Non-classical Weyl laws for Dirichlet tubes with diverging twisting

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November 20, 2026

Abstract

A twisted tube $\Omega \subset \mathbb{R}^3$ is swept out by a bounded planar cross-section ω rotating about an axis with angle $\theta(s)$. When the angular speed diverges at both ends and ω lies in an open half-plane bounded by a line through the axis, Krejčiřík showed that the Dirichlet Laplacian has compact resolvent although $|\Omega| = \infty$, and asked for its Weyl asymptotics. We prove $N(T^2) \sim P_{V_+}(T) + P_{V_-}(T)$, where P_V is a two-dimensional phase-space volume summed over the angular Dirichlet modes of ω, and the speed $b = |\theta'|$ of each end enters only through its sublevel measure $V(y) = |\{s : b(s) < y\}|$. The speed may be non-monotone: we assume divergence, small logarithmic oscillation on the rotation scale, bounded doubling of V and comparability of the active length with V. Cross-sections may consist of finitely many angular bands with pinching widths, several components and slits, including discs and ellipses, and may touch the axis. For regularly varying V of index $\kappa \geq 0$ the coefficient is a ζ-weighted angular-width moment $M_{1+\kappa}(\omega)$; log-periodic speeds give an explicit periodic Mellin profile. For power speeds the exponent $\frac{3}{2} + \frac{1}{2p}$ of the Barseghyan–Khrabustovskyi upper bound is attained, while the leading coefficient is a nonlinear width moment rather than a multiple of the area. We further prove stability under angle perturbations with finite limits, an $O(\Lambda^{7/4})$ remainder with a second periodic profile for constant-step sections and speeds whose inverse has bounded slope, recovery of width moments, a finite-data obstruction, and a connected cross-section whose leading count is blind to every odd logarithmic harmonic of the speed.

Keywords: Dirichlet Laplacian; twisted tube; Weyl law; infinite volume; non-classical asymptotics; regular variation; Mellin transform; inverse spectral problem.
2020 Mathematics Subject Classification: 35P20, 35J25, 81Q10.

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1 Introduction and main results

1.1 The question

Let $\omega \subset \mathbb{R}^2$ be bounded and open, let $\theta \in C^1(\mathbb{R})$, and let R_γ denote rotation of $\mathbb{R}^2$ by the angle γ. The twisted tube

$$\Omega_\theta = \{(R_{\theta(s)}y, s) : \ y \in \omega, \ s \in \mathbb{R}\} \subset \mathbb{R}^3 \tag{1.1}$$

has infinite volume, and its volume is locally preserved: every axial piece of length L has volume $L|\omega|$. Let H_θ be the Friedrichs Dirichlet Laplacian on Ω_θ and

$$N_\theta(\Lambda) = \operatorname{Tr} \mathbf{1}_{(-\infty,\Lambda)}(H_\theta)$$

its counting function, with multiplicity. Krejčiřík [15] proved that H_θ has purely discrete spectrum when ω lies in an open half-plane bounded by a line through the axis and $|\theta'(s)| \to \infty$ as $|s| \to \infty$. In the concluding section of [15] he proposed to establish "some sort of Weyl-type asymptotics for the discrete eigenvalues" in this quasi-bounded setting. The question is Open Problem 8 of his recent survey [16], which asks, for the quasi-bounded realisation of its Figure 8.5 (the elliptical cross-section rotating about an exterior axis of Figure 2 of [15]), "what are the Weyl-type asymptotics for the accumulation of the discrete eigenvalues at infinity?" The survey notes that the law must be non-standard because $|\Omega| = \infty$.

The only previous result on the counting function is the Berezin-type bound of Barseghyan and Khrabustovskyi [1]. For ω in an open half-plane bounded by a line through the axis and θ' nondecreasing with $\pm\theta' \geq 0$ on $\mathbb{R}_\pm$ (or the variant in their Section 4.1), their Theorem 2.1 with moment order $\sigma = 0$ gives

$$N_\theta(\Lambda) \leq \frac{L_0}{(1-\varepsilon)^{3/2}}|\omega| \int_{\mathbb{R}} [\Lambda - \varepsilon f(s)]_+^{3/2} \, \mathrm{d}s, \qquad 0 < \varepsilon < 1, \tag{1.2}$$

with the Cwikel–Lieb–Rozenblum constant L_0 and an explicit f which their Proposition 4.1 compares with θ'^2. For $|\theta'(s)| \sim c|s|^p$ the right-hand side is of order $|\omega|\Lambda^{3/2+1/(2p)}$. Our results show that this exponent is the true one (Corollary 1.5), while the true leading coefficient is not a function of the area: it is proportional to the moment $M_{1+1/p}(\omega)$ of the angular widths, which at fixed area can be made arbitrarily small by fragmenting the cross-section (Proposition 9.1).

1.2 Cross-sections and speeds

Write $y = (r\cos t, r\sin t)$ in polar coordinates and, for $r > 0$, let $\omega_r = \{t \in \mathbb{R}/2\pi\mathbb{Z} : (r\cos t, r\sin t) \in \omega\}$ be the angular fibre.

Definition 1.1 (Finite-band cross-section)**.** Let $0 \le r_- < r_+ < \infty$. A nonempty bounded open set $\omega \subset \mathbb{R}^2$ is a *finite-band cross-section on* (r_-, r_+) if $\omega \subset \{r_- < |y| < r_+\}$ and there is a partition $r_- = r_0 < r_1 < \cdots < r_M = r_+$ with the following property. On each band $I_m = (r_{m-1}, r_m)$,

$$\omega_r = \bigcup_{j=1}^{k_m} \big(l_{mj}(r), u_{mj}(r)\big) \pmod{2\pi}, \qquad r \in I_m,$$

where $k_m \ge 0$, the functions $l_{mj} < u_{mj}$ are continuous on I_m and extend continuously to $\overline{I}_m$, $u_{mj} < l_{m,j+1}$ on I_m, and all values on $\overline{I}_m$ lie in a closed interval J_m of length $A_m \le 2\pi$. Widths $\alpha_{mj} = u_{mj} - l_{mj}$ and gaps may tend to zero at band endpoints. We call ω *off-axis* if $r_- > 0$ and *reaching the axis* if $r_- = 0$.

No regularity of $\partial\omega$ is required. Slit annuli $\{r_- < r < r_+\} \setminus \{t = 0\}$ are admissible, with one band and $A_1 = 2\pi$. A full annulus centred on the axis is not: its fibres are full circles, and the corresponding tube is a straight cylinder with essential spectrum. If ω reaches the axis then $0 \notin \omega$, because the fibres of small circles are not full circles. The *width measure* μ_ω on $(0, 2\pi]$ and the *width moments* are

$$\int f \,\mathrm{d}\mu_\omega = \sum_m \int_{I_m} r \sum_j f(\alpha_{mj}(r))\,\mathrm{d}r, \qquad M_z(\omega) = \int \alpha^z \,\mathrm{d}\mu_\omega(\alpha) \quad (\Re z > 0). \tag{1.3}$$

Thus $M_1(\omega) = |\omega|$. Individual widths are raised to the power before they are summed.

Definition 1.2 (Admissible end)**.** The positive end of $\theta \in C^1(\mathbb{R})$ is *admissible* if there are $s_+ \in \mathbb{R}$ and $\sigma \in \{1, -1\}$ such that $\theta' = \sigma b$ on (s_+, ∞), where $b : [s_+, \infty) \to (0, \infty)$ is continuous and

(H1) $b(s) \to \infty$ as $s \to \infty$;

(H2) for every $K > 0$, $\lim\limits_{s\to\infty} \sup\Big\{\Big|\log\dfrac{b(t)}{b(s)}\Big| : t \ge s_+,\ |t - s| \le \dfrac{K}{b(s)}\Big\} = 0$;

(H3) the sublevel measure $V(y) = |\{s > s_+ : b(s) < y\}|$ satisfies $\limsup_{y\to\infty} V(2y)/V(y) < \infty$;

(H4) the active length $S(y) = \sup(\{s_+\} \cup \{s > s_+ : b(s) \le y\})$ satisfies $S(y) = O(V(y))$.

The negative end is admissible if $s \mapsto \theta(-s)$ has an admissible positive end; its sublevel measure is denoted V_-, and that of the positive end V_+.

Condition (H1) makes V finite, nondecreasing, left-continuous and divergent; changing s_+ changes V by a bounded amount. Condition (H3) holds in particular when V is intermediately regularly varying in the sense of Cline [6], i.e. $\lim_{\lambda\downarrow 1} \limsup_{y\to\infty} V(\lambda y)/V(y) = 1$; only the bounded doubling (H3) is used below. Condition (H2) says that b changes by a factor $1 + o(1)$

over the length on which the cross-section turns by a bounded angle. It holds when $b \in C^1$ and $b'/b^2 \to 0$, and it passes to every continuous b asymptotic to a speed satisfying it (Lemma 5.2). Speeds need not be monotone.

For a finite Borel measure μ on $(0, 2\pi]$ and an admissible end with sublevel measure V define the *reduced phase*

$$P_{V,\mu}(T) = \frac{1}{4\pi} \int \mathrm{d}\mu(\alpha) \sum_{n \geq 1} \int_{[0,\infty)} y \left[T^2 - \Big(\frac{\pi n y}{\alpha}\Big)^2 \right]_+ \mathrm{d}V(y), \tag{1.4}$$

where $\mathrm{d}V$ is the Stieltjes measure with $\mathrm{d}V([0, y)) = V(y)$, i.e. the image of Lebesgue measure on (s_+, ∞) under b.

1.3 Main results

Theorem 1.3 (Weyl law for off-axis cross-sections)**.** *Let ω be an off-axis finite-band cross-section, so that $\overline{\omega}$ stays at the positive distance $r_- > 0$ from the rotation axis. Let both ends of $\theta \in C^1(\mathbb{R})$ be admissible. Then H_θ has compact resolvent and*

$$N_\theta(T^2) = \big(P_{V_+,\mu_\omega}(T) + P_{V_-,\mu_\omega}(T)\big)\big(1 + o(1)\big), \qquad T \to \infty. \tag{1.5}$$

Moreover $P_{V_\pm,\mu_\omega}(T) \asymp T^3 V_\pm(T)$.

Theorem 1.4 (Axis contact)**.** *The conclusions of Theorem 1.3 hold for every finite-band cross-section reaching the axis, with the same formula* (1.5)*.*

The reduced phase is a semiclassical phase-space volume in two variables per angular mode. For a constant speed β and a sector of opening α, conjugation by the angular gauge of Section 3 turns the n-th angular Dirichlet mode $\sin(\nu_n t)$, $\nu_n = \pi n/\alpha$, into the effective symbol

$$p_r^2 + \frac{p_s^2}{1 + \beta^2 r^2} + (\beta^2 + r^{-2})\nu_n^2 \tag{1.6}$$

in the variables (r, s). With $\beta = b(s)$ frozen, the phase-space volume of $\{(1.6) < E\}$, divided by $(2\pi)^2$, is $\frac{1}{4\pi} \iint r\sqrt{b^2 + r^{-2}}\, \big[E - (b^2 + r^{-2})\nu_n^2\big]_+ \,\mathrm{d}r\,\mathrm{d}s$. Summing over n gives the physical phase (4.2) below, and removing the lower-order r^{-2} terms gives (1.4). The radial and longitudinal variables are thus treated semiclassically while the angular variable is kept quantized; this is the mechanism of non-classical Weyl asymptotics [23, 13], realised here uniformly in a speed that is itself of the order of $\sqrt{\Lambda}$ on the part of the tube that carries the count.

Corollary 1.5 (Regularly varying sublevel measures)**.** *In Theorems 1.3 and 1.4 suppose that V_e is regularly varying of index $\kappa_e \geq 0$ for $e = \pm$. Then*

$$N_\theta(\Lambda) \sim \sum_{e=\pm} c_{\kappa_e} M_{1+\kappa_e}(\omega) \Lambda^{3/2} V_e(\sqrt{\Lambda}), \qquad c_\kappa = \frac{\kappa\, \zeta(1+\kappa)}{2\pi^{2+\kappa}(\kappa+1)(\kappa+3)}, \quad c_0 = \frac{1}{6\pi^2}. \tag{1.7}$$

For continuous speeds $b_\pm(s) \sim c|s|^p$ with $c, p > 0$, which give admissible ends by Lemma 5.2, and in particular for $\theta'(s) = c(1 + s^2)^{p/2}$,

$$N_\theta(\Lambda) \sim \frac{p\, \zeta(1 + 1/p)\, c^{-1/p}}{\pi^{2+1/p}(p+1)(3p+1)} M_{1+1/p}(\omega) \Lambda^{\frac{3}{2} + \frac{1}{2p}}. \tag{1.8}$$

For the annular sector $\{r_- < r < r_+,\ 0 < t < \alpha\}$, $0 < \alpha \leq 2\pi$, and $p = 1$ this reads $N_\theta(\Lambda) \sim \alpha^2(r_+^2 - r_-^2)\Lambda^2/(96\pi c)$.

