Negative edge correlation in complete bipartite forests at every fugacity

Scott Schubr
Luminary Oddities

Abstract

For the random forest of the complete bipartite graph $K_{m,n}$ in which a forest F has probability proportional to $\lambda^{|F|}$, we prove that any two distinct edges are strictly negatively correlated, for all $m, n \geq 2$ and every fugacity $\lambda > 0$. Isolated vertices and the empty forest are included, and there is no restriction on the sizes or their ratio. Edges sharing a vertex are handled by exact Laguerre counts for the degree polynomial, which is real-rooted with unequal Bernoulli parameters. For disjoint edges the covariance deficit reduces exactly to a quadratic in one exterior Laguerre ratio. Continued-fraction bounds for that ratio settle the regimes $\lambda \geq 1$ and $\lambda \leq 1/\sqrt{mn}$, exact transfer-matrix identities settle all hosts with smaller part at most six, and a scalar barrier along the Riccati flow of the ratio then covers every remaining host and fugacity. The finite polynomial sign conditions are proved by complete lists of positive coefficients, which an accompanying exact-arithmetic checker regenerates from the formulas in the paper. The uniform case $\lambda = 1$ was proved independently by Cao and Fu.

Keywords. Random forests; arboreal gas; negative correlation; complete bipartite graph; Laguerre polynomials; continued fractions; computer-assisted proof.
2020 Mathematics Subject Classification. 60C05; 05C31; 05A15; 33C45.

1 Introduction

Let $\mathcal{F}_{m,n}$ be the set of forests of the complete bipartite graph $K_{m,n}$, that is, edge sets without cycles; isolated vertices and the empty set are allowed. For $\lambda > 0$ put

$$Z_{m,n}(\lambda) = \sum_{F \in \mathcal{F}_{m,n}} \lambda^{|F|}, \qquad \mathbb{P}_\lambda(F) = \frac{\lambda^{|F|}}{Z_{m,n}(\lambda)}.$$

This is the arboreal gas on $K_{m,n}$ at fugacity λ; at $\lambda = 1$ it is the uniform random forest. A conjecture of Kahn [9], stated by Grimmett and Winkler [6], asserts that any two edges of any finite graph are negatively correlated under the uniform forest measure, and its weighted form asks for the same at all positive weights. We prove the complete-bipartite case at every common fugacity.

Theorem 1.1. *Let $m, n \geq 2$. For every $\lambda > 0$ and any two distinct edges e, f of $K_{m,n}$,*

$$\mathbb{P}_\lambda(e, f \in F) < \mathbb{P}_\lambda(e \in F)\, \mathbb{P}_\lambda(f \in F).$$

On $K_{1,n}$ the edge indicators are independent Bernoulli variables with parameter $\lambda/(1+\lambda)$.

The last sentence holds because $K_{1,n}$ is a tree, so every edge set is a forest.

Related work. Cao and Fu [1] proved strict negative correlation for the uniform forest ($\lambda = 1$) and the uniform connected subgraph of every $K_{m,n}$, $m, n \geq 2$; Theorem 1.1 at $\lambda = 1$ is their theorem, and the present proof is independent. Stark [14] developed the enumeration of forests of $K_{m,n}$ and their asymptotics. Erickson's computer-assisted thesis [4] proves the Rayleigh inequality at arbitrary positive weights on every graph with at most seven vertices, which includes $K_{3,4}$, and the census of Grimmett and Winkler [6] includes uniform forests of $K_{3,n}$ for $n \leq 6$. Huang [8] proved incident negative correlation on every finite connected graph at sufficiently large common fugacity, and Çiçeksiz and Ravichandran [2] proved negative correlation at sufficiently large fugacity whenever one of the two leading coefficients of the Rayleigh difference is nonzero. For complete graphs K_n, Gupta [7] proved negative correlation for the uniform forest, and Shmalo [13] and the author [10] proved it at every common fugacity. The companion paper [10] also proves that the edges at any vertex of any weighted graph form a strongly Rayleigh family, which gives the weak inequality for incident pairs in general.

Method. Section 2 expresses all forest counts of $K_{m,n}$ through one finite polynomial $q_{m,n}$, a scaled associated Laguerre polynomial, by a two-variable Lagrange–Good inversion. Incident pairs follow from the degree polynomial in Section 3. For disjoint pairs, Section 4 reduces the covariance deficit exactly to $q^2 F(R)$, where F is an explicit quadratic and R is the exterior Laguerre ratio

$$R = \sum_i \frac{c_i}{x - c_i}, \qquad x = m + n + mn\lambda + \lambda^{-1},$$

over the Laguerre zeros c_i. Section 5 bounds R by finite continued fractions. These bounds decide the sign of $F(R)$ for $\lambda \geq 1$ and for $\lambda \leq 1/\sqrt{mn}$ (Section 6), and exact transfer matrices decide it on hosts with smaller part at most six (Section 7). For the remaining hosts and fugacities, Section 8 passes to compact coordinates in which the deficit is a polynomial in a single unknown V obeying a Riccati equation, and Section 9 proves that the deficit cannot reach zero along this flow, by a barrier argument run backward from the known positive regime $\lambda \geq 1$.

The finite sign conditions are proved by exhibiting polynomials whose coefficients are all positive after explicit substitutions that cover the required parameter domains. These lists contain several thousand coefficients. Section 10 specifies how each list is generated from the formulas in the paper, and the accompanying script `code/verify.py` regenerates every list in exact arithmetic and compares it with the stored data file.

2 An exact coefficient formula

Use exponential generating functions in labeled left and right vertices. The rooted bipartite tree series satisfy

$$U = xe^{\lambda V}, \qquad V = ye^{\lambda U}. \tag{1}$$

The unrooted tree series is $T = U + V - \lambda UV$, since a tree is counted once per vertex in the first two terms and once per edge in the last, and the forest series is $E = e^T$. The formal substitution $x = ue^{-\lambda v}$, $y = ve^{-\lambda u}$ has logarithmic Jacobian determinant $1 - \lambda^2 uv$, and the Lagrange–Good formula [5] gives

$$[x^p y^n] H(U, V) = [u^p v^n] H(u, v) e^{\lambda pv + \lambda nu} (1 - \lambda^2 uv). \tag{2}$$

For nonnegative integers p, n define

$$q_{p,n}(a, b, \lambda) = \sum_{k=0}^{\min(p,n)} (-\lambda)^k \binom{p}{k} (n)_k a^{p-k} b^{n-k}, \tag{3}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. It equals $p!\,n!\,[u^p v^n]e^{au+bv-\lambda uv}$. Subscripts on q denote partial derivatives in the continuous variables a, b, λ, with p, n fixed. Termwise differentiation gives

$$q_\lambda = -q_{ab}, \qquad \lambda q_{ab} = bq_b - nq, \qquad aq_a - bq_b = (p-n)q. \tag{4}$$

With $h = \min(p,n)$ and $\alpha = |n-p|$, the coefficient formula for associated Laguerre polynomials gives

$$q_{p,n} = (-\lambda)^h h!\, a^{p-h} b^{n-h} L_h^{(\alpha)}(ab/\lambda). \tag{5}$$

For $\alpha \geq 0$ the zeros of $L_h^{(\alpha)}$ are distinct and positive [3, Chapter 18].

