# Near-touching holes in a Dirichlet screen: the retained-segment heat profile, repulsive contact and the exact static Casimir constant

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November 2026

**Abstract**

As two holes in a Dirichlet screen approach each other, their connected scalar Casimir interaction is governed by a single planar observable: the heat integral $J(t)$ of Brownian bridges that hit both exterior rays of a retained Dirichlet segment of unit length while avoiding the segment. We determine this observable completely and use it to describe the near-touching interaction with and without surface contact. At short time $J(t) \sim t^{5/2}e^{-1/t}/(32\pi^{3/2})$, with its first correction, by a half-line factorization of the relative determinant; at long time $J(t) \to 1/8$ with an inverse-logarithmic expansion. A translation argument expresses $J$ through the gaps in the screen trace of the bridge and gives exact small- and large-gap counting laws. For two interval apertures we compute the static (zero-mode) heat integral in closed form,

$$Q_g^{\mathrm{stat}} = -2\sum_{n\geq 1} \log\tanh\Big(\pi n \frac{K(k')}{K(k)}\Big) = -\tfrac{1}{4}\log(1-q) - \log\frac{2K(k)}{\pi},$$

by mapping the loops that visit both apertures to the two seams of a flat cylinder. This fixes the finite static constant $\log\pi - \frac{1}{4}\log(\ell_-^{-1}+\ell_+^{-1})$ and attributes the entire double logarithm to loops of nonzero winding. We then add a common repulsive contact of strength $\rho_g$ on both apertures: bounded $g\rho_g$ gives a weighted bridge profile; strong contact gives explicit vacuum constants and a thermal sum of squared boundary kernels for curved and finite-type tangencies in every dimension; and for intervals an unbounded contact with $g\rho_g \to 0$ cuts off the static logarithm at length $1/\rho_g$ while retaining its double-logarithmic correction. The gap-scale free energy and both endpoints of its nonzero Matsubara profile are determined. Finally we let the two contact strengths scale independently. When one scaled strength diverges and the other has a finite limit, including zero, we prove the connected vacuum and thermal limits; with a transparent limiting side a causal half-line factorization gives exact momentum and heat formulas and explicit static and vacuum constants in every dimension. We also treat bounded and doubly divergent strengths, and for intervals with both scaled strengths vanishing a single-ray local-time bound gives the logarithmic crossover uniformly in the ratio of the strengths. We extend the strong-contact limits to contact profiles that vary with depth and to strengths vanishing like a power at the contact point, where three regimes appear and a quadratic interior zero produces an Airy coefficient, and we treat two independently strong degenerate contacts. For interval apertures with weak contact we obtain the joint static limit with an exact determinant, and the vacuum-to-temperature crossover. When the strong contact itself vanishes at a power-law zero, the weak static response splits exactly into a singly logarithmic term from the zero and a two-logarithm term from the far depths, with an explicit balance between them, a vacuum response linear in the weak amplitude and a

thermal limit that adds the static zero mode. Finally, at curved quadratic and separable finite-type tangencies in every dimension we determine the leading connected interaction of strong contacts that vanish at prescribed depths, with the gap and both strengths scaling independently: one strong side integrates its normal response against the frozen column of the opposite aperture, and two strong sides pair their two normal response measures against the squared separated kernel. When the zero of a strong contact moves periodically in depth on the scale of its normal confinement width, the leading connected trace per unit tangential area factors into a forced fractional cell response and a separated weak-side factor; the cell response decreases strictly with the ratio $r$ of confinement width to period, from the average of the local vertical responses to the response of the averaged potential, and its small-$r$ loss is quadratic for an interior zero and of exact order $r^2 \log(1/r)$ at the aperture edge, with both coefficients computed explicitly.

**Keywords.** Scalar Casimir effect, Dirichlet screen, near-contact asymptotics, Brownian loop measure, elliptic integrals, Robin contact, Matsubara sum.

## Overview

Sections 1–4 determine the retained-segment profile and the gap-scale free energy of transparent intervals. Sections 5–9 add repulsive contact and prove the exact static formula. Sections 10–13 let the two contact strengths scale independently. Sections 14–17 treat variable and degenerate contact, and Sections 18–21 weak contact and the temperature crossover at interval apertures. Section 22 combines a degenerate strong contact with weak contact, and Sections 27–31 carry degenerate strong contacts to curved tangencies in every dimension. Sections 32–37 treat a strong contact whose zero is periodically corrugated in depth. The near-contact law for transparent windows, of which $J$ is the local kernel, and the separated many-hole interaction are proved in the companion paper [2]; the underlying connected heat trace is that of [1].

## 1 The planar observable

Let $H_-$ and $H_+$ be the two open rays in the horizontal axis outside the retained segment $[-h/2, h/2]$. A planar bridge of duration $t$, generator $\Delta$, and base point $(S, X)$ contributes when it hits the axis in both rays and avoids the retained segment. Write $J_h(t)$ for the integral of this probability against $(4\pi t)^{-1} dS\, dX$, and write $J(t) = J_1(t)$. This is the local two-aperture kernel in the near-contact theorem of [2]. Brownian scaling gives $J_h(t) = J(t/h^2)$. At $h = 0$ the two slit-edge defects give $J_0(t) = 1/8$ [2, 9, 10]. In particular $0 \le J(t) \le 1/8$. Bridges that hit both rays have lateral range at least $h$, giving a Gaussian small-time bound up to a constant by one-dimensional bridge range and reflection estimates.

