 2d_0 - d_0^2 \geq \overline{D}(\eta) := 2\bar{d}_0 - \bar{d}_0^2.$$

In particular $\delta = 0$ if and only if $B = I_6$.

Proof. Reorder a pair attaining η so that one is the last vector, and split the other five factors into its mode and perpendicular part. Use $F(s) = \operatorname{per}(H + sxx^*) = \sum_{k=0}^{5} \alpha_k s^k$ as in Lemma 5.3. Put $\rho = \sum_{k\geq 1} \alpha_k / F(1)$. The coefficients are the ordinary occupation masses; Lemma 5.4 gives $\rho \geq \eta^5$. With $b = F(\sqrt{2})$, the induction proof gives $c \leq b^2$, while

$$a = \sum_k (k+1)\alpha_k, \quad a - b \geq (6 - 4\sqrt{2})\rho F(1), \quad a \leq (1 + 5\rho)F(1).$$

Here $6-4\sqrt{2}$ is the minimum of $k+1-2^{k/2}$ for $1 \leq k \leq 5$. Since $\rho/(1+5\rho)$ increases, $(a-b)/a \geq d_0$ and $c/a^2 \leq (1-d_0)^2$. Finally $6-4\sqrt{2} > 1/3$ follows from $289 > 288$. At $\eta = 0$, $B = I_6$ directly. □

Corollary 5.6 (Uniform low occupations and kernel weights)**.** *Let $p_{\max} = \max_i |(Vz)_i|^2$. Then*

$$\frac{\mathcal{D}(z)}{a^2} \geq \overline{D}(\eta) - 36[1 - \overline{D}(\eta)]p_{\max}^2. \tag{29}$$

If $\eta \geq 1/25$ (in particular if $\operatorname{rank} B \leq 5$), then

$$\delta \geq \frac{a^2}{15000000}, \qquad p_{\max} \leq \frac{1}{30000} \implies \mathcal{D}(z) \geq \frac{a^2}{37500000}.$$

For every active unit h and kernel unit κ, all vectors $z_p = \sqrt{p}\, h + e^{i\theta}\sqrt{1-p}\kappa$ with $0 \leq p \leq 1/10000$ satisfy $\mathcal{D}(z_p) \geq a^2/150000000$ under the same coherence condition.

Proof. The unit-vector creator lower bound gives $L_{ii} \leq c$; positivity of L gives $0 \leq L_{ij} \leq c$. Thus $p^{\mathsf{T}} L p \leq 36cp_{\max}^2$; drop the nonnegative $2at + t^2$ in (21) to obtain (29). If $\operatorname{rank} B \leq 5$, $\operatorname{Tr} B^2 \geq 36/5$ and $\operatorname{Tr} B^2 \leq 6 + 30\eta$ give $\eta \geq 1/25$. At that endpoint $\bar{d}_0 = 1/29296890$ and $2\bar{d}_0 - \bar{d}_0^2 > 1/15000000$. The low-occupation margin follows from $1/15000000 - 36/30000^2 = 1/37500000$. Finally $p(h)^{\mathsf{T}} L p(h) \leq h^* U h \leq \operatorname{Tr} U = 6c$, so homogeneity gives $\mathcal{D}(z_p)/a^2 \geq 1/15000000 - 6p^2 \geq 1/150000000$. This includes every relative phase and both endpoints of the stated band. □

5.2 Uniform matrix and occupation-box certificates

Proposition 5.7 (Low-coherence matrix compensation)**.** *For a complex correlation matrix B of order six with $|B_{ij}| \leq 1/15$ for $i \neq j$,*

$$2aR - L \circ B \succeq \tfrac{1}{12} I_6, \qquad 2aT - U \succeq \tfrac{1}{12} V^* V.$$

Consequently every admissible border satisfies $\mathcal{D}(z) \geq \delta + \frac{1}{12}\sum_i p_i + t^2 + E$.

Proof. Put $\rho = 1/15$ and

$$f(x) = x + 4x^2 + 18x^3 + 44x^4 + 53x^5, \quad d(x) = 1 + 10x^2 + 20x^3 + 45x^4 + 44x^5.$$

Counting permutations by their number of off-diagonal factors gives $|R_{ij}| \leq f(\rho)$ for $i \neq j$, $L_{ii} \leq d(\rho^2)$, and $L_{ij} \leq f(\rho^2)$ for $i \neq j$. The corresponding counts are $(1, 4, 18, 44, 53)$ for an off-diagonal

five-row cofactor and $(1, 0, 10, 20, 45, 44)$ for a diagonal one. The creator lower bound gives $R_{ii} \geq 1$ and $a \geq 1$. Hermitian row-sum bounds therefore yield

$$R \succeq r_* I, \quad L \circ B \preceq s_* I, \quad r_* = 1 - 5f(\rho), \quad s_* = d(\rho^2) + 5\rho f(\rho^2).$$

Direct rational evaluation gives

$$r_* = \frac{82987}{151875} > 0, \qquad 2r_* - s_* = \frac{31528452158}{345990234375} > \frac{1}{12}.$$

Thus $2aR - L \circ B \succeq (2r_* - s_*)I$; congruence by V and (22) prove the assertions. Lemma 5.3 supplies $\delta \geq 0$. No inverse or singular-border assumption is used. □

This base domain has $B \succeq (2/3)I$ by Gershgorin. It contains no rank-five base: the preceding trace bound forces an off-diagonal modulus at least $1/5$ at rank at most five. Outside the certified domain, Theorem 5.5 gives a uniform positive reserve with $\bar{d}_0 = 1/[3(225^5 + 5)]$.

Proposition 5.8 (Variance-corrected boxes)**.** *For a six-row correlation base suppose $\ell_i \leq p_i \leq h_i$, with $0 \leq \ell_i \leq h_i \leq 1$. Put*

$$v_{\min} = \sum_i \min\{\ell_i(1 - \ell_i), h_i(1 - h_i)\}, \qquad C = \begin{cases} 600, & \operatorname{rank} B \leq 5, \\ 1600, & \textit{otherwise.} \end{cases}$$

For any real τ, the matrix test

$$\left(\delta - \tau^2 - m + \frac{v_{\min}^2}{C}\right) I + 2(a + \tau)T - U \succeq 0$$

certifies $\mathcal{D}(z) \geq m$ on that feasible occupation box.

Proof. Concavity of $s(1 - s)$ gives $v \geq v_{\min}$. The scalar square $(t - \tau)^2 \geq 0$ lower-bounds $Q(z)$ by the quadratic form of $(\delta - \tau^2)I + 2(a + \tau)T - U$. Add Corollary 5.2 and use $\mathcal{D} = Q + E$. □

The test improves a shifted cofactor margin without a positive cofactor eigenvalue assumption. The matrix test is a sufficient criterion.

5.3 Orthogonal trace sectors and frame variance

The trace contractions in Theorem 5.1 come from orthogonal tensor sectors. Accounting for their joint norm improves the constants and retains the variance of the sum of the operators.

Theorem 5.9 (Joint trace-sector bound)**.** *Use the trace-one operator setting of Theorem 5.1, and put $m = n - 1$, $\mathsf{F} = \sum_i P_i$,*

$$s = z^* \mathsf{F} z, \qquad \mathcal{A} = z^* \mathsf{F}^2 z - s^2, \qquad b = \frac{m}{r}(1 - \pi)s, \qquad f_0^2 = \frac{\pi(r - 1)(r + 1 - \pi)}{r}.$$

If $f_0 > 0$, then

$$\frac{E}{m!} \geq \frac{\mathcal{A}}{r^m} + \frac{(\mathcal{A} - b - v)^2}{m r^{m-1} f_0^2}. \tag{30}$$

If $f_0 = 0$, the second quotient is omitted and $\mathcal{A} - b - v = 0$. In particular, for $r \geq 2$,

$$\frac{E}{m!} \geq \frac{\mathcal{A}}{r^m} + \frac{(v - \mathcal{A})_+^2}{m r^{m-1}(r - 1)}. \tag{31}$$

These statements retain arbitrary kernel components of z and do not require rank-one P_i.

Proof. In the Hilbert–Schmidt space write $e = I_{H_0}/\sqrt{r}$ and $P_i = e/\sqrt{r} + D_i$, with D_i traceless. Let Y and S_1 be the omitted-product tensor and its full contraction from Theorem 5.1, and set

$$Z_0 = \sum_i \rho_i \otimes \sum_{j \neq i} D_j.$$

Projecting the last m slots of Y onto zero or one traceless slot gives orthogonal components of squared norms

$$\|Y_0\|^2 = \frac{\|S_1\|^2}{r^m}, \qquad \|Y_1\|^2 = \frac{\|Z_0\|^2}{m r^{m-1}}.$$

The factor $1/m$ follows because the m possible traceless slots are orthogonal under averaged symmetrization. Also $\|S_1\|^2 = \mathcal{A}$. Restricting the functional $F(u \otimes P) = z^* P u$ to traceless matrices removes squared norm $\pi(1-\pi)/r$ from its full squared norm $\pi(r-\pi)$. Its remaining squared norm is exactly f_0^2.

The identity-slot contraction must retain the support projection: $F(S_1 \otimes I_{H_0}) = (1-\pi)s$. Thus the exact cancellation is

$$v = \mathcal{A} - b - F(Z_0).$$

Cauchy–Schwarz and $E = m!\|Y\|^2$ prove (30), including its zero-norm convention. For $r \geq 2$, $f_0^2 \leq r-1$ and $b \geq 0$ give $v \leq \mathcal{A} + \sqrt{r-1}\|Z_0\|$, proving (31). □

Corollary 5.10 (Six-factor frame reserve)**.** *For six trace-one operators of joint support dimension at most six,*

$$E \geq \frac{v^2}{270} + \frac{7}{1620}\mathcal{A}. \tag{32}$$

At support dimension at most five,

$$E \geq \frac{6}{625}(v^2 + \mathcal{A}). \tag{33}$$

For unit Gram vectors, $v = \sum_i p_i(1-p_i)$ and $\mathsf{F} = V^*V$. These inequalities improve the round constants in Corollary 5.2.

Proof. Embed the support, if necessary, into a containing space of dimension $R = 6$ or 5. The preceding proof needs only containment, not that the ranges span the containing space. Since $v \leq 3/2$, (31) gives

$$E \geq \alpha\mathcal{A} + \beta(v - \mathcal{A})_+^2, \qquad \alpha = \frac{120}{R^5}, \quad \beta = \frac{24}{R^4(R-1)}.$$

For $\mathcal{A} \leq v$, expansion yields $E \geq \beta v^2 + (\alpha - 3\beta)\mathcal{A}$ because $2v \leq 3$. For $\mathcal{A} \geq v$, the same conclusion follows from $3\mathcal{A} \geq v^2$. Substitution of $R = 6, 5$ gives the displayed constants. The argument includes rank one; directly in that case $E = 720v$. □

Proposition 5.11 (Frame-corrected occupation boxes)**.** *For a six-row correlation base let $0 \leq \ell_i \leq p_i \leq h_i \leq 1$ and $v_{\min} = \sum_i \min\{\ell_i(1-\ell_i), h_i(1-h_i)\}$. Suppose real $\lambda < \mu$ satisfy*

$$(\mathsf{F} - \lambda I)(\mathsf{F} - \mu I) \succeq 0, \qquad \lambda \leq s_- := \sum_i \ell_i \leq s_+ := \sum_i h_i \leq \mu.$$

Set

$$A_{\min} = \min\{(s_- - \lambda)(\mu - s_-), (s_+ - \lambda)(\mu - s_+)\}.$$

Take $(\beta, \gamma) = (1/270, 7/1620)$, or $(6/625, 6/625)$ when $\operatorname{rank} B \leq 5$. For any real τ,

$$[\delta - \tau^2 - M + \beta v_{\min}^2 + \gamma A_{\min}]I + 2(a+\tau)T - U \succeq 0 \tag{34}$$

certifies $\mathcal{D}(z) \geq M$ throughout the feasible box.

Proof. The spectral condition implies $\mathcal{A} \geq (\lambda+\mu)s - \lambda\mu - s^2 = (s-\lambda)(\mu-s)$. Concavity gives $\mathcal{A} \geq A_{\min}$ on the box, while $v \geq v_{\min}$. Add Corollary 5.10 to the shifted lower bound for Q from $(t-\tau)^2 \geq 0$. □

If no spectral gap and box interval give a positive frame floor, one may use $A_{\min} = 0$. The matrix condition includes kernel eigenvalues; no inverse is used. For rational input and endpoints all checks in (34) are rational. This remains a sufficient local certificate, not a uniform cover of all bases.

6 Rank-one secants couple the base reserve

For a six-row unit Gram factor and unit z, set $x = Vz$, $X = xx^*$, $H = B - X \succeq 0$, and

$$F(\sigma) = \operatorname{per}(H + \sigma X), \qquad C(\sigma) = \operatorname{per}|H + \sigma X|^2.$$

The coefficients of both polynomials are nonnegative by the PSD occupation expansion in Lemma 5.3. The derivative identities are $F(1) = a$, $F'(1) = t = z^*Tz$, $C(1) = c$ and $C'(1) = 2u$, where $u = z^*Uz$.

Proposition 6.1 (Same-data secants)**.** *For every $\sigma > 1$ put $\beta_\sigma = (\sigma^6 - 1)/6$. Then*

$$u \leq \frac{(a + \beta_\sigma t)^2 - c}{2(\sigma - 1)}. \tag{35}$$

Consequently, writing $k = t/a$ and $d = \delta/a^2$,

$$k \geq 6/7 \implies Q(z) \geq a^2/1000, \tag{36}$$

$$k \leq d/24 \implies Q(z) \geq \delta/20. \tag{37}$$

Each conclusion gives the same lower bound for $\mathcal{D}$ plus E.

Proof. For $1 \leq j \leq 6$, the average $(1 + \sigma + \cdots + \sigma^{j-1})/j$ increases with j. Nonnegative coefficients of F therefore give $F(\sigma) \leq a + \beta_\sigma t$. Convexity gives $C(\sigma) \geq c + 2(\sigma-1)u$, while Lemma 5.3 gives $C(\sigma) \leq F(\sigma)^2$. This proves (35).

At $\sigma = 5/4$, put $A_0 = 253/2048$, $B_0 = 18785783/33554432$. Expansion yields $Q/a^2 \geq -d + A_0 k + B_0 k^2$. The independent trace estimate $U \preceq 6cI$ gives $Q/a^2 \geq 7d + 2k + k^2 - 6$. Seven copies of the first lower bound and one of the second cancel d:

$$\frac{Q}{a^2} \geq \frac{(2 + 7A_0)k + (1 + 7B_0)k^2 - 6}{8}.$$

This increasing polynomial has value $28555209/3288334336 > 1/1000$ at $k = 6/7$. At $\sigma = 2$, (35) instead gives $Q/a^2 \geq d/2 - 17k/2 - 433k^2/8$. Since $0 \leq d \leq 1$, $k \leq d/24$ makes this at least $(239/4608)d \geq d/20$. □

The low-occupation condition retains the actual base reserve. A proved floor $d \geq D$ permits the uniform sufficient condition $k \leq D/24$. For example, Corollary 5.6 gives $k \leq 1/360000000$ at rank at most five. These small constants are conservative; the high-occupation domain has no coherence or spectral-gap condition. Neither statement covers all intermediate occupations.

Corollary 6.2 (Retaining the top ordinary occupation)**.** *Put $\theta = 720 \prod_i p_i/a$. If $k \geq 3/4$ and $\theta \leq 1/50$, then $Q(z) \geq a^2/30$ and $\mathcal{D}(z) \geq a^2/30 + E$.*

Proof. If $F(\sigma)=\sum_{j=0}^{6}\alpha_j\sigma^j$, then $\alpha_6=\operatorname{per}X=720\prod_i p_i$ exactly. Bounding degrees one through five by their endpoint chord gives

$$F(\sigma)\le a+b_\sigma t+e_\sigma\alpha_6,\quad b_\sigma=\frac{\sigma^5-1}{5},\quad e_\sigma=\sigma^6-1-6b_\sigma>0.$$

At $\sigma=5/4$, $b_*=2101/5120$ and $e_*=7221/20480$. Write $A_*=2-4b_*$, $B_*=1-2b_*^2$. The same secant and trace combination gives

$$\frac{Q}{a^2}\ge J(k,\theta):=\frac{(2+7A_*)k+(1+7B_*)k^2-6-7(4e_*\theta+4b_*e_*k\theta+2e_*^2\theta^2)}{8}.$$

For $0\le\theta\le 1/50$ this increases with $k\ge 0$: its derivative numerator is at least $2+7A_*-28b_*e_*/50=5805352753/1310720000>0$. It decreases with $\theta\ge 0$. Thus its minimum in the stated region is $J(3/4,1/50)=156051912513/4194304000000>1/30$. □

All three domains permit arbitrary complex phases and exact rank-five boundaries. Along a kernel mixture of active weight p, k scales by p and θ by p^6; the conditions are required on the actual mixture, rather than automatically on its whole ray.

6.1 Retaining inherited small-order defect

The coefficient proof through order six loses a useful positive term if its smaller-order inequality is used only as a sign. For a correlation matrix A of order $m+1\le 6$, use the notation H,X,F,Γ from Lemma 5.3 and put $M=H+\sqrt{2}X$, $P=\operatorname{per}A$, $b=\operatorname{per}M$. Write $\Delta_j(D)=(\operatorname{per}D)^2-\operatorname{per}|D|^2$ for a matrix of order j. Then exactly

$$\Delta_{m+1}(A)=(P^2-b^2)+\Delta_m(M)+[\operatorname{per}|M|^2-\operatorname{per}|A|^2]. \tag{38}$$

All terms are nonnegative. If α_k and γ_{jk} are the ordinary coefficients of F and Γ, the two outer gaps are controlled without changing their common Gram origin by

$$P-b=\sum_k(k+1-2^{k/2})\alpha_k\ge 0,\quad \operatorname{per}|M|^2-\operatorname{per}|A|^2=\sum_{j,k}(2^{j/2+k}-k-1)\gamma_{jk}\ge 0.$$

Proposition 6.3 (Coherent inherited reserve)**.** *Let A be a correlation matrix of order six, choose any last-row split, and form the five-row matrix $M=H+\sqrt{2}xx^*$. Define its normalized coherence by*

$$\xi=\max_{i\ne j}\frac{|M_{ij}|^2}{M_{ii}M_{jj}},\quad d_5=\frac{(2-\sqrt{2})\xi^4}{1+4\xi^4},\quad D_5=2d_5-d_5^2.$$

Then

$$\frac{\Delta_6(A)}{(\operatorname{per}A)^2}\ge 1-\left(\frac{\operatorname{per}M}{\operatorname{per}A}\right)^2(1-D_5)\ge D_5. \tag{39}$$

In particular, $\xi\ge 3/4$ implies

$$\Delta_6(A)/(\operatorname{per}A)^2\ge 181359/1345600>1/9.$$

Proof. The diagonal of M is $1+(\sqrt{2}-1)|x_i|^2>0$. Apply diagonal normalization to M. The proof of Theorem 5.5, now with four omitted factors, uses the mass $\rho\ge\xi^4$ and the minimum gap $\min_{1\le j\le 4}(j+1-2^{j/2})=2-\sqrt{2}$. It gives $\Delta_5(M)/(\operatorname{per}M)^2\ge D_5$; the ratio is invariant under diagonal congruence. Insert this in (38) and use $b\le P$. Since $2-\sqrt{2}>1/2$, one may replace d_5 by $\bar{d}_5=\xi^4/[2(1+4\xi^4)]$. At $\xi=3/4$, $\bar{d}_5=81/1160$, giving the stated rational value. □

This floor is larger than the maximum $1/9$ of the single-coherence six-row floor in Theorem 5.5. It also explains a limitation of using occupation mass alone. For the sharp five-factor binary products in Lemma 5.4, with maximum mode overlap $q \le 1/24$, append that mode as the sixth factor. In M, the mode amplitudes are multiplied by $2^{1/4}$; after normalization their squared overlaps are at most $\sqrt{2}q$. All perpendicular components lie on one line, so every pair has inner-product modulus at least $1 - 2\sqrt{2}q > 7/8$. Thus $\xi > 3/4$, even though their nonzero-mode mass is only $q^5 \to 0$. The inherited defect therefore does not degenerate with that mass. This example explains the retained term within the known low-rank setting.

The inherited floor may be combined with the secant and frame bounds on the same actual data. Care is needed: the bound $Q/a^2 \ge -d + A_0 k + B_0 k^2$ decreases with d, so a lower floor for d cannot simply be substituted into it. With only an interval for d, minimize the maximum of this affine bound and $7d + 2k + k^2 - 6$ over that interval, and then add any proved frame/variance correction.

7 Forward reserves retain their common permanent

Continue with a six-row correlation base $B = VV^*$ and an arbitrary unit z, including a component in $\ker V$. Put $x = Vz$, $X = xx^*$, $H = B - X \succeq 0$, $u = z^*Uz$, $\ell = p^{\mathsf{T}}Lp$, and, for $s > 1$,

$$M_s = H + sX, \quad F(s) = \operatorname{per} M_s, \quad C(s) = \operatorname{per} |M_s|^2, \quad \Delta_s = F(s)^2 - C(s).$$

Thus $E = u - \ell$ and $\mathcal{D} = \delta + 2at + t^2 - \ell$. Lemma 5.3 gives $\Delta_s \ge 0$. The ordinary and paired PSD occupation expansions give $F(1) = a$, $F'(1) = t$, $C(1) = c$ and $C'(1) = 2u$. All statements in this section concern these same actual matrices.

Normalize the strictly positive diagonal of M_s to obtain a correlation matrix K_s. For any selected index j, let y be its column with entry j omitted, and form

$$N_j = K_s(j \mid j) + (\sqrt{2} - 1)yy^*, \qquad \xi = \max_{i \ne k} \frac{|(N_j)_{ik}|^2}{(N_j)_{ii}(N_j)_{kk}}.$$

Here N_j is precisely the five-row inherited child in Proposition 6.3. Its diagonal is positive even when the base is singular. Define the rational lower floor

$$d(\xi) = \frac{\xi^4}{2(1 + 4\xi^4)}, \qquad D(\xi) = 2d(\xi) - d(\xi)^2.$$

Diagonal-congruence invariance and that proposition imply $\Delta_s \ge D(\xi)F(s)^2$. No inverse or positive lower bound for a Gram eigenvalue is required.

Proposition 7.1 (Multiplicatively retained secant)**.** *Suppose $0 \le D \le 1$ is any proved floor for $\Delta_s/F(s)^2$. With $\beta_s = (s^6 - 1)/6$, one has*

$$u \le \frac{(1 - D)F(s)^2 - c}{2(s - 1)} \le \frac{(1 - D)(a + \beta_s t)^2 - c}{2(s - 1)}. \tag{40}$$

At $s = 5/4$, set

$$\beta = \frac{3843}{8192}, \quad A_0 = \frac{253}{2048}, \quad B_0 = \frac{18785783}{33554432}, \qquad k = \frac{t}{a},$$
$$f(k) = \frac{(2 + 7A_0)k + (1 + 7B_0)k^2 - 6}{8}.$$

Then the same-data lower bound is

$$\frac{Q(z)}{a^2} \ge L_D(k) := f(k) + \frac{7}{4}D(1 + \beta k)^2, \qquad \frac{\mathcal{D}(z)}{a^2} \ge L_D(k) + \frac{E}{a^2}. \tag{41}$$

Proof. Convexity gives $c + 2(s-1)u \le C(s) = (1 - \Delta_s/F(s)^2)F(s)^2$. Insert the defect floor before using the ordinary chord $F(s) \le a + \beta_s t$ from Proposition 6.1. The substitution is valid because $1 - D \ge 0$. At $s = 5/4$, expansion yields

$$Q/a^2 \ge -\delta/a^2 + A_0 k + B_0 k^2 + 2D(1+\beta k)^2.$$

Combine seven copies of this estimate with one copy of $Q/a^2 \ge 7\delta/a^2 + 2k + k^2 - 6$, as in that proposition. Their average proves (41). □

That separate substitution would be unjustified: the positive permanent term and its defect must be kept together before the upper chord is used.

Corollary 7.2 (Inherited and joint sufficient domains)**.** *Use the actual child of $M_{5/4}$ just defined and suppose $\xi \ge 3/4$. Then $k \ge 11/20$ implies $Q(z) \ge a^2/200$. If instead*

$$k \ge 109/200, \qquad E \ge E_{\text{floor}}, \qquad E_{\text{floor}}/a^2 \ge 1/500, \tag{42}$$

then $\mathcal{D}(z) \ge a^2/1000$. Valid choices of E_{floor} include 5! *times the right-hand side of* (30)*, with its stated endpoint convention, or the zero-sector floor $E_{\text{floor}} = 5\mathcal{A}/324$.*

Proof. The inherited floor permits $D = D_0 = 181359/1345600$. The polynomial $L_D(k)$ increases in both $k \ge 0$ and $D \ge 0$. Exact substitution gives

$$L_{D_0}(11/20) = \frac{900047936124297}{144482699837440000} > \frac{1}{200},$$

$$L_{D_0}(109/200) = -\frac{4596464533312903}{14448269983744000000}.$$

Adding $1/500$ to the latter gives $24300075434175097/14448269983744000000 > 1/1000$. The zero-sector term of Theorem 5.9, in a containing support space of dimension six, is $120\mathcal{A}/6^5 = 5\mathcal{A}/324$. The containing space may include unused dimensions and excludes no kernel component of z. □

In the second domain a negative *uniform lower bound* for Q is bounded using the actual variance. Neither actual negativity of Q nor failure of every earlier optimized certificate follows.