Here $c_\kappa \to c_0$ as $\kappa \downarrow 0$. In the slowly varying case $\kappa = 0$, which contains exponential and double-exponential speeds, the coefficient is the area $M_1 = |\omega|$; for $\kappa > 0$ it is the nonlinear moment $M_{1+\kappa}$. Corollary 1.5 applies to non-monotone speeds such as $b(s) = s^p(2 + \sin s)$, for which $b(s)/s^p$ has no limit (Example 5.4).

The further results are the following.

- Section 5 evaluates P_V by a single Mellin-type formula: Karamata asymptotics for regularly varying V, explicit periodic profiles for log-periodic speeds (Theorem 5.7), and logarithmic corrections with the resulting trace-ideal thresholds.
- Section 6 proves angular stability: the law is unchanged when the angle is perturbed by a function with finite limits, with no control of its derivative (Theorem 6.2). It contains the disc and the ellipse of Open Problem 8 (Proposition 6.3).
- Section 8 proves an $O(\Lambda^{7/4})$ remainder for constant-step sections and inverse speeds of bounded slope, and a second periodic spectral profile (Theorems 8.1 and 8.2).
- Section 9 records what the leading coefficients detect: width moments, a finite-data obstruction, and geometric blindness of odd logarithmic harmonics (Theorems 9.3, 9.4 and Proposition 9.5).

The results cover finite-band cross-sections, on or off the axis, and speeds whose logarithm oscillates slowly on the rotation scale and whose sublevel measure grows at a doubling rate; this includes regularly, rapidly and log-periodically varying speeds.

1.4 Relation to the literature

Non-classical Weyl asymptotics for operators whose classical phase volume is infinite were found by Simon [23], for example for $-\Delta + x^2y^2$ and for horn-shaped regions; Solomyak [24] treated Schrödinger operators with non-regular homogeneous potentials. For Dirichlet Laplacians on horn-shaped regions of infinite volume, van den Berg [26] and van den Berg and Lianantonakis [27] obtained counting and heat-trace asymptotics in which transverse modes are retained and spectral-zeta coefficients appear; refined versions are in [11]. Rozenblum's estimate [22], with [7, 20], controls the number of eigenvalues by a phase-space integral and underlies (1.2). Geisinger and Weidl [12] proved sharp Berezin–Li–Yau-type estimates in domains of infinite volume, a point of reference already in [15]. Ivrii's survey [13] discusses the operator-valued viewpoint (Section 3.2.1) that keeps some variables as discrete modes. Barseghyan and Exner [2] compared spiral-shaped regions with horn-shaped ones. Chitour, Prandi and Rizzi [5] obtained slowly varying Weyl factors in singular Riemannian geometry.

For twisted waveguides, see the monograph of Exner and Kovařík [10]. Constant twisting raises the threshold of the essential spectrum [9]; twisting that vanishes at infinity yields a Hardy inequality [8]; a slowly decaying perturbation of constant twisting produces eigenvalue accumulation at the threshold of the essential spectrum [4], and perturbed periodic twisting is treated in [21]. Ruled strips with diverging twisting [17] and sheared waveguides [25] have other spectral mechanisms. The diverging regime with compact resolvent is the subject of [15, 1, 16].

Log-periodic leading terms and the cancellation of oscillations by zeros of a spectral transform are classical in the theory of fractal strings [19, 18] and in the theory of log-periodic functions [14]. In Section 9 every zero responsible for blindness comes from the width moment, since the zeta factor $\zeta(1 + z)$ is evaluated only on $\Re(1 + z) > 1$, where it has no zeros.

1.5 Prior versions

This paper supersedes three author-published editions of the Luminary Research Archive, listed below: W1, which proves (1.8) for C^1 polar strips and $\theta' = c(1+s^2)^{p/2}$; W2, which treats regularly and rapidly varying reference speeds, finite-band cross-sections and angular stability; and W3, which treats log-periodic inverse speeds, blindness, the quantitative remainder and the second profile.

W1 S. Schubr, *A Weyl law for rapidly twisted polar-strip tubes*, Luminary Research Archive, edition 2 (2026), `https://research.luminaryoddities.org/a-weyl-law-for-rapidly-twisted-polar-strip-tubes/v/2`.

W2 S. Schubr, *Weyl laws and angular stability for rapidly twisted tubes*, Luminary Research Archive, edition 2 (2026), `https://research.luminaryoddities.org/weyl-laws-angular-stability-rapidly-twisted-tubes/v/2`.

W3 S. Schubr, *Geometric blind modes in the Weyl asymptotics of twisted tubes*, Luminary Research Archive, edition 2 (2026), `https://research.luminaryoddities.org/geometric-blind-modes-weyl-asymptotics-twisted-tubes/v/2`.

The unified Theorem 1.3, the axis-contact Theorem 1.4, the Mellin evaluation of Section 5 and the non-monotone examples are new here. The closing section on provenance lists the origin of each section.

2 Forms, confinement, cores and caps

2.1 Coordinates

For a sector $Q = (r_-, r_+) \times (0, \alpha)$ with $0 < r_-$ and $0 < \alpha \le 2\pi$, and an axial interval I, the map

$$(r, t, s) \longmapsto (R_{\theta(s)}(r\cos t, r\sin t), s)$$

is injective on $Q \times I$, since t ranges in an open interval of length at most 2π. The opening $\alpha = 2\pi$ is allowed. Pullback is unitary from L^2 of the image onto $L^2(Q \times I, r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)$ and transforms $\int |\nabla u|^2$ into

$$q_\theta[v] = \int \Big(|v_r|^2 + r^{-2}|v_t|^2 + |X_\theta v|^2 \Big) r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s, \qquad X_\theta = \partial_s - \theta'(s)\partial_t. \tag{2.1}$$

Indeed, with physical polar angle $\varphi = t + \theta(s)$ one has $\partial_s|_\varphi = \partial_s - \theta'\partial_t$. Original Dirichlet form domains are closures of C_c^∞ in the form norm. We also use auxiliary forms with the same expression in which the radial faces $r = r_\pm$ (or internal radial cuts), or an axial cap $s = s_0$, are free: their domain is the set of restrictions of finite-energy functions that vanish on the remaining faces, i.e. the closure of smooth functions on the closed coordinate box vanishing near the Dirichlet faces. Angular faces are always Dirichlet. Since θ' is locally bounded, these domains are H^1-type on bounded axial pieces. A negative sign of θ' is reduced to a positive one by $t \mapsto \alpha - t$.

2.2 Localization

All localizations use the following principle; we write it out once.

Lemma 2.1 (IMS localization)**.** *Let q be a closed nonnegative form in $\mathcal{H}$, and let q_k, $k \in \mathcal{K}$ countable, be closed nonnegative forms in $\mathcal{H}_k$. Let $J_k : \mathcal{H} \to \mathcal{H}_k$ be linear with $\sum_k \|J_k v\|^2 = \|v\|^2$, $J_k(\mathrm{D}(q)) \subset \mathrm{D}(q_k)$ and*

$$\sum_k q_k[J_k v] \le q[v] + c\|v\|^2 \qquad (v \in \mathrm{D}(q)) \tag{2.2}$$

for some $c \ge 0$. Then $N(q, \Lambda) \le \sum_k N(q_k, \Lambda + c)$ for every Λ, where $N(\cdot, \Lambda)$ is the dimension of the spectral subspace of $(-\infty, \Lambda)$ of the associated operator.

If χ_k are real Lipschitz functions of s alone (respectively of the physical distance $|x_\perp|$ to the axis alone), locally finite, with $\sum_k \chi_k^2 = 1$, then for every v in an original or auxiliary form domain

$$\sum_k q_\theta[\chi_k v] = q_\theta[v] + \int \sum_k |\nabla \chi_k|^2 |v|^2. \tag{2.3}$$

Proof. Assume $n = \sum_k N(q_k, \Lambda + c) < \infty$; otherwise there is nothing to prove. Let E_k be the spectral subspace of q_k for $(-\infty, \Lambda + c)$ and P_k its orthogonal projection. If the spectral subspace of q for $(-\infty, \Lambda)$ had dimension greater than n, it would contain a subspace F of dimension $n+1$. The linear map $F \ni v \mapsto (P_k J_k v)_k \in \bigoplus_k E_k$ has a target of dimension n, so it annihilates some $v \in F \setminus \{0\}$. Then $J_k v \perp E_k$, hence $q_k[J_k v] \ge (\Lambda + c)\|J_k v\|^2$ for all k, and (2.2) gives $q[v] \ge \Lambda \|v\|^2$. This contradicts $q[v] < \Lambda \|v\|^2$, which holds for every nonzero v in the spectral subspace of $(-\infty, \Lambda)$.

For (2.3), every first-order derivative D occurring in the form (∂_r, $r^{-1}\partial_t$, X_θ, or the physical gradient) satisfies $D(\chi_k v) = \chi_k D v + (D\chi_k) v$; for cutoffs in s, $X_\theta \chi_k = \chi_k'$ because χ_k does not depend on t. Expanding the squares and using $\sum_k \chi_k \, D\chi_k = \frac{1}{2} D \sum_k \chi_k^2 = 0$ gives (2.3) for smooth v; closure gives the general case. □

The localization error in (2.3) contains no factor depending on θ'. This is the reason why axial localization at scales much shorter than the rotation length is harmless.

Squared partitions are built from translated bumps. For spacing $\ell > 0$ and $\eta \in (0, \frac{1}{2}]$ put $c_k = s_0 + k\ell$. Choose χ_k supported in $[c_k - \eta\ell/2, c_k + \ell + \eta\ell/2]$, with $\sum_k \chi_k^2 = 1$, at most two cutoffs overlapping, sine/cosine transitions on overlaps of length $\eta\ell$, and

$$\sum_k |\chi_k'|^2 \le C_\eta \ell^{-2}, \qquad C_\eta \le C\eta^{-2}. \tag{2.4}$$

We call $[c_k, c_k + \ell]$ the spacing cells and the supports the boxes.

2.3 Confinement along lines

Lemma 2.2 (Line confinement)**.** *Let $\theta' = b \ge B > 0$ on an interval I, and let v belong to a form domain on $(r_-, r_+) \times (0, \alpha) \times I$ with either radial convention. If v vanishes near both endpoints of I (or satisfies a Dirichlet condition there), then*

$$\int |X_\theta v|^2 r \, \mathrm{d}r \, \mathrm{d}t \, \mathrm{d}s \ge \frac{\pi^2 B^2}{\alpha^2} \int |v|^2 r \, \mathrm{d}r \, \mathrm{d}t \, \mathrm{d}s. \tag{2.5}$$

If one endpoint of I is a free cap, (2.5) holds with the constant $\pi^2 B^2/(4\alpha^2)$.

Proof. Use (r, φ, s) with $\varphi = t + \theta(s) \in \mathbb{R}$; the change $(t, s) \mapsto (\varphi, s)$ has Jacobian one and $X_\theta = \partial_s$ at fixed φ. No reduction modulo 2π is needed. For fixed (r, φ) the set $\{s \in I : 0 < \varphi - \theta(s) < \alpha\}$ is an interval, since θ is increasing, of length at most α/B; on it v vanishes at the endpoints coming from the angular walls and from Dirichlet ends of I. The one-dimensional Dirichlet (respectively Dirichlet–Neumann) Poincaré inequality on an interval of length L has constant π^2/L^2 (respectively $\pi^2/(4L^2)$). Integrate in $r \, \mathrm{d}r \, \mathrm{d}\varphi$. The inequality holds for smooth functions vanishing near the Dirichlet faces and passes to the closure by lower semicontinuity. The radial convention does not enter. □

2.4 Bounded pieces, caps and lateral extension

Lemma 2.3 (Cores and caps)**.** (a) *Let Ω' be an axial piece $\{s_1 < s < s_2\}$ of Ω_θ, with bounded ω, each cap Dirichlet or free, or a bounded coordinate box as in* (2.1) *with arbitrary radial and cap conventions. Then its count is at most $C(1+\Lambda^{3/2})$, and its form domain embeds compactly.*

(b) *Let N_D and N_F be the counts of an end $\{s > s_0\}$ of a tube or of a coordinate sector, with Dirichlet and with free cap at s_0, and the same lateral conventions. There is C with*

$$N_D(\Lambda) \le N_F(\Lambda) \le N_D(\Lambda + C) + C(1+\Lambda^{3/2}). \tag{2.6}$$

The same inequality compares two cap positions $s_0 < s_0'$, and the count of a full tube with the sum of the counts of its two Dirichlet ends and a bounded core.

Proof. (a) For a tube piece, extend form vectors by zero across the lateral Dirichlet boundary in physical coordinates; no extension is made through the caps. This maps the form domain isometrically, with equal energy, into H^1 of the box $(-r_+, r_+)^2 \times (s_1, s_2)$, whose Neumann count is an explicit lattice count $\le C(1+\Lambda^{3/2})$. For a bounded coordinate box, $|\theta'|$ is bounded there, so the form and norm are equivalent to the Euclidean H^1 form and norm on the box with Dirichlet angular faces; relaxing all faces to Neumann gives the same bound. Rellich's theorem gives compactness.