3 Degree counts and incident pairs

Fix a left vertex and put $p = m-1$. Its component consists of this vertex together with an unordered collection of right-rooted branches, each attached by one edge, and the other components contribute E. Marking each edge at the fixed vertex by z, its degree polynomial is

$$G(z) = p!\,n!\,[x^p y^n]E(x,y)e^{\lambda z V(x,y)}. \tag{6}$$

Each forest is counted once. By (2), with $a = 1+n\lambda$ and $b = 1+(p+z)\lambda$,

$$G(z) = q_{p,n} - \lambda^2 (q_{p,n})_{ab} = aq_{p,n} - \lambda b (q_{p,n})_b, \tag{7}$$

the last step by (4) and $a = 1+n\lambda$.

Lemma 3.1. *Let $Q(b) = b^k Q_1(b)$ have degree n, where Q_1 has $h = n-k$ distinct positive zeros. For $a = 1+n\lambda$ and $\lambda > 0$, the polynomial $aQ - \lambda bQ'$ has a zero of order k at 0 and h distinct positive zeros.*

Proof. The factor beyond b^k is $(1+h\lambda)Q_1 - \lambda b Q_1'$. With $P(t) = t^h Q_1(1/t)$ it equals $b^h(P(t)+\lambda t P'(t))$ at $t = 1/b$. If $0 < t_1 < \cdots < t_h$ are the zeros of P, then

$$\frac{P(t)+\lambda t P'(t)}{P(t)} = 1 + \lambda \sum_{i=1}^{h} \frac{t}{t-t_i}, \qquad \frac{d}{dt}\Big(1+\lambda\sum_i \frac{t}{t-t_i}\Big) = -\lambda \sum_i \frac{t_i}{(t-t_i)^2} < 0,$$

so there is one zero in $(0,t_1)$ and one in each (t_i, t_{i+1}), h in total. The case $h = 0$ is immediate. □

Proposition 3.2. *Let $m, n \geq 2$ and $\lambda > 0$. The degree of each vertex of $K_{m,n}$ under $\mathbb{P}_\lambda$ is distributed as a sum of independent Bernoulli variables whose parameters lie in $(0,1)$ and are not all equal, and any two edges sharing a vertex are strictly negatively correlated.*

Proof. By (5), $Q = q_{p,n}$ satisfies the hypotheses of Lemma 3.1 as a polynomial in b, so all zeros of (7) are real after the affine change from b to z. The forest interpretation gives nonnegative coefficients with positive constant and leading terms, so all zeros are strictly negative and

$$\frac{G(z)}{G(1)} = \prod_{i=1}^{n} (1 - p_i + p_i z), \qquad 0 < p_i < 1.$$

The incident edge indicators $X_1, \ldots, X_n$ at the vertex are exchangeable, so their common covariance satisfies $n(n-1)\operatorname{Cov}(X_1, X_2) = \operatorname{Var}(\sum X_i) - \sum \operatorname{Var}(X_i)$. Since $\sum X_i$ has the law of the Bernoulli sum, $\operatorname{Var}(\sum X_i) = \sum p_i(1-p_i)$, while $\operatorname{Var}(X_i) = \bar{p}(1-\bar{p})$ with $\bar{p} = n^{-1}\sum p_i$. Hence

$$\operatorname{Cov}(X_e, X_f) = -\frac{\sum_i (p_i - \bar{p})^2}{n(n-1)}. \tag{8}$$

For $m, n \geq 2$, either $h \geq 2$, giving distinct nonzero b-roots, or $h = 1$ together with a zero b-root. Thus the parameters are not all equal and (8) is strictly negative. Exchanging the two sides covers right vertices. □

4 The disjoint-edge quadratic

From now on $q = q_{m,n}$ and

$$W(a, b, \lambda) = q - \lambda^2 q_{ab} = (1 + n\lambda)q - \lambda b q_b.$$

At $a = 1+n\lambda$, $b = 1+m\lambda$, $W = Z_{m,n}(\lambda)$ by (2). Write $D = n\partial_a + m\partial_b + \partial_\lambda$; along the specialization this is the total derivative in λ. Let $N_e = \sum_{F \ni e} \lambda^{|F|}$ and $N_{ef} = \sum_{F \supseteq \{e,f\}} \lambda^{|F|}$ for a disjoint pair e, f. By edge transitivity,

$$N_e = \frac{\lambda DW}{mn}. \tag{9}$$

The degree-marked count at a fixed left vertex is $\frac{1}{m} W_a(a, b - \lambda + \lambda z, \lambda)$ at the specialized a, b: the coefficient in (6) has one fewer left label, and the derivative in a accounts for the factorial. Hence the counts of a fixed pair sharing a left vertex, respectively a right vertex, are

$$B_L = \frac{\lambda^2 W_{abb}}{mn(n-1)}, \qquad B_R = \frac{\lambda^2 W_{aab}}{nm(m-1)}.$$