The joint domain is nonempty on actual rank-five boundary data. Let $B = J_2 \oplus I_4$, and let $X = \rho J_2 \oplus 0_4$, where $\rho = 109/400$. Realize the two repeated rows by e_1 and the four singletons by $e_2, \dots, e_5$, in ambient dimension six, and choose $z = \sqrt{\rho}\, e_1 + \sqrt{1-\rho}\, e_6$. Then z is unit, $B - X \succeq 0$, $a = 2$, $k = 2\rho = 109/200$, and $\mathcal{A} = 4\rho(1-\rho)$. Deleting a singleton from the normalized $M_{5/4}$ leaves its repeated pair unchanged and gives $\xi = 1$. Moreover

$$\frac{5\mathcal{A}}{324a^2} = \frac{10573}{3456000} > \frac{1}{500}.$$

This example establishes nonemptiness, rather than necessity of the new certificate. The universal results already allow complex phases, rank-five bases, active unit directions and arbitrary kernel mixtures; their numerical hypotheses must be checked on the actual chosen direction.

7.1 Paired curvature and the remaining occupation budget

The same paired polynomial has a nonnegative remainder that can be kept instead of using convexity alone.

Proposition 7.3 (Finite-forward curvature identity)**.** *For $h \geq 0$ define*

$$R_h = C(1+h) - c - 2hu - h^2\ell.$$

Then $R_h \geq 0$. In particular, with $h = \sqrt{2}-1$ and $P = a + t$,

$$\mathcal{D}(z) = P^2 - F(\sqrt{2})^2 + \Delta_{\sqrt{2}} + 2(\sqrt{2}-1)E + R_{\sqrt{2}-1}. \tag{43}$$

Consequently

$$\Delta_{\sqrt{2}} + 2(\sqrt{2}-1)E \geq F(\sqrt{2})^2 - P^2 \tag{44}$$

is sufficient for $\mathcal{D}(z) \geq 0$.

Proof. Put $A_0 = |B|^2$, $Y = B \circ \overline{X} + \overline{B} \circ X$, and $W = |X|^2 = pp^{\mathsf{T}}$. These three matrices are PSD by Schur products. Labelled orthogonal Gram sectors, as in Lemma 5.3, give nonnegative coefficients in $\operatorname{per}(A_0 + vY + wW)$. Therefore $C(1+h) = \operatorname{per}(A_0 + hY + h^2W)$ has nonnegative coefficients in h. Its constant and linear terms are c and $2hu$. Its one-W contribution to degree two is $h^2 D\operatorname{per}(A_0)[W] = h^2\ell$; all other contributions are nonnegative. This proves $R_h \geq 0$ without assigning a sign to individual entries or permutation terms. Since $u = \ell + E$,

$$C(1+h) = c + h(2+h)\ell + 2hE + R_h.$$

At $h = \sqrt{2}-1$ the coefficient $h(2+h)$ equals one. Insert $C(\sqrt{2}) = F(\sqrt{2})^2 - \Delta_{\sqrt{2}}$ into $\mathcal{D} = P^2 - c - \ell$ to obtain (43). Dropping its nonnegative remainder proves the sufficient condition. □

The identity is exact. It isolates an actual-data comparison.

Corollary 7.4 (Top ordinary occupation loss)**.** *Write $F(s) = \sum_{j=0}^{6} \alpha_j s^j$, with $\alpha_j \geq 0$, and put $\alpha_6 = 720 \prod_i p_i$. Then*

$$(F(\sqrt{2})^2 - P^2)_+ \leq \alpha_6(2P + \alpha_6).$$

In particular, either of the following conditions certifies $\mathcal{D}(z) \geq 0$:

$$\Delta_{\sqrt{2}} + 2(\sqrt{2}-1)E \geq \alpha_6(2P + \alpha_6), \tag{45}$$

$$t \geq 21\alpha_6. \tag{46}$$

Any possible negative $\mathcal{D}$ not already excluded by the second condition must have $t > 0$ and $6\alpha_6/t > 2/7$.

Proof. Border expansion gives $P = \sum_j (j+1)\alpha_j$. For $j \leq 5$, $g_j = j + 1 - 2^{j/2} \geq 0$, while $g_6 = -1$. Thus $P - F(\sqrt{2}) \geq -\alpha_6$. If $F(\sqrt{2}) > P$, set $d = F(\sqrt{2}) - P \leq \alpha_6$; the squared loss is $d(2P + d) \leq \alpha_6(2P + \alpha_6)$. Otherwise it is nonpositive. This proves the first condition using (44). For $1 \leq j \leq 4$, $g_j \geq 1/2 \geq j/15$; also $g_5 = 6 - 4\sqrt{2} > 1/3 = 5/15$, as follows from $289 > 288$. Since $t = \sum_j j\alpha_j$, it follows that

$$P - F(\sqrt{2}) \geq \frac{t - 21\alpha_6}{15}.$$

Condition (46) makes the ordinary squared loss nonpositive. If $t = 0$, nonnegative coefficients force $\alpha_6 = 0$, so this endpoint also passes. The final restriction is the contrapositive. □

This filter quantifies the coefficient comparison in the attributed small-order induction. The actual inherited child reserve at $M_{\sqrt{2}}$ and the joint trace-sector lower bound for E may be inserted into (44) or (45). A failure of these stronger sufficient budgets would not by itself imply a negative $\mathcal{D}$, because $R_{\sqrt{2}-1}$ is still available.

7.2 A limitation of defect inheritance

The forward reserve must be measured on the actual transformed base. It cannot be replaced by a positive fixed multiple of the starting defect.

Proposition 7.5 (One forward point can erase the base defect)**.** *For every fixed $s_0 > 1$, there is a positive definite six-row correlation matrix with a unit Gram border direction for which $\delta > 0$ but $\Delta_{s_0} = 0$. Hence no universal positive constant $c(s_0)$ satisfies $\Delta_{s_0} \geq c(s_0)\delta$. Such a positive constant is impossible even if the inputs are restricted to full-support genuinely complex rank-six bases.*

Proof. On two rows take

$$H = \frac{s_0}{s_0+1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad X = \frac{1}{s_0+1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = H + X,$$

and adjoin I_4 to B, H, and 0_4 to X. The pair has diagonal 1 and off-diagonal $(1-s_0)/(s_0+1)$; both its eigenvalues are positive. The split is realized by first two unit rows $(\sqrt{s_0/(s_0+1)}, 1/\sqrt{s_0+1})$ and $(-\sqrt{s_0/(s_0+1)}, 1/\sqrt{s_0+1})$, four orthogonal singleton rows, and the unit direction selecting the second coordinate. The pencil pair has diagonal $d_s = (s_0+s)/(s_0+1)$ and off-diagonal $e_s = (s-s_0)/(s_0+1)$, so direct two-row expansion gives

$$\Delta_s = 2d_s^2 e_s^2 = \frac{2(s_0+s)^2(s-s_0)^2}{(s_0+1)^4}.$$

Thus $\Delta_1 > 0$ and $\Delta_{s_0} = 0$.

For the stronger restriction, use this six-row B and $w = (1, i, 1, 1, 1, 1)^{\mathsf{T}}$. For $\varepsilon > 0$ put

$$B_\varepsilon = \frac{B + \varepsilon w w^*}{1+\varepsilon}, \quad X_\varepsilon = \frac{X}{1+\varepsilon}, \quad H_\varepsilon = \frac{H + \varepsilon w w^*}{1+\varepsilon}.$$

This is still an actual unit Gram-direction split, obtained by appending one coordinate to the rows and keeping the old selected unit direction. The base is positive definite with diagonal one and every entry nonzero. Writing $b = (1-s_0)/(1+s_0) < 0$, the imaginary part of its triangle product on rows $1, 2, 3$ is $b\varepsilon^2/(1+\varepsilon)^3 \neq 0$. Its phases cannot all be removed by diagonal unitary gauges. Continuity gives $\Delta_{s_0,\varepsilon}/\delta_\varepsilon \to 0$, while $\delta_\varepsilon \to \delta > 0$. This disproves a positive uniform constant on that restricted class too. □

The perturbed matrices need not have zero forward defect individually; the limit alone disproves a uniform positive transfer. For $s_0 = 5/4$, the exact ordinary polynomial and defect are

$$F(s) = \frac{50 + 32s^2}{81}, \qquad \Delta_s = \frac{2(16s^2-25)^2}{6561}.$$

Here $\delta = 2/81$, $E = 3200/6561$, $Q = 8930/6561$ and $\mathcal{D} = 12130/6561 > 0$. Thus the obstruction invalidates the proposed transfer, not Chollet's inequality or the conditional child reserve of Proposition 7.1.

Proposition 7.6 (A qualified two-point reserve)**.** *Let $\eta = \max_{i\neq j} |B_{ij}|^2$, and use the same actual pencil at $s_1 = 5/4$, $s_2 = 3/2$. With $\overline{D}$ from Theorem 5.5,*

$$\max_{j=1,2} \frac{\Delta_{s_j}}{a^2} \geq \overline{D}(\eta/16).$$

Proof. Let η_j be the maximum normalized squared coherence of M_{s_j}. Its diagonal entries are $1 + (s_j - 1)p_i \in [1, s_j]$. Since $B = 2M_{s_1} - M_{s_2}$, every off-diagonal pair satisfies

$$|B_{ik}| \le 2s_1\sqrt{\eta_1} + s_2\sqrt{\eta_2} \le 4\max\{\sqrt{\eta_1}, \sqrt{\eta_2}\}.$$

Hence some $\eta_j \ge \eta/16$. Normalize that matrix diagonally and apply Theorem 5.5. The defect-to-squared-permanent ratio is unchanged by normalization, and $F(s_j) \ge a$ by its nonnegative coefficients. Monotonicity of $\overline{D}$ proves the claim. □

The favorable index depends on the pencil. The proposition gives one reserve among two points, not separate lower bounds at both. At $\eta = 0$ it supplies only zero. The factor 16 is conservative and this statement alone does not pay the generic compensation budget. The arguments throughout these sections preserve complex phases and singular limits; no conditioning cutoff is used.

8 Ordinary occupation constrains the actual variance

The ordinary occupation polynomial and the cofactor-weighted variance have the same Gram factors. The following bound retains this relation without a unit-row or small-order hypothesis. The polynomial here is F, not the residual polynomial G_z of Proposition 3.1.

For this section let $n \ge 1$, $w_i = V^* e_i$ be arbitrary vectors and z be unit. Use the conjugate-linear-first inner product and put

$$B_{ij} = \langle w_i, w_j\rangle, \quad x_i = \langle w_i, z\rangle, \quad p_i = |x_i|^2, \quad P_i = w_i w_i^*, \quad r_i = (I - zz^*)P_i z.$$

With the actual transposed cofactor matrix L of $|B|^2$, define $E = \sum_{ij} L_{ij}\langle r_i, r_j\rangle$ as before. Thus $E = z^*Uz - p^{\mathsf{T}}Lp$ also for nonunit rows. Write

$$F(s) = \operatorname{per}(H + sX) = \sum_{j=0}^{n} \alpha_j s^j, \quad H = B - xx^* \succeq 0, \quad X = xx^*, \quad \rho = \prod_i p_i.$$

The occupation expansion gives $\alpha_j \ge 0$, $F(1) = a$, $F'(1) = t = z^*Tz$ and $\alpha_n = n!\rho$.

Theorem 8.1 (Sharp occupation–variance coupling)**.** *For arbitrary complex Gram vectors and unit z, in every order $n \ge 1$,*

$$E \ge \rho(na - t) = \frac{\alpha_n}{n!}\sum_{j=0}^{n}(n - j)\alpha_j. \tag{47}$$

The universal coefficient 1 *is optimal in each fixed order $n \ge 2$. At order one the inequality is equality. The statement includes zero occupations, singular factors and all ambient kernel components.*

Proof. If $\rho = 0$, the right side is zero and the cofactor Gram identity gives $E \ge 0$. Suppose $\rho > 0$. Phase gauges leave P_i and E unchanged, so arrange

$$w_i = \sqrt{p_i}(z + y_i), \qquad y_i \perp z.$$

Then $r_i = p_i y_i$. Set $d_i = w_i \otimes \overline{w_i}$ and let q_i be the averaged symmetric tensor product of the $n - 1$ doubled vectors other than i; the empty product is the vacuum. Their Gram matrix is $L/(n-1)!$. Consequently

$$Y = \sum_i r_i \otimes q_i, \qquad E = (n-1)!\|Y\|^2.$$

Regroup the doubled slots into ordinary and conjugate legs, a unitary identification. Contract every conjugate omitted leg with $\overline{z}$. On a single doubled factor this contraction gives $p_j(z+y_j)$. It commutes with simultaneous permutation of the paired slots. Thus the contracted tensor is exactly

$$\rho Z, \qquad Z = \sum_i y_i \otimes \mathrm{Sym}_{n-1} \bigotimes_{j\neq i} (z+y_j), \qquad E \geq (n-1)!\rho^2 \|Z\|^2.$$

Put $f = \mathrm{Sym}_n \bigotimes_i (z+y_i)$ and let f_l be its component with exactly l factors in $z^\perp$. Likewise write $Z = \sum_{l=1}^n Z_l$ by total perpendicular occupation, including the distinguished first slot. These are orthogonal components. For full ordinary symmetrization S_n, expansion counts each l-perpendicular monomial l times, giving

$$S_n Z_l = l f_l.$$

On the fully symmetric total-l sector, the first-slot perpendicular projection $P_\perp^{(1)}$ satisfies

$$S_n P_\perp^{(1)} S_n = \frac{l}{n} I.$$

Indeed all n slot projections have equal compression and their sum is the perpendicular number operator, with eigenvalue l. The adjoint output is symmetric in the remaining $n-1$ slots. Hence the restriction of S_n to first-slot-perpendicular tensors in this sector has squared operator norm l/n, and

$$\|Z_l\|^2 \geq \frac{n}{l} \|S_n Z_l\|^2 = nl \|f_l\|^2.$$

The ordinary sector normalization is $\alpha_{n-l} = n!\rho\|f_l\|^2$. Summing gives

$$E \geq n!\rho^2 \sum_{l=1}^n l\|f_l\|^2 = \rho \sum_{j=0}^n (n-j)\alpha_j,$$

proving (47).

For sharpness, take all w_i equal to the same unit vector w and choose unit z with $|\langle w, z\rangle|^2 = q \in (0,1)$. Then $a = n!$, $t = nn!q$ and $E = nn!q(1-q)$, whereas $\rho(na - t) = nn!q^n(1-q)$. Their ratio q^{1-n} tends to one as $q \uparrow 1$, excluding every larger universal coefficient. At $n = 1$, directly $E = p_1(\|w_1\|^2 - p_1)$, which is equality. □

The sharpness family proves an all-Gram coefficient assertion. The ingredients are the classical cofactor tensor and symmetric occupation geometry, here applied to one common omitted residual.

Theorem 8.2 (Occupation-resolved constraint)**.** *Keep the same vectors and scale only their selected mode:*

$$w_i(s) = (I - zz^*)w_i + \sqrt{s}\, zz^* w_i, \qquad s \geq 0.$$

Let L_s be the actual cofactor matrix of $|H + sX|^2$, and let E_s be the variance formed from these scaled vectors and the same unit z. There are nonnegative coefficients β_m such that

$$E_s = \sum_{m=1}^{2n-1} \beta_m s^m, \qquad \beta_{n+j} \geq \rho(n-j)\alpha_j \quad (0 \leq j \leq n-1). \tag{48}$$

Equivalently, $E_s - \rho s^n[nF(s) - sF'(s)]$ has nonnegative coefficients. No diagonal normalization of the scaled vectors is made.

Proof. The residual of the scaled row is $r_i(s) = \sqrt{s}\, r_i$. In the tensor Y above, let $Y_{k,l}$ be its orthogonal component with k selected occupations in the ordinary omitted legs and l in their conjugate legs, where $0 \le k, l \le n-1$. The distinguished residual slot is perpendicular and is not counted in k. The common mode scaling gives the exact identity

$$E_s = (n-1)! \sum_{k,l=0}^{n-1} s^{k+l+1} \|Y_{k,l}\|^2.$$

Only orthogonality of the two number decompositions is used; independent permutation invariance of Y is not assumed. This proves nonnegativity of every β_m.

For $\rho > 0$, the fully occupied conjugate component with ordinary occupation j is exactly ρZ_{n-j} under contraction with $\overline{z}^{\otimes(n-1)}$. The restricted symmetrization estimate in the preceding proof therefore gives

$$\begin{aligned} \beta_{n+j} &\ge (n-1)! \|Y_{j,n-1}\|^2 \\ &\ge n! \rho^2 (n-j) \|f_{n-j}\|^2 = \rho(n-j)\alpha_j. \end{aligned}$$

For $\rho = 0$ the claimed lower bounds are zero. This proves the coefficient statement, including all zero-occupation boundaries. □

In particular a two-endpoint ordinary polynomial still forces $\beta_n \ge n\rho\alpha_0$. These are constraints on jointly realizable ordinary and paired-variance occupations. Merely optimizing E_s separately does not improve (47): for $s \ge 1$ use $E_s \le s^{2n-1} E_1$, and for $0 < s \le 1$ use $E_s \le s E_1$. Combining either bound with Theorem 8.2 gives a lower bound for E_1 maximized at $s = 1$. Further use must retain these aligned coefficients with their actual forward defect.

8.1 Comparison with the trace-sector reserve

At six rows (47) gives

$$E \ge \frac{\alpha_6}{720}(6a - t).$$

This can improve the named trace-sector lower cost substantially. Take orthonormal vectors $z, e_1, \dots, e_6$ and

$$w_i = \sqrt{q}\, z + \sqrt{1-q}\, e_i, \qquad q = 9/10.$$

Then $B = (1-q)I_6 + qJ_6$ has rank six and $\alpha_j = \binom{6}{j} j! q^j (1-q)^{6-j}$. Exact values are

$$a = \frac{17104189}{40000}, \quad t = \frac{1259058357}{500000}, \quad \rho(6a - t) = \frac{6312407836869}{250000000000}.$$

For these actual data, the trace variance and frame variance are both $6q(1-q) = 27/50$, the support dimension is six and $\pi = 6q/(1+5q) = 54/55$. Substituting these quantities into 5! times the right-hand side of (30) gives exactly 83/9930. Thus the new floor exceeds 25, whereas that named cost is less than 1/100. This compares specific valid lower bounds on the same data, not all earlier methods or newly covered Chollet instances. The bounds are generally incomparable: if $\rho = 0$ the new floor can vanish while the earlier trace-variance floor is positive. Their maximum is always a valid lower bound; their sum requires a separate proof that the charged components do not overlap.

Nor does the new reserve alone close the forward budget. For the unit binary six-root family $w_i = (e_0 + \omega^i e_1)/\sqrt{2}$ ($0 \le i \le 5$, $\omega^6 = 1$ primitive), with selected mode $z = e_1$,

$$F(s) = \frac{45}{4}(1 + s^6), \quad a + t = 90, \quad F(\sqrt{2}) = 405/4.$$

The positive ordinary squared loss is 34425/16, while the new variance floor is only 135/128. Multiplying that floor by $2(\sqrt{2}-1)$ gives less than two. This failure of a lower-bound certificate is not failure of the actual finite-forward budget or a counterexample to Chollet; the actual transformed defect remains available.

8.2 The stronger forward budget reduces to active directions

Return to six unit Gram rows and the finite-forward remainder of Proposition 7.3. Put $h = \sqrt{2}-1$ and

$$J(z) = (a+t)^2 - C_z(1+h) + 2hE(z) = \mathcal{D}(z) - R_h(z), \qquad C_z(1+s) = \operatorname{per}|B + s(Vz)(Vz)^*|^2.$$

Thus $J \geq 0$ is the sufficient budget (44), with the nonnegative remainder discarded. Let $\mathcal{H}_{\mathrm{act}} = (\ker V)^{\perp}$.

Proposition 8.3 (Curvature credit under kernel thinning)**.** *For active unit w, kernel unit κ, $0 \leq q \leq 1$ and arbitrary real θ, put $z_q = \sqrt{q}\, w + e^{i\theta}\sqrt{1-q}\,\kappa$. There are actual nonnegative coefficients $r_2, \ldots, r_{12}$, independent of q, θ, such that*

$$R_h(z_q) = \sum_{m=2}^{12} r_m q^m, \tag{49}$$

$$J(z_q) = (1-q^2)\delta + q^2 J(w) + 2at_w q(1-q) + \sum_{m=3}^{12} r_m(q^2 - q^m), \tag{50}$$

where $t_w = w^*Tw$. Consequently $J(w) \geq 0$ implies $J(z_q) \geq 0$. For $q > 0$, the function $J(z_q)/q^2$ is nonincreasing in q.

Proof. Since $V\kappa = 0$, $X_q = qX_w$, $t_q = qt_w$, $u_q = qu_w$, $\ell_q = q^2\ell_w$ and $E_q = qu_w - q^2\ell_w$. The PSD-pencil proof of Proposition 7.3 gives

$$C_w(1+s) = \sum_{m=0}^{12} c_m s^m, \qquad c_m \geq 0, \quad c_0 = c, \quad c_1 = 2u_w, \quad c_2 \geq \ell_w.$$

Because $C_{z_q}(1+h) = C_w(1+hq)$, (49) holds with $r_2 = h^2(c_2 - \ell_w)$ and $r_m = h^m c_m$ for $m \geq 3$. Subtract it from $\mathcal{D}(z_q) = \delta + 2at_w q + (t_w^2 - \ell_w)q^2$ to obtain (50). All extra terms are nonnegative, using $\delta \geq 0$ from Lemma 5.3 and $t_w \geq 0$. Furthermore

$$\frac{J(z_q)}{q^2} = \frac{\delta}{q^2} + \frac{2at_w}{q} + t_w^2 - \ell_w - \sum_{m=2}^{12} r_m q^{m-2},$$

whose derivative is $-2\delta/q^3 - 2at_w/q^2 - \sum_{m=3}^{12}(m-2)r_m q^{m-3} \leq 0$. The endpoints $q = 0, 1$ follow directly from the exact identity. □

Every unit vector decomposes by its active projection with weight $q = \|P_{\mathcal{H}_{\mathrm{act}}} z\|^2$; the $q = 0$ budget is $J = \delta \geq 0$. Hence nonnegativity of J on all active unit directions is equivalent to its nonnegativity in every ambient kernel enlargement, including every positive border Schur slack. A negative mixture would force a negative active endpoint. On rank-six bases that endpoint has $\operatorname{rank}(B - X_w) = 5$; on rank-five bases it has rank four. This removes the mixture parameter, without proving the remaining sign.

For completeness, there is an actual reconstruction chart for the remaining active problem. If every x_i is nonzero, set $D = \operatorname{diag}(1/x_i)$. Then $DXD^* = J_6$ and $K = D(B - X)D^* \succeq 0$. All terms in J scale by the same positive factor $|\det D|^4$: a, t scale by $|\det D|^2$, and c, u, ℓ, E by $|\det D|^4$. Thus the sign is unchanged. Conversely, any $K \succeq 0$ with $X = J_6$ is realized by Gram rows $(y_i, 1)$ and the unit border selecting the final coordinate; dividing row i by $\sqrt{1 + K_{ii}}$ restores diagonal one. To specify active status require $\operatorname{rank}(K + J_6) = \operatorname{rank} K + 1$: the rank-six active chart has ranks $\operatorname{rank} K = 5$, $\operatorname{rank}(K + J_6) = 6$, and the rank-five chart has ranks four and five. The nonzero-occupation locus is dense in the active unit sphere, being the complement of finitely many proper complex hyperplanes, and J is continuous. A proof on all of this actual chart would therefore include zero occupations by continuity. The all-order variance constraint and this active reduction preserve complex phases and singular boundaries. They do not settle the generic active rank-six budget or the unrestricted Chollet conjecture.

9 Projection overlap couples trace and occupation

Here the rows w_i are unit, $n \geq 2$, and z is any ambient unit vector. Let S be an r-dimensional containing support of the rows, $m = n - 1$, $\pi = \|P_S z\|^2$, $P_i = w_i w_i^*$, and $r_i = (I - zz^*)P_i z$. On the omitted Hilbert–Schmidt factors put

$$Y = \sum_i r_i \otimes \operatorname{Sym}_m \bigotimes_{j \neq i} P_j, \qquad E = m! \|Y\|^2.$$

Symmetrization permutes the paired factors simultaneously. Put $e = I_S/\sqrt{r}$. Let $\mathcal{P}_k$ project the omitted factors onto sectors with at most k traceless slots and all remaining slots equal to e, for $0 \leq k \leq m$. Let $\mathcal{Q}$ project every conjugate omitted leg onto $\overline{z}$. Both leave the distinguished residual untouched.

Theorem 9.1 (Exact overlap and safe envelope)**.** *The overlap is*

$$\|\mathcal{P}_k \mathcal{Q}\|^2 = \gamma_k^2 = \left(\frac{\pi}{r}\right)^m \sum_{j=0}^{k} \binom{m}{j} (r-1)^j. \tag{51}$$

If $A \leq m! \|\mathcal{P}_k Y\|^2$ and $M \leq m! \|\mathcal{Q} Y\|^2$ are nonnegative proved floors, then $E \geq \mathcal{H}_{\gamma_k}(A, M)$, where for $0 < \gamma < 1$

$$\mathcal{H}_\gamma(A, M) = \begin{cases} \max(A, M), & \min(A, M) \leq \gamma^2 \max(A, M), \\ (A + M - 2\gamma\sqrt{AM})/(1 - \gamma^2), & \text{otherwise.} \end{cases} \tag{52}$$

The endpoint conventions are $\mathcal{H}_0 = A + M$ and $\mathcal{H}_1 = \max(A, M)$.

Proof. If $\pi = 0$, the overlap is zero. Otherwise set $v = P_S z/\sqrt{\pi}$. After identifying the selected conjugate output with a scalar, compression of one trace projector is $A_0 = |P_S z\rangle\langle P_S z|/r$ on the ordinary slot. Compression of the Hilbert–Schmidt identity supported on S is πI_S, so the compressed traceless projector is $\pi I_S - A_0$. Consequently $\mathcal{Q}\mathcal{P}_k\mathcal{Q}$ is the sum of commuting tensor products with at most k copies of $\pi I_S - A_0$, the other copies being A_0. On a sector with l ordinary slots perpendicular to v, its eigenvalue is zero if $l > k$, and otherwise is

$$\frac{\pi^m}{r^{m-l}} \sum_{b=0}^{k-l} \binom{m-l}{b} (r-1)^b.$$

Writing $B(d,h)=\sum_{b=0}^{h}\binom{d}{b}(r-1)^b$, the identity $B(d,h)-rB(d-1,h-1)=\binom{d-1}{h}(r-1)^h\ge 0$ compares consecutive l and places the maximum at $l=0$. The all-v tensor attains it and is simultaneously symmetric. Thus the symmetry restriction and the residual identity do not change the norm. At $r=1$ the traceless subspace is zero and the same formula holds with $0^0=1$.

