Occupation bounds for coupled permanent cofactors

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Luminary Oddities

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Abstract

For a complex positive semidefinite Gram matrix $B = VV^*$ of order $n \geq 2$, we couple the permanent-cofactor contractions of B and $|B|^2$. Writing $a = \operatorname{per} B$, $c = \operatorname{per}|B|^2$, $T = V^*RV$ and $U = V^*(L \circ B)V$, with transposed permanent-cofactor matrices R, L, we prove the sharp inequality $0 \preceq U - aT/n! \preceq (n-1)(c - a^2/n!)I$. The proof identifies the difference as a one-mode occupation expectation in the orthogonal difference of two nested invariant subspaces. At order six, a central-transposition statistic gives a computable refinement involving fifteen mixed permanents. We apply it to a compensation operator, exhibit complex rank-five and rank-six examples with a strictly improved sufficient certificate, and retain all Gram kernel directions. We also give the residual occupation polynomial, its exact kernel-thinning law, a zero-safe top-coefficient cancellation, and actual complex examples showing that these occupation polynomials need not have only real zeros.

1 Introduction and notation

For a square matrix A, $\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_i A_{i,\sigma(i)}$; $A(j \mid i)$ deletes row j and column i. The symbol $\circ$ denotes entrywise multiplication, $*$ the conjugate transpose, and $|B|^2 = B \circ \overline{B}$, not the operator square.

Throughout, $V \in \mathbb{C}^{n \times d}$, $B = VV^*$, and d is arbitrary. Define

$$a = \operatorname{per} B, \qquad c = \operatorname{per}|B|^2, \qquad R_{ij} = \operatorname{per} B(j \mid i), \qquad L_{ij} = \operatorname{per}|B|^2(j \mid i), \tag{1}$$

$$T = V^*RV, \qquad U = V^*(L \circ B)V, \qquad r = c - \frac{a^2}{n!}, \qquad S = U - \frac{a}{n!}T. \tag{2}$$

All operator inequalities below act on $\mathbb{C}^d$ unless specified otherwise. No rank, normalization of the diagonal, or invertibility hypothesis is imposed.

Permanent-cofactor Gram matrices and their Schur products are classical. Paksoy [4], in particular, uses omitted symmetric tensors and a weighted cofactor matrix with constant permanent row sum. Symmetrizers and bosonic permanent states are standard [5, 6]; Schur weights and content eigenvalues are standard representation theory [7, 8]. Our focus is the *joint residual* S, its sharp norm bound, and a computable refinement that preserves its common Gram origin. Li, Zhao and Luo [3] represent coefficients of ordinary and doubled bordered permanent polynomials by squared norms over complete occupation sectors. The polynomial studied below instead comes from the joint residual projection $P_{\mathrm{diag}} - P_{\mathrm{sep}}$; its coefficients measure one shared residual state. The separate positivity of T and U, or a bound on either one alone, does not supply this residual estimate.

The motivation is the compensation in a bordered permanent inequality. Chollet asks whether $\operatorname{per}(A \circ D) \leq \operatorname{per} A \operatorname{per} D$ for complex positive semidefinite A, D; see [1, 2]. The results here concern coupled cofactor operators and sufficient border certificates, rather than an unrestricted solution. They are distinct from the author's rank-restricted scalar results [11] and cofactor-comparison limitations [12]. We consolidate the sharp residual estimate and its refinements in one note.

2 A sharp bound in every order

Theorem 2.1. *With* (1)–(2)*, for every* $n \geq 2$*,*

$$0 \preceq S \preceq (n-1) r I_d. \tag{3}$$

The coefficient $n-1$ *cannot be reduced for actual Gram factors. The conclusion includes arbitrary complex phases and singular factors.*