The interaction of two Dirichlet obstacles occupying the exterior rays involves bridges hitting both rays, as in the whisker geometry of [7, Section 3]. The event defining $J_h$ also excludes paths hitting the central segment. The closest classical comparison for finite apertures is the collinear *Neumann* needle calculation [3, Eq. (15)]. Reflection across the axis in the aperture problem leaves an unchanged odd Dirichlet sector and gives $N$ on the openings and $D$ elsewhere in the even sector. Neumann needles occupying the same intervals leave the even Neumann sector unchanged and yield the same mixed sector in the odd component. Thus the connected classical observable is the same in the two models, by cancellation of the unchanged sectors [5]. The known equal-needle force already gives the first two static logarithmic terms. For infinite rays the complementary Neumann half-plane scattering representation of [4, Eqs. (6)–(7)] contains both displayed positive-real-frequency coefficients of $Q$ in its lowest odd channel. We determine

the complex-frequency expansion needed for time inversion, the bridge gap law, and the nonzero-frequency thermal response, and match the arbitrary-length finite-interval expansion at the heat level.

## 2 Half-line determinant and exact short-time equivalent

**Theorem 1.** *With the principal root $w = \sqrt{z}$,*

$$Q(z) := \int_0^\infty e^{-zt} J(t)\,\frac{dt}{t} = \frac{e^{-2w}}{32\pi w^3}\left(1 - \frac{15}{4w} + O(|w|^{-2})\right)$$

*uniformly for* $\operatorname{Re} z > 0$*, as* $|z| \to \infty$*. Consequently*

$$J(t) = \frac{t^{5/2}}{32\pi^{3/2}} e^{-1/t}\left(1 - \frac{21}{4}t + O(t^2)\right) \qquad (t \downarrow 0).$$

*Proof.* For two bounded transparent apertures the relative determinant identity of [2] identifies their connected heat transform with minus the connected log determinant of the compressed response $2\sqrt{D^2 + z}$, where $D = -i\partial_u$ on $\mathbb{R}$. The factor two cancels. Put $r = |w|$ and $\mu = w/r$, and dilate the tangential coordinates by $r$. The isolated operators become, after removing the common scale factor, the form compression $A_\mu$ of $\sqrt{D^2 + \mu^2}$ to the positive half-line. The form domain is the subspace of $H^{1/2}(\mathbb{R})$ supported there. In coordinates $u, v > 0$ that increase into either ray, the equal off-diagonal blocks $B$ have kernel

$$-\frac{\mu}{\pi(r + u + v)} K_1\big(\mu(r + u + v)\big).$$

This follows by square-root subordination, with the identity term supported on the diagonal, or the Bessel formula in [2]. Uniform accretivity of $A_\mu$ in the form norm follows from $\operatorname{Re}\sqrt{\xi^2 + \mu^2} \geq c\sqrt{1 + \xi^2}$ for $|\arg \mu| \leq \pi/4$. Indeed the modulus of $\xi^2 + \mu^2$ is at least $(1 + \xi^2)/\sqrt{2}$ and the root argument has magnitude at most $\pi/4$.

On the positive axis, pass from bounded intervals to rays by increasing their lengths. The heat side converges by monotone convergence. For the determinant, variational approximation gives strong convergence of the extended inverses, since compact cutoffs approximate in the half-line form space and the restricted forms are uniformly coercive. The cross kernels converge in Hilbert–Schmidt norm by truncation. Uniform boundedness and strong convergence of the inverses and their adjoints preserve this norm convergence under multiplication, by finite-rank approximation. The small norm of $B$ at large $r$ permits the block determinant expansion

$$Q(z) = \sum_{j \geq 1} j^{-1} \operatorname{Tr}[(A_\mu^{-1} B A_\mu^{-1} B)^j].$$

For complex $z$, form the same product in the original unscaled coordinates. The inverse depends holomorphically on $z$ by local coercive form inversion. The separated Bessel kernel is Hilbert–Schmidt holomorphic on the right half-plane by local exponential domination. The scaled norm bounds below make the series holomorphic throughout its connected large-$|z|$ region. The identity theorem extends the equality from its positive real segment.

Set $f_\mu(u) = e^{-\mu u}$. Use the bilinear rank-one kernel $f_\mu(u) f_\mu(v)$. The fixed-order Bessel asymptotic [12, Eq. 10.40.2] gives

$$B = -\frac{\sqrt{\mu}\, e^{-w}}{\sqrt{2\pi}\, r^{3/2}} f_\mu \otimes f_\mu + E, \qquad \|E\|_2 \leq C e^{-\operatorname{Re} w} r^{-5/2}.$$

Indeed for $s = u + v$ the change in the algebraic factor $(1 + s/r)^{-3/2}$ is $O(s/r)$, the Bessel remainder is $O((r+s)^{-1})$, and $|e^{-\mu s}| \le e^{-s/\sqrt{2}}$. Terms with $j \ge 2$ are exponentially smaller. For the first trace it remains to compute $q(\mu) = \int_0^\infty f_\mu(u)(A_\mu^{-1} f_\mu)(u)\, du$.

For positive real $\mu$, the multiplier $L = (\mu + \partial_u)^{1/2}$ factors the full energy as $\|Lf\|_2^2$. Its inverse is convolution by $\mathbf{1}_{s>0} e^{-\mu s}/\sqrt{\pi s}$, which maps half-line $L^2$ into the supported form space. Conversely $L = (\mu + \partial_u) L^{-1}$ preserves half-line support in distributions. Thus $L$ is a bijective form isometry onto half-line $L^2$. The adjoint of the causal inverse maps $f_\mu$ to $f_\mu/\sqrt{2\mu}$. Taking its squared norm gives $q(\mu) = 1/(4\mu^2)$. Local form inversion and the holomorphic exponential extend the bilinear pairing to a sector containing the unit-circle arc $|\arg \mu| \le \pi/4$, and the identity theorem gives the same formula there. The leading trace is $\mu e^{-2w} r^{-3} q(\mu)^2/(2\pi)$. The Hilbert–Schmidt error and the uniform inverse norm give a relative $O(r^{-1})$ contribution since the leading coefficient is nonzero with constant modulus on the arc.