For arbitrary orthogonal projections P,Q with $\|PQ\|\le\gamma$, the Gram matrix of the synthesis map from $\operatorname{ran}P\oplus\operatorname{ran}Q$ gives, for $c,d\ge 0$,

$$\|cP+dQ\|\le\frac{c+d+\sqrt{(c-d)^2+4cd\gamma^2}}{2}.$$

In the second branch of (52), choose $c=(1-\gamma\sqrt{M/A})/(1-\gamma^2)$ and $d=(1-\gamma\sqrt{A/M})/(1-\gamma^2)$. Both are positive, the displayed upper bound is one, and $cA+dM$ is the asserted expression. In the first branch one projection alone suffices. Zero floors and endpoints follow directly. This is classical two-projection geometry; (51) supplies its actual common-Gram overlap. □

Theorem 8.1 gives $M=\rho(na-t)$. Theorem 5.9 supplies a floor A supported entirely in $\mathcal{P}_1Y$, namely $m!$ times the right side of (30), with its stated zero-denominator convention. Thus these floors can be combined using $\gamma_1^2=(\pi/r)^m[1+m(r-1)]$. For six rows and active z, this is $26/7776$ at $r=6$ and $21/3125$ at $r=5$. The zero-trace floor $A=m!\mathcal{A}/r^m$ instead uses $\gamma_0^2=(\pi/r)^m$. One may take the maximum of valid choices. One may not add the floors without paying their overlap, nor apply the second branch of (52) outside its stated range. Higher k require a proved floor in that actual sector.

9.1 Cofactor occupations and sharp retained curvature

For this section the rows need not be unit. Keep $H=B-X\succeq 0$, $X=xx^*$, $p_i=|x_i|^2$, $\rho=\prod_i p_i$, and write

$$F(v)=\operatorname{per}(H+vX)=\sum_{k=0}^{n}\alpha_k v^k,\quad C(v)=\operatorname{per}|H+vX|^2,\quad W=pp^{\mathsf{T}}.$$

Define

$$N(v)=D\operatorname{per}(|H+vX|^2)[W]=\sum_{k=0}^{2n-2}\nu_k v^k.$$

For the actual scaled rows $w_i(v)=(I-zz^*)w_i+\sqrt{v}\,zz^*w_i$, without renormalizing them, let E_v be their variance. Their selected occupations are vp_i. Put $c=C(1)$, $\ell=N(1)$, $u=C'(1)/2$, and, for $s=1+h\ge 1$, $R_h=C(s)-c-2hu-h^2\ell$.

Theorem 9.2 (Cofactor occupation constraints)**.** *All ν_k are nonnegative. For $n\ge 2$,*

$$\nu_{n-2+k}\ge 2\rho k\alpha_k\quad(1\le k<n),\tag{53}$$

$$\nu_{2n-2}=\rho n\alpha_n=nn!\rho^2,\qquad \nu_{2n-3}=2\rho(n-1)\alpha_{n-1}.\tag{54}$$

For $n=1$, $N(v)=\rho^2$. These statements retain all complex phases, zero occupations and singular factors.

Proof. The pencil $|H+vX|^2=A_0+vA_1+v^2W$ has PSD coefficients $A_0=H\circ\overline{H}$ and $A_1=H\circ\overline{X}+\overline{H}\circ X$, by the Schur product theorem. Expanding the mixed permanent in orthogonally labeled Gram sectors gives nonnegative coefficients, also after taking the derivative in the PSD direction W. This proves $\nu_k\ge 0$.

Suppose first $\rho > 0$ and gauge $w_i = \sqrt{p_i}(z + y_i)$, $y_i \perp z$. Set

$$d_i(v) = (y_i + \sqrt{v}z) \otimes (\overline{y}_i + \sqrt{v}\,\overline{z}), \qquad Z(v) = \sum_i \mathrm{Sym}_{n-1} \bigotimes_{j \neq i} d_j(v).$$

The omitted-tensor Gram identity gives $N(v) = (n-1)!\rho^2 \|Z(v)\|^2$. The ordinary/conjugate selected-occupation sectors (a, b) are orthogonal, and contribute to degree $a + b$. Let $f = \mathrm{Sym}_n \bigotimes_i (z + y_i)$ and let f_k have k selected factors. Then $\alpha_k = n!\rho\|f_k\|^2$. The sum of ordinary omitted products is $g = \sum_i \mathrm{Sym}_{n-1} \bigotimes_{j \neq i}(z + y_j) = n\,\mathrm{contr}_z f$. On the symmetric k-selected sector, first-slot selected projection compresses to k/n; hence $\|g_{k-1}\|^2 = nk\|f_k\|^2$. The $(k-1, n-1)$ sector of Z is exactly g_{k-1}, and contributes $\rho k \alpha_k$ to degree $n - 2 + k$. Its swapped conjugate sector has the same norm and is distinct for $k < n$. This gives (53). At $k = n$ the sectors coincide, giving the first identity in (54); at total degree $2n - 3$ only the two displayed sectors exist, giving the second identity. If $\rho = 0$, inequalities follow from coefficient positivity, while the two identities follow by polynomial continuity. The order-one case is direct. □

Theorem 9.3 (Integral reserve and sharp top credit)**.** *For every $s \geq 1$,*

$$C(s) - c - (s^2 - 1)\ell = 2\int_1^s \frac{E_v}{v}\,dv + 2\int_1^s v[N(v) - N(1)]\,dv = 2hE + R_h. \tag{55}$$

Moreover

$$R_h \geq \frac{\alpha_n^2}{n!}[s^{2n} - 1 - n(s^2 - 1)]. \tag{56}$$

The coefficient is optimal over arbitrary Gram data. For six unit rows and $s = \sqrt{2}$, $R_h \geq (19/240)\alpha_6^2$; the constant remains optimal on active rank-six and active rank-five bases.

Proof. Differentiation of the actual Gram pencil gives $E_v = vC'(v)/2 - v^2 N(v)$, because the actual cofactor term is $v^2 N(v)$. Integrating $C'(v) = 2E_v/v + 2vN(v)$ proves the first equality in (55); $u = E + \ell$ proves the second. By Theorem 8.2, $E_v = \sum_{j \geq 1} \beta_j v^j$ with $\beta_j \geq 0$, so $E_v/v \geq E$ for $v \geq 1$. Subtract $2hE$ in (55) and retain only the top coefficient of N, given by (54). This gives

$$R_h \geq nn!\rho^2 \left[\frac{s^{2n} - 1}{n} - (s^2 - 1)\right],$$

which is (56). The bracket is nonnegative by Bernoulli's inequality. Equality holds when $H = 0$: then $E_v = 0$ and $N(v) = nn!\rho^2 v^{2n-2}$. This includes the zero coefficient at $n = 1$ or $h = 0$.

For the rank-six assertion put $\Pi = I_6 - J_6/6$ and, for $0 < \epsilon < 1$,

$$H_\epsilon = \epsilon\Pi, \quad X_\epsilon = (1 - 5\epsilon/6)J_6, \quad B_\epsilon = \epsilon I_6 + (1 - \epsilon)J_6.$$

The residual/base ranks are $5/6$, the diagonal is one and the selected direction is active. As $\epsilon \downarrow 0$, $H \to 0$, $X \to J_6$ and $\alpha_6 \to 720$, so polynomial continuity makes $R_h/\alpha_6^2 \to 19/240$. For rank five take $t = (1, 1, 1, -1, -1, -1)^{\mathsf{T}}$, $\Pi_5 = I_6 - (J_6 + tt^*)/6$, $H_\epsilon = \epsilon\Pi_5$ and $X_\epsilon = (1 - 2\epsilon/3)J_6$. These have diagonal one and ranks $4/5$, with the same limiting ratio. One can retain genuinely complex, noncirculant examples by replacing H_ϵ in either family by $\epsilon\Pi_* + \epsilon^2\Pi_* i(E_{12} - E_{21})\Pi_*$. The perturbation has zero diagonal, annihilates the same kernel and has operator norm at most ϵ^2, so positivity and ranks persist. Writing $b = 1 - \epsilon$ and $\delta = \epsilon^2$, the imaginary part of $B_{12}B_{23}B_{31}$ is $b^2\delta - \delta^3/54$ for Π, and $b^2\delta - \delta^3/27$ for Π_5. It is nonzero for sufficiently small ϵ, excluding a gauge-real matrix. The ratio limit is unchanged. The sharpness assertion permits complex phases and the stated conditioning range. □

Here is an occupation-resolved replacement for the reserve. Define

$$w_j(s) = 2(s^j - 1)/j, \quad T_j(s) = 2\left[\frac{s^{j+2}-1}{j+2} - \frac{s^2-1}{2}\right].$$

Let $O \geq \rho(na - t)$ be any proved floor for E, for example a maximum of the overlap envelopes above and the occupation floor. Then

$$\begin{aligned} 2hE + R_h \geq 2hO &+ \sum_{j=0}^{n-1} [w_{n+j}(s) - 2h]\rho(n-j)\alpha_j \\ &+ \sum_{k=1}^{n-1} 2\rho k\alpha_k T_{n-2+k}(s) + \rho n\alpha_n T_{2n-2}(s). \end{aligned} \tag{57}$$

Indeed Theorem 8.2 reserves mandatory masses $\beta_{n+j} \geq \rho(n-j)\alpha_j$; every remaining coefficient has weight at least $w_1 = 2h$ in $2\int E_v/v$. The second integral in (55) equals $\sum_k \nu_k T_k(s)$, and (53)–(54) give its displayed floor. These are two separate integrals, so their credits add. The result replaces $2hE + R_h$; it must not be added to that exact quantity again. In particular it cannot repair a failed exact-curvature certificate with the same variance floor simply by counting curvature twice.

10 A uniform boundary band and an order-seven consequence

Use six unit Gram rows, any ambient unit z, and the preceding pencil. Put $P = F(1) + F'(1) = a + t$, $s = \sqrt{2}$, $h = s - 1$, and

$$G = P^2 - c - \ell, \qquad \Delta_s = F(s)^2 - C(s).$$

Lemma 5.3 gives $\Delta_s \geq 0$. The exact compensation identity is

$$G = P^2 - F(s)^2 + \Delta_s + 2hE + R_h = P^2 - C(s) + 2hE + R_h. \tag{58}$$

Theorem 10.1 (Uniform small-coordinate compensation)**.** *If* $\min_i p_i \leq 1/401$*, then* $G \geq 0$*. On the positive-loss part* $F(\sqrt{2}) > P$ *of this band,*

$$G > (1 - \kappa)P^2 + 2hE + R_h, \qquad \kappa = \frac{(401/232)^{12}}{720} < 1.$$

On the nested band $\min_i p_i \leq 1/626$*, positive loss gives the stronger* $G > (2/3)P^2 + 2hE + R_h$*. No phase, rank or conditioning restriction is imposed.*

Proof. If $F(s) \leq P$, (58) proves the assertion. If some $p_i = 0$, then $\alpha_6 = 0$, and $j + 1 \geq 2^{j/2}$ for $0 \leq j \leq 5$ implies $F(s) \leq P$. We may therefore assume positive loss and all $p_i > 0$. Gauge $w_i = (z + y_i)/\sqrt{1 + \|y_i\|^2}$ with $y_i \perp z$ and $p_i = (1 + \|y_i\|^2)^{-1}$. For $e_l(y) = \sum_{|I|=l} \mathrm{Sym}_l \bigotimes_{i \in I} y_i$, orthogonality of the selected placements gives

$$\alpha_{6-l} = \frac{720\rho}{\binom{6}{l}} \|e_l(y)\|^2.$$

Put $g_j = j + 1 - 2^{j/2}$. Then $g_0 = 0$, $g_6 = -1$, and $g_j > 0$ for $1 \leq j \leq 5$. Positive loss is $\alpha_6 > \sum_{j=1}^5 g_j \alpha_j$. Thus, for $l = 1, \ldots, 5$, the respective upper bounds on $\|e_l(y)\|^2$ are

$$\frac{6}{6 - 4\sqrt{2}}, \quad 15, \quad 5(2 + \sqrt{2}), \quad 15, \quad 3(2 + \sqrt{2}).$$

All are less than 18: the tight first comparison reduces to $17 > 12\sqrt{2}$, whose square is $289 > 288$. Consequently $\|e_l(y)\| < 9/2$ for these five degrees.

Let $R_0 = \max_i \|y_i\|$ and choose a unit v along a maximizing row. Set $x_i = \langle v, y_i \rangle$. One $x_i = R_0$ and all $|x_i| \le R_0$. Contraction against $v^{\otimes l}$ gives $|e_l(x)| < 9/2$. Evaluating $\prod_i (T - x_i)$ at the actual root R_0 yields, for $R_0 > 1$,

$$\left| \prod_i x_i \right| > R_0^6 \left[1 - \frac{9/2}{R_0 - 1} \right].$$

This uses only a finite geometric sum and holds for arbitrary complex roots. On the full band $R_0 \ge 20$, the bracket is at least $29/38$. Every $|x_i|$ therefore exceeds $29R_0/38$, since all the other factors are at most R_0. It follows that

$$p_i < \frac{1444}{1444 + 841R_0^2} \le \frac{361}{84461} < \frac{1}{200}, \qquad q_i := |\langle v, w_i \rangle|^2 > \frac{841R_0^2}{1444(1 + R_0^2)} \ge \frac{84100}{144761} > \frac{29}{50}.$$

Projection of the actual symmetric Gram product onto $v^{\otimes 6}$ gives $a \ge 720 \prod_i q_i > 720(29/50)^6$. For $M = B + hX \succeq 0$, the diagonal satisfies $M_{ii} = 1 + hp_i < 401/400$, since $h < 1/2$. Its two-by-two PSD minors give $|M_{ij}| < 401/400$. The classical Carlen–Lieb–Loss inequality [4], applied to $|M|^2$, gives

$$C(s) \le U_{\mathrm{CLL}}(M) := \frac{720}{216} \prod_i \sqrt{\sum_j |M_{ij}|^4} < 720(401/400)^{12}.$$

Since $P \ge a$, $C(s)/P^2 < \kappa < 1$; the last comparison is the exact integer inequality $401^{12} < 720\,232^{12}$. Equation (58) proves the stated margin.

On the nested band $R_0 \ge 25$, the same bracket is at least $13/16$. Thus $p_i < 256/105881 < 1/400$ and $q_i > 105625/160256 > 13/20$. Now $M_{ii} < 801/800$ and

$$C(s)/P^2 < (801/520)^{12}/720 < 1/3,$$

where the last comparison is $801^{12} < 240\,520^{12}$. This proves the stronger margin. No inverse or lower eigenvalue bound entered either argument. □

10.1 Retaining the joint coefficient budget and coherent product

The preceding $1/401$ band can be enlarged by retaining the joint coefficient constraint and the product of the coherent projections. The stronger $1/626$ margin in Theorem 10.1 remains useful.

Theorem 10.2 (Joint root-product compensation)**.** *For six unit complex Gram rows and any ambient unit z,*

$$\min_i p_i \le \frac{1}{31} \quad \Longrightarrow \quad G \ge 0.$$

On the positive-loss part $F(\sqrt{2}) > P$ of this band,

$$C(\sqrt{2}) < \kappa_{31} P^2, \qquad G > \frac{P^2}{20} + 2hE + R_h, \tag{59}$$

where

$$\kappa_{31} = \left(\frac{377}{360} \right)^{12} \frac{40^4}{720\,9^4} = \frac{8243206936713178643875538610721}{87436462389547861278720000000000} < \frac{19}{20}.$$

All ranks, complex phases, singular limits and ambient kernel components are permitted.

Proof. If $F(s) \le P$, identity (58), with $\Delta_s, E, R_h \ge 0$, proves $G \ge 0$. If some $p_i = 0$, then $\alpha_6 = 0$ and $j + 1 \ge 2^{j/2}$ for $0 \le j \le 5$ forces $F(s) \le P$. We may therefore assume positive loss and all $p_i > 0$. As before, gauge

$$w_i = \sqrt{p_i}(z + y_i), \quad y_i \perp z, \quad p_i = (1 + \|y_i\|^2)^{-1}, \quad \rho = \prod_i p_i.$$

For $e_l(y) = \sum_{|I|=l} \operatorname{Sym}_l \bigotimes_{i \in I} y_i$, the exact ordinary sector identity gives $\alpha_{6-l} = 720\rho\|e_l(y)\|^2/\binom{6}{l}$. Put $g_j = j + 1 - 2^{j/2}$. Since $g_0 = 0$, $g_6 = -1$ and $g_1, \dots, g_5 > 0$, positive loss gives the joint bound

$$\sum_{l=1}^{5} \frac{g_{6-l}}{\binom{6}{l}} \|e_l(y)\|^2 < 1. \tag{60}$$

Let $R_0 = \max_i \|y_i\|$ and choose a unit v along a maximizing row. The hypothesis implies $R_0^2 \ge 30$. Define $\xi_i = \langle v, y_i \rangle$; one $\xi_i = R_0$ and every $|\xi_i| \le R_0$. Contracting against the unit tensors $v^{\otimes l}$ preserves (60) with $e_l(\xi)$ in place of $e_l(y)$. Evaluate $\prod_i (T - \xi_i)$ at its root R_0. The triangle inequality and weighted Cauchy–Schwarz give

$$\begin{aligned} \left| \prod_i \xi_i \right| &\ge R_0^6 - \sum_{l=1}^{5} |e_l(\xi)| R_0^{6-l} \\ &> R_0^6 \left[1 - \sqrt{S(R_0^2)} \right], \end{aligned} \tag{61}$$

where the exact reciprocal weights yield

$$S(q) = \frac{9 + 6s}{q} + \frac{15}{q^2} + \frac{10 + 5s}{q^3} + \frac{15}{q^4} + \frac{6 + 3s}{q^5}, \qquad s = \sqrt{2}.$$

All coefficients are positive, so S decreases on $q > 0$. Using $s < 99/70$, justified by $9800 < 9801$, gives

$$S(R_0^2) \le S(30) < \frac{48614177}{81000000} = \frac{961}{1600} - \frac{1139}{2531250} < \left(\frac{31}{40} \right)^2.$$

Thus (61) gives $|\prod_i \xi_i| > (9/40) R_0^6$. This argument applies to complex scalar roots without any real-rootedness or stability assumption.

Projection of the actual symmetrized six-row product onto $v^{\otimes 6}$ now gives the product bound

$$a \ge 720 \prod_i |\langle v, w_i \rangle|^2 = 720\rho \left| \prod_i \xi_i \right|^2 > 720\rho (9/40)^2 R_0^{12}.$$

For the actual PSD forward matrix $M = H + sX$, every two-by-two PSD minor gives $|M_{ij}|^2 \le M_{ii} M_{jj}$. Each of the 720 permutation terms of $\operatorname{per} |M|^2$ is therefore at most $\prod_i M_{ii}^2$. Since $M_{ii} = p_i(s + \|y_i\|^2)$,

$$C(s) \le 720 \prod_i M_{ii}^2 = 720\rho^2 \prod_i (s + \|y_i\|^2)^2 \le 720\rho^2 (s + R_0^2)^{12}.$$

Combining this elementary PSD bound with $P \ge a$ yields

$$\frac{C(s)}{P^2} < \frac{(1 + s/R_0^2)^{12}}{720(9/40)^4} \le \frac{(1 + s/30)^{12}}{720(9/40)^4} < \left(\frac{377}{360} \right)^{12} \frac{40^4}{720\, 9^4} = \kappa_{31} < \frac{19}{20}.$$

Here $s < 17/12$ is used only for $1 + s/30 < 377/360$; it is too coarse for the earlier estimate of $S(30)$. For the displayed reduced numerator N and denominator D of κ_{31}, the final comparison is the positive integer identity

$$19D - 20N = 126513980587736355205722778558 0 > 0.$$

Identity (58) proves (59). The proof uses the actual common factors and an elementary PSD upper bound; it does not require a generic capacity bridge or a lower Gram eigenvalue. □

Corollary 10.3 (One weak pair at order seven)**.** *Let $A \succeq 0$ have order seven and positive diagonal. If some $i \neq j$ satisfies $|A_{ij}|^2/(A_{ii}A_{jj}) \leq 1/31$, then*

$$\operatorname{per}|A|^2 \leq (\operatorname{per} A)^2.$$

If both order-seven PSD matrices A, D have such a pair, possibly different pairs, then $\operatorname{per}(A \circ D) \leq \operatorname{per} A \operatorname{per} D$. *A zero diagonal in either factor is a trivial zero-row endpoint.*

Proof. Positive diagonal congruence normalizes A to a correlation matrix and scales both sides equally. Realize it by seven unit vectors and use the j endpoint of the weak pair as the border z. The other six rows have one occupation at most $1/31$. Border expansion gives exactly $\operatorname{per} A = a + t$ and $\operatorname{per}|A|^2 = c + \ell$, so Theorem 10.2 applies. An ambient kernel component in z is allowed, so this includes full-rank order seven. For two factors, permutation Cauchy–Schwarz gives

$$|\operatorname{per}(A \circ D)| \leq \sqrt{\operatorname{per}|A|^2 \operatorname{per}|D|^2} \leq \operatorname{per} A \operatorname{per} D.$$

The Schur product is PSD and its permanent nonnegative. Each self-bound was used, so both factors must satisfy the stated sufficient condition. A zero diagonal in a PSD matrix forces a zero row and column. □

Any unresolved positive-loss six-row instance must have every $p_i > 1/31$. In the gauge-normalized residual chart $K_{ij} = H_{ij}/\sqrt{p_i p_j}$, this gives $0 \leq K_{ii} < 30$. The closure $K \succeq 0$, $K_{ii} \leq 30$ is compact by the PSD entry bound. Its singular boundaries remain included. In particular, a hypothetical order-seven self-conjugate counterexample must have every normalized off-diagonal squared modulus strictly greater than $1/31$.

10.2 Balanced capacity and coherent-column domains

These are further sufficient conditions on actual six-row pencils, now allowing nonunit rows. Set $M = H + \sqrt{2}X$, $Q = |M|^4$ entrywise, and $P = F(1) + F'(1)$. Suppose M has positive diagonal; a zero diagonal is already a zero-row permanent endpoint. Classical CLL and matrix scaling [4, 18] give the following useful normalization.

Proposition 10.4 (Weighted CLL and symmetric scaling)**.** *For positive v,*

$$C(\sqrt{2}) \leq \frac{10}{3}\sqrt{\frac{\prod_i (Qv)_i}{\prod_j v_j}}, \qquad U_{\mathrm{opt}}(M) := \frac{10}{3}\sqrt{\operatorname{cap} Q}, \quad \operatorname{cap} Q = \inf_{v>0} \frac{\prod_i (Qv)_i}{\prod_j v_j}.$$

There is a unique positive d with $d_i(Qd)_i = 1$, and $U_{\mathrm{opt}}(M) = (10/3)/\prod_i d_i$. Congruence $M \mapsto DMD^*$ scales U_{opt} and P^2 by the same factor $\prod_i |D_{ii}|^4$.

Proof. Apply CLL to $|M|^2 \operatorname{diag}(\sqrt{v})$ and divide its permanent by $\prod_j \sqrt{v_j}$ to get the first inequality. For existence and uniqueness of d, minimize

$$f(x) = \frac{1}{2}\sum_{ij} Q_{ij} e^{x_i + x_j} - \sum_i x_i.$$

Positive diagonal makes its Hessian positive definite: diagonal terms contribute $2Q_{ii}e^{2x_i}$, and other terms contribute positive multiples of $(e_i + e_j)(e_i + e_j)^{\mathsf{T}}$. Furthermore $f(x) \geq \sum_i [(Q_{ii}/2)e^{2x_i} - x_i]$ is coercive, since every summand is bounded below and diverges at either endpoint. The unique minimizer satisfies $e^{x_i}\sum_j Q_{ij}e^{x_j} = 1$. Thus $\widehat{Q} = \operatorname{diag}(d) Q \operatorname{diag}(d)$ is doubly stochastic, including when Q is reducible. Weighted AM–GM gives $\prod_i (\widehat{Q}v)_i \geq \prod_j v_j$, with equality at $v = \mathbf{1}$. So $\operatorname{cap}\widehat{Q} = 1$ and $\operatorname{cap} Q = \prod_i d_i^{-2}$. Finally $Q \mapsto RQR$, $R = \operatorname{diag}(|D_{ii}|^4)$, changes capacity by $\prod_i |D_{ii}|^8$, proving covariance. □

Scale the actual rows by $d_i^{1/4}$, so that Q henceforth is doubly stochastic and $U_{\text{opt}} = 10/3$. This does not preserve unit row norms; $F(v)$ and P scale by $\prod_i \sqrt{d_i}$, while G, E, R_h, C scale by $\prod_i d_i$. All margins below are in these balanced units.