2.1 The two invariant projections

We use the conjugate-linear-first inner product. Regard row i of V as the ket $v_i = \sum_j V_{ij} e_j$ in $\mathcal{H} = \mathbb{C}^d$. Its physical Gram is $\overline{B}$. Put

$$f = v_1 \otimes \cdots \otimes v_n, \qquad W = f \otimes \overline{f}.$$

On $\mathcal{H}^{\otimes n} \otimes \overline{\mathcal{H}}^{\otimes n}$, let P_{sep} average independent permutations of the two legs, and let P_{diag} average equal simultaneous permutations. Both are orthogonal projections and $P_{\mathrm{sep}} P_{\mathrm{diag}} = P_{\mathrm{sep}}$. In particular $a = n! \| \operatorname{Sym} f \|^2 \geq 0$. Direct expansion gives

$$\| P_{\mathrm{sep}} W \|^2 = \frac{a^2}{(n!)^2}, \qquad \| P_{\mathrm{diag}} W \|^2 = \frac{c}{n!}. \tag{4}$$

For unit $z \in \mathbb{C}^d$, let $p_z = |\overline{z}\rangle\langle\overline{z}|$ and let $N_z = \sum_{j=1}^n p_z^{(j)}$ act on the first leg. The conjugation in p_z matches the row convention $B = VV^*$. This operator commutes with both projections.

Set $h = (P_{\mathrm{diag}} - P_{\mathrm{sep}}) W$. For $x = Vz$, differentiation of the two permanent norm formulas gives

$$\| h \|^2 = \frac{r}{n!}, \tag{5}$$

$$\langle h, N_z h \rangle = \frac{1}{n!} \left(z^* U z - \frac{a}{n!} z^* T z \right). \tag{6}$$

Here the conjugate physical Gram deformation is $B + t\, xx^*$, and

$$\frac{d}{dt} \operatorname{per}(B + t\, xx^*) \Big|_{t=0} = z^* T z.$$

The paired deformation is $|B|^2 + t(B \circ \overline{xx^*})$, whose permanent derivative is $z^* U z$. For example the latter derivative is $\sum_{ij} L_{ji} B_{ij} \overline{x_i} x_j$; L is real symmetric, yielding the defined U. The independent-projection expectation is $a(z^* T z)/(n!)^2$ and the diagonal one is $(z^* U z)/n!$. Their difference is (6); the cross terms vanish because N_z preserves the nested orthogonal components.

2.2 Excluding full occupation

Proof of Theorem 2.1. The spectrum of N_z is contained in $\{0, \ldots, n\}$. Its n-eigenspace has first leg $\overline{z}^{\otimes n}$. A diagonal-invariant vector in that eigenspace has a symmetric first leg; diagonal invariance then forces its second leg to be symmetric too. Thus the n-occupation part of range P_{diag} lies in range P_{sep} and is absent from h. Consequently

$$0 \leq \langle h, N_z h \rangle \leq (n-1) \| h \|^2.$$

Use (5)–(6) and vary unit z.

For sharpness, take $n-1$ equal unit rows in one coordinate and one unit row in an orthogonal coordinate. Then $B = J_{n-1} \oplus [1]$, $a = c = (n-1)!$ and, in the repeated-row direction, $z^* T z = (n-1) a$, $z^* U z = (n-1) c$. The upper bound is an equality. □

In particular $r \geq 0$. If $r = 0$ then $S = 0$. Both T and U vanish on $\ker V$. Therefore, with $\mathcal{A} = (\ker V)^\perp$ and its orthogonal projection $P_{\mathcal{A}}$,

$$0 \preceq S \preceq (n-1) r P_{\mathcal{A}}. \tag{7}$$

This statement retains the kernel rather than replacing the ambient space.

3 The occupation polynomial and a cancellation

For unit z, write $x = Vz$, $X = xx^*$ and $H = B - X \succeq 0$.

Proposition 3.1. *The polynomial*

$$G_z(s) = \operatorname{per}\big(|B|^2 + (s-1)(B \circ \overline{X})\big) - \frac{a}{n!}\operatorname{per}\big(B + (s-1)X\big) = \sum_{k=0}^{n} q_k(z)s^k \tag{8}$$

has $q_k(z) \geq 0$ and $q_n(z) = 0$. More precisely, if E_k is the k-occupation spectral projector of N_z, then $q_k(z) = n!\|E_k h\|^2$. Thus

$$G_z(1) = r, \qquad G_z'(1) = z^* S z, \tag{9}$$

$$q_0(z) = \operatorname{per}(H \circ \overline{B}) - \frac{a}{n!}\operatorname{per} H. \tag{10}$$