To one further order, the rank-one kernel after removal of its scalar prefactor is

$$f_\mu \otimes f_\mu + r^{-1}\left[\frac{3}{8\mu} f_\mu \otimes f_\mu - \frac{3}{2}(u f_\mu \otimes f_\mu + f_\mu \otimes u f_\mu)\right] + O_2(r^{-2}),$$

where $O_2$ is in Hilbert–Schmidt norm. The next Bessel error and the Taylor remainder of the distance factor are bounded after integration by the same exponential majorant times $C(1 + (u+v)^2)$. The adjoint of the causal inverse applied to $u f_\mu$ gives $(u + 1/(4\mu)) f_\mu/\sqrt{2\mu}$, first for positive real $\mu$. Consequently the pairing of $u f_\mu$ with $A_\mu^{-1} f_\mu$ is $3q(\mu)/(4\mu)$, and again extends holomorphically. The cross term in the first trace gives relative correction $r^{-1}(3/(4\mu) - 6 \cdot 3/(4\mu)) = -15/(4w)$.

The correction in the transform can be inverted term by term. The inverse of $z^{-p} e^{-2\sqrt{z}}$ for $p > 0$ is

$$\frac{1}{\sqrt{\pi}\Gamma(p)} \int_0^t u^{p-1}(t-u)^{-3/2} e^{-1/(t-u)}\, du = \frac{t^{2p-3/2} e^{-1/t}}{\sqrt{\pi}}(1 + p(1/2 - p)t + O(t^2)).$$

The integral is the convolution of the inverse of $z^{-p}$ with that of $e^{-2\sqrt{z}}$. To expand it, put $u = t^2 v$. The first-order factor multiplying $v^{p-1} e^{-v}$ is $1 + t(3v/2 - v^2)$. Taylor expansion on $v \le t^{-1/4}$ bounds the error by $t^2$ times an integrable polynomial against $e^{-v}$. On the omitted tail the exponential $e^{-v/(1-tv)}$ absorbs both the factor $(1 - tv)^{-3/2}$ and any fixed power of $1/t$.

For the transform remainder invert on $z = t^{-2} + iy$. Its vertical tail is integrable, and continuity of $J$, also given by the gap identity below, gives pointwise inversion. The exponent is bounded in modulus after taking out $e^{-1/t}$ by $\exp[-c(yt^2)^2/t]$ for $|yt^2| \le 1$ and by $\exp[-c|yt^2|^{1/2}/t]$ outside. This follows by taking the real part of $\sqrt{1 + iyt^2} - 1$. The factor $|z|^{-5/2} \le t^5$ and integration in $y$ give $O(t^{7/2} e^{-1/t})$ for the inverse remainder. Taking $p = 3/2$ and $p = 2$ in the two leading inverse terms and multiplying by $t$ proves the asserted expansion. □

## 3 Large times and the screen-trace gap law

**Theorem 2.** *Put $a = \gamma - \log 64$, where $\gamma$ is Euler's constant. At large time,*

$$J(t) = \frac{1}{8} - \frac{1}{\log t} - \frac{a}{(\log t)^2} - \frac{a^2 - \pi^2/6}{(\log t)^3} + O((\log t)^{-4}).$$

*The difference $1/8 - J(t)$ has the full inverse-logarithmic expansion of the planar isolated-segment heat defect.*

*Proof.* At duration one, $J_0(1)-J_h(1)$ counts loops hitting the short segment $[-h/2,h/2]$ and both zero-gap rays, up to polar endpoint events. Its difference from the isolated-segment heat defect counts loops hitting the segment while missing at least one zero-gap ray. Missing a zero-gap ray implies avoiding the corresponding exterior ray at gap $h$. The relevant loops therefore hit the disk of radius $h$ about the endpoint of that exterior ray while avoiding the ray. Time reversal and the first-half bridge density bound $p_{1/2}(z,q)/p_1(q,q)\le 2$ let us use the Brownian-motion probability to hit that disk before the slit boundary. In the slit exterior wedge of opening $2\pi$, for radius $r\ge 2h$ this probability is $O((h/r)^{1/2})$. Separation of variables gives terms $\sin(n\theta/2)(h/r)^{n/2}$ with bounded coefficients from the constant-one data on the circular arc. Independently, the free bridge reaching the disk has a Gaussian bound for $r$ above a fixed constant. The geometric mean of the two estimates is integrable over starting points and gives $O(h^{1/4})$. The inner disk has area $O(h^2)$. The other ray gives the same bound.

By scaling, the isolated-segment defect at this duration is the unit-segment defect at time $h^{-2}$, and the missing-ray error becomes $O(t^{-1/8})$. McGillivray's pinned-sausage expansion [6, Theorem 1.2 and Eq. (8.1)], divided by $4\pi t$, is the free-minus-killed heat trace. In its convention $C(K)=2\log\operatorname{cap}(K)$. The conformal map of the unit segment complement to the exterior unit disk has derivative 4 at infinity in the sense that its leading term is $4z$, hence the capacity is $1/4$. The second coefficient is $C(K)+\gamma-\log 4=a$, and the third is $a^2-\pi^2/6$. The arbitrary-order expansion gives the stated remainder and the full transfer. □