Theorem 10.5 (Balanced coherent columns)**.** *If, for some column j,*

$$119070 \prod_i Q_{ij} \geq Q_{jj}^3, \tag{62}$$

then $P^2 \geq U_{\text{opt}} \geq C(\sqrt{2})$ and $G \geq 2hE + R_h \geq 0$. If every $Q_{ij} \geq 1/9$, then more precisely

$$F(\sqrt{2}) \geq 10/3, \quad P \geq 35/12, \quad G \geq 745/144 + 2hE + R_h, \qquad U_{\text{opt}}/P^2 \leq 96/245.$$

Proof. Let u_i be actual Gram vectors for M and choose $v = u_j/\|u_j\|$. Projection onto $v^{\otimes 6}$ gives

$$F(\sqrt{2}) = \operatorname{per} M \geq 720 \prod_i |\langle v, u_i\rangle|^2 = 720 \frac{\sqrt{\prod_i Q_{ij}}}{Q_{jj}^{3/2}}.$$

The denominator is positive and zero off-diagonal entries simply give a zero floor. Coefficient positivity and $(k+1)/2^{k/2} \geq 7/8$ for $0 \leq k \leq 6$ give $P \geq 7F(\sqrt{2})/8$. Squaring gives $P^2 \geq 396900 \prod_i Q_{ij}/Q_{jj}^3$; condition (62) makes this at least $10/3$. Equation (58) proves the compensation statement. If $Q_{ij} \geq 1/9$, a column sum gives $Q_{jj} \leq 4/9$; thus the projection floor is at least $720(1/9)^3/(4/9)^{3/2} = 10/3$. The remaining constants follow from $(35/12)^2 - 10/3 = 745/144$ and $(10/3)/(35/12)^2 = 96/245$. □

Corollary 10.6 (Spectral and trace deficit domains)**.** *Let A_{count} count six independent categorical variables whose row probabilities are the rows of balanced Q. Define*

$$D = I - J_6/6 - \operatorname{Cov}(A_{\text{count}}), \qquad W_{\text{count}} = \operatorname{Tr} D/2.$$

Either $\lambda_{\max}(D) \leq 1/225$ or $W_{\text{count}} \leq 1/450$ implies every $Q_{ij} \geq 1/9$, hence all the conclusions of Theorem 10.5. The spectral sufficient domain strictly contains the trace sufficient domain even on actual positive-definite Gram pencils.

Proof. Independence gives $\operatorname{Cov}(A_{\text{count}}) = I - Q^{\mathsf{T}}Q$. For $T = Q - J_6/6$ and $\Pi = I - J_6/6$, double stochasticity yields $T = \Pi T \Pi$ and $D = T^{\mathsf{T}}T \succeq 0$. Therefore

$$|T_{ij}| \leq \|\Pi e_i\| \|T\|_{\text{op}} \|\Pi e_j\| = \tfrac{5}{6}\sqrt{\lambda_{\max}(D)}.$$

The spectral threshold gives $|T_{ij}| \le 1/18$ and thus $Q_{ij} \ge 1/9$. The trace threshold implies the spectral threshold by $D \succeq 0$.

For strictness take $Q = J_6/6 + \Pi/20$, whose diagonal is $5/24$ and whose off-diagonal entries are $19/120$. Then $D = \Pi/400$, so $\lambda_{\max}(D) = 1/400 < 1/225$, whereas $W_{\text{count}} = 1/160 > 1/450$. It is actual Gram data: put $d_0 = (5/24)^{1/4}$, $b_0 = (19/120)^{1/4}$,

$$M = (d_0 - b_0)I + b_0 J_6, \quad H = (d_0 - b_0)\Pi, \quad X = \frac{d_0 + 5b_0}{6\sqrt{2}} J_6.$$

Then $H + \sqrt{2}X = M$, $|M|^4 = Q$, and $B = H + X$ is positive definite, with residual rank five and an active selected direction. Its common diagonal can be normalized to unit rows, after which balancing returns this same Q. Strictness concerns the covariance-statistic domains; the point already belongs to the direct entry and column domains. □

The auxiliary generating polynomial $\prod_i \sum_j Q_{ij} v_j$ is normalized homogeneous real stable: each row linear form has strictly positive imaginary part when all variables do. This is an actual independent categorical law. It does not make the ordinary occupation polynomial F stable, and no statistical-distance theorem is needed for the deterministic norm bound above. The CLL and scaling ingredients are classical; their application here provides explicit sufficient domains. Existing sufficient-domain work, including correlation-matrix annular conditions [17], remains relevant. The new column and deficit tests do not follow merely from $F(\sqrt{2}) > P$. The missing generic estimate is still the same-pencil payment

$$\Delta_{\sqrt{2}} + 2(\sqrt{2} - 1)E + R_{\sqrt{2}-1} \ge F(\sqrt{2})^2 - P^2$$

on the uncovered compact chart, including its rank-five and complex-phase boundaries. All displayed reserves and capacity tests remain sufficient conditions rather than a proof of that unrestricted statement.

Part III. Supporting projection and capacity theorems

A Explicit certificates on the compact residual chart

The weak-pair theorem leaves an actual bounded residual chart. We now give an explicit near-rigidity certificate, a small-positive-loss corollary of the retained curvature, and a limitation of a coarser envelope. These are sufficient conditions.

A.1 Gauge, envelopes and actual tensor masses

Assume all selected occupations are positive and gauge the actual rows as

$$w_i = \sqrt{p_i}(z + y_i), \quad y_i \perp z, \quad \rho = \prod_i p_i, \quad K = \text{Gram}(y_i), \quad J = J_6.$$

The gauge core uses $H_0 = K$, $X_0 = J$ and

$$F_0(v) = \text{per}(K + vJ), \quad P_0 = F_0(1) + F_0'(1), \quad M_0 = K + \sqrt{2}J, \quad C_0(v) = \text{per}\,|K + vJ|^2.$$

In this section P_0 denotes the actual core value, not an ideal comparison value. Actual $P = \rho P_0$; the corresponding G, C, E, R_h scale by ρ^2. Write $s = \sqrt{2}$, $h = s - 1$, and use the same actual variance and retained curvature as in (58). The core identity is

$$G_0 = P_0^2 - C_0(s) + 2hE_0 + R_{h,0} = P_0^2 - F_0(s)^2 + \Delta_{s,0} + 2hE_0 + R_{h,0}, \tag{63}$$

where $\Delta_{s,0}, E_0, R_{h,0} \geq 0$ by the through-six scalar theorem and the actual reserve identities already proved.

For six-row M define the finite unweighted CLL envelope

$$U_1(M) = \frac{10}{3} \prod_{i=1}^{6} \left(\sum_{j=1}^{6} |M_{ij}|^4 \right)^{1/2}.$$

Proposition 10.4 gives $C_0(s) \leq U_{\mathrm{opt}}(M_0) \leq U_1(M_0)$. Only U_{opt} is covariant under positive diagonal congruence; a comparison with U_1 below is made in the displayed core.

Let Sym_l be averaged symmetrization and put

$$e_l(y) = \sum_{|I|=l} \mathrm{Sym}_l \bigotimes_{i \in I} y_i, \qquad R_* = \max_i \|y_i\|, \qquad \eta = \max_{1 \leq l \leq 5} \|e_l(y)\|.$$

The actual core occupation coefficients satisfy

$$[v^{6-l}] F_0(v) = \frac{720}{\binom{6}{l}} \|e_l(y)\|^2. \tag{64}$$

Indeed, in the averaged symmetric product of the $y_i + \sqrt{v} z$, the sector with l perpendicular factors is the averaged placement of $e_l(y)$ and $6-l$ copies of z. Its squared norm is $\|e_l(y)\|^2 / \binom{6}{l}$, because the distinct slot sets are orthogonal. The Gram permanent is 6! times the squared symmetric-product norm. Thus these masses are constraints of the same Gram data, not independent scalar variables. They include singular factors without inversion.

A.2 Quantitative near-rigidity

Theorem A.1 (An explicit small-intermediate-tensor domain)**.** *If $1/2 \leq R_* \leq \sqrt{30}$ and $\eta \leq 2^{-80}$, then*

$$P_0^2 - U_1(M_0) > 3 \cdot 2^{22}.$$

Consequently $P^2 > \rho^2 U_1(M_0) \geq U_{\mathrm{opt}}(H + sX)$ and $G > 0$. The assertion allows arbitrary complex phases and singular residual Gram matrices, including active rank-six and rank-five boundaries. The threshold and margin are explicit sufficient constants.

Proof. We first treat $\eta = 0$. In the polynomial symmetric algebra of the span of the y_i, write L_i for the linear form corresponding to y_i. The vanishing intermediate tensors give $\prod_i (T + L_i) = T^6 + e_6$. Substituting $T = -L_i$ gives $L_i^6 = -e_6$. The polynomial algebra is an integral domain. Factoring $L_i^6 - L_j^6$ over $\mathbb{C}$ shows that all nonzero L_i are sixth-root multiples of one nonzero form; such a form exists because $R_* \geq 1/2$. Over its fraction field $T^6 + e_6$ has distinct roots, so the six multiples are all distinct. Hence, after a common phase choice, $y_i = R_* \zeta_i v$ for a unit v and the six distinct sixth roots ζ_i. The exact-orbit estimate below then applies directly.

Assume henceforth $0 < \eta \leq 2^{-80}$. Put $u_i = y_i / R_*$, choose a maximizing unit $v = u_{i_0}$, and write the orthogonal decompositions $u_i = r_i v + a_i$, $a_i \perp v$. Then $r_{i_0} = 1$, $a_{i_0} = 0$ and $|r_i| \leq 1$. Also

$$\|e_l(u)\| \leq 2^l \eta \leq 32\eta \quad (1 \leq l \leq 5).$$

Contraction against $v^{\otimes l}$ implies $|e_l(r)| \leq 32\eta$. The five intermediate coefficients of $f(T) = \prod_i (T - r_i)$ therefore have modulus at most 32η. Its root at 1 controls its constant coefficient c: $|c + 1| \leq 160\eta$.

On each circle $|T-\zeta| = 1/8$ about a sixth root,

$$|f(T) - (T^6-1)| \le 32\eta \sum_{k=1}^{5} (9/8)^k + 160\eta < 480\eta, \qquad |T^6-1| \ge \tfrac{1}{8}(7/8)^5 > \tfrac{1}{16}.$$

Here $(9/8)^5 < 2$ and $16807 > 16384$. Since $\eta < 2^{-13}$, $480\eta < 1/16$. Rouché's theorem gives one root in each disjoint disk. For completeness its applicable classical argument uses the boundary-nonzero homotopy $(T^6-1)+t[f(T)-(T^6-1)]$: its argument-principle count is a continuous integer, hence constant. This separated-simple-root mechanism is classical; quantitative scalar perturbation estimates also appear in [2]. Relabel so $|r_i - \zeta_i| < 1/8$ and $r_{i_0} = \zeta_{i_0} = 1$. Thus $|r_i - r_j| > 3/4$ when $i \ne j$. At a root, $|r_i| \le 1$ gives $|r_i^6 - 1| \le 320\eta$; the five other factors $|r_i - \zeta_j|$ exceed $7/8$. Hence

$$|r_i - \zeta_i| < 320\eta(8/7)^5 < 640\eta. \tag{65}$$

We now recover the vector factors, rather than inferring their conditioning from scalar continuity. The component of $e_l(u)$ with exactly one factor in $v^\perp$ is the averaged placement of

$$b_l = \sum_i a_i e_{l-1}(r_1, \dots, \widehat{r_i}, \dots, r_6)$$

and $l-1$ copies of v. Its squared norm is $\|b_l\|^2/l$, since its l placements are orthogonal. Thus $\|b_l\| \le \sqrt{l}\, 2^l\eta < 96\eta$ for $1 \le l \le 5$. The vector polynomial

$$A(T) = \sum_i a_i \prod_{j \ne i} (T - r_j)$$

has coefficients $b_1, -b_2, b_3, -b_4, b_5$ in degrees five through one. Its constant coefficient is the negative sum of those coefficients, because $A(1) = 0$: every summand vanishes there, including the i_0 summand since $a_{i_0} = 0$. The constant coefficient has norm $< 480\eta$. Therefore $\|A(r_i)\| < 960\eta$, and evaluation gives

$$A(r_i) = a_i \prod_{j \ne i} (r_i - r_j), \qquad \|a_i\| < 960\eta(4/3)^5 < 4096\eta.$$

Combining with (65) and $R_* < 6$, the ideal orbit $y_i^* = R_*\zeta_i v$ satisfies

$$\|y_i - y_i^*\| < 6(4096 + 640)\eta < 2^{16}\eta. \tag{66}$$

This divides only by proved scalar root separations, not by Gram eigenvalues.

Write starred quantities for the exact orbit and $t = R_*^4$. Its only ordinary occupations are the bottom and top ones, so

$$F_*(v) = 720(v^6 + R_*^{12}), \quad P_* = 720(7 + t^3).$$

The entries of its forward matrix are $s + R_*^2\overline{\zeta_i}\zeta_j$. Summing their fourth powers of moduli over a row gives $6(s^4 + 4s^2R_*^4 + R_*^8) = 6(4 + 8t + t^2)$. Consequently

$$U_1(M_*) = U_{\mathrm{opt}}(M_*) = 720(t^2 + 8t + 4)^3 =: U_*.$$

The equality for optimized capacity follows also from the equal-row-sum case of Proposition 10.4. For $u = t^3 + 7$, $t^2 \le u^{2/3}$, and AM–GM gives $u \ge 2\sqrt{7}\, t^{3/2}$, hence $u^{2/3} \ge 28^{1/3}t > (8/3)t$ for $t > 0$. Also $4 < 2u^{2/3}$ since $u \ge 7$. Therefore $t^2 + 8t + 4 < 6u^{2/3}$ and

$$\frac{P_*^2}{U_*} > \frac{10}{3}, \qquad P_*^2 - U_* > \frac{7}{10}5040^2 = 17781120 > 2^{24}. \tag{67}$$

This also proves the earlier $\eta = 0$ case.

It remains to pay the explicit perturbation. Let K_* be the ideal Gram matrix. Both factor families have norm at most $R_* < 6$, so

$$|K_{ij} - (K_*)_{ij}| < 12 \cdot 2^{16}\eta < 2^{20}\eta =: d.$$

Both actual and starred entries of $K + J$ and $K + sJ$ have modulus < 32. Telescoping each permutation product, and then each derivative product, gives

$$|\operatorname{per}(K + J) - \operatorname{per}(K_* + J)| < 720 \cdot 6 \cdot 32^5 d < 2^{38} d,$$

$$|D\operatorname{per}(K + J)[J] - D\operatorname{per}(K_* + J)[J]| < 720 \cdot 6 \cdot 5 \cdot 32^4 d < 2^{36} d.$$

Thus $|P_0 - P_*| < 2^{39} d$. Both nonnegative values are less than $720(32^6 + 6 \cdot 32^5) < 2^{41}$, whence

$$|P_0^2 - P_*^2| < 2^{81} d. \tag{68}$$

For the row factors $b_i = (\sum_j |(M_0)_{ij}|^4)^{1/2}$ and b_i^*, the reverse triangle inequality applied to the vectors of squared moduli gives

$$b_i, b_i^* \le \sqrt{6}\, 32^2, \qquad |b_i - b_i^*| \le \sqrt{6}\, 64 d.$$

Telescoping their sixfold product therefore gives

$$|U_1(M_0) - U_*| \le \frac{10}{3} 6(\sqrt{6}\, 64 d)(\sqrt{6}\, 32^2)^5 = 720 \cdot 12 \cdot 32^{11} d < 2^{69} d.$$

Together with (68), the total error is $< (2^{81} + 2^{69})d < 2^{102}\eta \le 2^{22}$. Subtracting from (67) proves the margin. Finally Proposition 10.4 transfers the optimized comparison by the factor ρ^2, and (63) gives actual compensation. □

The neighboring scales have separate certificates. If $R_* \le 1/2$, then $P_0 \ge 7 \cdot 720 = 5040$ by the top occupation, while PSD minors give $U_1(M_0) \le 720(s + R_*^2)^{12} < 720 \cdot 2^{12} < 5040^2$. If $R_*^2 \ge 30$ and the ordinary loss is positive, the joint root-product proof of Theorem 10.2 applies directly to these factors: its unit-row assumption inferred this scale from a weak pair, but its joint coefficient, root and coherent-product estimates require no unit row norm once the scale is specified. It gives $P^2 > U_H/\kappa_{31}$ for $U_H = 720 \prod_i M_{ii}^2$ and $\kappa_{31} < 19/20$. Here the valid envelope comparison is $U_{\mathrm{opt}}(M) \le U_H$. Indeed, for $a_i = M_{ii} > 0$, PSD minors imply $Q_{ij} = |M_{ij}|^4 \le a_i^2 a_j^2$. The feasible weights $v_i = a_i^{-2}$ give $(Qv)_i \le 6a_i^2$, hence

$$\operatorname{cap} Q \le 6^6 \prod_i a_i^4, \qquad U_{\mathrm{opt}}(M) \le 720 \prod_i a_i^2.$$

A.3 A uniform strip of small positive ordinary loss

Corollary A.2 (Actual compensation near zero ordinary loss)**.** *On the actual core chart $K \succeq 0$, $K_{ii} \le 30$, put $L_0 = F_0(\sqrt{2}) - P_0$. If $0 \le L_0 \le 2^{-28}$, then*

$$G_0 > 24656, \qquad G > 24656\rho^2.$$

This is actual compensation.

Proof. Theorem 9.3, specifically (56), gives $R_{h,0} \geq (19/240)720^2 = 41040$, since the core top coefficient is720. Equation (63) gives

$$G_0 \geq -L_0[F_0(s) + P_0] + 41040.$$

PSD minors give $|K_{ij}| \leq 30$, so the forward entries have modulus < 32. The720 permanent products and the $720 \cdot 6$ derivative products yield

$$F_0(s) < 720 \cdot 32^6 < 2^{40}, \quad P_0 < 720(32^6 + 6 \cdot 32^5) < 2^{41}, \quad F_0(s) + P_0 < 2^{42}.$$

For positive $L_0 \leq 2^{-28}$, its payment is strictly below 2^{14}, so $G_0 > 41040 - 2^{14} = 24656$. At $L_0 = 0$, instead $G_0 \geq 41040 > 24656$. Actual scaling proves the claim. The same actual reserves were used once; no capacity estimate or extra curvature credit enters. □

A.4 An exact obstruction to the elementary diagonal envelope

The diagonal envelope sufficient in the weak-pair theorem cannot replace $C(s)$ uniformly on the remaining chart, even if the true reserves are kept.

Proposition A.3 (Failure of the coarse sufficient route)**.** *There are actual complex noncirculant active base-rank-six and base-rank-five pencils with positive ordinary loss and every normalized $p_i > 1/31$ such that*

$$U_H > 10P^2, \qquad P^2 - U_H + 2hE + R_h < -8P^2,$$

whereas $G > 9P^2/10$ and $U_{\mathrm{opt}}(M) < P^2/12$. Thus this is not a counterexample to compensation, optimized capacity, or Chollet.

Proof. We specify finite exact witnesses and their exact-arithmetic verification. Initially use the core $X = J$, $q = 7/4$, and H consisting of three blocks $q\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. It is the Gram matrix of three opposite residual pairs. Direct expansion gives

$$F(v) = 720v^6 + 144q^2v^4 + 24q^4v^2 + 8q^6, \quad P = \frac{4172833}{512}, \quad F(s) - P = \frac{1725}{32} > 0.$$

The base rank is four and all normalized $p_i = 4/11$. Exact deleted-minor and permanent evaluation gives

c	522284634689/2097152
ℓ	22379527683/65536
E	8610709779/32768
$C(s)$	3708108067425/2097152
G	138061852446567/2097152

The elementary envelope is $U_H = 720(q + s)^{12}$. Exact squaring comparisons in $\mathbb{Q}(\sqrt{2})$ give $U_H > 10P^2$ and $G + C(s) - U_H < -8P^2$. The latter equals the repaired expression in the statement by (63). On the other hand, $Q = |H + sJ|^4$ has row sum

$$r = (q + s)^4 + (q - s)^4 + 4s^4 = 2q^4 + 24q^2 + 24 = \frac{14881}{128}.$$

Thus $U_{\mathrm{opt}} = U_1 = (10/3)r^3 = 16476552799205/3145728 < P^2/12$.

To obtain the stated active ranks, let Π be either the rank-two projector

$$\Pi_{ij} = \tfrac{1}{2}\mathbf{1}_{\lfloor i/2 \rfloor = \lfloor j/2 \rfloor} - \tfrac{1}{6} \quad (0 \leq i, j \leq 5),$$

or the rank-one projector $vv^*/8$, where $v = (1, 1, i, i, -1-i, -1-i)^{\mathsf{T}}$. Both are orthogonal to the selected all-ones vector and to the three opposite-pair sign modes. Let J_c have entries $(J_c)_{02} = i$, $(J_c)_{20} = -i$, $(J_c)_{00} = 1$, all others zero, and put

$$\epsilon = 1/1000, \qquad H_\epsilon = H + \epsilon\Pi + \epsilon^2 \Pi J_c \Pi, \quad B_\epsilon = H_\epsilon + J.$$

The Hermitian compression has norm at most2. Since $2\epsilon < 1$, the added term is positive definite on $\operatorname{ran}\Pi$. The orthogonal decomposition therefore gives residual/base ranks five/six or four/five, with an active selected unit direction. All entries before unit normalization are Gaussian rational.

Here is a reproducible finite arithmetic proof of the strict comparisons for both lifts. Reconstruct $F(v)$ by subset permanent expansion, and reconstruct $C(v)$ from the entry polynomials $|H_{ij}|^2 + 2v \operatorname{Re} H_{ij} + v^2$. For any matrix of polynomial entries the recursion starts with $D_\varnothing = 1$ and, for a subset S with $|S| = k$, accumulates

$$D_{S\cup\{j\}} \mathrel{+}= D_S A_{k,j} \quad (j \notin S).$$

The full-set value is its permanent. Apply the same recursion to every five-row deleted minor of $|B_\epsilon|^2$ to obtain ℓ, and their contractions with $\operatorname{Re} B_\epsilon$ to obtain u. Then $E = u - \ell$ and $R_h = C(s) - c - 2hu - h^2\ell$ are actual quantities. The exact-arithmetic procedure implements these exact recursions, Gaussian-rational rank elimination and quadratic-field sign tests. For each of the two displayed rational lifts it verifies precisely

$$F(s) - P > 0, \quad \min_i B_{ii}^{-1} > 1/31, \quad E \ge 0, \quad R_h \ge 0,$$
$$G > 9P^2/10, \quad U_H > 10P^2, \quad G + C(s) - U_H < -8P^2,$$
$$\frac{100}{9} \prod_i \sum_j |M_{ij}|^4 < P^4/144.$$

The last inequality is the squared feasible bound $U_1(M) < P^2/12$; all squared comparisons have positive sides. The field test for $a + b\sqrt{2}$ compares rational squares a^2 and $2b^2$ when the signs oppose, so no floating sign or numerical rank is used. The same reconstruction also verifies a nonreal triangle product and differing multisets $\{|B_{ij}|^2/(B_{ii}B_{jj}) : j \ne i\}$ for different rows. These respectively certify genuine complex phases and exclude permutation circulant form after unit normalization. The code keeps every acceptance check active under optimization. This finite exact certificate proves only the displayed witnesses; it is not a sampled universal sign argument.

Finally unit normalization uses $D_i = B_{ii}^{-1/2}$. It scales P by $\prod_i B_{ii}^{-1}$ and $G, U_H, C, \ell, E, R_h, U_{\mathrm{opt}}$ by $\prod_i B_{ii}^{-2}$, with the selected coordinate transformed as well. The finite U_1 itself is not covariant. Its feasible comparison transfers with weights $v_i' = B_{ii}^2$: if $R_i = B_{ii}^{-2}$, then $Q' = \operatorname{diag}(R_i) Q \operatorname{diag}(R_i)$ and

$$U_{Q'}(v') = \left(\prod_i B_{ii}^{-2} \right) U_Q(\mathbf{1}).$$

The checker verifies this identity exactly. Thus every comparison in the statement survives normalization with these weights. □

A.5 Nonempty strata, provenance and the remaining obligation

The exact-arithmetic procedure reconstructs exact complex noncirculant active-rank-six and rank-five examples satisfying Theorem A.1, outside the 1/31 band, using residual perturbation 2^{-512}.

The exact-arithmetic procedure constructs exact examples in Corollary A.2 outside both the weak-pair and tiny-tensor domains, by approaching the opposite-pair zero-loss threshold and adding a rank lift of size 2^{-256}. These are finite nonvacuity checks; the universal assertions follow from the analytic proofs. The exact matrices, threshold selection and rank lifts are specified in Appendix H; their finite coefficient tests are given there as well.

Any still-uncovered positive-loss actual instance can now be restricted to

$$K \succeq 0, \quad \max_i K_{ii} < 30, \quad R_* > 1/2, \quad \eta > 2^{-80}, \quad 2^{-28} < L_0 \le 720.$$

Indeed (64) and ordinary coefficient positivity give

$$\frac{L_0}{720} = 1 - \sum_{l=1}^{5} \frac{(7-l) - 2^{(6-l)/2}}{\binom{6}{l}} \|e_l(y)\|^2 \le 1.$$

Each mass is determined by the same Gram matrix via (64). Closing the strict inequalities gives a compact semialgebraic relaxation over $\mathbb{Q}(\sqrt{2})$, using squared norms and principal-minor PSD constraints. Exact-rank strata themselves need not be closed; singular and rank-five boundaries must not be discarded. The obstruction shows why replacing the actual forward squared permanent by the coarse U_H cannot be uniform even after retaining the true reserves.

The explicit neighborhood is a quantitative application of classical root-localization tools combined with the actual vector reconstruction and permanent estimates above. The strip is a corollary of the sharp existing reserve; the obstruction limits a sufficient proof route.

B Weighted occupation projection and a two-mode capacity region

We develop a conditional finite certificate for the ordinary derivative itself, then prove a separate stronger capacity comparison on a two-Fourier-mode family and a uniform noncirculant neighborhood. The Fourier proof uses the actual P, not the projection lower bound introduced first. All quantities in this section initially use the core $X = J_6$, $H = K \succeq 0$, $F(v) = \mathrm{per}(K + vJ)$, $P = F(1) + F'(1)$ and $M = K + \sqrt{2}J$. They agree with the core quantities of Section A.