(b) The first inequality is domain inclusion. For the second, take a squared partition $\chi_1^2 + \chi_2^2 = 1$ in s, with χ_1 supported in $[s_0, s_0+2)$ and χ_2 vanishing near s_0. By (2.3) the error is $\sup(\chi_1'^2 + \chi_2'^2) = C$, $\chi_2 v$ lies in the Dirichlet domain, and $\chi_1 v$ in the bounded collar $[s_0, s_0+2]$; apply Lemma 2.1 and part (a). The other comparisons are identical. □

Lemma 2.4 (Lateral zero extension)**.** *Restrict an original form vector of Ω_θ to a radial slab $\rho_1 < |x_\perp| < \rho_2$ and an axial half-line. If, in the coordinates of a C^1 angle Θ, its support lies in a finite union of pairwise disjoint sector tubes $(\rho_1, \rho_2) \times (c_j, d_j) \times I$ with $\sum(d_j - c_j) \le 2\pi$ and disjoint closures modulo 2π up to common endpoints, then extension by zero is an isometric, energy-preserving map into the direct sum of the corresponding coordinate sector forms with free radial faces and a free or Dirichlet cap.*

Proof. On bounded axial pieces, the transverse zero extension of an original form vector is H^1, first for compactly supported smooth functions and then by completion. Restriction to the slab and the half-line produces no boundary term: the gradient is the original gradient on the support and zero elsewhere. The argument is in physical coordinates and differentiates no polar endpoint function. Exhaustion passes to finite energy on a half-line, and restriction to the disjoint sectors gives the direct sum. □

3 The frozen-box lemma

Fix ρ in a compact subinterval of $(0, \infty)$, constants $B > 0$, $h > 0$, $L > 0$ and $0 < \alpha \le 2\pi$. On $(0,h) \times (0,\alpha) \times (0,L)$ with Lebesgue measure consider

$$q_{\rho,B}[v] = \int \left(|v_r|^2 + \rho^{-2}|v_t|^2 + |v_s - Bv_t|^2 \right) \mathrm{d}r\, \mathrm{d}t\, \mathrm{d}s, \tag{3.1}$$

with Dirichlet angular and longitudinal faces and radial faces either both Dirichlet ($\mathsf{b} = \mathsf{D}$) or both Neumann ($\mathsf{b} = \mathsf{N}$). Let $D_\mathsf{b}(h, L, \rho, B; E)$ be its count below E. Put $\mathcal{J}_\mathsf{D} = \{1, 2, \dots\}$, $\mathcal{J}_\mathsf{N} = \{0, 1, \dots\}$, $\nu_n = \pi n/\alpha$, $a_0 = B^2 + \rho^{-2}$, $w_0 = \sqrt{1 + B^2\rho^2} = \rho\sqrt{a_0}$, and

$$J_\mathsf{b}(h, \rho, B; E) = \frac{w_0}{\pi} \sum_{j \in \mathcal{J}_\mathsf{b},\, n \ge 1} \left[E - (\pi j/h)^2 - a_0 \nu_n^2 \right]_+^{1/2}. \tag{3.2}$$

Counts and positive-part sums at negative energy are zero.

Lemma 3.1 (Frozen box)**.** (a) *For every $\eta \in (0, \frac{1}{2}]$ there is $C_\eta \le C\eta^{-2}$, independent of B, ρ, h, L, E, α and b, such that*

$$\frac{L}{1+\eta} J_{\mathsf{b}}\Big(h, \rho, B; E - \frac{C_\eta}{L^2}\Big) \le D_{\mathsf{b}}(h, L, \rho, B; E) \le L J_{\mathsf{b}}(h, \rho, B; E). \tag{3.3}$$

(b) *Uniformly in $B > 0$, $h > 0$, $E \ge 0$, b and α,*

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{h}{4\pi} w_0 \sum_{n \ge 1} \Big[E - a_0 \nu_n^2\Big]_+ + O(E). \tag{3.4}$$

Proof. (a) Replace the longitudinal condition by $v(L) = e^{i\vartheta} v(0)$, with frequencies $\xi = (2\pi k + \vartheta)/L$, $k \in \mathbb{Z}$. At fixed ξ, completing the square in the angular variable gives

$$\rho^{-2}|w_t|^2 + |i\xi w - B w_t|^2 = a_0 \Big| w_t - \frac{iB\xi}{a_0} w \Big|^2 + \frac{\xi^2}{1 + B^2\rho^2} |w|^2. \tag{3.5}$$

The gauge $w = e^{iB\xi t/a_0} g$ has constant modulus, preserves the angular Dirichlet conditions, and does not depend on r because ρ is frozen. Hence the quasiperiodic eigenvalues are exactly

$$(\pi j/h)^2 + a_0 \nu_n^2 + \frac{\xi^2}{1 + B^2\rho^2}. \tag{3.6}$$

The Dirichlet form domain is contained in every quasiperiodic domain, so D_{b} is at most each quasiperiodic count. For $c > 0$, $\frac{1}{2\pi} \int_0^{2\pi} \#\{k : ((2\pi k + \vartheta)/L)^2 < c^2\} \, \mathrm{d}\vartheta = Lc/\pi$ exactly; averaging over ϑ and summing over the bands of (3.6) gives the upper bound.

For the lower bound put $d = L/(1+\eta)$. On a circle of length Kd take K translates of a squared partition with spacing d, supports of length L, and (2.4) with $\ell = d$. By Lemma 2.1 ((2.3) with $X = \partial_s - B\partial_t$), $N_{\mathrm{per}}(Kd, E_0) \le K \, D_{\mathsf{b}}(h, L, \rho, B; E_0 + C_\eta L^{-2})$. At fixed parameters only finitely many bands of (3.6) are active, and each contributes a lattice count in a bounded frequency interval with error at most two. Hence $K^{-1} N_{\mathrm{per}}(Kd, E_0) \to d J_{\mathsf{b}}(h, \rho, B; E_0)$ as $K \to \infty$. Take $E_0 = E - C_\eta L^{-2}$.

(b) For $A \ge 0$, comparison of the decreasing function $x \mapsto \big[A - (\pi x/h)^2\big]_+^{1/2}$ with its integral $hA/4$ gives $\sum_{j \ge 1} \big[A - (\pi j/h)^2\big]_+^{1/2} = hA/4 + O(\sqrt{A})$, and $j = 0$ adds at most $\sqrt{A}$. Summed over n, the error is bounded by $\frac{w_0}{\pi} \sum_n \big[E - a_0\nu_n^2\big]_+^{1/2} \le \frac{w_0}{\pi} \cdot \frac{\alpha E}{4\sqrt{a_0}} = \frac{\rho \alpha E}{4\pi}$, by the same comparison in n. The identity $w_0 = \rho\sqrt{a_0}$ cancels the large factor B. □

4 Local clocks and the uniform phase comparison

Throughout this section we consider one admissible positive end with $\theta' = b > 0$ on (s_+, ∞) (the sign $\sigma = -1$ is reduced to this by $t \mapsto \alpha - t$), and a sector of opening $\alpha \in (0, 2\pi]$ on (r_-, r_+) with $r_- > 0$.

4.1 Local clocks

Lemma 4.1 (Local clock)**.** *Let I be an interval of length L in (s_+, ∞), $B = L^{-1} \int_I b$, and suppose $e^{-\epsilon} \le b/B \le e^{\epsilon}$ on I. Let $Z = (\rho, \rho + h) \times (0, \alpha) \times I$ with $r_- \le \rho < \rho + h \le r_+$. Then $u(r, t, x) = v(r, t, s(x))$, where $x(s) = B^{-1} \int_{\inf I}^{s} b$, is a bijection between the form domains on Z (any face conventions) and on $(\rho, \rho + h) \times (0, \alpha) \times (0, L)$, and with $1 + \tau = e^{2\epsilon}(1 + h/r_-)^3$,*

$$\frac{1}{1+\tau} \frac{q_{\rho,B}[u]}{\|u\|^2} \le \frac{q_\theta[v]}{\|v\|^2_{L^2(r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)}} \le (1+\tau) \frac{q_{\rho,B}[u]}{\|u\|^2}, \tag{4.1}$$

where $q_{\rho,B}$ and $\|u\|$ refer to (3.1) *on the shifted box with Lebesgue measure.*

Proof. The map x is a C^1 diffeomorphism of I onto $(0,L)$ with $g = \mathrm{d}s/\,\mathrm{d}x = B/b \in [e^{-\epsilon}, e^{\epsilon}]$, and $X_\theta v = g^{-1}(u_x - Bu_t)$. Exact substitution gives

$$\|v\|^2 = \int g|u|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x, \qquad q_\theta[v] = \int \Big\{ g(|u_r|^2 + r^{-2}|u_t|^2) + g^{-1}|u_x - Bu_t|^2 \Big\} r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x.$$

On the cell $\rho \le r \le \rho(1+h/r_-)$, so r and r^{-1} are within the factor $1+h/r_-$ of ρ and ρ^{-1}. Each term of the form and the norm is thus within the stated factors of its frozen counterpart. The map is bi-Lipschitz in s, preserves traces on all faces and maps compactly supported smooth functions to C^1 functions with the same supports; it therefore identifies the form closures. No derivative of g or of a density enters. □

Lemma 4.2 (Uniform windows)**.** *Let* (H1) *and* (H2) *hold and let* $C_1 \ge 1$. *There are* $K_T \to \infty$ *with* $K_T \le T^{1/2}$ *such that, with* $\ell_T = K_T/T$,

$$\mathcal{E}_T = \sup\Big\{ \Big| \log \frac{b(t)}{b(s)} \Big| : s, t > s_+,\ b(s) \le C_1 T,\ |t-s| \le 2\ell_T \Big\} \longrightarrow 0.$$

Proof. Let $e(K,T)$ be the same supremum with ℓ_T replaced by K/T. Fix K and $\epsilon > 0$. By (H2) with $2C_1K$ in place of K there is M such that for $s \ge M$ the oscillation of $\log b$ over $|t-s| \le 2C_1K/b(s)$ is below ϵ; if moreover $b(s) \le C_1T$ then $2K/T \le 2C_1K/b(s)$. For $s_+ < s < M$, uniform continuity of $\log b$ on $[s_+, M+1]$ bounds the oscillation over $|t-s| \le 2K/T$ by ϵ for large T. Hence $e(K,T) \to 0$ for each K. Choose $T_j \uparrow \infty$ with $e(j,T) < 1/j$ and $j \le T^{1/2}$ for $T \ge T_j$, and put $K_T = j$ on $[T_j, T_{j+1})$. □

4.2 Phases

Put $a(r,s) = b(s)^2 + r^{-2}$, $w = r\sqrt{a}$ and $\nu_n = \pi n/\alpha$. The *physical phase* of the end is

$$I_\alpha(E) = \frac{1}{4\pi} \int_{s_+}^{\infty} \int_{r_-}^{r_+} w \sum_{n \ge 1} \Big[E - a\nu_n^2 \Big]_+ \,\mathrm{d}r\,\mathrm{d}s, \tag{4.2}$$

and for a single opening the *reduced phase density* is

$$\Phi_V(\alpha, T) = \frac{1}{4\pi} \sum_{n \ge 1} \int y \Big[T^2 - (\pi n y/\alpha)^2 \Big]_+ \,\mathrm{d}V(y), \tag{4.3}$$

so that $P_{V,\mu}(T) = \int \Phi_V(\alpha, T)\,\mathrm{d}\mu(\alpha)$. Let

$$\psi(z) = z \sum_{n \ge 1} \Big[1 - n^2 z^2 \Big]_+ \quad (z > 0), \qquad \psi(0) = \tfrac{2}{3}. \tag{4.4}$$

Lemma 4.3 (Phase bounds)**.** (a) ψ *is continuous on* $[0,\infty)$, *vanishes on* $[1,\infty)$, *and satisfies* $0 \le \psi \le 1$, $|\psi(z) - \frac{2}{3}| \le z$, *and* $\psi \ge \frac{3}{16}$ *on* $[0, \frac{1}{2}]$. *Moreover*

$$\Phi_V(\alpha, T) = \frac{\alpha T^3}{4\pi^2} \int \psi\Big(\frac{\pi y}{\alpha T} \Big)\,\mathrm{d}V(y). \tag{4.5}$$

(b) $\frac{3\alpha}{64\pi^2} T^3 V(\frac{\alpha T}{2\pi}) \le \Phi_V(\alpha, T) \le \frac{\alpha}{4\pi^2} T^3 V(\frac{\alpha T}{\pi})$, *and for* $T \le T' \le 2T$,

$$0 \le \Phi_V(\alpha, T') - \Phi_V(\alpha, T) \le \frac{\alpha}{4\pi^2} (T'^2 - T^2) T'\, V\Big(\frac{\alpha T'}{\pi} \Big).$$

(c) *Under* (H3) *there are* C, T_0 *such that for* $T \ge T_0$ *and* $\alpha, \alpha' \in (0, 2\pi]$, $|\Phi_V(\alpha', T) - \Phi_V(\alpha, T)| \le C|\alpha' - \alpha|\, T^3 V(T)$, *and for every finite* μ *with* $\mu([\alpha_0, 2\pi]) > 0$, $c_\mu T^3 V(T) \le P_{V,\mu}(T) \le C(\int \alpha\,\mathrm{d}\mu) T^3 V(T)$.