Let $B_1,B_2$ be independent unit-duration one-dimensional bridges of generator $\Delta$ from zero to zero. For a fixed initial normal coordinate $X$, define

$$Z_X=\{t: X+B_2(t)=0\},\qquad Y_X=\{B_1(t): t\in Z_X\}.$$

Let $\mathcal{G}_X$ be the bounded complementary intervals of the compact set $Y_X$. Empty and singleton traces have no such gaps. Vary the initial lateral coordinate $S$. The translated trace $S+Y_X$ avoids the retained segment and meets both exterior rays exactly when the segment lies in a translated bounded gap, apart from zero-measure endpoint placements. A gap of length $\ell$ permits lateral shifts of measure $(\ell-h)_+$. Thus

$$4\pi J_h(1)=\int_{\mathbb{R}}\mathbb{E}\sum_{I\in\mathcal{G}_X}(|I|-h)_+\,dX=\int_h^\infty M(u)\,du,\qquad M(u)=\int_{\mathbb{R}}\mathbb{E}\#\{I\in\mathcal{G}_X:|I|>u\}\,dX. \tag{1}$$

Tonelli applies. For a normal bridge that reaches zero, the total gap length is bounded by the lateral bridge range, which is independent of the normal event. The latter has integrable probability $e^{-X^2}$. This proves continuity at $h>0$ by dominated convergence. Planar point polarity and Fubini also give $|Y_X|=0$ almost surely for every $X\neq 0$. At $h=0$, the integrated expected range of $Y_X$ therefore equals $4\pi J_0(1)=\pi/2$.

**Theorem 3.** *The integrated gap count satisfies*

$$M(h)\sim\frac{2\pi}{h\log^2(1/h)}\quad(h\downarrow 0),\qquad M(h)\sim\frac{h^{-4}}{4\sqrt{\pi}}e^{-h^2}\quad(h\to\infty).$$

*For the dimensional coefficient* $D_d=(4\pi)^{-(d-1)/2}(4\sqrt{\pi})^{-1}\int_0^\infty t^{-(d+2)/2}J(t)\,dt$ *of [2],*

$$D_d=\frac{(4\pi)^{-(d+1)/2}}{2\sqrt{\pi}\,d(d+1)}\int_{\mathbb{R}}\mathbb{E}\sum_{I\in\mathcal{G}_X}|I|^{d+1}\,dX,\qquad\int_0^\infty t^{-(d+2)/2}J(t)\,dt\sim\frac{\Gamma(d/2-5/2)}{32\pi^{3/2}}\quad(d\to\infty).$$

*Proof.* By (1), the two secants of the decreasing function $4\pi J_h(1)$ across $h$ bound $M(h)$, since $M$ is decreasing. At small $h$ use endpoints $h/q, h, qh$, with $q > 1$ fixed. In Theorem 2 and $J_h(1) = J(h^{-2})$, the second inverse-log term changes by $O(\log^{-3}(1/h))$; the leading difference across each secant is $\log q/(2\log^2(1/h))$ for $J_h(1)$. Send $q \downarrow 1$ after $h \downarrow 0$. At large $h$, Theorem 1 gives $J_h(1) \sim h^{-5}e^{-h^2}/(32\pi^{3/2})$. With endpoints $h - b/h, h, h + b/h$ for fixed $b > 0$, the ratios of the shifted values to the central value tend to $e^{2b}$ and $e^{-2b}$. Send $b \downarrow 0$ after $h \to \infty$.

For the moment identity, set $h = t^{-1/2}$ and integrate (1), using $2\int_0^\ell h^{d-1}(\ell - h)\,dh = 2\ell^{d+1}/[d(d+1)]$. For large $d$, instead set $v = 1/t$; the large-$v$ integrand is $v^{d/2-7/2}e^{-v}/(32\pi^{3/2})$ up to arbitrarily small relative error, by Theorem 1. The integral on a fixed bounded $v$-interval is negligible at the gamma scale. This proves both claims. □

## 4 Gap-scale free energy of finite intervals

Let the two transparent intervals have fixed lengths $\ell_-, \ell_+ > 0$, with gap $g$ centered at zero. Let $\mathcal{I}_g(t)$ denote their connected all-hit heat gain. At $g = 0$ retain the interval labels and disregard the polar common endpoint. Fix a common unit of length in logarithms. Define

$$\begin{aligned} C_* &= \lim_{L\to\infty}\left[\int_0^{L^2} J(t)\frac{dt}{t} - \frac{1}{4}\log L + \log\log L\right], \\ C_{\ell_-,\ell_+} &= C_* + \int_0^1 (\mathcal{I}_0(t) - 1/8)\frac{dt}{t} + \int_1^\infty \mathcal{I}_0(t)\frac{dt}{t}, \\ N_\lambda &= 2\sum_{n\ge 1} Q((2\pi n/\lambda)^2). \end{aligned}$$

**Theorem 4.** *At inverse temperature $\beta = \lambda g$, locally uniformly for $\lambda > 0$, the connected free energy is*

$$\mathcal{F}_{g,\lambda g} = -\frac{\log(1/g)}{8\lambda g} + \frac{\log\log(1/g)}{2\lambda g} - \frac{C_{\ell_-,\ell_+} + N_\lambda}{2\lambda g} + o(g^{-1}).$$

*The nonzero Matsubara function obeys*

$$\begin{aligned} &N_\lambda \sim \frac{\lambda^3}{128\pi^4}e^{-4\pi/\lambda} && (\lambda \downarrow 0), \\ &N_\lambda = 2\lambda D_1 - \tfrac{1}{4}\log\lambda + \log\log\lambda - C_* + \tfrac{\gamma}{8} - \frac{a+\gamma}{2\log\lambda} + O((\log\lambda)^{-2}) && (\lambda \to \infty). \end{aligned}$$

*Proof.* The bounded intervals are contained in the infinite rays of gap $g$. For small physical times,

$$0 \le J(t/g^2) - \mathcal{I}_g(t) \le Ce^{-c/t}$$

uniformly in sufficiently small $g$. A bridge in the difference visits a ray beyond an outer endpoint and also visits the opposite ray. Its contact-axis range in unit-duration coordinates is at least $\min(\ell_-, \ell_+)/\sqrt{t}$. Integrate over contact-axis starts using the lateral range to bound their measure, over the normal starts by the reflection probability, and then apply the Gaussian range tail. For each fixed $t > 0$, the all-hit event converges as $g \downarrow 0$, outside the zero-probability event of hitting one of the limiting endpoints. Gaussian localization gives convergence of the spatial integral. At large time use the enclosing-window $O(t^{-1})$ bound [2, 1].