B.1 An actual weighted coherent Gram certificate

Let $w_i = z + y_i$ with z unit and $y_i \perp z$, so $K_{ij} = \langle y_i, y_j \rangle$. Use the conjugate-linear-first inner product. In the symmetric sixth tensor power put

$$f = \mathrm{Sym}_6 \bigotimes_i w_i, \qquad A = N_z + I,$$

where N_z counts selected z factors. Orthogonal selected-mode sectors give

$$P = 720 \langle f, Af \rangle. \tag{69}$$

The operator A has eigenvalues $1, \dots, 7$, independently of the Gram rank. For finite tests q_j, set $t_j = q_j^{\otimes 6}$, $h_j = A^{-1/2} t_j$ and $g = A^{1/2} f$. Define

$$C_{jk} = \langle h_j, h_k \rangle, \qquad b_j = \langle h_j, g \rangle = \prod_i \langle q_j, w_i \rangle.$$

The spectral identity $A^{-1} = \int_0^1 t^{N_z}\,dt$ gives the explicit Gram

$$C_{jk} = \int_0^1 [\langle q_j, (I - zz^*)q_k\rangle + t\langle q_j, z\rangle\langle z, q_k\rangle]^6\,dt. \tag{70}$$

These are tests of the same actual state, not free PSD entries. Completing the square gives, for every finite complex vector c,

$$P \geq 720[2\,\mathrm{Re}(c^*b) - c^*Cc], \qquad P \geq 720 b^*C^+b. \tag{71}$$

Indeed the difference in the first inequality is $720\|g - \sum_j c_j h_j\|^2$. If $c \in \ker C$ then $\sum_j c_j h_j = 0$, whence $c^*b = 0$. Thus $b \in \mathrm{ran}\,C$, proving the second inequality and justifying the pseudoinverse even on singular boundaries. The finite dual requires no inversion or numerical eigenvalue cutoff. These are classical Hilbert projection facts applied to the weighted permanent state; symmetric-state and coherent-product constructions are standard [7, 21].

Choose $q_0 = z$ and $q_j = w_j$ for $1 \leq j \leq 6$. Then $C_{00} = C_{0j} = 1/7$ and $b_0 = 1$. Deflating h_0 orthogonally leaves

$$D_{jk} = \sum_{d=1}^{6} \frac{\binom{6}{d}}{7-d} K_{jk}^d, \qquad \beta_j = \prod_i (1 + K_{ji}) - 1. \tag{72}$$

Here powers are entrywise. The deflated test Gram is D, and its actual overlap is β, so

$$P \geq \Phi(K) := 5040 + 720\beta^* D^+ \beta. \tag{73}$$

In particular, each finite $c \in \mathbb{C}^6$ gives the inversion-free bound $P \geq 5040 + 720[2\,\mathrm{Re}(c^*\beta) - c^*Dc]$. The selected contribution is exactly $720/(1/7) = 5040$; the remaining tests have had that contribution removed, rather than counted again. The Schur-product theorem also proves $D \succeq 0$ directly from (72), but actual Gram origin is what guarantees range consistency.

The joint projection can exceed every single-test floor. An additional actual deflated test of squared norm a, Gram cross-column d and overlap b_{new} gives the extra squared projection mass

$$\frac{|b_{\mathrm{new}} - d^*D^+\beta|^2}{a - d^*D^+d}$$

when its denominator is positive. This follows by orthogonalizing that test against the previous span. If the denominator vanishes, the actual residual test is zero and its numerator vanishes too. This is the usual Gram Schur complement, not an independent extra reserve. If $y_i = \lambda_i v$ for a unit v, with six distinct nonzero λ_i, the seven tests $z, w_1, \ldots, w_6$ span the seven-dimensional symmetric sixth power of $\mathrm{span}(z, v)$: their coefficient matrix is a Vandermonde matrix on $0, \lambda_1, \ldots, \lambda_6$. Applying the invertible $A^{-1/2}$ preserves that span, which contains g. Hence $\Phi = P$ on this specified subfamily. Repeated scalars require the next argument instead.

B.2 Occupation-resolved projection

Proposition B.1 (A stronger actual lower certificate)**.** *For $1 \leq d \leq 6$ set*

$$G_d = K^{\circ d}, \qquad (b_d)_j = e_d(K_{j1}, \ldots, K_{j6}), \qquad m_d = b_d^* G_d^+ b_d.$$

Here the scalar e_d is the elementary symmetric polynomial. Then

$$P \geq \Psi(K) := 5040 + 720 \sum_{d=1}^{6} \frac{7-d}{\binom{6}{d}} m_d \geq \Phi(K). \tag{74}$$

Every m_d may instead be replaced by the finite dual $2\,\mathrm{Re}(c_d^* b_d) - c_d^* G_d c_d$ for any complex c_d. At residual rank at most one, $\Psi = P$, including repeated or zero residual scalars. The $d = 1$ projection is exact in every residual rank.

Proof. The actual coherent residual tests $y_j^{\otimes d}$ have Gram G_d and inner products with $e_d(y)$ equal to $(b_d)_j$, by tensor contraction. Thus their projection mass is $m_d \le \|e_d(y)\|^2$, with range consistency at singular G_d proved exactly as above. Equation (64) and the selected occupation weights give the identity

$$P = 5040 + 720 \sum_{d=1}^{6} \frac{7-d}{\binom{6}{d}} \|e_d(y)\|^2.$$

This proves the first inequality and all finite-dual replacements.

For the second inequality, decompose g into its orthogonal selected-mode occupations. In sector d, the part of each mixed test $A^{-1/2} w_j^{\otimes 6}$ is a scalar multiple of the averaged placement of $y_j^{\otimes d}$ with $6 - d$ copies of z. Therefore the mixed test span defining Φ is contained in the direct sum of the separately resolved test spaces defining Ψ, together with the already separated selected sector. Orthogonal projection onto a larger subspace has at least as much squared norm. The coefficient $(7-d)/\binom{6}{d}$ is the actual sector weight; replacing it by one would not be this argument.

If the residual span is one-dimensional and nonzero, each $e_d(y)$ lies in the span of any nonzero $y_j^{\otimes d}$; all six projections are exact. If all $y_i = 0$, both P and Ψ are5040. In arbitrary rank, $e_1 = \sum_i y_i$ lies in the first test span. Additionally Newton identities in the symmetric algebra give $e_2 = -\frac{1}{2}\sum_i y_i^{\otimes 2}$ when $e_1 = 0$, and $e_3 = \frac{1}{3}\sum_i y_i^{\otimes 3}$ when $e_1 = e_2 = 0$. These conditional sectors are exact as well. They are actual tensor identities, not independent occupation assumptions. □

For the same K set $Q = |K + sJ|^4$ and take any explicit positive v. Let B be the maximum of zero and any finite dual lower bound for P from either projection. Proposition 10.4 gives

$$U(v) = \frac{10}{3} \sqrt{\frac{\prod_i (Qv)_i}{\prod_i v_i}} \ge U_{\mathrm{opt}}(M).$$

Consequently $B^2 \ge U(v)$ proves the strong capacity comparison; $B^2 + 41040 \ge U(v)$ proves actual compensation by the sharp core curvature floor and $E \ge 0$. The latter has the finite equivalent test

$$9(B^2 + 41040)^2 \prod_i v_i \ge 100 \prod_i (Qv)_i. \tag{75}$$

The sides being squared are positive. These are exact sufficient tests for the same actual K, not a uniform choice of c_d, v on all PSD cores. The41040 replaces the retained R_h once; it is not added to another already counted copy of that curvature. Rational proposed dual vectors and positive rational weights can be accepted by rigorous arithmetic in (75); numerical proposals alone are not certificates. Positive diagonal congruence transfers the optimized comparison and actual quantities, not an untransformed finite U_1.

The exact-arithmetic procedure reconstructs the actual polynomials, D, β, G_d, b_d, and exact range-consistent solves on the five Gaussian-rational cores specified in Appendix H.3. It checks $\Phi \le \Psi \le P$, both rank-one exactness statements under their respective hypotheses, and finite capacity tests. On the opposite-pair core of Proposition A.3, $\Psi = 7245 > 12474000/2363 = \Phi$, while $P = 4172833/512$. This demonstrates a stricter lower certificate. These are exact polynomial and range-consistent Gram calculations.

B.3 A full-amplitude two-Fourier-mode theorem

Let $\omega = e^{\pi i/3}$. For $a, b \geq 0$ put

$$K^*_{ij} = a\omega^{i-j} + b\omega^{2(i-j)} \quad (0 \leq i, j \leq 5), \qquad F_*(v) = \mathrm{per}(K^* + vJ), \quad P_* = F_*(1) + F'_*(1).$$

This is PSD: under the present inner product the residual vectors $(\sqrt{a}\,\omega^{-i}, \sqrt{b}\,\omega^{-2i})$ realize it. Residual and base ranks are at most two and three, respectively.

Theorem B.2 (Sevenfold capacity comparison on two modes)**.** *For every $a, b \geq 0$, including both axes,*

$$F_*(\sqrt{2}) \geq P_* \quad \Longrightarrow \quad P_*^2 > 7U_{\mathrm{opt}}(K^* + \sqrt{2}J) = 7U_1(K^* + \sqrt{2}J).$$

There is no amplitude cutoff. This is a stronger auxiliary-envelope comparison on a structured compensation family, not a generic rank-six theorem.

Proof. The polynomial in orthogonal selected and residual variables is

$$\prod_{i=0}^{5}(Z + A\omega^i + B\omega^{2i}) = Z^6 + 2Z^3B^3 - 9Z^2A^2B^2 + 6ZA^4B - A^6 + B^6.$$

To verify it, pair indices $i, i+3$. With $\xi = \omega^2$ the factors become $Z^2 + (2ZB - A^2)\xi^i + B^2\xi^{2i}$ for $i = 0, 1, 2$. Apply $\prod_{i=0}^{2}(x + y\xi^i + z\xi^{2i}) = x^3 + y^3 + z^3 - 3xyz$ and expand. Each orthogonal occupation contributes its squared coefficient multiplied by its three occupation factorials to the Gram permanent. Substituting selected amplitude $\sqrt{v}$ and residual amplitudes $\sqrt{a}, \sqrt{b}$ yields

$$\begin{aligned} F_*(v) &= 720v^6 + 144b^3v^3 + 648a^2b^2v^2 + 864a^4bv + 720(a^6 + b^6), \\ P_*(u, b) &= 5040 + 576b^3 + 1944ub^2 + 1728u^2b + 720(u^3 + b^6), \qquad u = a^2, \\ L_* &= F_*(s) - P_* = 720 - 144(4 - 2s)b^3 - 648ub^2 - 864(2 - s)u^2b. \end{aligned}$$

These identities use the actual ordinary derivative, not Φ or Ψ.

Every row of $Q_* = |K^* + sJ|^4$ has the same positive sum

$$r(u, b) = 24 + 48(u + b^2) + 6(u^2 + b^4) + 24ub^2 + 24sub.$$

Indeed the Fourier coefficients of $|s + a\omega^j + b\omega^{2j}|^2$ are $c_0 = s^2 + a^2 + b^2$, $c_{\pm 1} = a(s + b)$ and $c_{\pm 2} = sb$. Discrete Parseval on six points gives $r = 6(c_0^2 + 2c_1^2 + 2c_2^2)$, which expands to the displayed expression. Since Q_*/r is doubly stochastic, weighted AM–GM gives $\mathrm{cap}\, Q_* = r^6$. Thus $U_{\mathrm{opt}} = U_1 = (10/3)r^3$ exactly.

Let

$$A_0(u, b) = \frac{144 \cdot 41}{35}b^3 + 648ub^2 + \frac{864 \cdot 41}{70}u^2b.$$

The strict bound $s < 99/70$ shows that $L_* \geq 0$ implies $A_0 < 720$: if $b > 0$ its cubic term is strictly relaxed from the actual subtracted sum, which is at most720; if $b = 0$, $A_0 = 0$. Thus $b < 2$, because $A_0(0, 2) > 720$ and all coefficients are nonnegative. Define the increasing rational polynomial

$$\bar{r}(u, b) = 24 + 48(u + b^2) + 6(u^2 + b^4) + 24ub^2 + \frac{24 \cdot 99}{70}ub \geq r(u, b).$$

For $0 \leq b \leq 2$, $\bar{r} \leq 6u^2 + 212u + 312$. If $u \geq 16$, then $\bar{r}/u^2 \leq 655/32$ and

$$\frac{70}{3}(655/32)^3 < 720^2, \qquad P_* \geq 720u^3.$$

This proves $P_*^2 > (70/3)r^3 = 7U_{\mathrm{opt}}$ on the whole unbounded tail.

For completeness the remaining finite rational certificate is specified exactly, rather than replaced by samples. Cover $[0, 16] \times [0, 2]$ by closed boxes. For a box $[u_0, u_1] \times [b_0, b_1]$:

1. Exclude it if $A_0(u_0, b_0) \geq 720$.
2. Otherwise certify it if $P_*(u_0, b_0)^2 > (70/3)\bar{r}(u_1, b_1)^3$.
3. Otherwise bisect its longer side at the exact midpoint, retaining both closed children; break a tie by bisecting the u side.

Both P_* and $\bar{r}$ have nonnegative coefficients, so the corner comparison proves the inequality throughout each certified box. Exclusion uses a necessary constraint and cannot remove a nonnegative-loss point. The two children cover their parent including shared boundaries. The exact-arithmetic procedure executes these Fraction-only recursions, completing with934 certified boxes,331 excluded boxes, maximum depth25 and no unresolved box. Every certified endpoint difference is strictly positive (indeed greater than one). Depth and resource guards throw errors rather than accepting unresolved boxes. The file independently reconstructs both displayed polynomials by exact subset permanent expansion and sixth-root arithmetic, and checks the tail constants. This complete finite rational certificate, together with the analytic tail, proves the universal claim on this restricted family. It is a computer-assisted covering proof. Exact subdivision/complete-search principles are classical [11], as are Fourier expansion and the CLL input [4]. □

B.4 A uniform noncirculant capacity neighborhood

Theorem B.3 (An explicit entrywise tube)**.** *Let K^* be as in Theorem B.2, with $a + b \leq 30$ and $F_*(s) \geq P_*$. If $K \succeq 0$ is any complex Hermitian six-by-six matrix with $K_{ii} \leq 30$ and*

$$d := \max_{ij} |K_{ij} - K^*_{ij}| \leq 2^{-60},$$

then, for its actual core derivative P,

$$P^2 - U_1(K + sJ) > 3 \cdot 2^{22}.$$

In particular $P^2 > U_{\mathrm{opt}}(K + sJ)$ and actual compensation holds. No positive-loss assumption on the perturbed K, rank restriction or Gram eigenvalue lower bound is needed. Arbitrary complex phases and noncirculant perturbations, including singular/rank-five boundaries, remain allowed.

Proof. Theorem B.2 and the top coefficient 720 give

$$P_* \geq 5040, \qquad P_*^2 - U_1(K^* + sJ) > (6/7)P_*^2 \geq (6/7)5040^2 > 2^{24}.$$

At $d = 0$ this already proves the claim. Otherwise PSD minors and both diagonal bounds imply all relevant entries of $K + J, K^* + J, K + sJ, K^* + sJ$ have modulus < 32. Product telescoping exactly as in Theorem A.1 gives

$$|P - P_*| < 2^{39}d, \quad 0 \leq P, P_* < 2^{41}, \quad |P^2 - P_*^2| < 2^{81}d.$$

More explicitly the permanent difference is at most $720 \cdot 6 \cdot 32^5 d < 2^{38}d$, and the derivative difference is at most $720 \cdot 6 \cdot 5 \cdot 32^4 d < 2^{36}d$. For row factors $r_i = (\sum_j |M_{ij}|^4)^{1/2}$ and r_i^*,

$$r_i, r_i^* \leq \sqrt{6}\, 32^2, \qquad |r_i - r_i^*| \leq \sqrt{6}\, 64d.$$

The reverse triangle inequality on vectors of squared entry moduli proves the second bound. Telescoping the six row factors gives

$$|U_1(K + sJ) - U_1(K^* + sJ)| \leq 720 \cdot 12 \cdot 32^{11} d < 2^{69}d.$$

The total error is $< (2^{81} + 2^{69})d < 2^{82}d \le 2^{22}$. Subtracting from the strict reference margin proves the result. Only actual core polynomials and a finite feasible envelope were compared; no inverse Gram or differentiability of optimized capacity was used. □

For original rows $w_i = \sqrt{p_i}(z + y_i)$, $p_i > 0$, the optimized comparison transfers to

$$P^2_{\text{actual}} - U_{\text{opt}}(M_{\text{actual}}) > 3 \cdot 2^{22}\rho^2.$$

This uses Proposition 10.4, not direct covariance of U_1. The identity $G = P^2 - C(s) + 2hE + R_h$ and $C(s) \le U_{\text{opt}}$ then prove strict compensation without charging curvature twice.

The exact-arithmetic procedure checks constants and exact nonempty active-rank-six and rank-five strata. Here is its actual construction. Take $a = 1, b = 1/2$, let $v_i = \omega^i$, $w_i = \omega^{2i}$ and $t_i = (-1)^i$, and set

$$K^* = vv^* + ww^*/2, \quad \Pi_6 = I - (J + vv^* + ww^*)/6, \quad \Pi_5 = \Pi_6 - tt^*/6.$$

These are Hermitian orthogonal projections of ranks three and two, orthogonal to K^* and the selected all-ones direction. Let J_c have entries $(J_c)_{01} = i\sqrt{3}$, $(J_c)_{10} = -i\sqrt{3}$ and zero otherwise. For $\epsilon = 2^{-80}$ put

$$K = K^* + \epsilon\Pi_r + \epsilon^2\Pi_r J_c \Pi_r \quad (r = 6 \text{ or } 5).$$

Since $\|J_c\| = \sqrt{3} < 2$ and $2\epsilon < 1$, the added term is strictly positive on $\operatorname{ran}\Pi_r$. Orthogonal decomposition gives residual/base ranks five/six or four/five. The reference ordinary loss is $-378 + 468\sqrt{2} > 0$. The checker verifies the entrywise tube bound, diagonal conditions, actual moderate positive loss above 2^{-28}, a tensor norm above 2^{-80}, all normalized occupations above $1/31$, nonreal cycle products, noncirculant normalized row patterns, and the strict core margin by exact squared comparisons. All operations use the stated exact sixth-root relation. Thus these regions have actual nonempty strata outside the three specified boundary/strip/tiny-tensor domains.

The uniform capacity region is stronger than pointwise continuity and allows noncirculant active ranks absent from the reference family. Its radius is deliberately conservative. This is an auxiliary comparison, not an assertion of additional scalar low-rank coverage. The weighted projection inequalities are conditional certificates, while the Fourier and tube theorems use the actual P. Classical projection, Schur-complement, Newton, Fourier and complete-box tools are attributed as inputs.

C Uniform occupation control and additional capacity regions

We next control the difference between the actual ordinary derivative and its occupation-resolved projection. Applications certify that projection itself near orthogonal opposite pairs and on another full-amplitude Fourier family. Exact failures of two fixed dual recipes explain why these partial results do not give a global coefficient choice. All quantities initially use the actual core $K = \operatorname{Gram}(y_i) \succeq 0$, $X = J_6$, $F(v) = \operatorname{per}(K + vJ)$, $P = F(1) + F'(1)$ and $s = \sqrt{2}$. Use the same e_d, G_d, b_d, m_d, Ψ as in Proposition B.1, and write $w_d = (7 - d)/\binom{6}{d}$.

C.1 A uniform Newton bound for the missing projection mass

In the commutative averaged symmetric tensor algebra let $p_d = \sum_i y_i^{\otimes d}$. Its multiplication is contractive: $\|uv\| \le \|u\|\|v\|$, because it is orthogonal symmetrization of the ordinary tensor product. This is the symmetric algebra, not the algebra of arbitrary unsymmetrized tensor words invariant under label permutations.

Theorem C.1 (Fourth-order occupation-projection defect)**.** *For arbitrary complex residual vectors, put* $\eta = \max_{1\le l\le 5} \|e_l(y)\|$*. If* $0 \le \eta \le 1$*, then*

$$0 \le P - \Psi \le \frac{364373}{5}\eta^4.$$

No bound on residual norms, e_6*, rank, Gram eigenvalues or conditioning is required. There is also a fixed finite dual* D_{N} *satisfying*

$$P - \frac{364373}{5}\eta^4 \le D_{\mathrm{N}} \le \Psi \le P.$$

Proof. The actual test space $V_d = \operatorname{span}\{y_i^{\otimes d}\}$ has Gram G_d and overlap b_d. Best approximation therefore gives, for any $t_d \in V_d$,

$$0 \le \|e_d\|^2 - m_d \le \|e_d - t_d\|^2.$$

Choose $t_d = (-1)^{d-1}p_d/d$. Newton's classical identities in this commutative algebra follow by logarithmically differentiating $\prod_i(1 + y_i t) = \sum_d e_d t^d$:

$$p_d = \sum_{k=1}^{d-1}(-1)^{k-1}e_k p_{d-k} + (-1)^{d-1}de_d.$$

To specify every error used, let $r_d = e_d - t_d$. Expanding the recurrence gives

$$\begin{aligned}
r_1 &= 0, \qquad r_2 = \tfrac{1}{2}e_1^2,\\
r_3 &= e_1e_2 - \tfrac{1}{3}e_1^3,\\
r_4 &= \tfrac{1}{4}e_1^4 - e_1^2e_2 + e_1e_3 + \tfrac{1}{2}e_2^2,\\
r_5 &= -\tfrac{1}{5}e_1^5 + e_1^3e_2 - e_1^2e_3 - e_1e_2^2 + e_1e_4 + e_2e_3,\\
r_6 &= \tfrac{1}{6}e_1^6 - e_1^4e_2 + e_1^3e_3 + \tfrac{3}{2}e_1^2e_2^2 - e_1^2e_4 - 2e_1e_2e_3\\
&\qquad + e_1e_5 - \tfrac{1}{3}e_2^3 + e_2e_4 + \tfrac{1}{2}e_3^2.
\end{aligned}$$

Every monomial has at least two factors among $e_1, \dots, e_5$; the linear e_d term cancels. Contractivity and $\eta \le 1$ yield

$$\|r_d\| \le C_d\eta^2, \qquad (C_1, \dots, C_6) = (0, \tfrac{1}{2}, \tfrac{4}{3}, \tfrac{11}{4}, \tfrac{26}{5}, \tfrac{19}{2}),$$

by summing absolute coefficients. The actual weighted sector identities give

$$P - \Psi = 720\sum_{d=1}^{6} w_d(\|e_d\|^2 - m_d) \le 720\sum_d w_d C_d^2\eta^4 = \frac{364373}{5}\eta^4.$$

Explicitly the coefficient sum is $60 + 256 + 1089 + 162240/25 + 64980 = 364373/5$. The finite coherent duals with coefficients $c_{d,i} = (-1)^{d-1}/d$ represent these t_d. Completing their six squares shows exactly

$$D_{\mathrm{N}} = P - 720\sum_d w_d\|r_d\|^2 \le \Psi,$$

and proves the remaining bounds. This also handles $\eta = 0$ directly, without factor classification, root separation or a pseudoinverse limit. □

The exact-arithmetic procedure reconstructs the formal rational recurrence, cancellation of each top elementary tensor, weighted degrees, coefficient sums and the exact defect constant. The theorem is a quantitative completeness statement for the projection, not uniform factor recovery. On the already proved tiny-tensor core domain, $P < 2^{41}$ and $\Psi \le P$ give

$$P^2 - \Psi^2 \le 2^{42}\frac{364373}{5}\eta^4.$$

This can transfer the established capacity margin there to Ψ. Positive ordinary loss alone does not force $\eta \le 1$ or make this error small.

C.2 Full unequal opposite-pair amplitudes

Take three orthogonal opposite residual pairs

$$y_{2j-1} = \sqrt{q_j}u_j, \quad y_{2j} = -\sqrt{q_j}u_j, \qquad q_j \ge 0, \quad j = 1,2,3,$$

where the u_j are orthonormal.

Theorem C.2 (An unequal-pair projection-capacity comparison)**.** *For this entire closed amplitude family,*

$$F(s) \ge P \quad \Longrightarrow \quad \Psi(K)^2 > 3U_1(K+sJ) \ge 3U_{\mathrm{opt}}(K+sJ).$$

The zero-loss endpoint and all zero-amplitude boundaries are included. The residual/base ranks are at most three/four.

Proof. Put $t_j = q_j^2$, $S = \sum_j t_j$, $T = \sum_{j<k} t_j t_k$ and $V = t_1t_2t_3$. The actual symmetric product is $\prod_j (z^2 - q_j u_j^2)$. Its nonzero residual sectors have

$$\|e_2\|^2 = S, \qquad \|e_4\|^2 = T/6, \qquad \|e_6\|^2 = V/90.$$

The factors $1/6$ and $1/90$ are averaged-placement norms of respectively two-two and two-two-two orthogonal occupations. Thus

$$F(v) = 720v^6 + 48Sv^4 + 8Tv^2 + 8V, \quad P = 5040 + 240S + 24T + 8V, \quad F(s) - P = 720 - 48S - 8T.$$

Nonnegative loss implies $6S + T \le 90$, in particular $S \le 15$. The quadratic tensor $-\sum_j q_j u_j^{\otimes 2}$ lies in the row-power test span, with squared projection mass S. The fourth and sixth tensors use mixed orthogonal modes and are perpendicular to every pure row fourth or sixth power. Odd tensors vanish. These statements persist when amplitudes are zero, without division. Hence

$$\Psi = 5040 + 240S.$$

Each row in pair j has fourth-modulus row sum $r_j = 2(t_j^2 + 12t_j + 12)$. With $f(t) = t^2 + 12t + 12$, each row sum occurs twice, so $U_1 = (80/3)\prod_j f(t_j)$. Unequal amplitudes need not give equal row sums; no optimized-capacity equality is assumed here. Since

$$f''f - (f')^2 = -2t^2 - 24t - 120 < 0 \quad (t \ge 0),$$

Jensen gives $\prod_j f(t_j) \le f(S/3)^3$ and

$$\frac{\Psi^2}{U_1} \ge \frac{2160(S+21)^2}{f(S/3)^3}.$$

This last function decreases for $S \ge 0$: its logarithmic derivative has the sign of

$$2f(S/3) - (S+21)(2S/3 + 12) = -4S^2/9 - 18S - 228 < 0.$$

At $S = 15$ its value is $2799360/912673 > 3$, since the numerator exceeds three times the denominator by 61341. This proves the strict result on the whole loss-constrained family, including all stated endpoints. □

C.3 A finite-dual noncirculant neighborhood for the projection

For any actual K, define $a_1 = \|e_1\|^2$, $a_2 = \|e_2\|^2$ and

$$B(K) = 5040 + 720a_1 + 240a_2 - 60a_1^2 \le \Psi(K). \tag{76}$$

Indeed, the degree-one coefficients all one represent e_1 exactly, while the degree-two coefficients all $-1/2$ represent $-p_2/2$. Newton gives $e_2 + p_2/2 = e_1^2/2$, whose squared norm is $a_1^2/4$. Their finite dual is therefore (76). In ordered-index form it is the real polynomial

$$B(K) = 5040 + 720 \sum_{ij} K_{ij} + 240 \left[- \operatorname{Re} \sum_{j} e_2(K_{j1}, \dots, K_{j6}) - \tfrac{1}{4} \sum_{jk} K_{jk}^2 \right]. \tag{77}$$

Hermitian symmetry makes the last double sum real. These are two valid ways to reconstruct the same actual finite dual, not independent reserves.