The count of ordered pairs of distinct edges is $\lambda^2 D^2 W$. Splitting it into the two incident types and the $m(m-1)n(n-1)$ ordered disjoint pairs gives

$$N_{ef} = \frac{\lambda^2 (D^2 W - W_{abb} - W_{aab})}{m(m-1)n(n-1)}, \tag{10}$$

with all derivatives taken before specialization. In addition to (4), differentiation gives

$$q_{bb} = -\frac{an}{\lambda b} q + \frac{ab - \lambda(m - n + 1)}{\lambda b} q_b, \tag{11}$$

so every derivative in (9)–(10) reduces to q and q_b. Put $s = m + n$, $p = mn$ and $R = bq_b/q - n$. Substitution gives

$$\frac{N_e^2 - Z N_{ef}}{q^2} = F(R) = \frac{A_2 R^2 - A_1 R + A_0}{m^2 n^2 (m-1)(n-1)(1+m\lambda)(1+n\lambda)}, \tag{12}$$

where

$$\begin{aligned}
A_0 &= \lambda p^2 \{\lambda^3 p + \lambda^2(2s - p) + 3\lambda + 1\}, \\
A_1 &= p\{2\lambda^5 p^2 + \lambda^4(3sp - p^2 + 2p) + \lambda^3(2s + 6p) + \lambda^2(3s + 2) + 4\lambda + 1\}, \\
A_2 &= \lambda\{\lambda^5 p^3 + \lambda^4(s + 2)p^2 + \lambda^3[p(s + 1) + 4p^2] \\
&\qquad + \lambda^2(4sp + s - s^2 - p^2 + 2p) + \lambda(5p + 1 - s) + p\}.
\end{aligned} \tag{13}$$

Since $\mathbb{P}(e)\mathbb{P}(f) - \mathbb{P}(e, f) = (N_e^2 - Z N_{ef})/Z^2$, a disjoint pair is strictly negatively correlated exactly when $F(R) > 0$ at the actual R. The reduction from (9)–(11) to (12) is a finite computation with rational functions; Section 10 specifies it, and the checker performs it independently of the closed coefficients (13).

5 Exterior Laguerre ratios and continued fractions

Let $h = \min(m, n)$, $\alpha = |m - n|$ and

$$x = \frac{(1 + n\lambda)(1 + m\lambda)}{\lambda} = m + n + mn\lambda + \lambda^{-1}.$$

By (5), with c_i the zeros of $L_h^{(\alpha)}$,

$$R = R_{m,n}(x) = \sum_{i=1}^{h} \frac{c_i}{x - c_i}. \tag{14}$$

The Jacobi matrix of $L_h^{(\alpha)}$ has diagonal entries $2j + \alpha + 1$ and off-diagonal entries $\sqrt{j(j+\alpha)}$, so its maximal absolute row sum is less than $m + n - 1 + 2\sqrt{mn}$, while $x \geq m + n + 2\sqrt{mn}$ by the arithmetic–geometric mean inequality. Thus x exceeds every zero, $q > 0$, and the same holds for all the lower-index matrices below. The Laguerre recurrence and derivative identity give

$$R_{m,n}(x) = \frac{mn}{x - (m + n - 1) - R_{m-1,n-1}(x)}, \qquad R_{0,n} = R_{m,0} = 0, \tag{15}$$

with x fixed as the indices decrease; each denominator is a ratio of consecutive exterior Laguerre values and is positive. For $m, n \geq 3$ put $d_j = x - (m + n - 2j - 1)$ and

$$L_3 = \cfrac{mn}{d_0 - \cfrac{(m-1)(n-1)}{d_1 - \cfrac{(m-2)(n-2)}{d_2}}}. \tag{16}$$

Replacing the nonnegative tail by zero and using that $t \mapsto c/(d - t)$ is increasing gives $R \geq L_3$.

For an upper bound let $c, d > 0$ be indices below m, n, and let $\nu_- < \nu_+$ solve $x = c + d + cd\nu + 1/\nu$. Positivity of the forest partition function of $K_{c,d}$ at fugacity ν_+, together with (2), gives

$$0 < \frac{Z_{c,d}(\nu_+)}{q_{c,d}(\nu_+)} = 1 - \nu_+ R_{c,d}(x), \qquad \text{so} \qquad R_{c,d}(x) < \nu_+^{-1} = cd\,\nu_- < cd\lambda,$$

the last step because $x > c + d + cd\lambda + 1/\lambda$ places λ strictly between the two roots. A zero index gives a zero tail. Hence, with $T = (m - 3)(n - 3)\lambda$,

$$R \leq U_3 = \cfrac{mn}{d_0 - \cfrac{(m-1)(n-1)}{d_1 - \cfrac{(m-2)(n-2)}{d_2 - T}}}. \tag{17}$$

Indeed $d_2 - T > 1/\lambda$, so the innermost fraction is at most $(m - 2)(n - 2)\lambda$; repeating this outward proves that every denominator is positive and justifies each monotone substitution.

6 Large and small fugacity

Corollary 6.1. *Let $m, n \geq 2$. Every pair of distinct edges of $K_{m,n}$ is strictly negatively correlated whenever $\lambda \geq 1$ or $0 < \lambda \leq 1/\sqrt{mn}$.*

The proof uses the following finite polynomial facts, each established by a complete list of coefficients (Section 10).

Lemma 6.2. *For $m, n \geq 3$:*

1. *if $\lambda \geq 1$, then $F(L_3) > 0$, $F'(L_3) > 0$ and $F'' > 0$;*
2. *if $0 < \lambda \leq 1/\sqrt{mn}$, then $F(U_3) > 0$ and $F'(0) < 0$.*

Proof. For the first assertion substitute $m = 3 + u$, $n = 3 + v$, $\lambda = 1 + w$ with $u, v, w \geq 0$. The reduced rational functions $F(L_3)$, $F'(L_3)$, F'' have numerators and denominators with respectively 208/960, 396/441 and 112/75 monomials, all with nonnegative integer coefficients and positive constant terms.

For the second put $m = \mathsf{a}^2$, $n = \mathsf{b}^2$, $\lambda = [\mathsf{ab}(1 + w)]^{-1}$ and $\mathsf{a} = \sqrt{3} + u$, $\mathsf{b} = \sqrt{3} + v$, with $u, v, w \geq 0$. This covers every real $m, n \geq 3$ and every $\lambda \in (0, 1/\sqrt{mn}]$. The reduced rational functions $F(U_3)$ and $-F'(0)$ have numerators and denominators with respectively 844/2468 and 93/454 monomials, every coefficient strictly positive in $\mathbb{Q}(\sqrt{3})$, positive constants included. □

Proof of Corollary 6.1. Incident pairs are covered by Proposition 3.2; consider a disjoint pair and recall $q > 0$.

Width two. If $m = 2$, the continued fraction terminates after two levels and direct substitution gives, for every $\lambda > 0$,

$$F(R) = \frac{\lambda^4(\lambda + 1)^2}{(4\lambda^4 n^2 + 8\lambda^3 n + 3\lambda^2 n + 4\lambda^2 + 4\lambda + 1)^2} > 0. \tag{18}$$

Exchanging the sides covers $n = 2$. Assume now $m, n \geq 3$.