Split the zero-mode integral as

$$\int_0^\infty \mathcal{I}_g(t)\frac{dt}{t} = \int_0^{g^{-2}} J(T)\frac{dT}{T} + \int_0^1 [\mathcal{I}_g(t) - J(t/g^2)]\frac{dt}{t} + \int_1^\infty \mathcal{I}_g(t)\frac{dt}{t}.$$

The last two terms converge by the preceding bounds. The first is $\frac{1}{4}\log(1/g) - \log\log(1/g) + C_* + o(1)$ by Theorem 2. In the nonzero modes the substitution $t = g^2T$ gives $\mathcal{I}_g(g^2T) \uparrow J(T)$. The Gaussian bounds on $J(T)$ at zero and on the nonzero-mode sum at infinity give dominated convergence locally uniformly in $\lambda > 0$. The free-energy prefactor is $-1/(2\lambda g)$ [2]. The first nonzero Matsubara pair dominates as $\lambda \downarrow 0$ by Theorem 1, which also bounds the higher-mode tail.

For large $\lambda$, Poisson summation as in [8] gives

$$2\lambda D_1 - N_\lambda = \int_0^\infty J(t)W(t/\lambda^2)\frac{dt}{t}, \qquad W(x) = (4\pi x)^{-1/2} - 2\sum_{n\geq 1} e^{-4\pi^2 n^2 x}.$$

Equivalently $W(x) = 1 - 2(4\pi x)^{-1/2}\sum_{n\geq 1} e^{-n^2/(4x)}$. Thus $k(x) = W(x) - \mathbf{1}_{x<1}$ has all finite absolute logarithmic moments against $dx/x$. Its zeroth moment is $-\gamma$, by

$$\int_0^\infty x^{s-1}W(x)\,dx = -2\Gamma(s)(2\pi)^{-2s}\zeta(2s) \qquad (0 < \operatorname{Re} s < 1/2).$$

Indeed the positive $n$-sum has its termwise Mellin transform for $\operatorname{Re} s > 1/2$. Subtract $(4\pi x)^{-1/2}$ on $(0,1)$ and continue its integral $1/(\sqrt{4\pi}(s-1/2))$. In the indicated strip this term equals minus the integral of the same power over $(1,\infty)$. The constant term at $s = 0$ of the displayed expression is $-\gamma$, since $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. The pole $1/s$ is exactly the Mellin transform of the indicator.

The sharp cutoff contribution is

$$\int_0^{\lambda^2} J(t)\frac{dt}{t} = \tfrac{1}{4}\log\lambda - \log\log\lambda + C_* + \frac{a}{2\log\lambda} + O((\log\lambda)^{-2}).$$

Set $t = \lambda^2 x$ in the difference. For $\lambda^{-1} < x < \lambda$, Theorem 2 gives

$$J(\lambda^2 x) = \frac{1}{8} - \frac{1}{2\log\lambda} + O\Big(\frac{1 + |\log x|}{(\log\lambda)^2}\Big).$$

The bounds on $k$ and $J$ make the omitted tails $O(\lambda^{-1/2})$. Thus the difference is $-\gamma/8 + \gamma/(2\log\lambda) + O((\log\lambda)^{-2})$, giving the stated large-period law. □

For comparison, the noncompact Dirichlet–Neumann boundary four-point function [11, Appendix A] gives the normalized expression

$$(1-q)^{-1/8}[2K(\sqrt{q})/\pi]^{-1/2}, \qquad q = \frac{\ell_-\ell_+}{(\ell_- + g)(\ell_+ + g)},$$

where $K$ is the complete elliptic integral of the first kind. Identifying twice its logarithm with the static heat integral would give $C_{\ell_-,\ell_+} = \log\pi - \frac{1}{4}\log(\ell_-^{-1} + \ell_+^{-1})$. Theorem 4 gives a heat-level definition and proof of the finite limit. Theorem 8 below proves this identification and the stated value of $C_{\ell_-,\ell_+}$.

## 5 Contact and local geometry

Let two bounded apertures in the plane of a Dirichlet screen in $\mathbb{R}^{d+1}$ have gap $g > 0$. In each aperture impose a common repulsive surface form $\rho_g \int |f|^2$, $\rho_g \geq 0$. Across the screen the

openings allow transmission. Reflection leaves an odd full-Dirichlet half-space problem and an even half-space problem with Robin coefficient $\rho_g/2$ on the openings and Dirichlet on the rest. The factor $1/2$ follows from the unitary restriction of an even full-space function to the half-space. It also appears as the ratio of the contact term to the two-arm response $2\sqrt{|D|^2+z}$.