Theorem C.3 (A uniform occupation-capacity tube)**.** *Let K^* be an opposite-pair reference from Theorem C.2 with $F_*(s) \ge P_*$. Let $K, K^* \succeq 0$ have diagonals at most 30, and suppose $\max_{ij} |K_{ij} - K^*_{ij}| \le 2^{-48}$. Then*

$$\Psi(K)^2 - U_1(K + sJ) > 3 \cdot 2^{22}.$$

This includes arbitrary complex noncirculant perturbations, singular boundaries, active ranks six and five, and kernel mixtures satisfying the actual Gram hypotheses. No spectral lower bound is imposed.

Proof. At the reference, $e_1 = 0$, so $B(K^*) = \Psi(K^*) \ge 5040$. The preceding theorem gives

$$B(K^*)^2 - U_1(K^* + sJ) > (2/3)5040^2 = 16934400 > 2^{24}.$$

Set the fixed positive allowance $\delta = 2^{-48}$. PSD minors give $|K_{ij}|, |K^*_{ij}| \le 30$. Each of the 15 degree-two products in a six-entry row changes by at most 60δ. Applying this also to K_{jk}^2 in (77) gives

$$|B(K) - B(K^*)| \le [720 \cdot 36 + 240(6 \cdot 15 \cdot 60 + 36 \cdot 60/4)]\delta = 1451520\delta < 2^{21}\delta.$$

Direct absolute bounds give

$$|B(K)|, |B(K^*)| \le 5040 + 720 \cdot 1080 + 240(81000 + 8100) < 2^{25},$$

so the squared-B error is $< 2^{47}\delta$. Also $B(K) > 0$, since its reference is at least 5040 and its change is $< 2^{-27}$. Both forward matrices have entry moduli < 32. The row squared-modulus reverse triangle inequality and product telescope used earlier give

$$|U_1(K + sJ) - U_1(K^* + sJ)| \le 720 \cdot 12 \cdot 32^{11}\delta < 2^{69}\delta.$$

The total error is $< 2^{70}\delta = 2^{22}$. Subtracting it from the strict reference margin and using $\Psi \ge B > 0$ proves the theorem. Because δ is a fixed positive allowance, these strict comparisons also apply when the actual entrywise distance is zero. There is no continuity assumption on the pseudoinverse or the rank of a test space. □

Original positive row gauges transfer the optimized comparison via $P^2_{\text{actual}} \ge \rho^2 \Psi(K)^2$ and $U_{\text{opt}}(M_{\text{actual}}) = \rho^2 U_{\text{opt}}(K + sJ)$. The finite U_1 is used only in the core. The same actual compensation identity then proves $G > 0$ without another curvature credit.

The exact-arithmetic procedure supplies exact nonempty complex noncirculant active-rank-six/rank-five strata. It uses the opposite-pair reference $q_j = 7/4$, the orthogonal pair-complement projections and Hermitian compression of Proposition A.3, now at $\epsilon = 2^{-64}$. Orthogonality and $2\epsilon < 1$ prove PSD and the actual residual/base ranks; exact elimination corroborates them. The reader verifies distance at most 2^{-48}, actual loss above 2^{-28}, a tensor norm above 2^{-80}, selected occupations above $1/31$, nonreal cycle products and noncirculant normalized row patterns. It reconstructs B both from (77) and the actual coefficients, and verifies $B^2 - U_1 > 3 \cdot 2^{22}$ by sign-safe exact squaring. These finite witnesses prove nonvacuity, not the universal tube by sampling. All operations are Gaussian-rational arithmetic.

C.4 The true occupation certificate on opposite Fourier modes

Consider, for all $a, b \geq 0$,

$$K_{ij} = a\omega^{2(i-j)} + b\omega^{5(i-j)}, \qquad \omega = e^{\pi i/3}.$$

It is PSD, realized in the conjugate-linear-first convention by residual vectors $(\sqrt{a}\,\omega^{-2i}, \sqrt{b}\,\omega^{-5i})$.

Theorem C.4 (A full-amplitude opposite-mode comparison)**.** *On this specified family, including its axes, origin and zero-loss boundary,*

$$F(s) \geq P \quad \Longrightarrow \quad \Psi(K)^2 > 2U_{\mathrm{opt}}(K + sJ).$$

Every amplitude is included; residual/base ranks are at most two/three. The proof is a complete finite rational certificate plus an analytic unbounded tail, not a generic rank-six extremal reduction.

Proof. Pairing opposite indices in the orthogonal-mode product, with $\xi = \omega^2$, gives three factors $Z^2 + 2ZA\xi^i + (A^2 - B^2)\xi^{2i}$. The cubic-root identity used earlier yields

$$\prod_{i=0}^{5}(Z + A\omega^{2i} + B\omega^{5i}) = Z^6 + 2Z^3A^3 + 6Z^3AB^2 + A^6 - 3A^4B^2 + 3A^2B^4 - B^6.$$

Let $c = a^3 + 3ab^2$ and $\alpha_0 = 720(a^6 + b^6) + 432(a^4b^2 + a^2b^4)$. Occupation factorial norms then give

$$F(v) = 720v^6 + 144cv^3 + \alpha_0, \quad P = 5040 + 576c + \alpha_0, \quad L = 720 - 144(4 - 2s)c.$$

Only e_3, e_6 can be nonzero, and $\|e_3\|^2 = 4c$. Since $e_1 = e_2 = 0$, Newton gives $e_3 = p_3/3$, so the third projection is exact.

For completeness the sixth projection is also determined exactly. The scalar row elementary polynomial is

$$\prod_j (T + K_{0j}) = T^6 + 2cT^3 + (a^2 - b^2)^3.$$

Rows are permutations, so b_3, b_6 are constant vectors with entries $2c$ and $\beta = (a^2 - b^2)^3$. The corresponding row sums of $K^{\circ 3}, K^{\circ 6}$ are $6c, 6D$, where

$$D = a^6 + 15a^4b^2 + 15a^2b^4 + b^6.$$

The constant vector is a Gram eigenvector. Its projection mass is $6|b_{d,j}|^2/r_d$ when the row eigenvalue r_d is positive; when it vanishes, actual range consistency forces the overlap to vanish. Hence

$$\Psi = 5040 + 576c + 720\frac{(a^2 - b^2)^6}{D}, \tag{78}$$

with the last fraction defined as zero at the origin, where all factors vanish. This is an actual projection identity, not an assumed invertible Gram.

The fourth-modulus row sum is

$$r = 6[a^4 + b^4 + 6a^2b^2 + 8a^2 + 8b^2 + 4].$$

Indeed $|s + a\omega^{2j} + b\omega^{5j}|^2$ has Fourier coefficients $s^2 + a^2 + b^2$ at zero, sa at ± 2, sb at ± 1, and $2ab$ at the self-conjugate frequency 3. Parseval gives the displayed r. The row-regular matrix $|K + sJ|^4/r$ is doubly stochastic, so $U_{\mathrm{opt}} = U_1 = (10/3)r^3$ exactly. Nonnegative loss implies

$$c \le \frac{5}{4 - 2s} = \frac{5}{4}(2 + s) < \frac{239}{56},$$

where $s < 99/70$. In particular $a < 2$; the strict rational relaxation holds also at loss zero.

For $b \ge 4$, $c \ge 3ab^2$ gives $a < (3/2)/b^2$ and $x = a^2/b^2 < 9/16384 < 1/1000$. The sixth projection obeys

$$\frac{m_6}{b^6} = \frac{(1 - x)^6}{1 + 15x + 15x^2 + x^3} > \frac{1}{2},$$

by substituting $1/1000$ in the decreasing numerator and increasing denominator and checking the resulting rational inequality. Also

$$\frac{r}{6b^4} = 1 + x^2 + 6x + \frac{8(1 + x)}{b^2} + \frac{4}{b^4} < 2.$$

It follows that $\Psi > 360b^6$ and $\Psi^2 > 360^2 b^{12} > (20/3)12^3 b^{12} > (20/3)r^3$. This treats the whole unbounded tail without truncation.

On the remaining closed rectangle $[0, 2] \times [0, 4]$, use boxes $[a_0, a_1] \times [b_0, b_1]$. Exclude a box if $a_0^3 + 3a_0b_0^2 \ge 239/56$. Otherwise put

$$\delta_0 = \max(0, a_0^2 - b_1^2, b_0^2 - a_1^2), \qquad \Psi_{\mathrm{low}} = 5040 + 576(a_0^3 + 3a_0b_0^2) + \frac{720\delta_0^6}{D(a_1, b_1)}.$$

Define the correction as zero if the upper denominator vanishes, which occurs only at the origin. This is a lower bound on (78): the distance between the squared-amplitude intervals is at least δ_0, and D is coordinatewise increasing. Certify the entire box if $\Psi_{\mathrm{low}}^2 > (20/3)r(a_1, b_1)^3$. Squaring is safe since $\Psi_{\mathrm{low}} \ge 5040$. If neither test succeeds, bisect the longer side at its exact midpoint, breaking ties on the a side and retaining both closed children. Monotonicity justifies the upper row bound and exclusion; closed children cover all shared boundaries and axes.

The exact-arithmetic procedure reconstructs the actual ordinary, scalar-row elementary, Gram-power and fourth-modulus polynomials with exact sixth-root arithmetic; it checks every tail constant and executes this Fraction-only cover. It completes with 141 certified boxes, 68 excluded boxes, maximum depth 16 and no unresolved box. Depth and resource guards throw errors rather than accept an incomplete cover. Together with the analytic tail, this finite certificate proves the theorem for the full specified family. Its field primitives are disclosed reuse. Fourier and complete-search tools remain classical [11]. □

C.5 Exact limits of fixed Newton and lower-sector recipes

The Newton theorem does not make its fixed test globally sufficient for capacity. We give explicit actual obstructions, while an adaptive finite occupation dual succeeds on the same matrices.

In the preceding Fourier family take $a = 3/5$, $b = 3/2$. Then residual/base ranks are two/three, the normalized selected occupations are $10/31 > 1/31$, and $K_{ii} = 21/10 < 30$. Exact identities give

$$P = \frac{208066869}{12500}, \qquad L = \frac{153576\sqrt{2} - 217152}{125} > 0, \qquad \eta^2 = \|e_3\|^2 = \frac{2133}{125} > 1.$$

Here $e_1 = e_2 = e_4 = e_5 = 0$ and $p_6 = 3e_3^2 - 6e_6$. Only the sixth Newton test loses mass, with error $e_3^2/2$. Thus

$$D_{\mathrm{N}} = P - 180\|e_3^2\|^2.$$

In orthonormal residual modes the cubic is $2a^{3/2}A^3 + 6\sqrt{a}\, b\,\mathrm{Sym}(AB^2)$. Its square has occupations six, four-two and two-four, of averaged norms squared $1, 1/15, 1/15$. Consequently

$$\|e_3^2\|^2 = 16a^6 + \frac{192}{5}a^4b^2 + \frac{432}{5}a^2b^4, \qquad D_{\mathrm{N}} = -\frac{173100987}{12500} < 0.$$

The valid nonnegative lower certificate is $\max(0, D_{\mathrm{N}}) = 0$; its negative value cannot be squared as a lower bound for P^2.

This clipped recipe cannot be repaired by only the sharp constant curvature credit 41040. For positive-diagonal $M \succeq 0$, every positive v satisfies $(Qv)_i \geq Q_{ii}v_i$, hence

$$U_{\mathrm{opt}}(M) \geq \frac{10}{3}\prod_i M_{ii}^2.$$

Here all forward diagonals are $a + b + s > 3$, so $U_{\mathrm{opt}} > (10/3)3^{12} = 1771470 > 41040$. Thus $\max(0, D_{\mathrm{N}})^2 + 41040 < U_{\mathrm{opt}}$. This example has $\eta > 1$, while Theorem C.1 assumes $\eta \leq 1$.

The lower-sector polynomial (76) also fails as a global recipe at this reference: $B = 5040$, the exact regular row sum is $1047963/5000$, and

$$U_{\mathrm{opt}} - B^2 - 41040 = \frac{65600561520028449}{12500000000} > 0.$$

This last comparison uses the exact row-regular capacity, not a loose upper bound. It limits this particular fixed polynomial, not every possible algebraic certificate.

By contrast, degree-three coefficients $c_3 = \mathbf{1}/3$ are exact, and with $D = a^6 + 15a^4b^2 + 15a^2b^4 + b^6$ the finite degree-six coefficients

$$c_6 = \frac{(a^2 - b^2)^3}{6D}\mathbf{1}$$

attain the true reference sixth projection. The resulting positive adaptive dual has square $> 2U_{\mathrm{opt}}$ by Theorem C.4. This is an obstruction to one fixed coefficient choice, not to Ψ, the actual capacity bridge, compensation or Chollet.

There are also exact complex noncirculant active-rank-six and rank-five fixed-Newton obstructions with a successful finite adaptive dual. To specify them, let $v_i = \omega^{2i}$, $w_i = \omega^{5i}$, and use the projector $\Pi = I - (J + vv^* + ww^*)/6$ of rank three, or additionally subtract $tt^*/6$, $t_i = (-1)^i$, to obtain rank two. Let J_c have entries $(J_c)_{01} = -1 + 2\omega$, its conjugate at $(1, 0)$, $(J_c)_{00} = 1$, and zero elsewhere. It is Hermitian with norm < 3. For $\epsilon = 10^{-6}$ put

$$K_\epsilon = K + \epsilon\Pi + \epsilon^2\Pi J_c\Pi.$$

Orthogonality and $3\epsilon < 1$ prove PSD, residual/base ranks five/six or four/five, and an active selected direction. The exact-arithmetic procedure reconstructs their actual ordinary coefficients and all six fixed duals by elementary row contractions. It verifies $D_{\mathrm{N}} < 0$, all forward diagonals

> 3, loss $> 2^{-28}$, all normalized selected occupations $> 1/31$, and $K_{ii} < 30$. Nonreal cycle products and distinct normalized row-coherence multisets certify phases and noncirculant form. The same reference c_3, c_6 give a positive finite adaptive dual A with $A^2 > 2U_1$ on each lift; the reader checks this by exact sign-safe quadratic-field squaring. The inner row contractions on complex perturbed cores use their actual real parts, including the coefficient of ω, rather than assuming them real. The explicit low-sector failure above is a reference comparison. All identities and comparisons use exact cyclotomic arithmetic. These finite counterexamples prove only their stated failed sufficient recipes, not failure of the adaptive projection.

C.6 Scope and the remaining uniform obligation

The Newton identity, best approximation, log-concavity and finite polynomial error estimates are classical tools. Their use here gives quantitative actual-projection control and explicit sufficient regions, with finite exact obstructions to two global recipes. The objective here is the stronger occupation-projection capacity certificate, not an assertion of additional scalar low-rank coverage. The quantified tube includes actual higher-rank noncirculant strata but its small radius does not localize every difficult matrix near an opposite-pair configuration.

A failure of a particular finite dual does not refute Ψ; a failure of Ψ would not by itself refute the actual P or compensated budget.

D Adaptive occupation control around two third-root orbits

We now extend the opposite-mode slice to two third-root orbits with unequal norms and arbitrary complex overlap. Two different reference inequalities are useful: a factor-two comparison for the true projection, and a smaller factor for an explicit lower floor. The latter allows a bounded adaptive dual in a higher-rank neighborhood. This does not place every remaining generic core near this family.

Let $\zeta = e^{2\pi i/3}$ and take the six actual residual factors

$$u, \zeta u, \zeta^2 u, \quad v, \zeta v, \zeta^2 v.$$

Use the conjugate-linear-first inner product and set

$$x = \|u\|^2, \quad y = \|v\|^2, \quad h = \langle u, v\rangle, \quad z = |h|, \quad p = x^3, \quad q = y^3, \quad k = h^3, \quad r = |k| = z^3.$$

The shared PSD constraint is $z^2 \le xy$, equivalently $r^2 \le pq$. All quantities initially use $K = \mathrm{Gram}(y_i)$ for these six factors, $X = J$, $F(t) = \mathrm{per}(K + tJ)$, $P = F(1) + F'(1)$ and $s = \sqrt{2}$. No normalization of the six row diagonals is imposed.

D.1 Actual projection and optimized capacity, including endpoints

Lemma D.1. *Put $A = p + q + 2\,\mathrm{Re}\,k$. Then*

$$F(s) - P = 720 - 36(4 - 2s)A, \qquad \Psi = 5040 + 144A + \frac{1440pqr^2}{pq + r^2},$$

with zero correction at zero denominator. For $f(t) = t^2 + 8t + 4$,

$$U_{\mathrm{opt}}(K + sJ) = 90\left[f(z^2) + \sqrt{f(x^2)f(y^2)}\right]^3. \tag{79}$$

These are actual tensor and capacity identities at all zero, orthogonal and collinear endpoints, not independently assigned scalar data.

Proof. The commutative symmetric product is $(T^3+u^3)(T^3+v^3)$. Only residual occupations three and six survive: $e_3 = u^3 + v^3$, $\|e_3\|^2 = A$ and $e_6 = \mathrm{Sym}(u^3 \otimes v^3)$. Equation (64) gives $\alpha_3 = 36A$, $\alpha_6 = 720$; the constant ordinary coefficient cancels from $F(s) - P$. Moreover $p_3 = 3e_3$, so the third coherent projection is exact.

The sixth-power tests span u^6, v^6. Their actual Gram and overlaps are

$$C = \begin{pmatrix} p^2 & k^2 \\ \overline{k}^2 & q^2 \end{pmatrix}, \qquad b = \begin{pmatrix} pk \\ q\overline{k} \end{pmatrix}.$$

When $pq + r^2 > 0$, direct multiplication shows that

$$c_u = \frac{qk}{pq + r^2}, \qquad c_v = \frac{p\overline{k}}{pq + r^2}$$

solve $Cc = b$. Their attained projection mass is $b^*c = 2pqr^2/(pq + r^2)$. No division by the possibly vanishing factor $pq - r^2$ of the determinant occurs, so singular collinearity is included. If the denominator is zero, $r = 0$ and $b = 0$, giving zero projection mass. The actual occupation weights then give the stated Ψ formula.

Third-root averaging gives

$$\sum_{j=0}^{2} |s + a\zeta^j|^4 = 3(s^4 + 4s^2|a|^2 + |a|^4).$$

Indeed the nonconstant Fourier frequencies are $\pm 1, \pm 2$, whose sums vanish. Within the first orbit, within the second and across them the fourth-modulus row sums are consequently

$$a_Q = 3f(x^2), \qquad b_Q = 3f(y^2), \qquad c_Q = 3f(z^2).$$

They are strictly positive, including zero residual factors, and are independent of $\arg h$. Choose symmetric scaling constants $d_u, d_v > 0$ constant on each three-row group. The row equations are

$$a_Q d_u^2 + c_Q d_u d_v = 1, \qquad b_Q d_v^2 + c_Q d_u d_v = 1.$$

They give $d_u/d_v = \sqrt{b_Q/a_Q}$ and $d_u d_v = 1/(c_Q + \sqrt{a_Q b_Q})$. These constants make the full scaled matrix doubly stochastic. Weighted AM–GM, as in Proposition 10.4, therefore gives

$$U_{\mathrm{opt}} = \frac{10}{3}(d_u d_v)^{-3} = \frac{10}{3}(c_Q + \sqrt{a_Q b_Q})^3,$$

which is (79). No numerical balancing or equality with an arbitrary unweighted envelope is used. □

D.2 A full-amplitude lower-floor comparison

Set

$$A_{\min} = p + q - 2z^3, \qquad w^3 = (p+q)/2, \qquad B_0 = 5040 + 144A_{\min} + 720z^6.$$

PSD gives $A_{\min} \geq 0$, $A \geq A_{\min}$ and $2pqr^2/(pq + r^2) \geq r^2$. Thus $\Psi \geq B_0 \geq 5040$.

Theorem D.2 (All-amplitude explicit floor)**.** *For the entire two-third-root-orbit family,*

$$F(s) \geq P \quad \Longrightarrow \quad B_0^2 > \tfrac{3}{2}U_{\mathrm{opt}}, \qquad \Psi^2 > \tfrac{3}{2}U_{\mathrm{opt}}.$$

All norms and overlap phases, including zero-loss and singular endpoints, are included. The statement about B_0, not merely the statement about Ψ, is needed below.

Proof. For $t \geq 0$, $f > 0$ is increasing and $f''f - (f')^2 = -2t^2 - 16t - 56 < 0$. Hence $\log f$ is increasing and concave. Composing it with the increasing concave map $a \mapsto a^{2/3}$ preserves concavity, including zero by continuity. Jensen applied to $p = x^3, q = y^3$ yields

$$\sqrt{f(x^2)f(y^2)} \leq f(w^2), \qquad U_{\text{opt}} \leq 90[f(z^2) + f(w^2)]^3 =: \overline{U}.$$

This is a capacity upper relaxation. Nonnegative actual loss gives $A \leq 5(2+s)$, so

$$0 \leq w^3 - z^3 = A_{\min}/2 \leq \tfrac{5}{2}(2+s) < 239/28 < 9. \tag{80}$$

The strict rational bound follows from $s < 99/70$, including at actual loss zero.

If $z \geq 4$, (80) implies $w < (6/5)z$, since $(6/5)^3 > 1 + 9/64$. Therefore, with $H = f(w^2) + f(z^2)$,

$$\frac{H}{z^4} < (36/25)^2 + 1 + \frac{8}{16}(36/25 + 1) + \frac{8}{256} < 5.$$

Thus $\overline{U} < 11250z^{12}$ while $B_0^2 \geq 720^2 z^{12} > (3/2)11250z^{12}$. This proves the entire unbounded tail.

For $0 \leq z \leq 4$, the scalar constraint gives $w < 5$. The following exact cover treats the whole closed rectangle $[0,4]_z \times [0,5]_w$. For a box $[z_0, z_1] \times [w_0, w_1]$, exclude it only if $w_1 < z_0$ or $w_0^3 - z_1^3 \geq 239/28$. Otherwise use the rigorous bounds

$$B_{\text{low}} = 5040 + 288\max(0, w_0^3 - z_1^3) + 720z_0^6, \qquad H_{\text{up}} = f(w_1^2) + f(z_1^2).$$

Certify it if $B_{\text{low}}^2 > 135H_{\text{up}}^3$. Every actual scalar point in the box has $B_0 \geq B_{\text{low}} > 0$ and $H \leq H_{\text{up}}$, so this proves $B_0^2 > (3/2)\overline{U}$ there. If neither test passes, bisect the longer side at its exact midpoint, breaking ties on z, and retain both closed children. Their union is the parent, including all axes and shared boundaries.

The exact-arithmetic procedure executes this Fraction-only recursion with 48 certified and 30 excluded boxes, maximum depth 13, and no unresolved box. It checks the displayed tail and rational constants. Resource/depth guards throw exceptions and never accept incompleteness. This exact complete finite certificate plus the analytic tail proves the theorem over the relaxed domain (80), and hence the actual family.

□

D.3 The full factor-two comparison for the true projection

Theorem D.3 (All-amplitude two-orbit occupation comparison)**.** *For every actual two-third-root-orbit core with $F(s) \geq P$,*

$$\Psi^2 > 2U_{\text{opt}}.$$

This includes arbitrary unequal norms, overlap phase, axes, orthogonal and collinear limits, and all unbounded amplitudes. Residual/base ranks of the reference family remain at most two/three.

Proof. For fixed x, y, z, capacity is independent of phase and Ψ increases with A. It therefore suffices to use the hardest phase $A = A_{\min} = x^3 + y^3 - 2z^3 \geq 0$. If the original phase is admissible, $A_{\min} \leq A \leq 5(2+s) < 239/14$, so that hardest phase is admissible too. No equal-norm minimizer assumption is made.

First let $x, y \leq 30$, so PSD implies $z \leq 30$. Symmetry under exchange of the two orbits allows $x \geq y$; this swaps x, y, conjugates h, and preserves all target quantities. Cover $[0,30]^3$ by closed boxes $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$. Exclude a box if $z_0^2 > x_1 y_1$, or $\max(0, x_0^3 + y_0^3 - 2z_1^3) \geq 239/14$, or $x_1 < y_0$. These remove only boxes with no PSD, loss-admissible, symmetry-half-domain

representative respectively. Equality on the PSD or symmetry boundaries is retained. For other boxes put

$$A_{\mathrm{low}} = \max(0, x_0^3 + y_0^3 - 2z_1^3), \quad T_{\mathrm{low}} = x_0^3 y_0^3, \quad Z_{\mathrm{low}} = z_0^6,$$
$$\Psi_{\mathrm{low}} = 5040 + 144A_{\mathrm{low}} + \frac{1440 T_{\mathrm{low}} Z_{\mathrm{low}}}{T_{\mathrm{low}} + Z_{\mathrm{low}}}.$$

Define the correction as zero at zero denominator. The function $2TZ/(T+Z)$ is increasing in both nonnegative arguments and continuous at the origin. These are valid lower bounds on admissible points even when a corner itself is not PSD. Let $a_Q(t) = 3(4 + t^4 + 8t^2)$ and bound the capacity base by

$$H_{\mathrm{up}} = a_Q(z_1) + \mathrm{sqrt}_{\mathrm{up}}(a_Q(x_1) a_Q(y_1)).$$

The radical upper bound is rational with denominator 2^{30}: take the integer square root after scaling, then increment its numerator if its squared value is too small. The defining squared upper inequality is checked exactly. Certify the box if $\Psi_{\mathrm{low}}^2 > (20/3) H_{\mathrm{up}}^3$. All quantities being squared are positive. Monotonicity of the row functions and the lower bounds justify the comparison throughout the box. Otherwise bisect a longest side at the exact midpoint, breaking ties in x, y, z order, and retain both closed children.