Large fugacity. For $\lambda \geq 1$, Lemma 6.2 shows that F is increasing on $[L_3, \infty)$, so $F(R) \geq F(L_3) > 0$.

Small fugacity. Direct algebra gives

$$A_1^2 - 4A_2A_0 = p^2\mathcal{L}\mathcal{B}, \qquad \mathcal{L} = \lambda^2(p - s) - 2\lambda - 1, \tag{19}$$

where

$$\begin{aligned}\mathcal{B} = {} & \lambda^6 p^2(p - s) + \lambda^5(6p^2 - 8sp + 4p) + \lambda^4(-3p^2 + 10sp - 8s^2 - 8p + 4s) \\ & + \lambda^3(8p - 16s + 4) + \lambda^2(3p - 5s - 8) - 6\lambda - 1.\end{aligned}$$

On $0 < \lambda \leq 1/\sqrt{p}$ we have $\mathcal{L} = p\lambda^2 - 1 - s\lambda^2 - 2\lambda < 0$. Under the substitution of Lemma 6.2, let $\widetilde{\mathcal{B}}(w) = [\mathsf{ab}(1 + w)]^6\mathcal{B}$, a polynomial of degree six in w. The negatives of its coefficients of w^j, $j = 1, \ldots, 6$, have respectively $39, 46, 49, 49, 49, 49$ coefficients in u, v, all strictly positive in $\mathbb{Q}(\sqrt{3})$. Hence $\widetilde{\mathcal{B}}$ is strictly decreasing on $w \geq 0$ and tends to $-\infty$. As λ increases from 0 to $1/\sqrt{p}$, w decreases from ∞ to 0, so

$$I = \{\lambda \in (0, 1/\sqrt{p}] : \mathcal{B}(\lambda) \leq 0\}$$

is an initial interval containing all small $\lambda > 0$. The discriminant of F is nonnegative on I and strictly negative on its complement.

The function $H(\lambda) = -F'(U_3)$ is continuous, the continued-fraction denominators being positive. As $\lambda \downarrow 0$, $U_3 = p\lambda + O(\lambda^2)$ and $F'(U_3) \to -1/[mn(m - 1)(n - 1)] < 0$. Suppose H had a first zero in I. There $F'(0) < 0 = F'(U_3)$ with $U_3 > 0$, so the affine function F' has positive slope $2A_2/\mathcal{D}$, where $\mathcal{D} > 0$ is the denominator in (12), and

$$\operatorname{disc}(F) = F'(U_3)^2 - 4\frac{A_2}{\mathcal{D}}F(U_3) = -4\frac{A_2}{\mathcal{D}}F(U_3) < 0,$$

contrary to the nonnegative discriminant on I. Hence $F'(U_3) < 0$ on I, so F decreases on $[0, U_3]$ and $F(R) \geq F(U_3) > 0$ there. On the complement of I the quadratic has negative discriminant and $F(U_3) > 0$, so it is positive on the whole line. Thus $F(R) > 0$ for $0 < \lambda \leq 1/\sqrt{p}$, and (12) gives the claim. □

6.1 The uniform specialization

At $\lambda = 1$ the lower fraction is

$$L_3 = \frac{mn(m^2n^2 + 9mn + 2m + 2n + 20)}{\mathsf{B}},$$
$$\mathsf{B} = m^3n^3 + 10m^2n^2 + 3m^2n + 3mn^2 + 31mn + 10m + 10n + 34,$$

and $F(L_3) = 2\mathsf{P}/[(m+1)(n+1)\mathsf{B}^2]$ with

$$\mathsf{P} = m^3n^3 + 6m^3n^2 - 5m^3n + 2m^3 + 6m^2n^3 + 10m^2n^2 + 44m^2n - 8m^2 - 5mn^3 + 44mn^2 + 87mn + 50m + 2n^3 - 8n^2 + 50n + 132.$$

Its positivity for $m, n \geq 1$ is visible by grouping $6m^3n^2 - 5m^3n$, $6m^2n^3 - 5mn^3$, $44m^2n - 8m^2$ and $44mn^2 - 8n^2$; the other terms are positive. The numerator of $F'(L_3)$ has 36 positive coefficients after shifting m, n by three, and $F'' = 2A_2/\mathcal{D} > 0$ is evident at $\lambda = 1$ since $p \geq s$ for $m, n \geq 2$. Together these give a short proof of the uniform case for $m, n \geq 3$.

7 Finite widths

Corollary 7.1. *If $2 \leq \min(m, n) \leq 6$, every pair of distinct edges of $K_{m,n}$ is strictly negatively correlated for every $\lambda > 0$.*

7.1 Width three

Process the right vertices one at a time and record the connectivity partition of the three left vertices. A set S of left neighbors of a new right vertex is allowed exactly when its members lie in distinct blocks, since an old path between two chosen neighbors would close a cycle through the new vertex; the blocks meeting S are then merged. Choices with $|S| \leq 1$ have total weight $A = 1 + 3\lambda$ and preserve the partition, and the remaining transfer N lowers the number of blocks, so $N^3 = 0$ and

$$(AI + N)^r = A^r I + rA^{r-1}N + \binom{r}{2}A^{r-2}N^2.$$

This counts unrooted forests exactly, isolated right vertices included, and summing over states gives

$$Z_{3,n} = A^n + n(3\lambda^2 + \lambda^3)A^{n-1} + 3n(n-1)\lambda^4 A^{n-2}.$$

There are three orbits of edge pairs: L, sharing a left vertex; R, sharing a right vertex; D, disjoint. Mark representatives $(0,0),(0,1)$, respectively $(0,0),(1,0)$ and $(0,0),(1,1)$, in the first two columns by variables z_1, z_2. Enumerating the 64 neighbor-set choices in the first two columns and applying the transfer to the remaining $r = n-2$ columns gives exact marked polynomials, and differentiating in z_1, z_2 at one gives

$$N_e N_f - Z N_{ef} = c_O \lambda^4 A^{2n-6} Q_O(r, \lambda), \qquad c_L = 2, \quad c_R = r + 1, \quad c_D = 1, \tag{20}$$

where the coefficients of λ^k in Q_O are:

k	Q_L	Q_R	Q_D
0	1	1	1
1	12	12	12
2	$5r+59$	$6r+58$	$7r+58$
3	$36r+154$	$46r+144$	$44r+144$
4	$5r^2+100r+230$	$12r^2+130r+193$	$7r^2+101r+193$
5	$18r^2+132r+192$	$48r^2+162r+132$	$18r^2+102r+132$
6	$18r^2+72r+72$	$9r^3+45r^2+72r+36$	$9r^2+36r+36$

Every coefficient is positive for $r \geq 0$ and $A > 0$, so (20) proves the width-three case.