Karabali and Nair [13, Section 3] develop the square-root boundary action for general boundary conditions and apply it to a plate and a parallel half-plate. Fosco and Mazzitelli [14] study variable delta interactions between parallel thin plates. The derivative expansion for facing imperfect mirrors [15] and the curved delta arc in a wedge [16, Section 6] treat different geometries. Frank and Geisinger [17] and Sher [18] analyze large Robin parameters at smooth boundaries, including the scale $\rho\sqrt{t}$ in the heat kernel. We use the connected interaction of apertures in one screen. Inclusion–exclusion cancels the individual aperture terms, and the relevant $\rho_g\sqrt{t}$ scale becomes $g\rho_g$ at the gap diffusion time. The previous transparent near-contact laws [2] answer the approach question of [19, Section V] in several geometries. Here we vary the contact through the competing gap scale and obtain a separate squared-kernel limit when $g\rho_g$ diverges.

Take $s \in \mathbb{R}$ along the gap and $y \in \mathbb{R}^{d-1}$ tangential to it in the screen, with $x$ normal to the screen. Before translating the apertures by $\mp g/2$ in $s$, suppose their closures have unique contact at zero and lie on opposite sides of $s = 0$. For $d \geq 2$, assume bounded convex apertures with facing graphs $s = -a_-(y)$ and $s = a_+(y)$ near zero. The interiors are respectively on the left and right. We use either of the following hypotheses from [2].

- The graphs are $C^2$ with positive definite sum $B_- + B_+$ of their Hessians at zero. Put $\alpha = (d-1)/2$ and $H(w) = 1 + \frac{1}{2}w^{\mathsf{T}}(B_- + B_+)w$.
- For $p_j > 1$, $a_i(y) = \sum_j b_{ij}|y_j|^{p_j} + o(\sum_j |y_j|^{p_j})$ in orthonormal coordinates, where $b_{ij} \geq 0$ and $b_{-j} + b_{+j} > 0$. Set $\alpha = \sum_j p_j^{-1}$ and $H(w) = 1 + \sum_j (b_{-j} + b_{+j})|w_j|^{p_j}$.

In $d = 1$ take fixed interval lengths $\ell_-, \ell_+ > 0$, put $\alpha = 0$ and $H = 1$, and interpret the zero-dimensional integral as evaluation.

Let $\mathcal{I}_{g,\rho}(t)$ be the connected heat gain obtained by the two-opening relative heat trace minus the two one-opening relative traces, each relative to the cut half-spaces. Denote by $L_t$ the normal local time at the plane, in the occupation-density convention for generator $\Delta$. The connected gain is the free diagonal integral of bridges that avoid the retained screen and hit both openings, with weight $e^{-\rho L_t}$ [1]. The weight is common to all surviving subsets, so the same inclusion–exclusion applies. The scalar vacuum and thermal observables in units $\hbar = c = k_{\mathrm{B}} = 1$ are [2]

$$\mathcal{E}_g = -\frac{1}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \mathcal{I}_{g,\rho_g}(t)\, dt, \tag{2}$$

$$\mathcal{F}_{g,\beta} = -\frac{1}{2\beta} \int_0^\infty \mathcal{I}_{g,\rho_g}(t) \sum_{n\in\mathbb{Z}} e^{-t(2\pi n/\beta)^2}\, \frac{dt}{t}. \tag{3}$$

## 6 Planar estimates and bounded scaled contact

Let $J_h^b(t)$ be the planar all-hit gain for the exterior rays of a retained segment of length $h$, with bridge weight $e^{-bL_t}$. For $b = 0$, write $J_h(t)$ and $J(t) = J_1(t)$. The zero-gap rays have $J_0(t) = 1/8$ [2] and Sections 1–4. Brownian scaling gives

$$J_h^b(t) = J_{h/\sqrt{t}}^{b\sqrt{t}}(1) = J_1^{bh}(t/h^2) \quad (h > 0).$$

For later use set $Q_b(z) = \int_0^\infty J_1^b(t) e^{-zt}\, dt/t$ for $z > 0$, also for $z = 0$ when $b > 0$.

At duration one, vary the lateral base point of the ray bridge while keeping the lateral increments and normal path fixed. The available lateral length is bounded by the full lateral bridge range Sections 1–4, which has finite expectation and is independent of the normal bridge. The occupation local time of a generator-$\Delta$ normal unit bridge from $X$ to $X$ satisfies

$$\Pr_X(L_1 > u) = e^{-(|X|+u)^2} \quad (u \ge 0).$$

Indeed the reflected joint bridge density for variance-one Brownian motion with local time above $l$ is $p_1(|x|+|y|+l)$ before conditioning on the endpoint $y$. Divide by $p_1(y-x)$ at $x = y = X/\sqrt{2}$ and use that variance-two semimartingale local time is twice its occupation local time. Integration over $X$ gives $2e^{-u^2}\, du$ for the positive part of the local-time density. Hitting the line and acquiring positive local time are almost surely the same event for these bridges. The bound on the lateral range and $1 - e^{-s} \le s$ give

$$J_h^b(t) \le C \min(1, (b\sqrt{t})^{-1}), \qquad 0 \le J_h(t) - J_h^b(t) \le C \min(1, b\sqrt{t}). \tag{4}$$

These estimates are uniform in $h \ge 0$ and apply to the respective interval observables by event inclusion. The separate Gaussian estimate $J_1(T) \le Ce^{-\eta/T}$ at $T \le 1$ follows from the lateral range needed to meet both rays.