(d) *For* $T^2/2 \le E_1 \le E_2 \le 2T^2$, $0 \le I_\alpha(E_2) - I_\alpha(E_1) \le C(E_2 - E_1)\, T\, V(4T)$ *and* $I_\alpha(E) \le C T^3 V(4T)$. *The integrand of* $I_\alpha(E)$ *vanishes where* $b(s) \ge \alpha\sqrt{E}/\pi$ *and is at most* $r\alpha E^{3/2}/\pi$.

Proof. (a) Each term of ψ is continuous and vanishes for $nz \geq 1$, so ψ is continuous on $(0,\infty)$ and zero on $[1,\infty)$. Since $f(x) = [1-x^2]_+$ is decreasing, $\int_z^\infty f \leq z\sum_{n\geq 1} f(nz) \leq \int_0^\infty f = \frac{2}{3}$ and $\int_0^z f \leq z$, which gives $|\psi - \frac{2}{3}| \leq z$ and continuity at 0; also $\psi(z) \leq z\,\#\{n : nz < 1\} \leq 1$. For $z \leq \frac{1}{2}$, the at least $\lfloor 1/(2z)\rfloor \geq 1/(4z)$ indices with $nz \leq \frac{1}{2}$ contribute at least $\frac{3}{4}z$ each. Formula (4.5) follows from $y\,[T^2 - (\pi n y/\alpha)^2]_+ = T^3\frac{\alpha}{\pi} z\,[1 - n^2z^2]_+$ with $z = \pi y/(\alpha T)$.

(b) The bounds follow from (a), since $\psi(\pi y/(\alpha T))$ vanishes for $y \geq \alpha T/\pi$ and is at least $\frac{3}{16}$ for $y \leq \alpha T/(2\pi)$. The derivative of $y\sum_n [T^2 - (\pi n y/\alpha)^2]_+$ with respect to T^2 is $y\,\#\{n : \pi n y/\alpha < T\} \leq \alpha T/\pi$.

(c) Φ_V is increasing in α, and scaling gives $\Phi_V(\lambda\alpha, T) = \lambda^{-2}\Phi_V(\alpha, \lambda T)$. Let $\alpha \leq \alpha' = \lambda\alpha$. If $\lambda \geq 2$, both values are at most $\frac{\alpha'}{4\pi^2}T^3V(2T) \leq 2C(\alpha' - \alpha)T^3V(T)$ by (b) and (H3). If $\lambda \leq 2$, write $\Phi_V(\alpha', T) - \Phi_V(\alpha, T) = \lambda^{-2}[\Phi_V(\alpha, \lambda T) - \Phi_V(\alpha, T)] - (1-\lambda^{-2})\Phi_V(\alpha, T)$ and use (b): both terms are $O((\lambda - 1)\alpha T^3 V(4T)) = O((\alpha' - \alpha)T^3V(T))$. The bounds on $P_{V,\mu}$ follow from (b) and (H3), which gives $V(\alpha_0 T/(2\pi)) \geq cV(T)$ and $V(2T) \leq CV(T)$ for large T.

(d) The energy derivative of $w\sum_n [E - a\nu_n^2]_+$ is $w\,\#\{n : a\nu_n^2 < E\} \leq r\sqrt{a}\,\alpha\sqrt{E}/(\pi\sqrt{a})$, and the integrand vanishes unless $b(s) < \alpha\sqrt{E}/\pi \leq 2\sqrt{2}T < 4T$. The measure of such s is at most $V(4T)$. □

Lemma 4.4 (Reduction)**.** *Under* (H1) *and* (H3)*,* $I_\alpha(T^2) = \frac{r_+^2 - r_-^2}{2}\Phi_V(\alpha, T) + o(T^3V(T))$.

Proof. Since $\mathrm{d}V$ is the image of $\mathrm{d}s$ under b,

$$\frac{r_+^2 - r_-^2}{2}\Phi_V(\alpha, T) = \frac{1}{4\pi}\iint r b(s) \sum_n \left[T^2 - b(s)^2\nu_n^2\right]_+ \mathrm{d}r\,\mathrm{d}s.$$

Fix $B_0 \geq 1$. Where $b(s) < B_0$ both integrands are $O(T^3)$ by Lemma 4.3(d), on a set of measure $V(B_0)$. Where $b \geq B_0$, put $\delta = r_-^{-2}B_0^{-2}$; then $1 \leq a/b^2 \leq 1 + \delta$ and $1 \leq w/(rb) \leq (1+\delta)^{1/2}$. Hence the physical integrand lies between the reduced integrand at energy $T^2/(1+\delta)$ and $(1+\delta)^{1/2}$ times the reduced integrand at T^2. By Lemma 4.3(b),(c) the difference is $O(\delta T^3 V(T)) + O_{B_0}(T^3)$. Since $V(T) \to \infty$, let $T \to \infty$ and then $B_0 \to \infty$. □

4.3 The one-ended sector law

Proposition 4.5 (One-ended sector law)**.** *Let the positive end be admissible,* $0 < r_- < r_+$ *and* $0 < \alpha \leq 2\pi$*. The count* N_{end} *of the coordinate sector* $(r_-, r_+) \times (0, \alpha) \times (s_+, \infty)$ *with form* (2.1)*, either radial convention and either cap, is finite for every energy and*

$$N_{\mathrm{end}}(T^2) = \frac{r_+^2 - r_-^2}{2}\,\Phi_V(\alpha, T)\,(1 + o(1)). \tag{4.6}$$

Proof. By Lemma 2.3(b) and Lemma 4.3(b),(c) it suffices to take a Dirichlet cap: a fixed energy shift C changes $\Phi_V(\alpha, T)$ by $O(T\,V(T))$ and the collar costs $O(T^3)$, both $o(T^3V(T))$. Fix $\eta \in (0, \frac{1}{2}]$, let $C_1 = 4$, and take K_T, $\ell = \ell_T = K_T/T$ and $\mathcal{E}_T$ from Lemma 4.2. Put $h = T^{-1/2}$ and split (r_-, r_+) into radial cells of length h (the last one possibly shorter). Let $1 + \gamma_T = e^{4\mathcal{E}_T}(1 + h/r_-)^3$, so $\gamma_T \to 0$.

Upper bound. Take the squared partition (2.4) with $s_0 = s_+$ and spacing ℓ. For v in the form domain, $\chi_k v$ restricted to the box $\mathrm{supp}\chi_k \cap (s_+, \infty)$ has Dirichlet axial ends (from the cutoff or the cap), and its further restrictions to the radial cells lie in the forms with free radial cell faces. These restrictions split $q_\theta[\chi_k v]$ exactly, because (2.1) has no mixed radial derivative. By (2.3) and Lemma 2.1,

$$N_{\mathrm{end}}(T^2) \leq \sum_{k,j} N_{kj}(T^2 + C_\eta \ell^{-2}), \tag{4.7}$$

where N_{kj} is the count of the box k and radial cell j, and $C_\eta \ell^{-2} = C_\eta T^2/K_T^2 = o(T^2)$.

If $b \geq 4T$ on box k, Lemma 2.2 gives energy at least $16\pi^2 T^2/\alpha^2 \geq 4T^2$, so $N_{kj}(T^2 + C_\eta \ell^{-2}) = 0$ for large T. Every other box contains s with $b(s) < 4T$ and lies within 2ℓ of s; hence it lies in $(s_+, S(4T) + 2\ell)$, and b oscillates on it by at most $e^{\pm 2\mathcal{E}_T}$. There are at most $(S(4T) - s_+)/\ell + 3$ such boxes. On each, Lemma 4.1 with the box average B_k and the left cell endpoint ρ_j, followed by Lemma 3.1, gives with $E_+ = (1 + \gamma_T)(T^2 + C_\eta \ell^{-2})$

$$N_{kj}(T^2 + C_\eta \ell^{-2}) \leq D_{\mathsf{N}}(h_j, L_k, \rho_j, B_k; E_+) \leq L_k \frac{h_j}{4\pi} w_0 \sum_n \left[E_+ - a_0 \nu_n^2\right]_+ + C L_k E_+,$$

where $L_k \leq (1 + \eta)\ell$ is the box length, $h_j \leq h$, and a_0, w_0 are frozen at (ρ_j, B_k). For (r, s) in the spacing cell $[c_k, c_k + \ell]$ times the radial cell, $a_0 \geq a(r, s)/(1 + \gamma_T)$ and $w_0 \leq (1 + \gamma_T) w(r, s)$, so

$$w_0 \sum_n \left[E - a_0 \nu_n^2\right]_+ \leq w \sum_n \left[(1 + \gamma_T) E - a \nu_n^2\right]_+ \tag{4.8}$$

for every E, including all threshold crossings. Integrating (4.8) over the spacing cell and the radial cell, and summing,

$$N_{\mathrm{end}}(T^2) \leq (1 + \eta) I_\alpha\big((1 + \gamma_T) E_+\big) + C\ell T^3 + C\Big(\frac{S(4T)}{\ell} + 3\Big)\Big(\frac{r_+ - r_-}{h} + 1\Big)\ell T^2.$$

The term $C\ell T^3$ accounts for the box with $k = -1$, whose spacing cell lies outside the end. By (H4) and (H3) the last term is $O(V(T) T^{5/2})$. By Lemma 4.3(d), replacing $(1 + \gamma_T) E_+ = T^2(1 + o(1))$ by T^2 costs $o(T^3 V(T))$. Hence, by Lemma 4.4 and Lemma 4.3(c), $N_{\mathrm{end}}(T^2) \leq (1 + \eta)\frac{r_+^2 - r_-^2}{2} \Phi_V(\alpha, T) + o(T^3 V(T))$.

Lower bound. Let $\mathcal{K}$ be the set of $k \geq 0$ such that the cell $[c_k, c_k + \ell]$ contains a point with $b < 4T$. Insert Dirichlet walls on all cell boundaries and all radial cell faces; by domain inclusion $N_{\mathrm{end}}(T^2) \geq \sum_{k \in \mathcal{K}, j} N_{kj}^{\mathsf{D}}(T^2)$. On these cells b oscillates by at most $e^{\pm 2\mathcal{E}_T}$, and Lemmas 4.1 and 3.1 give

$$N_{kj}^{\mathsf{D}}(T^2) \geq \frac{\ell}{1 + \eta} \frac{h_j}{4\pi} w_0 \sum_n \left[E_- - a_0 \nu_n^2\right]_+ - C\ell T^2, \qquad E_- = \frac{T^2}{1 + \gamma_T} - C_\eta \ell^{-2}.$$

Now $a_0 \leq (1 + \gamma_T) a$ and $w_0 \geq w/(1 + \gamma_T)$ on the cell, so $w_0 \sum_n \left[E - a_0 \nu_n^2\right]_+ \geq (1 + \gamma_T)^{-1} w \sum_n \left[E/(1 + \gamma_T) - a \nu_n^2\right]_+$. Cells outside $\mathcal{K}$ carry no phase at energies below $2T^2$, by Lemma 4.3(d). Summing,

$$N_{\mathrm{end}}(T^2) \geq \frac{I_\alpha(E_-/(1 + \gamma_T))}{(1 + \eta)(1 + \gamma_T)} - O(V(T) T^{5/2}),$$

and as before $N_{\mathrm{end}}(T^2) \geq (1 + \eta)^{-1} \frac{r_+^2 - r_-^2}{2} \Phi_V(\alpha, T) - o(T^3 V(T))$.

Since $\Phi_V(\alpha, T) \asymp T^3 V(T)$ by Lemma 4.3(c), letting $\eta \downarrow 0$ after $T \to \infty$ proves (4.6). The upper bound (4.7) is a finite sum of finite counts, so N_{end} is finite at every energy. The radial convention enters only through the outer radial faces of the extreme cells, and both bounds hold for both conventions. □

Remark 4.6. The proof uses (H3) only as bounded doubling and (H4) only through $S(4T) = o(T^{1/2} V(T))$. It uses no monotonicity of b, no derivative of b, and no regular variation: all energy shifts $T^2(1 + o(1))$ are removed by the Lipschitz bound of Lemma 4.3, and the fixed core costs $O(T^3) = o(T^3 V(T))$ because $V(T) \to \infty$.