7.2 Widths four, five and six

Fix $m \in \{4,5,6\}$ and $n \geq m$, and put $A = 1 + m\lambda$, $e = n - m - 1$. The same transfer works on m left vertices: a column is allowed exactly when its neighbor set meets each block at most once, empty and singleton sets have weight A and preserve the state, and every other allowed set lowers the block count, so $T = AI + N$ with $N^m = 0$. Marking a disjoint pair in the first two columns, the marked enumerator has the form $A^e P_m$ by

$$T^{n-2} = \sum_{j=0}^{m-1} \binom{n-2}{j} A^{n-2-j} N^j,$$

so the Rayleigh numerator is $A^{2e} D_m(n-m, \lambda)$ for a polynomial D_m.

The polynomial D_m is computed from the formula of Section 2 as follows. With $a = 1 + n\lambda$, the polynomial $P = A^{-e} Z$ is

$$P = \sum_{k=0}^{m} (-\lambda)^k \binom{m}{k} (n)_k \big[a^{m-k} A^{m+1-k} - \lambda^2 (m-k)(n-k) a^{m-k-1} A^{m-k} \big],$$

the second term omitted when $k = m$. Put

$$J = AP_\lambda + emP, \qquad T_2 = A^2 P_{\lambda\lambda} + 2emAP_\lambda + e(e-1)m^2 P, \qquad L_R = A^{2-e}(W_{abb} + W_{aab}),$$

the derivatives in L_R being taken before specialization. Differentiating $Z = A^e P$ and using (10) gives the polynomial identity

$$\lambda^2 \big[(m-1)(n-1) J^2 - mnP(T_2 - L_R) \big] = A^2 m^2 n^2 (m-1)(n-1)\, D_m(n-m, \lambda). \tag{21}$$

The polynomials D_4, D_5, D_6 have respectively 45, 77 and 117 nonzero coefficients, all positive integers; their degrees in $(\lambda, n-m)$ are $(16,4)$, $(20,6)$ and $(24,8)$, and the coefficient of $\lambda^4 (n-m)^0$ is one. Hence $D_m > 0$ for $n \geq m$ and $\lambda > 0$. At $n = m$ the factor $A^e = A^{-1}$ is harmless: the term $j = m-1$ of the transfer has a zero binomial coefficient, every surviving term of P_m contains A, and $D_m(0, \lambda)$ is divisible by A^2. Together with Proposition 3.2, symmetry and (18), this proves Corollary 7.1.

8 A regular compactification

For the remaining hosts we pass to coordinates in which the deficit becomes a polynomial with bounded coefficients. Put

$$r = \sqrt{mn}, \quad \gamma = \frac{m+n}{r}, \quad \zeta = \frac{1}{\gamma+2}, \quad \varepsilon = \frac{\gamma+2}{r} = \Big(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \Big)^2, \quad b = 1 - 2\zeta.$$

Thus $0 < \zeta \le 1/4$, $\varepsilon \le 4/\min(m,n)$, $r^{-1} = \varepsilon\zeta$ and $\gamma = b/\zeta$. Write $t = \lambda r$, $y = t + t^{-1}$, $D = t^2 - 1$, $L = t + \zeta(t-1)^2$ and $Z_0 = L/t = 1 + \zeta(y-2)$, and define $f(y) = R(r(\gamma + y))/r$, so that the actual ratio is $R = rf(t + t^{-1})$.

Since $R = xP'/P - \min(m,n)$ for $P = L^{(|m-n|)}_{\min(m,n)}$, the Laguerre equation $xP'' + (|m-n| + 1 - x)P' + \min(m,n)P = 0$ gives $xR_x = (x - m - n)R - mn - R^2$, and therefore

$$\varepsilon Z_0 f_y = yf - 1 - f^2, \qquad 0 < f < t^{-1}, \qquad f_y < 0 \qquad (t \ge 1). \tag{22}$$

The upper bound is the partition-function positivity $Z/q = 1 - \lambda R > 0$ at the larger root $\lambda = t/r$, and the sign of f_y follows from (14); at $t = 1$ the argument x still exceeds every Laguerre zero. Set

$$V = \frac{t^{-1} - f}{\varepsilon}, \qquad J_0 = 1 - \varepsilon b + \varepsilon^2\zeta^2 = (1 - 1/m)(1 - 1/n) > 0, \qquad \mathcal{H} = \frac{\zeta\,(A_2R^2 - A_1R + A_0)}{r^2}.$$

The probability deficit is

$$\mathbb{P}_\lambda(e \in F)\mathbb{P}_\lambda(f \in F) - \mathbb{P}_\lambda(e, f \in F) = \frac{\varepsilon^2\zeta^4\mathcal{H}}{J_0 L t^2 V^2}, \tag{23}$$

because the denominator in (12) equals $r^6 J_0 L/\zeta$ and $Z/q = t\varepsilon V$; in particular $V > 0$ for every host. Substitution into (13) cancels all negative powers of ζ and gives the polynomial

$$\mathcal{H} = \sum_{j=0}^{4} \varepsilon^j E_j(V), \tag{24}$$

where

$$\begin{aligned}
E_0 &= L - D^2V,\\
E_1 &= -b^2t - \zeta b(t^2+1) + [b(t^4 - 5t^2) - \zeta(2t^3 + 6t)]V + (t - t^3)V^2,\\
E_2 &= \zeta^2 L + [2\zeta bt + 2b^2t^2 - 2\zeta^2(t^2 + t^4)]V + [\zeta(5t^2 + 4t^4 + t^6) + b(4t^3 + t^5)]V^2,\\
E_3 &= [-2\zeta^3(t + t^3) - 2\zeta^2 bt^2]V + [-\zeta bt^2 + (2\zeta^2 - b^2)t^3 + \zeta bt^4 + 2\zeta^2 t^5]V^2,\\
E_4 &= \zeta^2 t^2 L V^2.
\end{aligned}$$

9 A barrier for every host

We complete the proof of Theorem 1.1. Proposition 3.2 handles incident pairs and Corollary 7.1 all hosts with $\min(m,n) \le 6$, so it remains to treat disjoint pairs with $\min(m,n) \ge 7$. In this section write $a = \varepsilon$, $q = \zeta$ (these a, q, b are unrelated to Section 2), $b = 1 - 2q$, and $F(V) = \mathcal{H}(a, q, t, V)$, holding a, q fixed when differentiating. The parameters of hosts with $\min(m,n) \ge 7$ lie in the admissible region

$$0 < q \le 1/4, \qquad 0 < a \le 4/7, \qquad ab = \frac{1}{m} + \frac{1}{n} \le \frac{2}{7}, \qquad J_0 = 1 - ab + a^2q^2 > 0. \tag{25}$$

By (23) the deficit has the sign of F at the actual V.