Set $c_g = g\rho_g$. For $b \ge 0$ define

$$D_d(b) = \frac{(4\pi)^{-(d-1)/2}}{4\sqrt{\pi}} \int_0^\infty T^{-(d+2)/2} J_1^b(T)\, dT.$$

**Theorem 5.** *If $c_g \to c < \infty$, then*

$$g^{d-\alpha} \mathcal{E}_g \longrightarrow - \int_{\mathbb{R}^{d-1}} H(w)^{-d} D_d(cH(w))\, dw.$$

*For $d \ge 2$, and also $d = 1$ when $c > 0$, one has locally uniformly for $\lambda > 0$*

$$g^{d-\alpha} \mathcal{F}_{g,\lambda g} \longrightarrow -\frac{(4\pi)^{-(d-1)/2}}{2\lambda} \int_{\mathbb{R}^{d-1}} \int_0^\infty T^{-(d+1)/2} J_{H(w)}^c(T) \sum_{n \in \mathbb{Z}} e^{-T(2\pi n/\lambda)^2}\, dT\, dw. \tag{5}$$

*Proof.* For $t = g^2 T$, dilation in the two fast bridge coordinates changes the weight to $e^{-c_g L_T}$. The slow tangential freezing in [2, proof of the near-contact theorem] uses the uniform continuity of a scaled bridge and point polarity at the two limiting endpoints. Multiply the converging all-hit indicators by this bounded weight, which converges pointwise for fixed $T$. Every spatial and time majorant of the transparent proof still applies, since the weight is at most one. For the additional $d = 1, c > 0$ zero mode use (4) at large $T$ with $c_g$ bounded below. The small-time Gaussian bound and the Matsubara sum control the other terms uniformly for $\lambda$ in compact positive sets. For the vacuum integral use $J_h^c(T) = J_1^{ch}(T/h^2)$ and substitute $T = h^2 u$. This gives the factor $h^{-d}$ and the asserted result. The limiting integrals are strictly positive, by a positive-probability bridge tube that forces two visits and by strict positivity of the local-time factor. □

## 7 Strong scaled contact

Write $n_{\kappa^2}$ for the off-diagonal kernel of $2\sqrt{|D|^2 + \kappa^2}$ in dimension $d$, in the convention of [2]. Define, for $h > 0$,

$$V_d(\kappa, h) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{d-1}} |n_{\kappa^2}(h + u + v, x)|^2\, dx\, du\, dv, \qquad A_d = \frac{2\Gamma(d/2+1)^2 \Gamma(d/2+2)}{\pi^{d/2+2} \Gamma(d+4)}.$$

**Theorem 6.** *If $c_g \to \infty$, then*

$$g^{d-\alpha} c_g^2 \mathcal{E}_g \longrightarrow -A_d \int_{\mathbb{R}^{d-1}} H(w)^{-d-2}\, dw,$$
$$g^{d-\alpha} c_g^2 \mathcal{F}_{g,\lambda g} \longrightarrow -\frac{1}{2\lambda} \int_{\mathbb{R}^{d-1}} \sum_{n\in\mathbb{Z}} V_d(2\pi|n|/\lambda, H(w))\, dw.$$

*The second limit is locally uniform for $\lambda > 0$. The static kernel integral is*

$$V_d(0,h) = \frac{4\Gamma((d+1)/2)^2\Gamma((d+3)/2)}{\pi^{(d+3)/2}\Gamma(d+3)}\, h^{-d-1}.$$

*In particular $A_1 = 1/(64\pi)$ and $V_1(0,1) = 2/(3\pi^2)$.*

*Proof.* Let $\gamma_* = d - 1 - \alpha$ and let $K_{i,g}$ be the apertures dilated by $1/g$ in the screen. At scaled momentum $\kappa$ the relative response, after removing a common factor, has isolated diagonal forms $A_{i,g} + c_g$, with $A_{i,g}$ the compression of $2\sqrt{|D|^2 + \kappa^2}$. The cross block $B_g$ acts by $n_{\kappa^2}$ between the two scaled domains. Their separation is at least one. Subordination gives

$$|n_{\kappa^2}(x)| \le C|x|^{-d-1} e^{-b\kappa|x|} \quad (|x| \ge 1).$$

Differentiating its integral gives $|\nabla n_{\kappa^2}(x)| \le C_\kappa |x|^{-d-1}$ there for fixed $\kappa$. The Schur test on $|x - y| \ge 1$ also gives $\|B_g\| \le Ce^{-b\kappa}$ uniformly in $g$.

In the facing charts use the physical tangential coordinates $y_j = g^{1/p_j} w_j$ and $y' = y + gx$, with $p_j = 2$ in the quadratic case. The two inward depths are $gu, gv$. The pair separation divided by $g$ is $(1 + (a_-(y) + a_+(y'))/g + u + v, x)$, which tends to $(H(w) + u + v, x)$. In the finite-type case $a_+$ is convex and $a_+(y) = o(|y|)$. Its Lipschitz constants on shrinking centered neighborhoods tend to zero: bound its slope in any direction between points in a small ball by its oscillation on a concentric ball of twice the radius divided by that radius. This gives $(a_+(y') - a_+(y))/g \to 0$ at fixed chart variables. For a fixed local chart, bounded Lipschitz constants and $a_-(y) + a_+(y) \gtrsim \sum_j |y_j|^{p_j}$ give a lower bound for the scaled pair distance by

$$C^{-1}\left(1 + u + v + |x| + \sum_j |w_j|^{p_j}\right).$$

In fact the nonnegative graph heights give the bound in $u, v$, the lateral difference gives the bound in $|x|$, and replacing $a_+(y')$ by $a_+(y)$ costs at most $Cg|x|$.