5 Evaluating the phase

By (4.5),

$$P_{V,\mu}(T) = \frac{T^3}{4\pi^2} \int \alpha \, \mathrm{d}\mu(\alpha) \int \psi\Big(\frac{\pi y}{\alpha T}\Big) \, \mathrm{d}V(y). \tag{5.1}$$

The function ψ is a fixed compactly supported profile, and all dependence on the speed is in the Stieltjes integral. For $\Re z > 0$, Fubini and $\int_0^1 v^z(1-v^2)\,\mathrm{d}v = \frac{2}{(z+1)(z+3)}$ give the Mellin transform

$$\widehat{\psi}(z) := z\int_0^\infty \psi(u) u^{z-1}\,\mathrm{d}u = \sum_{n\geq 1} n^{-1-z} z \int_0^1 v^z(1-v^2)\,\mathrm{d}v = \frac{2z\,\zeta(1+z)}{(z+1)(z+3)}. \tag{5.2}$$

Thus, for a complex power, $P_{y^z,\mu}(T) = K_\mu(z) T^{3+z}$ with

$$K_\mu(z) = \frac{z\,\zeta(1+z)\,M_{1+z}}{2\pi^{2+z}(1+z)(3+z)}, \qquad M_{1+z} = \int \alpha^{1+z}\,\mathrm{d}\mu. \tag{5.3}$$

5.1 Karamata: regularly varying sublevel measures

Proposition 5.1. *Let V be nondecreasing and regularly varying of index $\kappa \geq 0$, and let μ be a finite measure on $(0, 2\pi]$. Then $P_{V,\mu}(T) \sim c_\kappa M_{1+\kappa} T^3 V(T)$ with c_κ as in (1.7).*

Proof. Fix α and put $c = \alpha T/\pi$. The measures ν_c on $[0,1]$ defined by $\nu_c([0,u)) = V(cu)/V(c)$ are probability measures (the integrand of (5.1) vanishes for $y \geq c$). Their distribution functions converge to u^κ at every $u \in (0,1]$ (to 1 if $\kappa = 0$), so ν_c converges weakly to $\kappa u^{\kappa-1}\,\mathrm{d}u$, or to δ_0 if $\kappa = 0$. Since ψ is continuous on $[0,1]$,

$$\int \psi\Big(\frac{y}{c}\Big)\,\mathrm{d}V(y) = V(c)\int \psi\,\mathrm{d}\nu_c \sim V(c)\, m_\kappa, \qquad m_\kappa = \frac{2\kappa\zeta(1+\kappa)}{(\kappa+1)(\kappa+3)}, \quad m_0 = \frac{2}{3},$$

by (5.2) and $\psi(0) = \frac{2}{3}$. Also $V(\alpha T/\pi)/V(T) \to (\alpha/\pi)^\kappa$. The integrand in α is bounded by $\alpha \sup\psi\, V(2T)/V(T)$, which is uniformly bounded, and μ is finite; dominated convergence gives $P_{V,\mu}(T) \sim \frac{m_\kappa}{4\pi^{2+\kappa}} M_{1+\kappa} T^3 V(T)$, and $m_\kappa/(4\pi^{2+\kappa}) = c_\kappa$. □

Proof of Corollary 1.5. The first statement follows from Theorems 1.3, 1.4 and Proposition 5.1, with $T = \sqrt{\Lambda}$. If $b(s) \sim cs^p$ then $V(y) \sim (y/c)^{1/p}$ (Lemma 5.2(c) below), so $\kappa = 1/p$; since $c_{1/p} = p\zeta(1+1/p)/(2\pi^{2+1/p}(p+1)(3p+1))$, two equal ends give (1.8). For the sector, $M_2 = \alpha^2(r_+^2 - r_-^2)/2$ and $\zeta(2) = \pi^2/6$. □

Lemma 5.2 (Verifying the hypotheses)**.** *Let $b : [s_+, \infty) \to (0,\infty)$ be continuous with $b \to \infty$.*

(a) If $b \in C^1$ and $b'/b^2 \to 0$, then (H2) *holds.*

(b) If $b/b_0 \to 1$ and b_0 satisfies (H2)*, then b does.*

(c) If $b/b_0 \to 1$, where b_0 is eventually increasing with inverse R_0 and R_0 is regularly varying of index $\kappa \geq 0$ or, more generally, intermediately regularly varying, then $V(y) \sim R_0(y)$, (H3) *holds, and $S(y) = O(R_0(2y))$, so* (H4) *holds.*

(d) Every continuous $b \in \mathrm{RV}_p$, $p > 0$, satisfies (H1)–(H4) *with $V \in \mathrm{RV}_{1/p}$.*

Proof. (a) Let $\epsilon_s = \sup_{t \geq s - K/b(s)} |(1/b)'(t)|$, which tends to 0 because $s - K/b(s) \to \infty$. For $|t-s| \leq K/b(s)$, $|1/b(t) - 1/b(s)| \leq \epsilon_s K/b(s)$, so $b(s)/b(t) \in [1-\epsilon_s K, 1+\epsilon_s K]$.

(b) $|\log(b(t)/b(s))| \leq |\log(b/b_0)(t)| + |\log(b_0(t)/b_0(s))| + |\log(b_0/b)(s)|$, and $K/b(s) \leq 2K/b_0(s)$ for large s.

(c) For $\epsilon > 0$ there is s_ϵ with $(1-\epsilon)b_0 \le b \le (1+\epsilon)b_0$ on (s_ϵ, ∞), where b_0 is increasing. Thus $\{b < y\}$ contains $\{s > s_\epsilon : b_0(s) < y/(1+\epsilon)\}$ and is contained in $(s_+, s_\epsilon] \cup \{b_0 < y/(1-\epsilon)\}$, so $R_0(y/(1+\epsilon)) - C_\epsilon \le V(y) \le R_0(y/(1-\epsilon)) + C_\epsilon$ and $S(y) \le \max\{s_\epsilon, R_0(y/(1-\epsilon))\}$. Regular or intermediate regular variation of R_0 gives the claims, since $R_0 \to \infty$.

(d) Choose a smooth nonnegative $\phi \not\equiv 0$ supported in $(1,2)$, put $C_\phi = \int t^p \phi(t)\,\mathrm{d}t$ and $b_0(x) = C_\phi^{-1} \int_1^2 b(xt)\phi(t)\,\mathrm{d}t$. The uniform convergence theorem [3, Theorem 1.5.2] gives $b_0/b \to 1$. Writing $b_0(x) = C_\phi^{-1} \int b(u)\phi(u/x)\,\mathrm{d}u/x$ and differentiating, $x b_0'(x) = -C_\phi^{-1} \int_1^2 b(xt)(\phi + t\phi')\,\mathrm{d}t$; dividing by $b(x)$ and integrating by parts gives $x b_0'/b_0 \to p$. Hence b_0 is eventually increasing, its inverse lies in $\mathrm{RV}_{1/p}$ [3, Theorem 1.5.12], and $(\log b_0)' \sim p/x$, so the oscillation of $\log b_0$ over $|t-s| \le K/b_0(s)$ is $O(K/(s b_0(s))) \to 0$. Apply (b) and (c). $\square$

Example 5.3 (Recovering the earlier classes)**.** (i) $\theta' = c(1+s^2)^{p/2}$: $b'/b^2 \to 0$ and $V(y) \sim (y/c)^{1/p}$; this is W1. (ii) Continuous $b_\pm \in \mathrm{RV}_{p_\pm}$ at the two ends, independently: Lemma 5.2(d) and (1.7) give $N \sim \Lambda^{3/2} \sum_e c_{1/p_e} M_{1+1/p_e} V_e(\sqrt{\Lambda})$; this is the type (P) law of W2. Logarithmic factors are treated in Corollary 5.5. (iii) b increasing, C^1, rapidly varying, with $b'/b^2 \to 0$ (type (Q) of W2): (H2) holds by Lemma 5.2(a), and V differs by a constant from the inverse R, which is slowly varying [3, Section 2.4]; indeed for $0 < \epsilon < 1$, rapid variation gives $b((1 \mp \epsilon)R(T))/T \to 0, \infty$, and monotonicity places $R(cT)/R(T)$ in $[1-\epsilon, 1+\epsilon]$ for large T. Thus $\kappa = 0$ and the coefficient is $|\omega|/(6\pi^2)$. For $b_e(s) = c_e \exp(\kappa_e s)$, $R_e(T) \sim \kappa_e^{-1} \log T$ and

$$N(\Lambda) \sim \frac{|\omega|}{12\pi^2} (\kappa_+^{-1} + \kappa_-^{-1}) \Lambda^{3/2} \log \Lambda;$$

for $b_e(s) = c_e \exp(\exp(\kappa_e s))$, $N(\Lambda) \sim \frac{|\omega|}{6\pi^2} (\kappa_+^{-1} + \kappa_-^{-1}) \Lambda^{3/2} \log\log \Lambda$. In both, $b'/b^2 \to 0$ by differentiation.

Example 5.4 (A non-monotone speed)**.** Let $p > 0$ and $b(s) = s^p(2 + \sin s)$ on $(1, \infty)$. Then $b'/b^2 = O(s^{-p})$, so (H2) holds; $b(s) \ge s^p$ gives $S(y) \le y^{1/p}$, and $b(s) \le 3s^p$ gives $V(y) \ge (y/3)^{1/p} - 1$, so (H4) holds. With $\lambda = y^{1/p}$ and $s = \lambda u$,

$$V(y) = \lambda\Big(3^{-1/p} - \lambda^{-1} + \int_{3^{-1/p}}^1 \mathbf{1}\{\sin(\lambda u) < u^{-p} - 2\}\,\mathrm{d}u\Big),$$

because $u^p(2 + \sin) < 1$ always holds for $u < 3^{-1/p}$ and never for $u > 1$. On a subinterval $[u_1, u_2]$, bracket $u^{-p} - 2$ between its values at the endpoints; for a constant c, $\int_{u_1}^{u_2} \mathbf{1}\{\sin(\lambda u) < c\}\,\mathrm{d}u \to (u_2 - u_1)F(c)$ by periodicity, where $F(c) = \frac{1}{2} + \frac{1}{\pi}\arcsin c$ for $|c| \le 1$. Since F is continuous, refining the subdivision gives $V(y) \sim \kappa_0 y^{1/p}$, so (H3) holds, with

$$\kappa_0 = 3^{-1/p} + \int_{3^{-1/p}}^1 F(u^{-p} - 2)\,\mathrm{d}u = \frac{1}{2\pi} \int_0^{2\pi} (2 + \sin\varphi)^{-1/p}\,\mathrm{d}\varphi$$

by Fubini; for $p = 1$, $\kappa_0 = 1/\sqrt{3}$. Hence, for a tube whose two ends both have this speed,

$$N(\Lambda) \sim \frac{p\,\zeta(1 + 1/p)}{\pi^{2+1/p}(p+1)(3p+1)}\, \kappa_0\, M_{1+1/p}(\omega)\, \Lambda^{\frac{3}{2} + \frac{1}{2p}}.$$

The speed b is not monotone, and $b(s)/s^p$ oscillates between 1 and 3. The coefficient sees b only through κ_0; for $b = cs^p$ one has $\kappa_0 = c^{-1/p}$.

Corollary 5.5 (Logarithmic factors and trace ideals)**.** *Let $b_\pm$ be continuous with $b_\pm(x) \sim c x^p (\log x)^q$, $c, p > 0$, $q \in \mathbb{R}$, and ω as in Theorem 1.3 or 1.4. Put $d = \frac{3}{2} + \frac{1}{2p}$ and $C_{p,q,c,\omega} = C_{p,c,\omega}(2p)^{q/p}$, where $C_{p,c,\omega}$ is the coefficient in* (1.8)*. Then*

$$N(\Lambda) \sim C_{p,q,c,\omega} \Lambda^d (\log \Lambda)^{-q/p}, \qquad \mathrm{Tr}(e^{-tH_\theta}) \sim C_{p,q,c,\omega} \Gamma(d+1)\, t^{-d} (\log(1/t))^{-q/p} \quad (t \downarrow 0).$$

For $z > 0$, $\mathrm{Tr}(H_\theta^{-z}) < \infty$ if $z > d$ and $= \infty$ if $z < d$; and $\mathrm{Tr}(H_\theta^{-d}) < \infty$ if and only if $q > p$.

Proof. A smooth equivalent of $cx^p(\log x)^q$ has $b_0'/b_0^2 \to 0$, so Lemma 5.2 applies with $R_0(T) \sim c^{-1/p}p^{q/p}T^{1/p}(\log T)^{-q/p}$; the factor $2^{q/p}$ comes from $\log\sqrt{\Lambda} = \frac{1}{2}\log\Lambda$. The bottom of the spectrum is positive, and Tonelli gives $\operatorname{Tr}(H^{-z}) = z\int_0^\infty N(\Lambda)\Lambda^{-z-1}\,\mathrm{d}\Lambda$; the integral test gives the trace statements, including divergence at $q = p$. For the heat trace, $\operatorname{Tr}(e^{-tH}) = t\int_0^\infty e^{-t\Lambda}N(\Lambda)\,\mathrm{d}\Lambda$; after $x = t\Lambda$, Potter bounds [3, Theorem 1.5.6] give a majorant $Ce^{-x}\max(x^{d-\epsilon}, x^{d+\epsilon})$ on the high-energy part, the bounded-energy part is negligible, and dominated convergence gives $\Gamma(d+1)$. □

The same arguments give the standard consequences of every regularly varying law.