9.1 Interval exclusion and a discriminant switch

The continued-fraction tail in (15) is positive, so

$$R > \frac{mn}{x-(m+n-1)}, \qquad f > \frac{1}{t+t^{-1}+aq}, \qquad 0 < V < V_1 := \frac{1+aqt}{at(t^2+1+aqt)}.$$

With $K = (1-ab)t^2 - 2aqt - 1$, substitution gives

$$F(0) = J_0 L > 0, \qquad F(V_1) = -\frac{t^3 J_0 K}{a(aqt+t^2+1)^2}, \qquad \operatorname{disc}_V(F) = K\mathcal{T}, \tag{26}$$

where $\mathcal{T}$ is a polynomial in a, q, t that is positive on (25) for $t \geq 1$ (Section 9.3). Since $1-ab \geq 5/7$ and $K(1) = -a < 0$, K has exactly one zero $t_c > 1$ and changes sign from negative to positive there; explicitly $t_c = (\sqrt{J_0} - aq)^{-1}$, with positive denominator because $J_0 - a^2q^2 = 1 - ab > 0$. For $1 \leq t < t_c$ the discriminant is negative and $F(0) > 0$, so F is positive on the whole line. At $t = t_c$, $F(V_1) = 0$ and the discriminant vanishes; the leading coefficient is $F(0)/V_1^2 > 0$ and the double root is V_1, which the actual $V < V_1$ avoids. Hence the actual F is positive for $1 \leq t \leq t_c$. For $t > t_c$, $F(0) > 0 > F(V_1)$, so F has exactly one root in $(0, V_1)$; we control the derivative of F along the flow at that root.

9.2 The flow and a positive-root coordinate

Equation (22) with $dy/dt = 1 - t^{-2}$ gives, for $t > 1$,

$$aLt^2 V_t = -L + (t^2-1)^2 V + at(t^2-1)V^2.$$

Define

$$P = aLt^2 F_t + F_V\big[-L + (t^2-1)^2 V + at(t^2-1)V^2\big], \tag{27}$$

with F_t taken at fixed V. Along the actual ratio, $aLt^2\, dF/dt = P$. Write $F = AV^2 + BV + C$ and $P = p_3V^3 + p_2V^2 + p_1V + p_0$, and put

$$N_1 = 1 + aqt, \qquad B_0 = at(t^2+1+aqt), \qquad X = \frac{V}{V_1 - V} > 0.$$

The transformed quadratic $B_0^2(1+X)^2 F(N_1X/[B_0(1+X)])$ has coefficients

$$g_0 = CB_0^2, \qquad g_1 = 2CB_0^2 + BN_1B_0, \qquad g_2 = CB_0^2 + BN_1B_0 + AN_1^2 = B_0^2 F(V_1) < 0,$$

so for $t > t_c$ it has exactly one positive root. The transformed cubic $Q(X) = B_0^3(1+X)^3 P(N_1X/[B_0(1+X)])$ has coefficients

$$\begin{aligned} Q_0 &= p_0B_0^3, & Q_1 &= 3p_0B_0^3 + p_1N_1B_0^2, \\ Q_2 &= 3p_0B_0^3 + 2p_1N_1B_0^2 + p_2N_1^2B_0, & Q_3 &= p_0B_0^3 + p_1N_1B_0^2 + p_2N_1^2B_0 + p_3N_1^3. \end{aligned}$$

Reducing Q modulo the quadratic, by replacing X^2 and X^3, leaves the linear remainder $(R_1X + R_0)/g_2^2$ with

$$\begin{aligned} R_1 &= Q_1g_2^2 - Q_2g_1g_2 + Q_3(g_1^2 - g_0g_2), \\ R_0 &= Q_0g_2^2 - Q_2g_0g_2 + Q_3g_0g_1. \end{aligned} \tag{28}$$

These factor as

$$R_1 = a^5 t^9 (aqt+1)^2 (aqt+t^2+1)^2 J_0 LK\, H_L,$$
$$R_0 = -\, a^6 t^{10} (aqt+1)^2 (aqt+t^2+1)^3 J_0^2 LK\, H_C, \tag{29}$$

with polynomials $H_L < 0$ and $H_C > 0$ on the admissible region for $t \geq 1$ (Section 9.3). For $t > t_c$ every other factor is positive, so $R_1, R_0 < 0$. At the root $X > 0$ of the quadratic, therefore, $Q < 0$, and hence $P < 0$.

9.3 Sign certificates

For $-H_L$ and H_C substitute

$$a = \frac{4}{7+u}, \qquad q = \frac{1}{4(1+v)}, \qquad t = 1+s, \qquad u, v, s \geq 0,$$

which covers $0 < a \leq 4/7$, $0 < q \leq 1/4$, $t \geq 1$, and multiply by $(7+u)^{d_a}[4(1+v)]^{d_q}$, where d_a, d_q are the degrees in a, q. The resulting polynomials in u, v, s have 341 and 85 nonzero coefficients, all positive integers, with positive constant terms. For $\mathcal{T}$ use the sharper restriction $ab \leq 2/7$:

$$q = \frac{1}{4(1+v)}, \qquad a = \frac{4(1+v)}{(7+u)(1+2v)}, \qquad t = 1+s,$$

so that $b = (1+2v)/[2(1+v)]$ and $ab = 2/(7+u)$; every admissible pair q, ab arises, with $v = (4q)^{-1} - 1$ and $u = 2/(ab) - 7$. After multiplication by $(7+u)^{d_a}[4(1+v)]^{d_q}(1+2v)^{d_a}$ the expansion has 175 nonzero coefficients, all positive integers, with positive constant. These 601 coefficients prove $H_L < 0$, $H_C > 0$ and $\mathcal{T} > 0$, including at $u = v = s = 0$.