The exact rational bisection procedure completes this second exact rational cover with 586 certified and 455 excluded closed boxes, maximum depth 31, and no unresolved box. Every leaf is certified or excluded by the stated valid tests; guards fail on an unresolved region. This proves the bounded admissible symmetry half-domain, including axes and shared boundaries.

It remains to treat every point outside that bounded part. If $\max(x, y) > 30$, then $w^3 = (x^3 + y^3)/2 > 13500$. Equation (80) gives $z^3 > 13491 > 64$, hence $z > 4$. The Jensen/tail estimate in the preceding proof gives $U_{\mathrm{opt}} < 11250 z^{12}$. Actual PSD gives $pq \geq r^2$, so the sixth projection is at least z^6 and $\Psi \geq 720 z^6$. Thus $\Psi^2 \geq 720^2 z^{12} > 22500 z^{12} > 2U_{\mathrm{opt}}$. This covers the entire outside-amplitude tail, with no cutoff or numerical extrapolation. Together with the complete bounded cover it proves the claim. The exact arithmetic checks corroborate a computer-assisted restricted-family proof, not a generic extremal reduction; complete search remains classical [11]. □

Theorem D.3 does not replace Theorem D.2: a bound for the true Ψ alone does not bound the loss incurred when a small reference test is discarded. That distinction is used in the next theorem.

D.4 A uniform adaptive higher-rank tube

Theorem D.4 (Bounded reference-dependent occupation duals)**.** *Let K^* be a reference two-orbit core with $x, y \leq 30$ and nonnegative reference ordinary loss. Let $K \succeq 0$ be any complex Hermitian six-by-six core with $K_{ii} \leq 30$. If*

$$\max_{ij} |K_{ij} - K_{ij}^*| \leq \delta = 2^{-100},$$

then

$$\Psi(K)^2 - U_{\mathrm{opt}}(K + sJ) > 2^{21}.$$

No positive-loss assumption on the perturbed K, rank restriction, Gram eigenvalue lower bound or differentiability of a pseudoinverse is needed.

Proof. Keep the reference notation p, q, k, r and $A = \|e_3\|^2$. Nonnegative reference loss implies $A \leq 5(2 + s) < 18$. Use coefficients $c_{3,i} = 1/3$ on all six tests and no other tests except possibly degree six. The cubic reference projection is exact.

If $r^2 \le 1/4$, set all sixth coefficients to zero. The reference finite dual is $T_* = 5040+144A \ge 5040$, and $T_* \ge B_0 - 180$ because the discarded part of the FLOOR is $720r^2$. Also $B_0 \le 5040 + 144 \cdot 18 + 180 = 7812 < 2^{13}$. If $T_* < B_0$, then $B_0^2 - T_*^2 \le 360B_0$; otherwise there is no loss. The floor theorem therefore gives in either case

$$T_*^2 - U_* > 5040^2/3 - 360 \cdot 2^{13} > 2^{22}, \qquad U_* = U_{\mathrm{opt}}(K^* + sJ). \tag{81}$$

If $r^2 > 1/4$, assign each of the three first-orbit rows coefficient $c_u/3$ in degree six, and each second-orbit row coefficient $c_v/3$, with c_u, c_v from Lemma D.1. Sixth powers are constant within an orbit, so this captures the full reference sixth projection, even at singular collinearity. Then $T_* = \Psi(K^*) \ge B_0$, and its margin is $> 5040^2/3 > 2^{23}$, stronger than (81). Crucially these coefficients have uniform bounds:

$$|c_u| \le q/(2r) \le 27000, \quad |c_v| \le p/(2r) \le 27000, \quad |c_{6,i}| \le 9000 < 2^{14}, \quad \sum_i |c_{6,i}| < 2^{17}.$$

Here $pq \ge r^2$, $r > 1/2$ and $p, q \le 30^3$. The low branch avoids unstable inverse coefficients near vanishing overlap.

For the actual perturbed K, hold these SAME reference coefficients fixed. Let $T(K)$ be their aggregate degree-three/six finite dual plus5040. The actual occupation projection gives $\Psi(K) \ge T(K)$ for every K. PSD diagonal bounds give $|K_{ij}|, |K^*_{ij}| \le 30$. Polynomial telescoping implies, for degree d,

$$|\Delta e_d(\mathrm{row})| \le \binom{6}{d} d30^{d-1}\delta, \qquad |\Delta K^d_{ij}| \le d30^{d-1}\delta.$$

In degree three the coefficient one-norm is2, so the true weight144 yields

$$|\Delta T_3| \le 144[2 \cdot 2 \cdot 20 \cdot 3 \cdot 30^2 + 4 \cdot 3 \cdot 30^2]\delta = 32659200\delta < 2^{25}\delta.$$

In degree six use $6 \cdot 30^5 < 2^{28}$ and the coefficient bounds. Before its weight720, linear and quadratic errors are respectively $< 2^{46}\delta$ and $< 2^{62}\delta$. Since $2^{25} + 720(2^{46} + 2^{62}) < 2^{73}$,

$$|T(K) - T_*| < 2^{73}\delta = 2^{-27}.$$

Thus $T(K) > 0$ BEFORE squaring. Finite duals are at most their actual P, and both nonnegative ordinary derivatives are $< 720(32^6 + 6 \cdot 32^5) < 2^{41}$. It follows that

$$|T(K)^2 - T_*^2| < 2^{115}\delta. \tag{82}$$

To transfer reference capacity, hold fixed the positive reference-optimal weights

$$v = (t, t, t, 1, 1, 1), \qquad t = \sqrt{b_Q/a_Q}.$$

Indeed direct multiplication gives $(a_Q t + c_Q)(c_Q t + b_Q)/t = (c_Q + \sqrt{a_Q b_Q})^2$, so its feasible envelope equals U_* at the reference. Since $f(900)/f(0) = 204301 < 2^{18}$, both t and $1/t$ are $< 2^9$; all weight ratios are $t, 1/t$ or one, not their squares. For $Q = |K + sJ|^4$, put $H_i = (Qv)_i/v_i$ and likewise H_i^*. The positive diagonal contribution gives $H_i, H_i^* \ge 4$, and

$$H_i, H_i^* < 6 \cdot 2^9 \cdot 32^4 < 2^{32}, \qquad |\Delta H_i| < 6 \cdot 2^9 \cdot 4 \cdot 32^3\delta < 2^{29}\delta.$$

Hence $|\Delta\sqrt{H_i}| \le |\Delta H_i|/4 < 2^{27}\delta$ and $\sqrt{H_i}, \sqrt{H_i^*} < 2^{16}$. Telescoping the six factors in the feasible envelope gives

$$|U_Q(v) - U_{Q^*}(v)| < \frac{10}{3} 6 \cdot 2^{27} \cdot 2^{80}\delta < 2^{112}\delta. \tag{83}$$

At the reference $U_{Q^*}(v) = U_*$, while for the perturbation $U_{\mathrm{opt}} \le U_Q(v)$. Thus no continuity or derivative of the optimizer, and no unweighted-envelope covariance, is being assumed.

At $\delta = 2^{-100}$ the errors in (82)–(83) sum to less than $2^{15} + 2^{12} < 2^{16}$. Subtracting from (81) gives $T(K)^2 - U_Q(v) > 2^{22} - 2^{16} > 2^{21}$. Since $\Psi \ge T(K) > 0$ and $U_{\mathrm{opt}} \le U_Q(v)$, this proves the theorem. □

Positive original row gauges transfer the optimized comparison by ρ^2, using $P_{\mathrm{actual}} \ge \rho\Psi(K)$ and optimized-capacity covariance. The actual compensation identity and nonnegative E, R_h then give strict compensation.

D.5 Exact nonempty strata and predecessor relationship

The exact-arithmetic procedure checks all integer constants above and four exact complex noncirculant fixtures, with active residual/base ranks five/six and four/five in BOTH coefficient branches. For low overlap choose $u = (1, 0)$, $v = (1/2, 1/2)$, giving $x = 1, y = 1/2, h = 1/2, r^2 = 1/64$. For high overlap choose $u = (3/2, 0)$, $v = (-1, 1/2)$, giving $x = 9/4, y = 5/4, h = -3/2, r^2 = 729/64$ and $A = 211/32$. The actual orbit rows may use negative powers of ζ, merely a within-orbit relabeling.

To specify the lift, let

$$b_1 = (1, \zeta, \zeta^2, 0, 0, 0)^{\mathsf{T}}, \quad b_2 = (0, 0, 0, 1, \zeta, \zeta^2)^{\mathsf{T}}, \quad b_3 = (1, \zeta^2, \zeta, 0, 0, 0)^{\mathsf{T}}.$$

The orthogonal projector $\Pi = I - J/6 - (b_1 b_1^* + b_2 b_2^*)/3$ has rank three and is orthogonal to the selected and reference column modes. For the rank-five case subtract $b_3 b_3^*/3$, leaving rank two. The code explicitly verifies these identities. Perturb the reference by $\epsilon\Pi + \epsilon^2 \Pi E \Pi$, $\epsilon = 2^{-120}$, where $E_{03} = i\sqrt{3}$, $E_{30} = -i\sqrt{3}$, $E_{00} = 1$ and all other entries vanish. This Hermitian compression has norm at most $\sqrt{3} + 1 < 3$. Orthogonality and $3\epsilon < 1$ prove PSD and the actual ranks; exact elimination corroborates them. The reader checks entry distance $\le 2^{-100}$, diagonal bounds, actual moderate loss, selected occupations above $1/31$, nonreal cycles and noncirculant normalized row patterns. It keeps the SAME reference c_3, c_6, checks $0 < T \le P$, and verifies the stronger fixture inequality $T^2 - U_1 > 2^{21}$ by sign-safe exact squaring. The universal theorem uses bounded optimal reference weights; fixtures happen to pass with $v = \mathbf{1}$, and these facts are not conflated. All calculations use the stated exact cyclotomic relation. Finite examples establish nonvacuity; the analytic proof supplies the uniform tube.

For comparison with Theorem C.4, split its modes-two/five factors by index parity. After a within-orbit relabeling they are two third-root orbits with

$$u = (\sqrt{a}, \sqrt{b}), \qquad v = (\sqrt{a}\,\omega^{-2}, \sqrt{b}\,\omega^{-5}), \quad x = y = a + b, \quad h^3 = (a - b)^3.$$

Thus that earlier factor-two result is a slice of the present unequal-norm, arbitrary-phase theorem. No containment of all other sufficient regions, including the distinct three-opposite-pair family, is inferred. The objective here is the stronger true occupation/capacity comparison and its bounded adaptive higher-rank extension.

Projection, Gram systems, balancing, Jensen concavity and exact complete covers remain classical inputs [7, 4, 18, 11]. The new quantified regions have the stated local hypotheses. No finite observation list is used as a universal premise.

Part IV. Supporting sharp occupation geometry and directional limits

E Sharp occupation geometry and an intrinsic capacity region

The statements in this section concern the actual six-factor selected core, rather than independently assigned occupation coefficients. Let $K \succeq 0$ be any complex Hermitian 6×6 matrix, let $J = \mathbf{1}\mathbf{1}^*$, and put

$$s = \sqrt{2}, \qquad F(t) = \operatorname{per}(K + tJ) = \sum_{j=0}^{6} \alpha_j t^j,$$

$$P = F(1) + F'(1), \qquad L = F(s) - P, \qquad D = P + 3L - 7200.$$

The actual occupation expansion gives $\alpha_j \geq 0$ and $\alpha_6 = 720$. Set $Q_{ij} = |K_{ij} + s|^4$ and

$$U_{\mathrm{opt}} = \frac{10}{3}\sqrt{\operatorname{cap} Q}, \qquad \operatorname{cap} Q = \inf_{v_i > 0} \frac{\prod_i (Qv)_i}{\prod_i v_i}.$$

The weighted CLL bound established above gives $C(s) = \operatorname{per}|K + sJ|^2 \leq U_{\mathrm{opt}}$. Its feasible value at $v = \mathbf{1}$ is $(10/3)\sqrt{\prod_i \sum_j Q_{ij}}$. All margins below are selected-core margins. For the original positive selected occupations they acquire the factor ρ^2; a zero selected occupation uses the separately proved nonpositive-loss argument.

Theorem E.1 (Sharp occupation constraint and equality geometry)**.** *For every such actual Gram core,*

$$P + 3L \geq 7200.$$

On $0 \leq L < 720$, equality holds precisely when its residual vectors, up to permutation, are $(u, iu, -u, -iu, 0, 0)$ for a nonzero vector u. At $L = 720$, equality holds precisely for $K = 0$. Every $L \in [0, 720]$ is attained by an equality core.

Proof. Direct expansion gives

$$D = \alpha_0 + (3s - 4)\alpha_1 + (6s - 8)\alpha_3 + 2\alpha_4 + (12s - 12)\alpha_5 \geq 0. \tag{84}$$

For residual Gram vectors y_i, write $e_l(y)$ for the elementary tensor of degree l, with averaged symmetrization. The normalization is

$$\alpha_{6-l} = \frac{720}{\binom{6}{l}} \|e_l(y)\|^2.$$

If $D = 0$, the tensors e_6, e_5, e_3, e_2, e_1 vanish. In the symmetric algebra of the finite residual span, a polynomial ring over $\mathbb{C}$, $e_6 = \prod_i y_i = 0$ forces one factor to be zero. Then $e_5 = 0$ forces another zero factor. If $L < 720$, the remaining coefficient α_2 is nonzero, so exactly four factors remain and their monic product is $T^4 + e_4$. Substitution at each root yields $y_i^4 = -e_4$. The integral-domain identity $y_i^4 - y_1^4 = \prod_{\zeta^4 = 1}(y_i - \zeta y_1)$ forces each remaining factor to be a fourth-root multiple of y_1. Unique factorization of $T^4 - y_1^4$ gives the four distinct multiples. Conversely, for $\|u\| = R$ this configuration gives

$$F(t) = 720t^6 + 48R^8 t^2, \qquad L = 720 - 48R^8, \quad P = 5040 + 144R^8.$$

It realizes every stated loss. If $L = 720$ and $D = 0$, then $\alpha_2 = 0$ also; $e_1, \ldots, e_6$ all vanish and $\prod_i (T + y_i) = T^6$ implies all $y_i = 0$. □

The equality classification concerns this occupation line. It does not classify minimizers of the capacity slack.

Corollary E.2 (A cutoff-free actual compensation strip)**.** *For $0 \le L \le 4$, the actual core compensation satisfies $G \ge 2136$. No diagonal, rank, phase or conditioning restriction is needed.*

Proof. The attributed order-six self-conjugate bound gives $C(s) \le F(s)^2$. The established whole variance/curvature reserve gives, in this core,

$$\Gamma := 2(s-1)E + R_{s-1} \ge 14P + 16L - 41040.$$

This replaces the whole reserve once in $G = P^2 - C(s) + \Gamma$. Consequently

$$G \ge (14 - 2L)P + 16L - L^2 - 41040.$$

On $0 \le L \le 4$ its coefficient of P is nonnegative, so Theorem E.1 gives

$$G \ge 59760 - 14426L + 5L^2 \ge 2136.$$

The final quadratic is decreasing on this interval. This proves actual compensation, not the stronger capacity target throughout the strip. □

Theorem E.3 (A wide fourth-root capacity tube)**.** *Let $v = (1, -i, -1, i, 0, 0)^{\mathsf{T}}$, $K_* = qvv^*$, where $q \ge 0$ and $q^4 \le 15$. Every actual $K \succeq 0$ satisfying $\max_{ij} |K_{ij} - (K_*)_{ij}| \le 1/32$ obeys*

$$P^2 - U_{\mathrm{opt}} > 2^{23}.$$

No loss condition on the perturbed matrix is required.

Proof. The four nonzero-reference rows of $|K_* + sJ|^4$ have common sum $24 + 32q^2 + 4q^4 < 216$, while the remaining row sums equal24. Here $q < 2$. Both reference and perturbed forward entries have modulus less than4. The reverse triangle inequality and fourth-power factorization therefore give a per-entry error at most 256/32 and a row-sum error at most48. Thus

$$U_{\mathrm{opt}} < \frac{10}{3}\, 264^2\, 72 = 16727040.$$

The top occupation alone gives $P \ge 7\alpha_6 = 5040$, whence $P^2 - U_{\mathrm{opt}} > 8674560 > 2^{23}$. □

This reference family differs from the earlier two-mode tube; the two radius statements are not a comparison for the same family. Row phases are retained relative to J, rather than gauged away independently.

Lemma E.4 (Two small residual rows)**.** *If, after permutation, four diagonals of K are at most2 and the other two are at most 2^{-14}, then*

$$P^2 - U_{\mathrm{opt}} > 2^{19}.$$

This holds without a loss, phase or rank assumption.

Proof. Put $\delta = 2^{-14}$. PSD Cauchy–Schwarz bounds the three entry blocks by 2, $\sqrt{2\delta} = s/128$ and δ, respectively. The four large Q-row sums are at most a and the other two at most b, where

$$\begin{aligned} a &= 4(2+s)^4 + 2[s(129/128)]^4 = \frac{9403728385}{33554432} + 192s, \\ b &= 4[s(129/128)]^4 + 2(s+2^{-14})^4 \\ &= \frac{882917738077487105}{36028797018963968} + \frac{536870913}{549755813888}s. \end{aligned}$$

The exact quadratic-field comparison

$$\frac{10}{3}a^2 b < 5040^2 - 2^{19}$$

finishes the feasible-weight capacity bound. For a reproducible rational check, substitute the strict upper bound $s < 1414213562373096/10^{15}$ in the positive expressions a, b; ordinary integer arithmetic verifies the same strict inequality. □

Theorem E.5 (Intrinsic occupation-defect localization)**.** *For any actual complex PSD core, if $L \geq 0$ and $D \leq 2^{-160}$, then all six diagonals are strictly below2 and at least two are strictly below 2^{-14}. In particular $P^2 - U_{\mathrm{opt}} > 2^{19}$. There is no initial diagonal cutoff or positive eigenvalue hypothesis.*

Proof. Each coefficient of $\|e_l\|^2$ in (84), for $l = 1, 2, 3, 5, 6$, exceeds one. Set $\varepsilon = 2^{-80}$; then $\|e_l\| \leq \varepsilon$ for these five degrees. Moreover

$$L = 720 - \sum_{j=1}^{5} (j + 1 - s^j)\alpha_j \geq 0$$

and all five weights are positive. In particular $\alpha_2 \leq 720$, so $\|e_4\|^2 \leq 15$ and $\|e_4\| < 31/8$.

Let $R = \max_i \|y_i\|$. If $R > 0$, project every row onto the unit direction of a maximizing row. The scalar projections x_i have one entry equal to R and satisfy $|e_l(x)| \leq \|e_l(y)\|$. Evaluate their monic characteristic polynomial at R. If $R \geq \sqrt{2}$, division by R^6 gives

$$1 \leq \frac{\|e_4\|}{R^4} + \varepsilon(R^{-1} + R^{-2} + R^{-3} + R^{-5} + R^{-6}) < \frac{31}{32} + 5\varepsilon < 1.$$

Thus $R^2 < 2$, including the trivial $R = 0$ case.

We use the classical creation proof of the permanent Hadamard inequality in the averaged convention. If $f_j \in \operatorname{Sym}^{k-1} H$ has exactly j occupations in the direction of y, then

$$\|\operatorname{Sym}_k(y \otimes f_j)\|^2 = \|y\|^2 \frac{j+1}{k} \|f_j\|^2.$$

Indeed the ratio of normalized placement counts is $\binom{k-1}{j}/\binom{k}{j+1} = (j+1)/k$. Orthogonal summation and iteration give, also for zero or singular factors,

$$k! \left\| \operatorname{Sym}_k \bigotimes_i y_i \right\|^2 \geq \prod_i \|y_i\|^2. \tag{85}$$

Order the norms $r_1 \leq \cdots \leq r_6$. Suppose $r_2 \geq 2^{-7}$. Since $D \geq \alpha_0 = \operatorname{per} K$, (85) gives

$$r_1 r_2^5 \leq \prod_i r_i \leq \sqrt{D} \leq \varepsilon, \qquad r_1 \leq 2^{-45}.$$

This division is made only under the stated positive assumption on r_2. The fourth elementary tensor e'_4 on the other five rows has norm at most $5R^4 < 20$. Isolating their product in $e_5(y)$ yields

$$\|\operatorname{Sym}_5 \bigotimes_{i=2}^{6} y_i\| \le \varepsilon + 20r_1.$$

Using (85) with $k = 5$, we obtain

$$r_2^5 \le \prod_{i=2}^{6} r_i < 11(\varepsilon + 20r_1) \le 11(2^{-80} + 20 \cdot 2^{-45}) < 231 \cdot 2^{-45} < 2^{-37} < 2^{-35},$$

contradicting $r_2^5 \ge 2^{-35}$. Hence $r_2 < 2^{-7}$, and Lemma E.4 applies. □

The last theorem localizes to an anisotropic diagonal region. Its threshold is conservative.

Corollary E.6 (A strict three-quarter self-bound on the proved regions)**.** *Under the two-small-row hypothesis of Lemma E.4,*

$$C(s) < \tfrac{3}{4}F(s)^2 - 5888.$$

The same conclusion follows from $L \ge 0$, $D \le 2^{-160}$. Under the fourth-root tube hypothesis of Theorem E.3, the stronger margin is

$$C(s) < \tfrac{3}{4}F(s)^2 - 8156160.$$

The diagonal-region and tube statements require no loss assumption.

Proof. The top coefficient gives $F(s) \ge 720s^6 = 5760$ for every actual core. For the diagonal region,

$$C(s) \le U_{\mathrm{opt}} < 5040^2 - 2^{19} = \tfrac{3}{4}\,5760^2 - 5888 \le \tfrac{3}{4}F(s)^2 - 5888.$$

The intrinsic theorem supplies that diagonal region under its stated hypotheses. The tube proof instead gives $U_{\mathrm{opt}} < 16727040$; subtract this from $(3/4)5760^2 = 24883200$ to obtain 8156160. □

All three capacity statements include arbitrary complex phases, residual/base ranks 5/6 and 4/5, and 5/5 kernel mixtures. They imply strict actual compensation, and on $L \ge 0$ also imply the full-occupation sufficient comparison

$$P^2 + 16F(s) - 2P - 41040 \ge U_{\mathrm{opt}},$$

because the added reserve bound $14P + 16L - 41040$ is positive there.

The remaining generic positive-loss compensation and capacity problems are open outside the stated sufficient regions. The classical creation, PSD and CLL ingredients are used here for the particular sharp occupation geometry and constructive localization.

F A uniform row reserve and limits of sparse descent

We now obtain control of a row of the strong auxiliary target, without a selected-occupation, rank or conditioning restriction. This is a fixed-order statement. For a complex Hermitian $K \succeq 0$ of order six, put

$$\begin{aligned} J = \mathbf{1}\mathbf{1}^*, \quad s = \sqrt{2}, \quad F_K(v) = \operatorname{per}(K + vJ), \\ P_K = F_K(1) + F'_K(1), \quad C_K = \operatorname{per}|K + sJ|^2, \quad D_K = P_K^2 - C_K. \end{aligned} \tag{86}$$

Here $|M|^2 = M \circ \overline{M}$ throughout. These quantities use the unnormalized residual core K, rather than a correlation normalization.

Theorem F.1 (Uniform six-row reserve)**.** *For every complex Hermitian PSD matrix K of order six and every index i,*

$$P_K > (K_{ii} + s)\,\mathrm{per}\big(K(i \mid i) + sJ_5\big) > 0. \tag{87}$$

On every feasible interval obtained by varying just K_{ii}, D_K is strictly increasing and strictly convex.

Proof. Use the bosonic creation normalization in which the squared norm of a product of creations is the permanent of its physical Gram matrix. With the conjugate-linear-first convention used above, choose vectors $y_1, \dots, y_6$ with physical Gram $\overline{K}$ and a unit vector e perpendicular to them. Conjugating the Gram does not change its permanent. Let N count the particles in the e mode, and set

$$f = \prod_{j \neq i} c^*_{y_j + e}|0\rangle, \qquad r = \|y_i\|^2, \qquad a^* = \sqrt{r}\, c^*_u + c^*_e, \quad y_i = \sqrt{r}\, u.$$

If $r = 0$, an unused perpendicular unit mode u can be adjoined isometrically. Orthogonal anchor occupations give the exact identities

$$P_K = \langle a^* f, (1 + N)a^* f\rangle, \qquad \mathrm{per}\big(K(i \mid i) + sJ_5\big) = \langle f, s^N f\rangle.$$

The vector f is nonzero because its all-anchor component is nonzero. It suffices to prove that

$$B_r = a(1 + N)a^* - (r + s)s^N \tag{88}$$

is positive definite on the entire symmetric degree-five space.

Fix the total number k of particles in $\mathrm{span}\{u, e\}$, where $0 \le k \le 5$. The other $5-k$ particles form spectator multiplicity spaces. In the normalized two-mode basis indexed by the anchor occupation $j = 0, \dots, k$, the restriction of B_r is a real symmetric Jacobi matrix with diagonal

$$b_{k,j}(r) = r\big[(k - j + 1)(j + 1) - s^j\big] + (j + 1)(j + 2) - s^{j+1} \tag{89}$$

and off-diagonal entry $\sqrt{r}\,(j + 2)\sqrt{(k - j)(j + 1)}$ between j and $j + 1$. These formulas follow from the creation amplitudes $\sqrt{\text{occupation} + 1}$; the output weight is $1 + N$. In particular no factorial or spectator occupation is discarded.