Corollary 5.6 (Eigenvalues and heat trace)**.** *Suppose that $N_\theta(\Lambda) \sim F(\Lambda)$ with F regularly varying of index $d > 0$, as in Corollary 1.5, where $d = \frac{3}{2} + \frac{\kappa}{2}$ when both ends have index κ. Let $\lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues with multiplicity and $F^{\leftarrow}$ an asymptotic inverse of F. Then*

$$\lambda_k \sim F^{\leftarrow}(k) \quad (k \to \infty), \qquad \operatorname{Tr}(e^{-tH_\theta}) \sim \Gamma(d+1)F(1/t) \quad (t \downarrow 0).$$

For $N_\theta(\Lambda) \sim C\Lambda^d$ this reads $\lambda_k \sim (k/C)^{1/d}$ and $\operatorname{Tr}(e^{-tH_\theta}) \sim C\Gamma(d+1)t^{-d}$.

Proof. For $\epsilon > 0$, $(1-\epsilon)F(\Lambda) \le N_\theta(\Lambda) \le (1+\epsilon)F(\Lambda)$ for large Λ. Since $N_\theta(\lambda_k) \le k-1 < k \le N_\theta(\lambda_k(1+\delta))$ for every $\delta > 0$, and $F(\lambda(1+\delta))/F(\lambda) \to (1+\delta)^d$, we get $F(\lambda_k) \sim k$; as $F \in \mathrm{RV}_d$ with $d > 0$, this gives $\lambda_k \sim F^{\leftarrow}(k)$ [3, Theorem 1.5.12]. The heat trace follows as in the proof of Corollary 5.5. □

5.2 Mellin: log-periodic speeds

Fix $\gamma, L > 0$ and a real smooth L-periodic h with

$$h > 0, \qquad q := \gamma h + h' > 0. \tag{5.4}$$

Then $R(y) = y^\gamma h(\log y)$ is smooth and strictly increasing on $(0,\infty)$, with $R'(y) = y^{\gamma-1}q(\log y)$; let $b = R^{-1}$. Put $\tau = 2\pi/L$, $z_k = \gamma + ik\tau$ and $h_k = L^{-1}\int_0^L h(t)e^{-ik\tau t}\,\mathrm{d}t$.

Theorem 5.7 (Log-periodic speeds)**.** *Let ω be a finite-band cross-section, on or off the axis, and let an end of θ have $|\theta'(s)| = b(s)$ for $s > s_+$, with $b = R^{-1}$ as above. Then that end is admissible, and*

$$N_{\mathrm{end}}(T^2) = T^{3+\gamma}F_h(\log T) + o(T^{3+\gamma}), \qquad F_h(t) = \sum_{k\in\mathbb{Z}} h_k K_{\mu_\omega}(z_k)e^{ik\tau t}, \tag{5.5}$$

where F_h is smooth, L-periodic and bounded between positive constants. Two identical ends double the leading term; independently specified ends add.

Proof. Since $R'(y) \asymp y^{\gamma-1}$, $(\log b)' = 1/(bR'(b)) = O(b^{-\gamma})$ is bounded, so (H2) holds; V differs from R by a constant for large y, $R(y) \asymp y^\gamma$, and $S = R$ eventually, so (H1)–(H4) hold. Proposition 6.1, and for cross-sections reaching the axis the proof of Theorem 1.4 applied to one end, reduce the claim to the phase. The difference between $\mathrm{d}V$ and $\mathrm{d}R$ is supported in a bounded set, on which (5.1) contributes $O(T^3)$. For $\mathrm{d}R$, the Fourier series $h = \sum h_k e^{ik\tau t}$ and that of h' converge absolutely, so $\mathrm{d}R(y) = \sum_k h_k z_k y^{z_k-1}\,\mathrm{d}y$ with $\sum|h_k z_k| < \infty$; since ψ is bounded with support in $[0,1]$ and $\gamma > 0$, Fubini and (5.2) give $\int\psi(\pi y/(\alpha T))\,\mathrm{d}R = \sum_k h_k(\alpha T/\pi)^{z_k}\widehat{\psi}(z_k)$, and integration in α gives $P_{R,\mu}(T) = T^{3+\gamma}F_h(\log T)$ exactly. The moments are bounded on the lattice and the rational factor in (5.3) is $O((1+|k|)^{-1})$, so the series converges absolutely and smoothly. Since q is bounded between positive constants, $\mathrm{d}R$ is comparable with $\mathrm{d}(y^\gamma)$, and so F_h is bounded between positive multiples of $K_\mu(\gamma) > 0$. □

6 Finite-band cross-sections and angular stability

Proposition 6.1 (One-ended finite-band law)**.** *Let ω be an off-axis finite-band cross-section and let the positive end of θ be admissible. The count N_{end} of $\Omega_\theta \cap \{s > s_+\}$ with a Dirichlet or free cap satisfies $N_{\mathrm{end}}(T^2) = P_{V,\mu_\omega}(T)(1 + o(1))$.*

Proof. Work in the coordinates of θ itself. Fix $\epsilon > 0$ and trim radial neighbourhoods of the partition radii r_m of total length at most ϵ. On each retained closed subband all widths and gaps have positive minima, and uniform continuity gives a fixed mesh of cells C on which every l_{mj} and u_{mj} oscillates by less than ϵ', where ϵ' is smaller than half of all these minima. The intervals

$$(\max_C l_{mj}, \min_C u_{mj}) \subset (\min_C l_{mj}, \max_C u_{mj})$$

are then nonempty (inner) and pairwise disjoint (outer); all lie in J_m. Denote their lengths $\alpha^-_{Cj} \le \alpha^+_{Cj}$, so that $|\alpha^\pm_{Cj} - \alpha_{mj}(r)| \le 2\epsilon'$ for $r \in C$.

Lower bound. The inner sector tubes over the cells are disjoint subdomains of the end. Inserting Dirichlet walls and discarding the trimmed layers, domain inclusion and Proposition 4.5 give

$$N_{\mathrm{end}}(T^2) \ge (1 - o(1)) \sum_{C,j} \int_C r \,\mathrm{d}r\, \Phi_V(\alpha^-_{Cj}, T).$$

Upper bound. Restrict form vectors to the radial slabs of the cells and of the trimmed layers; this is isometric and splits the energy exactly. On a cell, Lemma 2.4 embeds the slab piece into the outer sector tubes; on a trimmed layer, first restrict further to its parts on either side of the partition radius r_m (this again splits the energy exactly), and embed the part lying in I_m into the envelope sector over that part with opening $\operatorname{int} J_m$. Both embeddings have free radial faces. Min–max and Proposition 4.5 give

$$N_{\mathrm{end}}(T^2) \le (1 + o(1)) \Big[\sum_{C,j} \int_C r \,\mathrm{d}r\, \Phi_V(\alpha^+_{Cj}, T) + \sum_{\mathrm{trim}} \int r \,\mathrm{d}r\, \Phi_V(A_m, T) \Big].$$

Comparison. By Lemma 4.3(c), for large T,

$$\Big| \sum_{C,j} \int_C r \,\mathrm{d}r\, \Phi_V(\alpha^\pm_{Cj}, T) - \sum_m \int_{I_m^{\mathrm{ret}}} r \sum_j \Phi_V(\alpha_{mj}(r), T) \,\mathrm{d}r \Big| \le C\epsilon' k r_+^2 \, T^3 V(T),$$

with $k = \max k_m$, while the trimmed layers carry at most $C\epsilon r_+ T^3 V(T)$ both in the envelope bound and in P_{V,μ_ω}. Since $P_{V,\mu_\omega}(T) \ge c_\omega T^3 V(T)$, letting $T \to \infty$, then $\epsilon' \to 0$ (refining the mesh), then $\epsilon \to 0$, gives the claim. The cap conventions are equivalent by Lemma 2.3(b). □

Proof of Theorem 1.3. Take a fixed squared partition $\chi_-^2 + \chi_0^2 + \chi_+^2 = 1$ in s with $\chi_\pm$ supported in the admissible ranges of the two ends and χ_0 compactly supported. By Lemma 2.1 and Lemma 2.3,

$$N_+^{\mathsf{D}}(T^2) + N_-^{\mathsf{D}}(T^2) \le N_\theta(T^2) \le N_+^{\mathsf{D}}(T^2 + C) + N_-^{\mathsf{D}}(T^2 + C) + C(1 + T^3),$$

where $N_\pm^{\mathsf{D}}$ are the Dirichlet end counts. Apply Proposition 6.1 to each end (the negative end after $s \mapsto -s$), and remove the fixed shift by Lemma 4.3(b),(c). All counts are finite, so the spectrum is discrete and H_θ has compact resolvent. The bounds $P \asymp T^3 V$ are Lemma 4.3(c). Since each end's error is o of its own phase, the relative error of the sum is at most the larger of the two relative errors; no comparison of V_+ and V_- is needed. □

Theorem 6.2 (Angular stability)**.** *Let ω be an off-axis finite-band cross-section with all $A_m < 2\pi$, let $\Theta \in C^1(\mathbb{R})$ have admissible ends with sublevel measures $V_\pm$, and let $\theta \in C^1(\mathbb{R})$ satisfy $\theta(s) - \Theta(s) \to \gamma_\pm \in \mathbb{R}$ as $s \to \pm\infty$. Then H_θ has compact resolvent and $N_\theta(T^2) = (P_{V_+,\mu_\omega}(T) + P_{V_-,\mu_\omega}(T))(1 + o(1))$. No control of $\theta' - \Theta'$ is required.*

Proof. Absorb γ_e into a rigid rotation of the comparison sectors; sector counts are invariant under it. Use the mesh of the preceding proof and fix $\delta > 0$ smaller than the minimal inner widths, smaller than half the minimal outer gaps, and with $A_m + 2\delta < 2\pi$. Choose fixed caps beyond which $|\theta - \Theta - \gamma_e| < \delta$. In the coordinates of Θ the actual fibre over (r, s) is the original fibre rotated by an angle within δ of γ_e. Hence the inner sectors contracted by δ at each endpoint lie in Ω_θ, and the actual slab pieces lie in the outer sectors enlarged by δ, which remain disjoint and span less than 2π, so Lemma 2.4 applies; in the trimmed layers, split each layer at its partition radius r_m as in the proof of Proposition 6.1 and embed the part lying in I_m into the envelope sector over that part with opening $\operatorname{int} J_m$ enlarged by δ. The proof of Proposition 6.1 goes through with widths $\alpha^-_{Cj} - 2\delta$ and $\alpha^+_{Cj} + 2\delta$, which change the phase by $O(\delta T^3 V(T))$ by Lemma 4.3(c); the bounded core is handled by Lemma 2.3. Let $T \to \infty$, then $\delta \to 0$, then refine the mesh, then remove the trimming. □

For example, take $\Theta' = c(1 + s^2)^{p/2}$ and $\theta(s) = \Theta(s) + (1 + s^2)^{-A/2} \sin((1 + s^2)^{B/2})$ with $A > 0$ and $B > A + p + 1$. The angle error tends to zero while its derivative has leading oscillatory size $B|s|^{B-A-1}$, which dominates Θ' with both signs at successive extrema. Thus θ' has infinitely many zeros and sign changes, and Theorem 6.2 still gives (1.8). Likewise, if Θ has the log-periodic ends of Theorem 5.7 and $\theta - \Theta$ has finite limits, then Theorem 6.2 and the exact evaluation in the proof of Theorem 5.7 give (5.5) for θ on every off-axis finite-band cross-section with all $A_m < 2\pi$: the periodic profile is stable under angle perturbations with finite limits.

6.1 Discs and ellipses

A disc of radius ϱ centred at distance $d > \varrho$ from the axis has the single band $d - \varrho < r < d + \varrho$ and width

$$\alpha(r) = 2 \arccos \frac{r^2 + d^2 - \varrho^2}{2dr}, \tag{6.1}$$

which pinches at both ends with unbounded derivative; it lies in a wedge of opening $2 \arcsin(\varrho/d) < \pi$. For $d = \varrho$ the disc touches the axis, $\alpha(r) = 2 \arccos(r/(2\varrho))$ on $(0, 2\varrho)$, and $\alpha(r) \to \pi$ as $r \to 0$; it reaches the axis in the sense of Definition 1.1.