9.4 Conclusion of the proof

Fix a host with $\min(m, n) \geq 7$. Corollary 6.1 gives $F > 0$ at the actual V for $\lambda \geq 1$, that is, $t \geq r$. Suppose $F \leq 0$ at some $t > t_c$. By continuity and the positivity for $t \geq r$, the zeros of F in $[t, r]$ form a nonempty compact set; let t_* be its largest element. Immediately to the right of t_* the values are positive, so $dF/dt \geq 0$ at t_*. But at t_* the actual V is the root of F in $(0, V_1)$, so $aLt^2\, dF/dt = P < 0$, a contradiction. Hence $F > 0$ for all $t > t_c$, and with the previous subsection, for all $t \geq 1$. Corollary 6.1 covers $0 < t \leq 1$, that is, $\lambda \leq 1/\sqrt{mn}$. By (23) every disjoint pair is strictly negatively correlated, and with Proposition 3.2 and Corollary 7.1 this proves Theorem 1.1.

The comparison of a Riccati flow with the zero set of a polynomial along it belongs to the classical theory of Riccati and recurrence comparison for ratios of orthogonal polynomials; see Segura [12].

10 Exact verification

The finite polynomial signs used above are exact statements about explicit polynomials. This section specifies how each is generated from formulas printed in the paper; the script `code/verify.py` carries out every step in exact arithmetic with SymPy and compares the result with the data file `data/certificates.json`.

10.1 Conventions

A polynomial is a finite map from exponent tuples to rational numbers, or to elements $\alpha+\beta\sqrt{3}$ of $\mathbb{Q}(\sqrt{3})$ stored as pairs of rationals. A coefficient $\alpha+\beta\sqrt{3}$ is positive exactly when $\alpha,\beta\geq 0$ are not both zero, or $\alpha>0>\beta$ and $\alpha^2>3\beta^2$, or $\beta>0>\alpha$ and $3\beta^2>\alpha^2$; no decimal approximation is used. A polynomial is certified positive on the nonnegative orthant when every nonzero coefficient is positive and the constant coefficient is positive. For a polynomial $P(x,y,z)=\sum p_{ijk}x^iy^jz^k$ of degrees d_x,d_y, the substitution $x=A/B$, $y=C/D$, $z=1+s$ with positive denominators is cleared as $\sum p_{ijk}A^iB^{d_x-i}C^jD^{d_y-j}(1+s)^k$, the multiplier being exactly $B^{d_x}D^{d_y}$.

10.2 The steps

1. *Identities for q.* The identities (4) and (11), which hold termwise for all p,n, are checked on the finite sums (3) for $0\leq p,n\leq 5$.

2. *The quadratic.* Represent $Aq+Bq_b$ by pairs (A,B). The derivative matrices in the variables a,b,λ, acting on row vectors from the right, are

$$\begin{pmatrix}(m-n)/a & b/a\\ -n/\lambda & b/\lambda\end{pmatrix},\qquad \begin{pmatrix}0 & 1\\ c & d\end{pmatrix},\qquad \begin{pmatrix}n/\lambda & -b/\lambda\\ -bc/\lambda & (n-1)/\lambda-bd/\lambda\end{pmatrix},$$

   with $c=-an/(\lambda b)$ and $d=(ab-\lambda(m-n+1))/(\lambda b)$, and each coefficient pair is also differentiated. Starting from $W=(1+n\lambda)q-\lambda bq_b$, form (9) and (10), specialize $a=1+n\lambda$, $b=1+m\lambda$, $q_b=(n+R)q/b$, and cancel q^2. The result equals (12) with (13).

3. *Large fugacity.* Build L_3 and U_3 by starting the tail at 0, respectively $(m-3)(n-3)\lambda$, and replacing it for $j=2,1,0$ by $(m-j)(n-j)/(x-(m+n-2j-1)-\text{tail})$. With $m=3+u$, $n=3+v$, $\lambda=1+w$ the numerator and denominator supports of $F(L_3)$, $F'(L_3)$, F'' are 208/960, 396/441, 112/75, all coefficients nonnegative integers with positive constants. The width-two formula (18) and the uniform formulas of Section 6, including the 36-term derivative numerator, are checked as identities.

4. *Small fugacity.* With $m=(\sqrt{3}+u)^2$, $n=(\sqrt{3}+v)^2$, $\lambda=[\sqrt{mn}(1+w)]^{-1}$ the supports of $F(U_3)$ and $-F'(0)$ are 844/2468 and 93/454. The identity (19) is checked, $\mathcal{B}$ is cleared by $[\sqrt{mn}(1+w)]^6$, and the negated coefficients of $w^1,\dots,w^6$ have supports $39,46,49,49,49,49$; all these coefficients are strictly positive in $\mathbb{Q}(\sqrt{3})$.

5. *Width three.* A state is a partition of $\{1,2,3\}$. For each new column enumerate the eight subsets of rows, reject a subset with two rows in one block, otherwise merge the selected blocks and multiply by $\lambda^{|S|}$ and by the relevant marks. The two marked columns followed by $T^r=A^{r-2}[A^2I+rAN+\binom{r}{2}N^2]$ give the identities (20) with the printed tables, and $Z_{3,n}$.

6. *Widths four to six.* With $n=m+t$, $\alpha=1+n\lambda$, $A=1+m\lambda$, $e=n-m-1$ and, for the needed i,j,o,

$$W_{ij}^{[o]}=\sum_{k=0}^{m}(-\lambda)^k\binom{m}{k}(n)_k\big[(m-k)_i(n-k)_j\alpha^{m-k-i}A^{m+1-k-j-o}\\ -\lambda^2(m-k)_{i+1}(n-k)_{j+1}\alpha^{m-k-i-1}A^{m-k-j-o}\big],$$

   omitting a summand whose falling factorial in $m-k$ vanishes, set $P=W_{00}^{[0]}$, $J=AP_\lambda+meP$, $T_2=A^2P_{\lambda\lambda}+2meAP_\lambda+m^2e(e-1)P$ and $L_R=W_{12}^{[-2]}+W_{21}^{[-2]}$. Exact division of the left

side of (21) by $A^2m^2(m-1)n^2(n-1)$ has zero remainder and yields D_m, whose supports, degrees, positivity, $[\lambda^4 t^0]D_m = 1$ and divisibility of $D_m(0,\lambda)$ by $(1+m\lambda)^2$ are checked.