Proof. For real $s \geq 0$ multiply the selected first-leg mode by $\sqrt{s}$. The resulting operator is $s^{N_z/2}$ and commutes with both projections. The diagonal projected squared norm is the first permanent in (8) divided by $n!$; the independent projected squared norm is $a\ \operatorname{per}(B+(s-1)X)/(n!)^2$. Their difference is $\|s^{N_z/2}h\|^2$. Polynomial identification proves the coefficient formula. The full-occupation exclusion proves $q_n = 0$. The identities at $s = 1, 0$ follow directly. □

Proposition 3.2 (Exact kernel thinning)**.** *Let $w \in \mathcal{A}$ and $\kappa \in \ker V$ be unit vectors. For $0 \leq p \leq 1$ and real θ put $z_p = \sqrt{p}\, w + e^{i\theta}\sqrt{1-p}\,\kappa$. Then*

$$G_{z_p}(s) = G_w(1 - p + ps), \tag{11}$$

$$q_k(z_p) = \sum_{j=k}^{n-1} \binom{j}{k} p^k (1-p)^{j-k} q_j(w). \tag{12}$$

In particular $G'_{z_p}(1) = pG'_w(1)$ and $G''_{z_p}(1) = p^2 G''_w(1)$. The formulas include the endpoints and every relative complex phase.

Proof. $Vz_p = \sqrt{p}\, Vw$, and substitution into (8) gives the first identity. Binomial expansion gives the rest. □

Proposition 3.3 (Top occupation)**.** *Put $b_i = \prod_{j \neq i} x_j$, $y = \overline{b}$, $J = \mathbf{1}\mathbf{1}^*$ and*

$$F_0 = R \circ \overline{B} - \frac{a}{n} J.$$

Then $F_0 \succeq 0$, $F_0\mathbf{1} = 0$, and

$$q_{n-1} = y^*\left(R \circ |B|^2 - \frac{a}{n}B\right) y = y^*(B \circ F_0)y = y^*(H \circ F_0)y. \tag{13}$$

These are polynomial identities, including when some $x_i = 0$.

Proof. The usual omitted symmetric-tensor Gram construction gives $R \succeq 0$. The Schur product theorem gives $F = R \circ \overline{B} \succeq 0$. Column-$i$ Laplace expansion gives $\sum_j R_{ij}\overline{B_{ij}} = a$. Thus $F\mathbf{1} = a\mathbf{1}$ and removing that Hermitian eigencomponent gives $F_0 \succeq 0$. This construction is classical; compare [4].

In powers of $t = s - 1$, the coefficient of t^{n-1} in the first permanent in (8) is $\sum_{ij} |B_{ij}|^2 \operatorname{per} B(i \mid j)\overline{b_i} b_j$. Exchanging i, j gives $y^*(R \circ |B|^2)y$. The coefficient in the second permanent is $(n-1)!\, y^* B y$. Both leading t^n terms in G_z equal $a\prod_i |x_i|^2$ and cancel. Translating $t = s - 1$ therefore leaves the next coefficient unchanged, proving the first equality in (13). Finally $\operatorname{diag}(\overline{x})y = \overline{\prod_i x_i}\,\mathbf{1}$, so $y^*(X \circ F_0)y = 0$. This proves the last equality without division. □

For later directional tests at $n = 6$, set $r_+ = r - q_0$ and $m_2 = G_z''(1)$. The elementary moment bound is

$$z^* S z \le \min\left\{5r_+,\ \frac{m_2}{2j} + \frac{j+1}{2} r_+ :\ j = 1, \dots, 4\right\}. \tag{14}$$

Indeed $(k-j)(k-j-1) \ge 0$ at integer $1 \le k \le 5$. Multiplication by q_k and summation proves each affine bound. For fixed feasible r_+, m_2, mixtures on consecutive occupations attain its segments, so this is the exact finite-support relaxation. It is an elementary instance of discrete moment linear programming [9]. Retaining q_5 also gives $z^* S z \le 4(r - q_0) + q_5$. Neither estimate alone couples its inputs to the compensation budget.