Proposition 6.3 (Ellipses)**.** *Let $\mathcal{E} \subset \mathbb{R}^2$ be an open elliptical disc with $0 \notin \mathcal{E}$. Then $\mathcal{E}$ is a finite-band cross-section, with every $A_m \le \pi$; it is off-axis if $0 \notin \overline{\mathcal{E}}$ and reaches the axis if $0 \in \partial\mathcal{E}$. In particular Theorems 1.3, 1.4, 6.2 and Corollary 1.5 apply to the elliptical tubes of Open Problem 8 of [16] and Figure 2 of [15], and give* (1.8) *with $M_{1+1/p}(\mathcal{E})$.*

Proof. By convexity and separation, $\mathcal{E}$ lies in an open half-plane bounded by a line through 0, so all angles lie in an interval of length π. Parametrize $\partial\mathcal{E}$ by $t \mapsto x(t) = x_0 + e_1 \cos t + e_2 \sin t$; then $|x(t)|^2$ is a trigonometric polynomial of degree two, nonconstant because $\partial\mathcal{E}$ is not a circle centred at 0. Its derivative has finitely many zeros, so the distance to 0 has finitely many critical values on $\partial\mathcal{E}$; let these and $r_\pm = \min, \max |x|$ be the partition radii. On each open band every circle meets $\partial\mathcal{E}$ transversally in a fixed finite number of points, at most four, each lying on a fixed arc of $\partial\mathcal{E}$ on which $|x(t)|$ is strictly monotone, so that its parameter, and hence its polar angle, is a continuous function of r extending continuously to the band ends. The fibre is the union of the open arcs between consecutive intersection points that lie in $\mathcal{E}$, which gives the band structure with positive widths and gaps inside each band. □

The coefficient (1.8) for an off-axis disc is $M_{1+1/p} = \int r\alpha(r)^{1+1/p} \, \mathrm{d}r$ with (6.1); the centreline and the derivatives of the endpoints do not enter.

7 Axis contact

The inner part of a cross-section reaching the axis is controlled by the argument of Barseghyan and Khrabustovskyi [1], localized to one end: a Poincaré inequality along lines parallel to the axis, followed by the Cwikel–Lieb–Rozenblum inequality [22, 7, 20]. We replace the monotonicity of θ' assumed in [1] by (H2).

Lemma 7.1 (Inner confinement)**.** *Let ω reach the axis, let $0 < \varepsilon < r_1$, and let $\omega_\varepsilon = \omega \cap \{|y| < \varepsilon\}$. Let the positive end of θ be admissible. There are s_1 and C, depending only on b (and the CLR constant), such that the Dirichlet count of $\Omega_{\theta,\varepsilon} = \{(R_{\theta(s)}y, s) : y \in \omega_\varepsilon,\ s > s_1\}$ satisfies*

$$N(\Lambda) \le C|\omega_\varepsilon|\,\Lambda^{3/2} V(15\sqrt{\Lambda}). \tag{7.1}$$

Proof. By (H2) with $K = 12\pi$ choose $s_1 > s_+$ such that $b(t) \ge b(s)/2$ whenever $s, t > s_1$ and $|t-s| \le 12\pi/b(s)$. Let u be smooth with compact support in $\Omega_{\theta,\varepsilon}$, extended by zero to $\mathbb{R}^3$. Fix a transverse physical point $x_\perp \neq 0$ and let (c,d) be a component of $\{s > s_1 : (x_\perp, s) \in \Omega_{\theta,\varepsilon}\}$. Along it, the polar angle of $R_{-\theta(s)}x_\perp$ stays in one component of the lift of $(\omega_\varepsilon)_{|x_\perp|} \subset J_1$, an interval of length at most 2π, and θ is monotone there; hence $\int_c^d b \le 2\pi$. If $d-c > 10\pi/b(s)$ for some $s \in (c,d)$, then one side of s contains an interval of length $5\pi/b(s)$ on which $b \ge b(s)/2$, whence $\int_c^d b \ge 5\pi/2$, a contradiction. So $\pi^2/(d-c)^2 \ge b(s)^2/100$ for every $s \in (c,d)$, and the Dirichlet Poincaré inequality gives $\int_c^d |\partial_s u|^2\,\mathrm{d}s \ge \int_c^d \frac{b(s)^2}{100}|u|^2\,\mathrm{d}s$. Integrating over $x_\perp$ and passing to the form domain by Fatou,

$$\int |\nabla u|^2 - \Lambda \int |u|^2 \ge \frac{1}{2}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} W|u|^2 \Big), \qquad W = -2\left[\Lambda - \frac{b(s)^2}{200}\right]_+ \mathbf{1}_{\Omega_{\theta,\varepsilon}}.$$

Hence $N(\Lambda)$ is at most the number of negative eigenvalues of $-\Delta + W$ on $\mathbb{R}^3$, and the Cwikel–Lieb–Rozenblum inequality gives $N(\Lambda) \le L_0 \int |W|^{3/2} \le 2^{3/2} L_0 |\omega_\varepsilon| \int_{s_1}^\infty \left[\Lambda - b^2/200\right]_+^{3/2} \mathrm{d}s \le C|\omega_\varepsilon|\Lambda^{3/2} V(\sqrt{200\Lambda})$, because the cross-section of $\Omega_{\theta,\varepsilon}$ at each s has area $|\omega_\varepsilon|$. □

Proof of Theorem 1.4. For $0 < \varepsilon < r_1$ the set $\omega \cap \{|y| > \varepsilon\}$ is an off-axis finite-band cross-section, with partition $\varepsilon < r_1 < \cdots < r_M$. The width moments converge: $\mu_{\omega\cap\{|y|>\varepsilon\}} \uparrow \mu_\omega$, and by Lemma 4.3(b),(c),

$$P_{V,\mu_\omega}(T) - P_{V,\mu_{\omega\cap\{|y|>\varepsilon\}}}(T) \le C\,|\omega_\varepsilon|\,T^3 V(T), \tag{7.2}$$

since $\int \alpha\,\mathrm{d}\mu$ restricted to $r < \varepsilon$ equals $|\omega_\varepsilon|$.

Lower bound. Ω_θ contains the tube over $\omega \cap \{|y| > \varepsilon\}$, so Theorem 1.3 and (7.2) give $\liminf N_\theta(T^2)/P(T) \ge 1 - C|\omega_\varepsilon|/c_\omega$, where $P = P_{V_+,\mu_\omega} + P_{V_-,\mu_\omega}$ and $c_\omega T^3(V_+ + V_-) \le P$.

Upper bound. Take smooth $\chi_{\mathrm{in}}, \chi_{\mathrm{out}}$ of the physical distance $|x_\perp|$ with $\chi_{\mathrm{in}}^2 + \chi_{\mathrm{out}}^2 = 1$, $\chi_{\mathrm{in}} = 1$ on $|x_\perp| \le \varepsilon/2$, $\chi_{\mathrm{in}} = 0$ on $|x_\perp| \ge \varepsilon$, and $|\nabla\chi|^2 \le C\varepsilon^{-2}$. Then $\chi_{\mathrm{out}} v$ lies in the Dirichlet domain of the tube over $\omega \cap \{|y| > \varepsilon/2\}$ and $\chi_{\mathrm{in}} v$ in that over ω_ε. By Lemma 2.1, with $\Lambda' = T^2 + C\varepsilon^{-2}$, $N_\theta(T^2) \le N_{\mathrm{out}}(\Lambda') + N_{\mathrm{in}}(\Lambda')$. Split the inner tube by a fixed axial partition into a bounded core and two ends beyond $\pm s_1$: Lemmas 2.3 and 7.1 give $N_{\mathrm{in}}(\Lambda') \le C(1+T^3) + C|\omega_\varepsilon|T^3(V_+(15T') + V_-(15T'))$, $T' = \sqrt{\Lambda'}$, which is at most $C|\omega_\varepsilon|P(T) + O(T^3)$ by (H3) and Lemma 4.3(c). Theorem 1.3 for $\omega \cap \{|y| > \varepsilon/2\}$, monotonicity of P in μ and Lemma 4.3(b),(c) give $N_{\mathrm{out}}(\Lambda') \le P(T)(1+o(1))$. Hence $\limsup N_\theta(T^2)/P(T) \le 1 + C|\omega_\varepsilon|/c_\omega$.

Since $|\omega_\varepsilon| \to 0$, letting $\varepsilon \downarrow 0$ gives (1.5). The upper bound is finite at every energy, so H_θ has compact resolvent. □

The disc tangent to the axis, $d = \varrho$ in (6.1), the ellipses with $0 \in \partial\mathcal{E}$ of Proposition 6.3, and the slit disc $\{0 < |y| < \varrho\} \setminus \{t = 0\}$ centred on the axis (one band, $A_1 = 2\pi$) are covered. For the slit disc and $b \sim cs^p$ at both ends, (1.8) holds with $M_{1+1/p} = (2\pi)^{1+1/p}\varrho^2/2$.

8 Quantitative remainder and the second profile

A *constant-step section* is an off-axis finite-band cross-section whose fibres are constant on each band: on each of finitely many radial bands the fibre is a fixed finite union of intervals, of openings at most 2π. Sectors may join across radial interfaces.

Theorem 8.1 (Quantitative remainder)**.** *Let ω be a constant-step section. On an end, let $|\theta'| = b$ be C^1 and strictly increasing on (s_+, ∞) with inverse R satisfying $0 < c \le R'(y) \le C < \infty$ for $y \ge y_0 > 0$. Then, for one end and either fixed cap,*

$$N_{\mathrm{end}}(T^2) = P_{V,\mu_\omega}(T) + O(T^{7/2}). \tag{8.1}$$

Replacing V by any increasing C^1 extension of R to $[0,\infty)$ with bounded derivative changes P by $O(T^3)$. For two ends, add the phases and the errors. Constants depend on the geometry, the derivative bounds and the core.

Proof. Here $b' = (R' \circ b)^{-1}$ is bounded above and below, $b(s) \asymp s$, and $(\log b)'$ is bounded beyond a fixed cap. Both phases are $O(T^4)$, and the proof of Lemma 4.3(d) gives

$$|I_\alpha(E_2) - I_\alpha(E_1)| \le C|E_2 - E_1|T^2 \qquad (T^2/2 \le E_i \le 2T^2), \tag{8.2}$$

also over arbitrary axial subsets, with integrand at most CT^3 per unit axial length; the same holds for the reduced phase. The partitions (2.4) have $C_\eta \le C\eta^{-2}$, also in Lemma 3.1.

Put $B = T^{1/2}$. The region $b \le 2B$ has axial length $O(B)$. On it use a squared partition into Dirichlet supports of length at most 2 with uniformly bounded derivative sum, and a fixed radial mesh. On these supports b varies by a bounded factor, so Lemma 4.1 gives a fixed quotient distortion independent of the speed; the frozen upper bound and (3.4) give $O(T^3)$ eigenvalues per support at energy CT^2. There are $O(B)$ supports, so this region costs $O(T^{7/2})$. Use squared low/high cutoffs across $B \le b \le 2B$; since b' is bounded above and below, the transition has length comparable to B and derivatives $O(B^{-1})$. A free cap uses a fixed collar, with count $O(T^3)$.

On $b \ge B$ take axial spacing $\ell = 1$, overlap $\eta = T^{-1/2}$ and radial mesh $h = T^{-1/2}$. The relative speed variation on a support is $O(B^{-1})$ and radial freezing costs $O(h)$; the IMS and frozen-box energy errors are $O(\eta^{-2}) = O(T)$. Lemma 2.2 confines the active supports to an axial range of length $O(T)$. The quotient errors shift energy by $O((B^{-1} + h)T^2) = O(T^{3/2})$, costing $O(T^{7/2})$ by (8.2); overlap factors cost $O(\eta T^4) = O(T^{7/2})$; IMS shifts cost $O(T^3)$; and there are $O(T/h)$ boxes, so the summed radial error from (3.4) is $O(T^3/h) = O(T^{7/2})$. For the lower bound insert Dirichlet boxes of length 1 and Dirichlet radial cells on $\{b \ge B\}$ and discard $\{b < B\}$, whose phase is $O(BT^3) = O(T^{7/2})$; missing end intervals cost $O(T^3)$. The squeeze (4.8) controls all angular thresholds. This gives $N_{\mathrm{end}}(T^2) = I_\alpha(T^2) + O(T^{7/2})$ for each constant sector with either radial convention. At a step interface, insert Dirichlet walls for the lower comparison and free traces on entire radial faces for the upper comparison; this is an exact domain enlargement even where sectors join along part of a face. Each comparison splits into the same finite list of sectors.

Finally, for $y = b(s) \ge \max(1, y_0)$ the relative differences of $y^2 + r^{-2}$ from y^2 and of the weight from ry are $O(y^{-2})$. The phase per unit length is $O(T^3)$ with energy derivative $O(T)$, so the positive-part squeeze costs $O(T^3 y^{-2})$ pointwise; since $\mathrm{d}s = R'(y)\,\mathrm{d}y$ with bounded R', $|I_\alpha(T^2) - \frac{r_+^2 - r_-^2}{2}\Phi_V(\alpha, T)| \le CT^3(1 + \int_1^\infty y^{-2}\,\mathrm{d}y) = O(T^3)$. Fixed cap and core comparisons cost $O(T^3)$. □

Write $F_\sigma[f]$ for the profile in (5.5) with γ replaced by $\sigma > 0$ and h replaced by a smooth L-periodic f; it is linear in f and defined for signed f.