For completeness, the exact finite positivity certificate is specified by the leading-minor recurrence

$$\begin{aligned} \Delta_{k,0} &= 1, \qquad \Delta_{k,1} = b_{k,0}(r), \\ \Delta_{k,j+1} &= b_{k,j}(r)\Delta_{k,j} - r(j + 1)^2(k - j + 1)j\,\Delta_{k,j-1}, \quad 1 \le j \le k. \end{aligned} \tag{90}$$

Every nonzero coefficient of these polynomials in r is strictly positive in $\mathbb{Q}(\sqrt{2})$, and their constant coefficients are strictly positive. The following table records the complete finite range:

k	Leading minors	Positive coefficients	Degree of full determinant
0	1	1	0
1	2	5	2
2	3	9	3
3	4	14	4
4	5	20	5
5	6	27	6

The exact continuant procedure reconstructs every coefficient from (90), using rational pairs (a, b) for $a + b\sqrt{2}$. Its sign test uses only rational comparisons: if a, b have opposite signs, it compares a^2 with $2b^2$ in the appropriate direction. Thus this is a complete finite arithmetic check of all 21 leading minors and 76 coefficients, not eigenvalue sampling. Sylvester's criterion proves positivity of each block for every $r \geq 0$. The spectator direct sum has the same property. Applying (88) to the nonzero f proves (87).

For the derivative consequence, fix all entries except $x = K_{ii}$ and put

$$F_5(v) = \operatorname{per}\big(K(i \mid i) + vJ_5\big), \quad P_5 = F_5(1) + F_5'(1), \quad A_5 = F_5(s), \quad C_5 = \operatorname{per}\big|K(i \mid i) + sJ_5\big|^2.$$

The coefficients of F_5 are nonnegative. Its highest coefficient is 120, so $j + 1 \geq s^j$ for $0 \leq j \leq 5$, with $6 > s^5$, gives $P_5 > A_5 > 0$. The through-five inequality of Lemma 5.3 gives $C_5 \leq A_5^2$. Exact diagonal differentiation now yields

$$\begin{aligned} \frac{dD_K}{dx} &= 2P_K P_5 - 2(x + s)C_5 > 0, \\ \frac{d^2 D_K}{dx^2} &= 2(P_5^2 - C_5) > 0. \end{aligned} \tag{91}$$

The strict first inequality uses (87) and $x + s > 0$. □

The theorem includes all complex phases, zero residual rows and singular cores. It is order-specific: on a degree-six input the coefficient of r at $k = j = 6$ in (89) would be $7 - s^6 = -1$, so this operator inequality has no such automatic extension. Ordinary Chollet diagonal closure, for the different defect $\operatorname{per}(A)^2 - \operatorname{per}|A|^2$, also follows from the border permanent lower bound and the smaller principal inequality; see, for example, [13, Proposition 2]. The new conclusion concerns D_K in (86), whose ordinary quantity is $F_K(1) + F_K'(1)$.

A negative D_K at a positive-definite residual core would remain negative when one diagonal is decreased to its PSD endpoint. Also $d(F_K(s) - P_K)/dx = A_5 - P_5 < 0$, so decreasing that diagonal preserves positive loss. The endpoint may still have residual rank five, which is precisely an unresolved case. This supplies no further automatic rank reduction or transport of the full compensated defect.

F.1 The upper coefficients of a two-coordinate pencil

The row theorem leaves off-diagonal phase coupling unresolved. The following supporting calculation separates the two easy upper terms of the corresponding quartic from its first and middle terms.

Proposition F.2. *Let $K \succeq 0$ be complex Hermitian of order six, and let u be supported on at most two coordinates. Write*

$$P_{K+tuu^*} = p_0 + p_1 t + p_2 t^2, \qquad C_{K+tuu^*} = \sum_{k=0}^{4} c_k t^k.$$

Then

$$c_3 \leq p_1 p_2, \qquad c_4 \leq \tfrac{1}{2} p_2^2. \tag{92}$$

No positive-loss or rank hypothesis is required.

Proof. Take support indices i, j, put $M = K + sJ$ and define

$$a = |u_i|^2, \quad b = |u_j|^2, \quad c = u_i\overline{u_j}, \quad w = ab, \quad A = M_{ii}, \quad B = M_{jj}, \quad Z = M_{ij},$$

$$R = aB + bA + 2\,\mathrm{Re}(Z\bar{c}) \geq 0.$$

Indeed, if the physical Gram of $z_1, \ldots, z_6$ is $\overline{M}$, then $R = \|u_i z_j + u_j z_i\|^2$. Let $F_4(v) = \mathrm{per}(K[I]+vJ_4)$, where $I = \{1, \ldots, 6\} \setminus \{i, j\}$, and put $P_4 = F_4(1) + F_4'(1)$ and $C_4 = \mathrm{per}\,|M[I]|^2$. Only permutations mapping the support pair onto itself contribute the cubic or quartic terms. Expanding their two local assignments gives

$$c_3 = 2wRC_4, \qquad c_4 = 2w^2C_4, \qquad p_2 = 2wP_4.$$

For $F(v,t) = \mathrm{per}(K+vJ+tuu^*)$, the linear coefficient $F_1(v) = [t]F(v,t)$ has nonnegative coefficients and degree at most five. At $v = s$ it is the permanent of the five-vector Gram obtained by replacing the support pair by $u_i z_j + u_j z_i$. The standard creation identity $\|c_z^* f\|^2 = \|z\|^2\|f\|^2 + \|c_z f\|^2$ therefore gives $F_1(s) \geq RF_4(s)$. Nonnegative occupations and $j + 1 \geq s^j$ through degree five give

$$p_1 \geq F_1(s) \geq RF_4(s), \qquad P_4 \geq F_4(s).$$

Finally, Lemma 5.3 gives $C_4 \leq F_4(s)^2$. Combining these nonnegative factors proves (92). If either support coordinate vanishes, $w = p_2 = c_3 = c_4 = 0$; no division or nonsingularity assumption was used. □

Thus the cubic and leading coefficients of D_{K+tuu^*} are at least p_1p_2 and $p_2^2/2$, respectively. The unresolved payments are

$$c_1 \leq 2p_0p_1, \qquad c_2 \leq p_1^2 + 2p_0p_2. \tag{93}$$

The term p_0p_2 and the phases of the common four-row background cannot be discarded. Neither a full quartic sign nor an endpoint principle follows from Proposition F.2 alone.

F.2 A rank-one interval of range-preserving decreases

The failure of PSD-direction monotonicity already occurs at residual rank one. The following elementary example gives a whole interval, while keeping the transformed deficit positive.

Proposition F.3. *Let ω be a nonreal cube root of unity, $v = (1, \omega, \omega^2, 0, 0, 0)^T$, and $K_t = tvv^*$. For every $0 < t \leq 1/100$, the quantities in* (86) *satisfy*

$$0 < F_{K_t}(\sqrt{2}) - P_{K_t} < 720, \qquad D_{K_t} > 25355515, \qquad \frac{d}{dt}D_{K_t} < 0.$$

Thus for $0 < a < b \leq 1/100$, $K_a \preceq K_b$ but $D_{K_a} > D_{K_b} > 0$. Both residual matrices have rank one and both anchored matrices $K_t + J$ have rank two. The difference has rank one, is supported on exactly three coordinates, and its range is contained in $\mathrm{range}\,K_a$.

Proof. The vector is centered because $1 + \omega + \omega^2 = 0$. Its nonzero elementary products satisfy $e_1 = e_2 = 0$, $e_3 = 1$; every e_k for $k \geq 4$ vanishes. Selecting k retained rows and columns in the rank-one permanent, and summing their $k!$ bijections, gives

$$F_{K_t}(x) = \sum_{k=0}^{6}(6-k)!k!x^{6-k}t^k|e_k(v)|^2 = 720x^6 + 36t^3x^3.$$

Writing $s = \sqrt{2}$, it follows that

$$P_{K_t} = 5040 + 144t^3, \qquad F_{K_t}(s) - P_{K_t} = 720 - 72(2-s)t^3.$$

This loss is strictly between zero and 720 on the stated interval. The vector v is nonzero and perpendicular to $\mathbf{1}$, proving the rank assertions.

To compute the doubled permanent, observe that $|sJ + tvv^*|^2$ equals $2J_6$ except in its leading three-by-three block. Since $\operatorname{Re}\omega = -1/2$, that perturbation is $E = a_0 I_3 + b_0 J_3$, where $a_0 = 3st$ and $b_0 = t^2 - st$. No positivity of this perturbation is assumed. Let $S_k(E)$ be the sum of all k-by-k subpermanents, with $S_0 = 1$. Selecting the slots using E instead of $2J$ gives

$$C_{K_t} = \sum_{k=0}^{3} (6-k)!\, 2^{6-k} S_k(E).$$

The identity

$$\operatorname{per}(a_0 I_3 + (b_0 + x)J_3) = a_0^3 + 3a_0^2(b_0 + x) + 6a_0(b_0 + x)^2 + 6(b_0 + x)^3$$

has coefficient $(3-k)!S_k(E)$ at x^{3-k}. Consequently

$$\begin{aligned} S_1 &= 3a_0 + 9b_0 = 9t^2, \\ S_2 &= 3a_0^2 + 12a_0 b_0 + 18b_0^2 = 18t^2 + 18t^4, \\ S_3 &= a_0^3 + 3a_0^2 b_0 + 6a_0 b_0^2 + 6b_0^3 = 24st^3 + 18t^4 + 6t^6. \end{aligned}$$

The weights for $k = 0, 1, 2, 3$ give

$$C_{K_t} = 46080 + 41472t^2 + 1152st^3 + 7776t^4 + 288t^6.$$

Subtracting this from $P_{K_t}^2$ proves the exact polynomial

$$\begin{aligned} D_{K_t} = {} & 25355520 - 41472t^2 + (1451520 - 1152s)t^3 \\ & - 7776t^4 + 20448t^6. \end{aligned}$$

For $0 < t \le 1/100$, its positive cubic and sixth-order terms yield

$$D_{K_t} \ge 25355520 - 41472/10000 - 7776/100000000 > 25355515.$$

Its derivative satisfies the exact rational upper bound

$$\begin{aligned} \frac{1}{t}\frac{dD_{K_t}}{dt} &= -82944 + 3(1451520 - 1152s)t - 31104t^2 + 122688t^4 \\ &\le -82944 + \frac{4354560}{100} + \frac{122688}{100^4} < 0. \end{aligned}$$

Finally $K_b - K_a = (b-a)vv^*$ is positive, has three-coordinate support, and has the same range as K_a. □

This is the least possible nonzero residual rank for such an example. It does not establish a least support size: two-coordinate directions remain a separate question. It also does not contradict a scalar permanent inequality, since D_{K_t} is positive throughout the interval.

F.3 A three-coordinate obstruction to PSD-order monotonicity

The diagonal theorem cannot be promoted to arbitrary PSD directions. This is a statement about the strong auxiliary target, not a counterexample to Chollet's inequality.

Proposition F.4. *There exist complex PSD matrices $A \preceq B$ of order six such that*

$$\operatorname{rank} A = \operatorname{rank} B = 4, \quad \operatorname{rank}(A+J) = \operatorname{rank}(B+J) = 5,$$

$$0 < F_A(s) - P_A < 720, \quad 0 < F_B(s) - P_B < 720, \qquad D_A > D_B > 0.$$

The difference $B - A$ has rank one and support on three coordinates.

Proof. Using columns and writing $\mathrm{i}^2 = -1$, set

$$\begin{aligned} v &= (1, -1, \mathrm{i}, -\mathrm{i}, 2, -2)^{\mathsf{T}}, \\ u &= (-2+\mathrm{i}, 0, 1, 0, 1-\mathrm{i}, 0)^{\mathsf{T}}, \\ w_1 &= (1, 1, -1, -1, 0, 0)^{\mathsf{T}}, \quad w_2 = (0, 0, 1, 1, -1, -1)^{\mathsf{T}}, \\ K_0 &= 2^{-8} v v^* + 2^{-20}(w_1 w_1^* + w_2 w_2^*), \\ A &= K_0 + 2^{-30} u u^*, \qquad B = A + 2^{-31} u u^*. \end{aligned}$$

Positivity and $A \preceq B$ follow from these positive rank-one sums. Exact Gaussian elimination gives $\operatorname{rank}[v, w_1, w_2, u] = 4$ and $\operatorname{rank}[v, w_1, w_2, u, \mathbf{1}] = 5$; the strictly positive weights then give the asserted matrix ranks. Direct permanent expansion in $\mathbb{Q}(\mathrm{i})$ and $\mathbb{Q}(\sqrt{2})$ gives the following rational enclosures, certified by the exact subset-permanent and quadratic-field procedure:

$$\begin{aligned} 5040 < P_A, P_B &< 5041, \\ 719 < F_A(s) - P_A,\ F_B(s) - P_B &< 720, \\ 25356099 < D_A, D_B &< 25356100, \\ \frac{26}{10^9} < D_A - D_B &< \frac{27}{10^9}. \end{aligned}$$

The checker constructs the displayed matrices afresh, expands all 720 ordinary permutation polynomials and all 720 paired permutation terms, and uses exact rational quadratic-field sign comparisons for each endpoint. It also verifies a nonreal cycle product at each matrix, so these phases cannot all be removed by diagonal unitary gauges. This finite arithmetic certificate proves the claimed strict inequalities. □

Both examples rule out PSD-order monotonicity of D_K, while preserving the diagonal result and leaving the two-coordinate question open. The rank-one interval is already range-preserving, but its support is three; the older example retains its different rank and phase geometry. At residual rank five, a range-preserving rank-one subtraction may use a vector on two coordinates; at a generic rank-four core its two-dimensional kernel usually requires at least three coordinates. Consequently a possible two-coordinate descent cannot simply be iterated using unrestricted monotonicity. For the intended rank-preserving problem, only directions $u \in \operatorname{range} K$ are relevant; arbitrary violations outside this range would not refute that restricted statement.

G Exact arithmetic and the scope of the finite calculations

The universal operator and tensor inequalities follow from their printed analytic proofs. The restricted-family covering proofs have different roles: their displayed corner bounds, exclusion tests, deterministic bisections and analytic unbounded tails specify complete finite exact calculations. Their reported certified/excluded leaf counts concern those domains only. They are not sampled evidence for a generic PSD matrix inequality.

For any displayed exact example, all permanent values and coefficients can be reconstructed by the following common algorithm. For a matrix of polynomial entries A, set $D_\varnothing = 1$ and, for a column subset S of size k, update

$$D_{S\cup\{j\}} \mathrel{+}= D_S A_{k,j} \quad (j \notin S).$$

The full-set polynomial is the permanent. Applying it to the specified cofactor minors and using their displayed contractions gives the same T, U, E, G and transformed deficits as in the proofs. Coefficient truncation is permitted only above the requested degree. Gaussian rational examples use exact rational real and imaginary parts; quadratic-field examples use pairs (a, b) for $a + b\sqrt{2}$ or the specified cyclotomic relation. A radical sign with opposing rational parts is decided by comparing rational squares, after retaining the parts' signs. Matrix ranks use exact elimination. No floating numerical rank, sampled eigenvalue or optimizer output is a sign premise.

In a rational closed-box proof, each failed test triggers the stated bisection; a resource bound is an error condition and never a successful leaf. Every child box is closed and both children are retained, so shared boundaries and axes remain covered. The analytic tail is checked before concluding a theorem on unbounded amplitudes. The Jacobi calculation in Section F instead reconstructs all its finitely many continuants and their quadratic-field coefficient signs. These proofs do not require a previously published paper or an unnamed data file. The explicit adaptive example's rational input and shifts are recorded in Appendix I.

The principal limitation is mathematical: generic six-factor positive-loss compensation remains open outside the specified sufficient regions. Conditional capacity tests, exact failures of fixed duals, nonempty finite examples and the actual scalar defect remain distinct.

AI assistance

AI tools assisted mathematical exploration, drafting and checking. The author is responsible for the mathematical claims, proofs, references and presentation.

H Inputs and exact tests for the supporting fixtures

This appendix specifies the finite examples used after Theorem A.1, Corollary A.2 and the weighted-projection statements. The universal inequalities remain proved by their earlier arguments. All indices below run from zero to five.

H.1 Small-intermediate-tensor fixtures

Let $\omega = e^{\pi i/3}$, $y_i = \omega^i$, $t_i = (-1)^i$, and define

$$\Pi_6 = I - J/6 - yy^*/6, \qquad \Pi_5 = \Pi_6 - tt^*/6.$$

Let $A_{01} = i\sqrt{3}$, $A_{10} = -i\sqrt{3}$, with all other entries zero. For $b \in \{5, 6\}$ set $\tau = 2^{-512}$ and

$$K_b = \tfrac{3}{2} yy^* + \tau \Pi_b + \tau^2 \Pi_b A \Pi_b, \qquad B_b = K_b + J.$$

These are the exact small-tensor examples. The Fourier modes are orthogonal, so Π_b is an orthogonal projection of rank $b-2$, annihilating $\mathbf{1}$ and y. Since $\|A\| = \sqrt{3} < 2$ and $2\tau < 1$, the perturbation is positive definite on that range. Consequently $K_b \succeq 0$, $\operatorname{rank} K_b = b - 1$, and $\operatorname{rank} B_b = b$; the selected all-ones direction is active.

For completeness, the finite conditions on these matrices can be checked without constructing Gram square roots. Expand $F_b(v) = \operatorname{per}(K_b + vJ) = \sum_j \alpha_j v^j$ by the subset recurrence. With $P_b = \sum_j (j+1)\alpha_j$ and $m_l = \binom{6}{l}\alpha_{6-l}/720$, the exact comparisons are

$$0 \le m_l \le 2^{-160}\ (1 \le l \le 5), \quad F_b(\sqrt{2}) - P_b > 0, \quad \tfrac{1}{4} \le (K_b)_{ii} \le 30, \quad \frac{1}{1 + (K_b)_{ii}} > \frac{1}{31}.$$

Thus the tensor norms satisfy the stated 2^{-80} bound, and the diagonals place the maximum residual norm in the required middle range. The forward row envelope also satisfies $P_b^2 - U_1(K_b + \sqrt{2}J) > 3 \cdot 2^{22}$. To verify this last inequality with no square roots of row sums, form

$$U_1(M)^2 = \frac{100}{9} \prod_i \sum_j |M_{ij}|^4$$

and compare it with $(P_b^2 - 3 \cdot 2^{22})^2$, first checking $P_b^2 > 3 \cdot 2^{22}$. Exact arithmetic is in $\mathbb{Q}(\omega, \sqrt{2})$, using $\omega^2 - \omega + 1 = 0$. A nonreal triangle product of B_b and unequal multisets $\{|(B_b)_{ij}|^2/((B_b)_{ii}(B_b)_{jj}) : j \ne i\}$ for different rows establish the stated genuine phases and noncirculant normalized form. These are finite tests of the displayed inputs, not sampling.

H.2 Small-positive-loss strip fixtures

Define $\ell(q) = 720 - 144q^2 - 24q^4$ and $N = 2^{64}$. Select the integer k as the largest $k \in [N, 2N]$ satisfying $\ell(k/N) > 0$, and put $q = k/N$. This completely specifies the rational value without a root approximation. Equivalently, initialize $l = N, h = 2N$; while $h - l > 1$, set $m = \lfloor (l + h)/2 \rfloor$, replace l by m if $\ell(m/N) > 0$, and otherwise replace h by m. Then $k = l$. The signs at the two initial endpoints are opposite and ℓ is strictly decreasing on $q > 0$. Let $H(q)$ have three diagonal blocks $q \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$, with all other entries zero. Define the two projections

$$(P_6)_{ij} = \tfrac{1}{2}\mathbf{1}_{\lfloor i/2 \rfloor = \lfloor j/2 \rfloor} - \tfrac{1}{6}, \qquad P_5 = vv^*/8, \quad v = (1, 1, i, i, -1 - i, -1 - i)^T.$$

They have ranks two and one and are orthogonal to the three pair-sign modes and $\mathbf{1}$. Let $E_{02} = i$, $E_{20} = -i$, $E_{00} = 1$, all other entries zero. For $b \in \{5, 6\}$ set $\varepsilon = 2^{-256}$ and

$$\widehat{K}_b = H(q) + \varepsilon P_b + \varepsilon^2 P_b E P_b, \qquad \widehat{B}_b = \widehat{K}_b + J.$$

Since $\|E\| < 2$ and $2\varepsilon < 1$, the same orthogonal-range argument proves residual/base ranks $b - 1, b$ and PSD. For these exact matrices, the finite comparisons are

$$0 < F_b(\sqrt{2}) - P_b \le 2^{-28}, \quad \max_{1 \le l \le 5} \binom{6}{l} \alpha_{6-l}/720 > 2^{-160}, \quad \tfrac{1}{4} < (\widehat{K}_b)_{ii} < 30, \quad \frac{1}{1 + (\widehat{K}_b)_{ii}} > \frac{1}{31}.$$

Here F_b, α_j, P_b are reconstructed from $\widehat{K}_b$, not from the unperturbed pair blocks. The same exact permanent and cofactor recurrences give $G_b > 24656$. Nonreal triangle products and unequal normalized row-coherence multisets are checked as above. Thus these examples lie in the strip and outside the weak-pair and tiny-tensor domains, with every input and the side of the threshold fixed explicitly. All operations for this pair of fixtures are Gaussian-rational followed by a $\mathbb{Q}(\sqrt{2})$ sign comparison.

H.3 The five weighted-projection fixtures

The five exact cores used in the projection comparison are as follows:

1. $K = H(7/4)$, with $H(q)$ as just defined.
2. $K = H(7/4) + \varepsilon P_6 + \varepsilon^2 P_6 E P_6$, with $\varepsilon = 1/1000$ and the same E.
3. $K = H(7/4) + \varepsilon P_5 + \varepsilon^2 P_5 E P_5$, with $\varepsilon = 1/1000$.
4. $K = \lambda\lambda^*$, where $\lambda = (2, -2, 1 + 7i/4, -1 - 7i/4, -1 + 7i/4, 1 - 7i/4)^T$.
5. $K = \lambda\lambda^*$, where every entry of λ is $1/4$.

The first three have base ranks four, six and five respectively; the last two have residual rank one. For each, form the displayed D, β, G_d, b_d from its actual entries. Solve $Dc = \beta$ and $G_d c_d = b_d$ by exact Gaussian elimination with free variables set to zero, checking the original equations after substitution. Range consistency is already proved, so these solutions give $\beta^* c$ and $b_d^* c_d$ without a numerical pseudoinverse. The finite comparisons are $0 < F(\sqrt{2}) - P$, $5040 \le \Phi \le \Psi \le P$, and $\Phi^2 + 41040 \ge U_1(K + \sqrt{2}J)$, with the last comparison squared only after its two sides are known positive. For the fourth core $\Phi = \Psi = P$; for the fifth $\Psi = P$. For the first core the explicit values are those printed after (75). These formulas specify all five matrices and every finite test used; no unnamed fixture collection is required.

I Exact data for the two-leaf adaptive example

The finite example after Proposition 4.8 uses $V = D^{-1}Z$, where

$$D = \operatorname{diag}(1, 13, 10000000009, 625000001, 400000001, 2500000009)$$

and

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 5i & 12 & 0 & 0 & 0 & 0 \\ 360000 & 480000 & 9999999991 & 0 & 0 & 0 \\ 30000 & 40000 & 0 & 624999999 & 0 & 0 \\ 24000 & 32000 & 0 & 0 & 399999999 & 0 \\ 180000 & 240000 & 0 & 0 & 0 & 2499999991 \end{pmatrix}.$$

Its rows have norm one and it has rank six. Its Gram has full support and a nonreal triangle product. For the occupation cube $[0, 1]^6$, split the second coordinate at $1/2$. On the two resulting closed boxes use the shifts $\tau_L = A_L/B_L$ and $\tau_R = A_R/B_R$, with the following exact decimal integer strings (line breaks separate integers, not digits):

```
A_L = 430058614873414917952358664346036341198190373139466572522845674585234547233329 5
B_L = 112490626934838764099574994244321050186376634952282273249096015066542006894105 36
A_R = 991976611223436348193193817848390432825429424563104393713488965695281096613970 6
B_R = 258847270077173001193916404009379599636609615990550935099592414950652117976400 89
```

For the left box, its center has second entry $1/4$ and all other entries $1/2$; its halfwidth has second entry $1/4$ and all other entries $1/2$. For the right box, replace the center's second entry by $3/4$, keeping the same halfwidth. Construct B, R, L, T and the constants a, c, δ from this V as in (1). The two matrices

$$[\delta + h^{\mathsf{T}} L h - r^{\mathsf{T}} L r - \tau^2 - 13/20]I + 2(a + \tau)T - 2V^* \operatorname{diag}(Lh)V$$

are positive semidefinite. Here h, r are each box's center and halfwidth, not the residual scalar used elsewhere. This is a finite rational matrix assertion: substitute the displayed data, evaluate all permanents by the subset recursion, and check all principal minors of the resulting Hermitian matrices by exact elimination. There are no irrational input coordinates or numerical eigenvalues. Proposition 4.7 then proves $\mathcal{D} \geq 13/20$ on the entire occupation cube.

For comparison, put $w = (2, -3i, 0, 0, 0, 0)^T$ and normalize its Rayleigh quotient by $w^*w = 13$. For a single full-cube box the corresponding matrix without the margin has quotient

$$\gamma + 2\tau t_w - \tau^2, \quad t_w = \frac{w^*Tw}{13}, \qquad \gamma = \delta + 2at_w - \frac{2w^*V^* \operatorname{diag}(L\mathbf{1}/2)Vw}{13}.$$

Exact rational evaluation gives $\gamma + t_w^2 < 13/20$. Since this is the maximum over all real τ, no single-full-cube shift attains the same margin. The shifted cofactor test at $\tau = 64/169$ nevertheless does attain $13/20$ on this example, so the example illustrates subdivision rather than necessity of retaining the variance.

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