4 A central-transposition refinement at order six

In this section $n = 6$. On the first tensor leg let $C = \sum_{i<j} U_{(ij)}$ be the sum of transpositions and put

$$v = 720\,\langle h, Ch\rangle, \qquad M = \frac{v + 21r}{6}.$$

Theorem 4.1. *For every complex Gram factor of order six,*

$$0 \le S \le M I_d \le 5r I_d. \tag{15}$$

For each transposition π form $A^{\pi}_{ij} = B_{\pi(i),j}\overline{B_{ij}}$. Then

$$v = \sum_{\pi \text{ transposition}} \operatorname{per} A^{\pi} - \frac{15a^2}{720}. \tag{16}$$

The conclusion holds in every ambient dimension and at singular boundaries.

Proof. On a first-leg Schur sector of shape $\lambda \vdash 6$, the largest one-mode occupation is λ_1: column strictness in semistandard tableaux permits a fixed letter at most once in each column. The central transposition sum acts by the content sum $\kappa_\lambda = \sum_{(i,j)\in\lambda}(j - i)$; see [8, Example 3.15]. The full comparison is

λ	λ_1	κ_λ	$(\kappa_\lambda + 21)/6$
(6)	6	15	6
(5, 1)	5	9	5
(4, 2)	4	5	13/3
(4, 1, 1)	4	3	4
(3, 3)	3	3	4
(3, 2, 1)	3	0	7/2
(3, 1, 1, 1)	3	−3	3
(2, 2, 2)	2	−3	3
(2, 2, 1, 1)	2	−5	8/3
(2, 1, 1, 1, 1)	2	−9	2
(1^6)	1	−15	1

so $N_z \le (C + 21I)/6$. The separately symmetric first-leg sector is absent from h: first-leg symmetry and diagonal invariance force second-leg symmetry. All remaining sectors have $\kappa_\lambda \le 9$. Hence

$$720\,\langle h, N_z h\rangle \le (v + 21r)/6 \le 5r.$$

Use (6). Nonnegativity follows from Theorem 2.1.

The class sum C commutes with both projections, and acts as 15 on their separately symmetric range. Consequently

$$\langle h, Ch\rangle = \langle W, CP_{\mathrm{diag}} W\rangle - 15\|P_{\mathrm{sep}} W\|^2.$$

Expand the diagonal average and the fifteen swaps. Summing over its permutation index gives per A^π for each π (equivalently its complex conjugate under the opposite permutation convention). The sum is real since CP_{diag} is self-adjoint. Equations (4) give (16). □

More precisely, if $r_\lambda = 720\|E_\lambda h\|^2$, then $S \preceq (\sum_\lambda \lambda_1 r_\lambda) I$ and $v = \sum_\lambda \kappa_\lambda r_\lambda$. The single statistic v compresses these masses. Since S vanishes on $\ker V$, (15) also gives $S \preceq M P_{\mathcal{A}}$.

5 Compensation and exact examples

For any n define $\delta = a^2 - c$ and $D_0 = \delta I_d + 2aT - U$. Let P_i be the rank-one matrix $(P_i)_{\alpha\beta} = \overline{V_{i\alpha}} V_{i\beta}$, and set

$$\mathcal{K} = \delta I_{d^2} + a(T \otimes I + I \otimes T) + T \otimes T - \sum_{ij} L_{ij} P_i \otimes P_j. \tag{17}$$

Lemma 5.1. *$D_0 \succeq 0$ implies $\mathcal{K} \succeq 0$. For the bordered Gram matrix $\widetilde{B}_z = \begin{pmatrix} B & Vz \\ z^* V^* & \|z\|^2 \end{pmatrix}$,*

$$\operatorname{per}(\widetilde{B}_z)^2 - \operatorname{per}|\widetilde{B}_z|^2 = \langle z \otimes z, \mathcal{K}(z \otimes z)\rangle.$$