Theorem 8.2 (Second periodic profile)**.** *On a constant-step section suppose that, on an end, eventually*

$$R(y) = yh(\log y) + y^{\delta} g(\log y), \qquad \tfrac{1}{2} < \delta < 1,$$

with h, g real, smooth and L-periodic, $h > 0$ and $h + h' > 0$. Then

$$N_{\mathrm{end}}(T^2) = T^4 F_1[h](\log T) + T^{3+\delta} F_\delta[g](\log T) + O(T^{7/2}), \tag{8.3}$$

and both periodic profiles are determined by this asymptotic. For the connected two-width section of Proposition 9.5 and $L = 2\log 2$, every Fourier mode of g is recovered from the second profile, with exactly one Sobolev derivative lost, although the odd modes of h are invisible in the first profile.

Proof. $R' = h + h' + y^{\delta-1}(\delta g + g')$ is eventually bounded above and away from zero, so Theorem 8.1 applies to any increasing extension. Linearity of (5.1) in $\mathrm{d}R$ and the computation in the proof of Theorem 5.7, with the majorants $\sum n^{-1-\sigma}$ for $\sigma = 1, \delta$, give the two profiles; cutting off on a fixed interval costs $O(T^3)$. With $t = \log T$ sampled along $t_0 + nL$, division by T^4 determines the first profile; subtracting it, division by $T^{3+\delta}$ determines the second, since $\frac{7}{2} < 3 + \delta$. For the stated section, $M_{1+\delta+ik\tau} = 4 + 2^{1+\delta}(-1)^k$ and $4 - 2^{1+\delta} > 0$, so the moments are bounded above and away from zero, and the argument of Theorem 9.4 gives $c_\delta(1+|k|)^{-1} \le |K_\mu(\delta + ik\tau)| \le C_\delta(1+|k|)^{-1}$, which proves the recovery estimate with exactly one derivative lost. □

For instance, with $h = h_0 + u$, u smooth with $u(t + \log 2) = -u(t)$ and (5.4) preserved, and $g = a\cos(\tau t + \phi)$, on that section

$$N_{\mathrm{end}}(T^2) = \frac{h_0 T^4}{12\pi} + aT^{3+\delta} \Re\{e^{i\phi} K_\mu(\delta + i\tau) e^{i\tau \log T}\} + O(T^{7/2}),$$

with $K_\mu(\delta + i\tau) \neq 0$: a lower-order spectral signature at the logarithmic frequency of a blind leading mode.

9 Spectral information in the leading term

9.1 Width moments and area-only bounds

Proposition 9.1. *Let ω have at most m components at each radius, all widths at most $L \le 2\pi$, and let $\mathcal{V}_\omega = \int_{\{r:\omega_r \neq \emptyset\}} r\,\mathrm{d}r$. For $q > 1$,*

$$\frac{|\omega|^q}{(m\mathcal{V}_\omega)^{q-1}} \le M_q(\omega) \le L^{q-1}|\omega|.$$

The upper bound is attained by slit annuli with $L = 2\pi$. Splitting each fibre into m equal components multiplies the local moment by $m^{1-q} = m^{-1/p}$ for $q = 1 + 1/p$. Consequently, in (1.8) *the ratio of the leading coefficient to $|\omega|$ equals $\frac{p\zeta(1+1/p)c^{-1/p}}{\pi^{2+1/p}(p+1)(3p+1)}\, M_{1+1/p}(\omega)/|\omega|$, which is at most $(2\pi)^{1/p}$ times the prefactor, attains this value, and tends to 0 along fragmented cross-sections of fixed area.*

Proof. Pad missing widths with zeros and apply Jensen's inequality to m copies of $r\,\mathrm{d}r$; the upper bound is $\alpha^q \le L^{q-1}\alpha$, with equality for $\alpha \equiv 2\pi$; the splitting identity is $m(\alpha/m)^q = m^{1-q}\alpha^q$. □

Thus the exponent $\frac{3}{2} + \frac{1}{2p}$ of (1.2) is exact, while a bound proportional to $|\omega|$ captures the true coefficient only up to the factor $M_{1+1/p}(\omega)/((2\pi)^{1/p}|\omega|) \in (0, 1]$.

Remark 9.2 (Width distribution). Let ν_ω be the finite measure on $[0, 2\pi]$ given by $\int f \, \mathrm{d}\nu_\omega = \int \alpha f(\alpha) \, \mathrm{d}\mu_\omega(\alpha)$, of mass $|\omega|$. Then $M_q = \int \alpha^{q-1} \, \mathrm{d}\nu_\omega$. The leading coefficients (1.8) for known c and $p = 1/n$, $n = 1, 2, \ldots$, give all moments $\int \alpha^n \, \mathrm{d}\nu_\omega$, i.e. all moments of $\alpha \, \mathrm{d}\nu_\omega$; by density of polynomials in $C([0, 2\pi])$ they determine $\alpha \, \mathrm{d}\nu_\omega$, hence ν_ω (which has no atom at 0) and μ_ω. The recovery uses the family of tubes indexed by p and determines exactly the weighted width distribution, the datum shared by all rearrangements of ω that rotate fibres or reorder bands.

Theorem 9.3 (Finite-data obstruction). *Fix $0 < r_- < r_+$, $\mathcal{V} = (r_+^2 - r_-^2)/2$, an area $0 < A < 2\pi\mathcal{V}$ and finitely many $p_1, \ldots, p_k > 0$. There are two connected finite-band cross-sections in $\{r_- < |y| < r_+\}$ with area A and distinct width measures whose leading coefficients in* (1.8) *coincide for every p_i and common c.*

Proof. Put $\bar{\alpha} = A/\mathcal{V}$ and choose $\bar{\alpha} < L < 2\pi$ and $k + 3$ distinct $x_j \in (0, L)$ on both sides of $\bar{\alpha}$. Choose positive weights w_j with $\sum w_j = \mathcal{V}$ and $\sum w_j x_j = A$, possible since $\bar{\alpha}$ is interior to the convex hull of the x_j. With $q_i = 1 + 1/p_i$, the matrix with rows $(1)_j, (x_j)_j, (x_j^{q_i})_j$ has $k+2$ rows and $k+3$ columns, so it has a nonzero real kernel vector z. For small $\epsilon > 0$ the weights $w_j^\pm = w_j \pm \epsilon z_j$ are positive, and the two atomic width distributions have equal mass, area and selected moments but differ. Realize them by radial interfaces $r_j^\pm = (r_-^2 + 2\sum_{i \le j} w_i^\pm)^{1/2}$ and centred intervals $(-x_j/2, x_j/2)$ on successive bands, and take the interior of the closure of the union of these sectors. Adjacent centred intervals are nested, so the result is connected; it is a finite-band cross-section, and interfaces at finitely many radii do not change the moments. □

9.2 Blind modes

The following concerns the log-periodic class of Theorem 5.7 on a fixed known cross-section.

Theorem 9.4 (Detectability). *Two modulations h_1, h_2 satisfying* (5.4) *with the same γ, L have the same leading profile if and only if*

$$(h_{1,k} - h_{2,k}) M_{1+\gamma+ik\tau}(\omega) = 0 \qquad (k \in \mathbb{Z}). \tag{9.1}$$

Recovery of h from the leading profile is injective in this class exactly when no lattice moment $M_{1+\gamma+ik\tau}$ vanishes. If $|M_{1+\gamma+ik\tau}| \ge c(1 + |k|)^{-a}$ on the visible modes, for some $a \ge 0$, then after projection away from the blind modes

$$\|h_1 - h_2\|_{H^s} \le C \|F_{h_1} - F_{h_2}\|_{H^{s+a+1}} \qquad (s \in \mathbb{R}), \tag{9.2}$$

with periodic Sobolev norms. For a single constant width the loss of one derivative is sharp.

Proof. A continuous periodic profile is determined by the asymptotic, since two such profiles differing by $o(1)$ agree along every progression $t_0 + nL$. On $\Re z = \gamma > 0$ the factors of K_μ other than the moment are nonzero; in particular $\zeta(1 + z) \ne 0$ by the Euler product. Fourier uniqueness gives (9.1). The zero mode is visible since $M_{1+\gamma} > 0$. If a nonzero moment vanishes, small real cosine or sine perturbations of a positive constant h preserve (5.4) and are invisible. Absolute convergence of the Dirichlet series of ζ and $1/\zeta$ gives $\zeta(1 + \gamma)^{-1} \le |\zeta(1 + \gamma + it)| \le \zeta(1 + \gamma)$, and the rational factor in (5.3) is comparable with $(1 + |k|)^{-1}$ on the lattice. Hence $|K_\mu(z_k)| \ge c'(1 + |k|)^{-a-1}$ on visible modes, which gives (9.2). For a constant width the multiplier is comparable with $(1 + |k|)^{-1}$, and testing single high modes of small amplitude shows that one derivative is lost. □

Proposition 9.5 (A connected blind cross-section). *Let ω be the interior of the closure of the union of the polar sectors $(1, 3) \times (0, 1)$ and $(3, \sqrt{11}) \times (0, 2)$. It is a connected constant-step section with $\mu_\omega = 4\delta_1 + \delta_2$. For $\gamma = 1$ and $L = 2 \log 2$, $M_{2+ik\tau} = 4 + 2^{2+ik\tau} =$*

$4(1+(-1)^k)$, *so exactly the odd harmonics are blind, and the invisible modulations form the infinite-dimensional space of smooth* g *with* $g(t+\log 2) = -g(t)$. *For* $h_\epsilon(t) = 1+\epsilon\cos(\tau t)$ *with* $0 < |\epsilon| < (1+\tau^2)^{-1/2}$, *the inverse speeds* $R_\epsilon(y) = yh_\epsilon(\log y)$ *are smooth and strictly increasing, neither* R_ϵ *nor* $b_\epsilon = R_\epsilon^{-1}$ *is regularly varying, and on two identical ends*

$$N_\epsilon(\Lambda) = \frac{\Lambda^2}{6\pi} + O(\Lambda^{7/4}), \tag{9.3}$$

the same as for $\epsilon = 0$.

Proof. The sectors share an open angular interface, so ω is connected, and its widths are 1 on $(1,3)$ and 2 on $(3,\sqrt{11})$, with $\int_1^3 r\,\mathrm{d}r = 4$ and $\int_3^{\sqrt{11}} r\,\mathrm{d}r = 1$. Since $2^{ik\tau} = e^{ik\pi}$, the moment formula follows. Next, $h_\epsilon + h_\epsilon' = 1+\epsilon(\cos\tau t - \tau\sin\tau t) > 0$ and $h_\epsilon > 0$. Since $R_\epsilon' = h_\epsilon + h_\epsilon'$ is bounded above and below, Theorem 8.1 applies; the only nonzero Fourier modes of h_ϵ are $k = 0, \pm 1$, and $k = \pm 1$ are blind, so $P = T^4 h_0 K_\mu(1) = T^4/(12\pi)$ per end ($M_2 = 8$). Finally $R_\epsilon(2y)/R_\epsilon(y) = 2(1-\epsilon\cos(\tau\log y))/(1+\epsilon\cos(\tau\log y))$ has distinct subsequential limits; and if b_ϵ were regularly varying, comparability with y would force index 1, and inversion [3, Theorem 1.5.12] would make R_ϵ regularly varying. □

The example keeps the speed smooth and strictly increasing and exhibits a persistent order-one modulation of the speed that is invisible in the leading count. By Theorem 8.2, a lower-order modulation at the same frequency is visible in the second profile on the same section.

Provenance of the sections

Section 1: problem statement from [15, 16]; literature map and Theorems 1.3, 1.4 new; constants of Corollary 1.5 from W1 and W2. Section 2: tail estimate from W1–W3 (here in line form, without reduction modulo 2π), cores and caps from W2/W3, lateral extension from W2/W3, the single IMS write-up new. Section 3: W1, reproduced in W2 and W3, stated once. Section 4: local clock from W2/W3; window lemma, phase bounds in terms of V, and the one-ended law under (H1)–(H4) new, replacing the regular-variation and log-derivative arguments of W1–W3; reduction lemma from W3. Section 5: Mellin representation and Karamata evaluation new; smoothing in Lemma 5.2(d), examples and Corollary 5.5 from W2; Corollary 5.6 from W1; Theorem 5.7 from W3; Example 5.4 new. Section 6: finite-band passage and angular stability from W2, reorganized around the Lipschitz bound in the opening; disc from W2; Proposition 6.3 new. Section 7: new, adapting the argument of [1]. Section 8: W3. Section 9: Proposition 9.1, Remark 9.2 and Theorem 9.3 from W2 (the former Theorem 7.1 of W2 is now Remark 9.2); Theorem 9.4 and Proposition 9.5 from W3.

AI assistance

OpenAI Codex assisted with literature comparison, proof review and drafting of the earlier versions W1–W3. Anthropic Claude (through Claude Code) assisted with the referee review that led to this merger, the unified proofs of Theorems 1.3 and 1.4, the Mellin evaluation, the verification of the bibliography, and the preparation of this manuscript. The author takes responsibility for the content, accuracy and attribution of the work.

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