7. *Compactification.* The polynomial (24) is reconstructed from (13); the normalization (23), the identity $J_0 = (1-1/m)(1-1/n)$ and the Riccati equation behind (22) are checked.

8. *Barrier.* From $F = \mathcal{H}(a,q,t,V)$ and (27) the script checks (26), the flow equation and the formula for V_1, forms the transformed quadratic and cubic, takes the pseudo-remainder, divides by the prefactors in (29) with polynomial quotients, and generates the three coefficient lists of Section 9.3.

9. *Corroboration.* For $1 \le m,n \le 4$, direct enumeration of all 74,954 edge subsets of the sixteen graphs $K_{m,n}$ confirms the partition function, the degree polynomial (7) and the reduction (12) at $\lambda = 1/2, 1, 2$.

10.3 Output

Running `python code/verify.py` with Python 3.15 and SymPy 1.14 completes in about two minutes and prints:

```
q identities (q_lambda, lambda q_ab, Euler, q_bb) on 36 index pairs PASS
disjoint-edge quadratic F(R) from the finite sum PASS
large regime F(L3): 208/960 nonnegative integer coefficients, positive constants PASS
large regime F'(L3): 396/441 nonnegative integer coefficients, positive constants PASS
large regime F'': 112/75 nonnegative integer coefficients, positive constants PASS
width-two closed form PASS
uniform specialization: L3, F(L3)=2P/[(m+1)(n+1)B^2], 36-term derivative PASS
discriminant identity A1^2-4A2A0 = p^2 L B PASS
small regime F(U3): 844/2468 strictly positive Q(sqrt3) coefficients PASS
small regime -F'(0): 93/454 strictly positive Q(sqrt3) coefficients PASS
small regime: -[w^j]B~, j=1..6, have 39,46,49,49,49,49 strictly positive
  coefficients PASS
width three: marked transfer identities for orbits L, R, D, positive tables,
  Z_{3,n} PASS
width 4: D_4 has 45 positive integer coefficients, degrees [16, 4], [1^4]=1,
  A^2 | D(0) PASS
width 5: D_5 has 77 positive integer coefficients, degrees [20, 6], [1^4]=1,
  A^2 | D(0) PASS
width 6: D_6 has 117 positive integer coefficients, degrees [24, 8], [1^4]=1,
  A^2 | D(0) PASS
compactification: H = sum eps^j E_j, probability normalization, denominator PASS
Laguerre Riccati equation and R = bq_b/q - n = xP'/P - h on 16 index pairs PASS
barrier: F(0)=J0L, F(V1), disc=KT, flow equation, V1 PASS
barrier: pseudo-remainder factorizations R1, R0 PASS
barrier: -H_L cleared has 341 strictly positive integer coefficients PASS
barrier: H_C cleared has 85 strictly positive integer coefficients PASS
barrier: T cleared has 175 strictly positive integer coefficients PASS
direct enumeration: 16 hosts K_{m,n}, 1<=m,n<=4, 74954 edge subsets,
  lambda=1/2,1,2 PASS
total certificate coefficients regenerated: 7172
saved data/certificates.json matches the regenerated data exactly
PASS: all exact certificates for the paper
```

Data availability. The files `code/verify.py`, the checker, and `data/certificates.json`, the coefficient data, accompany this article. Every coefficient list is also determined by the specifications in Section 10, and the checker regenerates all of them from the printed formulas.

Use of AI tools. OpenAI ChatGPT/Codex and Anthropic Claude assisted with literature discovery, proof exploration, drafting, editing and exact verification code. Separate AI-assisted internal checks were performed. The author takes full responsibility for the content.

Prior version. This article revises the complete-bipartite part of a manuscript first posted in the Luminary Research Archive on 24 October 2026 [11].

References

[1] Y. Cao and H. Fu, Negative edge correlation in complete bipartite graphs, arXiv:2610.36749 (2026). `https://arxiv.org/abs/2610.36749`.

[2] R. A. Çiçeksiz and M. Ravichandran, Negative correlations for forests and the $q < 1$ random cluster model, arXiv:2610.00549 (2026). `https://arxiv.org/abs/2610.00549`.

[3] NIST Digital Library of Mathematical Functions, Chapter 18, Orthogonal Polynomials. `https://dlmf.nist.gov/18`.

[4] A. Erickson, *Negative Correlation Properties for Matroids*, M.Math. thesis, University of Waterloo, 2008. `http://hdl.handle.net/10012/4165`.

[5] I. J. Good, Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes, *Proc. Cambridge Philos. Soc.* **56** (1960), 367–380. doi:10.1017/S0305004100034666.

[6] G. R. Grimmett and S. N. Winkler, Negative association in uniform forests and connected graphs, *Random Structures Algorithms* **24** (2004), 444–460. doi:10.1002/rsa.20012.

[7] A. Gupta, Edges of the uniform random forest of K_n are pairwise negatively correlated for every n, arXiv:2609.26134 (2026). `https://arxiv.org/abs/2609.26134`.

[8] X. Huang, On negative correlation of Arboreal Gas for specific parameters, *Statist. Probab. Lett.* **213** (2024), 110174. doi:10.1016/j.spl.2024.110174; arXiv:2311.00965.

[9] J. Kahn, A normal law for matchings, *Combinatorica* **20** (2000), 339–391.

[10] S. Schubr, Stable stars, rank-one replacement and negative edge correlation in random forests, companion manuscript (2026).

[11] S. Schubr, Stable star marginals and strict edge correlations for random forests, Luminary Research Archive, first edition 24 October 2026, edition 4 (2026). `https://research.luminaryoddities.org/degree-laws-and-edge-correlations-for-random-forests/v/4`.

[12] J. Segura, On bounds for solutions of monotonic first order difference-differential systems, preprint (2011), Sections 2 and 4.2. `https://arxiv.org/abs/1110.0870`.

[13] Y. Shmalo, Negative edge correlation for weighted spanning forests of complete graphs, preprint, Zenodo (8 November 2026). doi:10.5281/zenodo.24069903.

[14] D. Stark, The asymptotic number of spanning forests of complete bipartite labelled graphs, *Discrete Math.* **313** (2013), 1256–1261. doi:10.1016/j.disc.2012.10.026.