Proof. L is real symmetric PSD by the omitted-tensor Gram construction. Factor $L = CC^T$ over the reals and put $A_k = \sum_i C_{ik} P_i$. Then A_k are Hermitian and $\sum_k A_k^2 = U$, since $P_i P_j$ has entries $\overline{V_{i\alpha}} B_{ij} V_{j\beta}$. Expansion gives the identity

$$\mathcal{K} = \frac{D_0 \otimes I + I \otimes D_0}{2} + T \otimes T + \frac{1}{2} \sum_k (A_k \otimes I - I \otimes A_k)^2.$$

Every displayed term is PSD when $D_0 \succeq 0$. Finally border expansion gives $\operatorname{per} \widetilde{B}_z = a\|z\|^2 + z^* T z$ and $\operatorname{per}|\widetilde{B}_z|^2 = c\|z\|^4 + \sum_{ij} L_{ij} |(Vz)_i|^2 |(Vz)_j|^2$. Subtracting proves the formula. □

Theorem 2.1 gives, in every order,

$$D_0 \geq \left[\left(1 + \frac{n-1}{n!}\right) a^2 - nc \right] I_d + \left(2 - \frac{1}{n!}\right) aT.$$

At order six this certifies $D_0, \mathcal{K} \succeq 0$ whenever $a > 0$ and $c/a^2 \leq 145/864$. Theorem 4.1 improves the sufficient test to

$$D_0 \succeq (\delta - M) I_d + \frac{1439a}{720} T, \qquad M \leq \delta. \tag{18}$$

This is a condition on the actual mixed-permanent statistic.

Here are exact factors witnessing strict improvement of the scalar test. Let $W \in \mathbb{C}^{6\times 6}$ have first three rows $(1, \frac{1}{2}, 0, 0, 0, 0)$, $(1, \frac{i}{2}, 0, 0, 0, 0)$, $(1, -\frac{1}{2}, 0, 0, 0, 0)$, and last three rows e_2^T, e_3^T, e_4^T, using coordinates $0, \ldots, 5$. For $0 \leq i \leq 5$, $0 \leq j \leq 4$ set

$$E_{ij} = \frac{((i+2j) \bmod 3) - 1 + i_c(((2i+j) \bmod 3) - 1)}{10000}, \qquad E_{i5} = 0,$$

where $i_c = \sqrt{-1}$ is distinguished from the row index. Put $V_5 = W + E$ and $V_6 = V_5 + 10^{-4} e_2 e_5^T$. The accompanying exact-rational verifier proves, for $B_j = V_j V_j^*$,

$$\operatorname{rank} V_j = j, \quad B_j \text{ has no zero entries}, \qquad \frac{c}{a^2} > \frac{145}{864}, \quad \frac{\delta - M}{a^2} > \frac{17}{1000} \quad (j = 5, 6).$$

Both have a triangle $B_{ij} B_{jk} B_{ki}$ with nonzero imaginary part, which cannot be removed by diagonal unitary row gauges. Numerically c/a^2 is about 0.20174. Strict margins persist on relative open neighborhoods of the corresponding rank strata. These examples distinguish two sufficient scalar certificates.

For unit z, writing $t = z^*Tz$, the remaining directional target is

$$z^*Sz \le \delta + \frac{1439a}{720}t + t^2.$$

Moment bounds such as (14) can certify individual directions. For a rank-five factor in six ambient coordinates, Proposition 3.2 retains every active/kernel mixture.

6 Actual residual polynomials need not be real-rooted

The positivity in Proposition 3.1 does not imply real stability. This limitation matters when importing probability-law inequalities. Consider the Gaussian-integer factor

$$V_6 = \begin{pmatrix} -1 & -i & 1-i & 1-i & -1+i & 0 \\ -1-i & 1 & i & -1 & -1 & -1 \\ i & -1-i & -1-i & 1+i & -1-i & 1-i \\ -1-i & 1+i & i & -1+i & -1 & -1-i \\ -1-i & 1-i & 1-i & -i & 1+i & -1-i \\ -1+i & -1-i & -i & 1+i & 1 & -1+i \end{pmatrix}.$$

Let V_5 be the same matrix with its last column zero, and take $z = e_0$. Here the names V_5, V_6 denote these obstruction factors only. Exact elimination gives ranks five and six; both Gram matrices have full support and nonzero imaginary triangle products. The coefficient lists $(q_0, \dots, q_5)$ are

$$\begin{aligned} \operatorname{rank} V = 6: &\quad (20542128507952/45,\ 5821468652200/9,\ 18083778219968/45,\\ &\qquad 2049107101804/15,\ 127236053248/5,\ 6211082816/3),\\ \operatorname{rank} V = 5: &\quad (403462265536/5,\ 127255137024,\ 458165154544/5,\\ &\qquad 187047753024/5,\ 8625123712,\ 878481472). \end{aligned}$$

All entries are positive and $q_6 = 0$. Nevertheless

$$4q_1^2 - 10q_0q_2 = \begin{cases} -651686468368265612895710 72/405, & \operatorname{rank} V = 6,\\ -458273045040072984596 48/5, & \operatorname{rank} V = 5. \end{cases}$$

If a degree-five polynomial with positive coefficients had only real roots, they would be negative and it could be written $q_0 \prod_{\ell=1}^{5}(1 + \rho_\ell s)$ with $\rho_\ell > 0$. Cauchy–Schwarz then gives the necessary Newton inequality $4q_1^2 \ge 10q_0q_2$. The exact reverse inequalities prove nonreal roots. This elementary condition belongs to the classical theory of real-rooted probability generating polynomials; see [10].

In the rank-five example $e_5 \in \ker V_5$. For every $p > 0$, a nonreal root ρ is carried by Proposition 3.2 to $(\rho - 1 + p)/p$, still nonreal. Thus every nontrivial active/kernel mixture along this ray retains the obstruction, independently of relative complex phase. At $p = 0$ the polynomial is constant. These are not compensation counterexamples: the exact directional budgets $(a + t)^2 - c - z^*Uz$ are respectively 906092392444480 and 337221858201088.

7 Scope and reproducibility

The sharp theorem is analytic. The sector refinement uses classical Schur weights and content eigenvalues; its specialization and mixed-permanent evaluation produce the stated coupled bound. The moment relaxation and thinning algebra are classical tools. The cofactor PSD construction in Proposition 3.3 is credited to the existing omitted-tensor method; its role here is the residual coefficient cancellation.

The standalone standard-library program `verify.py` reconstructs the displayed factors, exact ranks, phase witnesses, mixed permanents, occupation coefficients, Newton defects and strict scalar margins. It uses rational arithmetic and explicit checks that remain active under `python -O`. No private computational files are needed.

The conclusions in [7] concern Haar averages.

AI assistance

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References

[1] J. Chollet, Problem 6482, *Amer. Math. Monthly* 89 (1982), 57–58.

[2] K. Rodtes, Chollet's permanent conjecture for 4×4 matrices, *Linear Multilinear Algebra* 72 (2024), 2633–2638, arXiv:2211.02839.

[3] Q. Li, Q. Zhao and W. Luo, Chollet's permanent conjecture through order six via border induction, Preprints.org 202608.2196v1 (31 August 2026), doi:10.20944/preprints202608.2196.v1.

[4] V. E. Paksoy, A permanent inequality for positive semidefinite matrices, *Electron. J. Linear Algebra* 39 (2023), 71–77, publisher's article.

[5] A. W. Harrow, The church of the symmetric subspace (2013), arXiv:1308.6595.

[6] J. M. Zhang, H. F. Song and Y. Liu, Permanent variational wave functions for bosons (2021), arXiv:2106.14679.

[7] S. Geller and E. Knill, Measuring multiparticle indistinguishability with the generalized bunching probability, arXiv:2509.04550v2 (2025).

[8] C. Ryba, Stable centres of wreath products, *Algebraic Combinatorics* 6 (2023), 413–455, doi:10.5802/alco.264.

[9] A. Prékopa and G. Alexe, Dual methods for the numerical solution of the univariate power moment problem, RUTCOR Research Report 14-2003, report.

[10] W. Tang and F. Tang, The Poisson binomial distribution—old & new, arXiv:1908.10024 (2019).

[11] S. Schubr, Chollet's inequality for positive semidefinite matrices of rank at most four (2026), author-published preprint, edition 1.

[12] S. Schubr, Sharp limitations of two permanental cofactor comparisons (2026), author-published preprint, edition 1.