The local Jacobian is $g^{d+1+\alpha}$ for the two physical variables together, hence $g^{-\gamma_*}$ in the scaled aperture spaces. The aperture indicators tend to one at fixed positive $u, v$ by the local interior graph description. The squared radial decay exponent $2(d+1)$ exceeds the effective volume exponent $d + 1 + \alpha$ of the chart variables. Outside the local charts compactness and unique contact give a positive physical distance lower bound. This part gives $O(g^2 e^{-b\kappa})$ in the squared scaled kernel integral, using the bounded physical volumes. Dominated convergence gives

$$g^{\gamma_*}\|B_g\|_2^2 \longrightarrow \int_{\mathbb{R}^{d-1}} V_d(\kappa, H(w))\, dw, \qquad g^{\gamma_*}\|B_g\|_2^2 \le Ce^{-b\kappa}. \tag{6}$$

It remains to control the diagonal inverse in this same limit. Put $R_{i,g} = c_g(A_{i,g} + c_g)^{-1}$, a positive contraction. For either output side of the cross block, multiply its kernel in the output variable by

$$\chi_{i,g,\delta}(x) = \min(1, \operatorname{dist}(x, K_{i,g}^c)/\delta), \qquad \delta > 0.$$

At a fixed chart tuple with positive output depth, the scaled boundary distance is bounded below by a positive constant times that depth for small $g$, by the graph's bounded Lipschitz constant. The majorant for (6) therefore shows the removed kernel has negligible normalized squared Hilbert–Schmidt norm when $g \downarrow 0$ and then $\delta \downarrow 0$. The cutoff columns have zero extensions in $H^1$. Their normalized integrated squared gradients are bounded by $C_{\kappa,\delta}$ by differentiating the kernel in the output variable and using the bound on the cutoff gradient. The normalized integrated $A_{i,g}$ form of these columns is bounded by the same type of constant, since $2\sqrt{|\xi|^2+\kappa^2} \le C_\kappa(1+|\xi|^2)$. The spectral bound

$$(I - R_{i,g})^2 \le c_g^{-1} A_{i,g}$$

then shows that the action of $I - R_{i,g}$ on the cutoff columns is negligible in the normalized Hilbert–Schmidt norm. First send $g \downarrow 0$ and then $\delta \downarrow 0$, using contractivity for the removed part.

Define $q_g(\kappa) = \int_0^\infty e^{-\kappa^2 T} \mathcal{I}_{g,\rho_g}(g^2 T)\, dT/T$. The pair relative determinant identity from [2] expands this as a positive trace logarithm. Its first term times $c_g^2$ is $\mathrm{Tr}(R_{-,g} B_g R_{+,g} B_g^*)$. The preceding cutoff bound and Hilbert–Schmidt Cauchy–Schwarz show its difference from $\|B_g\|_2^2$ is $o(g^{-\gamma_*})$. After multiplication by $c_g^2$, the remaining trace series is bounded by $C c_g^{-2} \|B_g\|_2^2 \|B_g\|^2$, by bounding all but the first normalized cross-pair factor in operator norm. Thus $g^{\gamma_*} c_g^2 q_g(\kappa)$ has the pointwise limit in (6) and is bounded by $Ce^{-b\kappa}$. It is a decreasing function of $\kappa$ by the positive heat representation. Continuity of the limit and a finite mesh give uniform convergence on compact $\kappa$-intervals. The exponential bound controls the upper tails when integrating in $\kappa$ or summing Matsubara frequencies. In (2) the corresponding momentum formula is $-(2\pi g)^{-1} \int_0^\infty q_g(\kappa) d\kappa$; in (3) the prefactor at $\beta = \lambda g$ is $-(2\lambda g)^{-1}$. Both have scaling $g^{\alpha-d}$ once $g^{-\gamma_*}$ is included.

Tangential Plancherel gives

$$V_d(\kappa, h) = (2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} V_1(\sqrt{\kappa^2 + |k|^2}, h)\, dk, \qquad V_1(\kappa, h) = h^{-2} V_1(h\kappa, 1).$$

The planar kernel is $-2\kappa K_1(\kappa r)/(\pi r)$ at separation $r$. Radially integrating the $\kappa, k$ variables gives the vacuum factor $h^{-d-2}$ times

$$\frac{|S^{d-1}|}{2(2\pi)^d} \frac{4}{\pi^2} \int_0^\infty \frac{s\, ds}{(1+s)^{d+4}} \int_0^\infty q^{d+1} K_1(q)^2\, dq.$$

The first integral is $1/[(d+2)(d+3)]$. The second is $2^{d-1}\Gamma(d/2+1)^2\Gamma(d/2)\Gamma(d/2+2)/\Gamma(d+2)$. To see this, use $K_1(q) = \frac{q}{4}\int_0^\infty x^{-2} e^{-x-q^2/(4x)} dx$ twice and integrate in $q$ first. In the remaining two variables $x, y$, substitute $x+y$ and $x/(x+y)$. Their gamma and beta integrals give the stated moment. This yields $A_d$. At $\kappa = 0$ the kernel is $-2\Gamma((d+1)/2)\pi^{-(d+1)/2}|x|^{-d-1}$. Integrate its square over the $d-1$ tangential coordinates first, by the radial beta integral, and then integrate $\int_0^\infty s(h+s)^{-d-3} ds = h^{-d-1}/[(d+1)(d+2)]$. This proves the static formula. □

In Theorems 5 and 6, the leading law at fixed $\beta > 0$ is the vacuum law. Indeed the scaled momentum step is $\Delta = 2\pi g/\beta$. For a positive decreasing function, the half-weighted mesh sum after multiplication by $\Delta$ differs from its integral by at most $\Delta$ times its value at zero. Use $g^{\gamma_*} c_g^2 q_g$ in the strong case, bounded at zero by (6). In the bounded case use $g^{\gamma_*} q_g$. Its value at zero is bounded by the transparent Matsubara sum at the scaled period one, which is $O(1)$ in $d \ge 2$ and $O(\log(1/g))$ in $d = 1$ by [2]. Multiplication by $\Delta$ makes the error negligible in both cases. Positivity of the limiting vacuum constants permits comparison of equivalents.

## 8 Interval zero mode with weak scaled contact

For $d = 1$, write

$$Z_g = \int_0^\infty \mathcal{I}_{g,\rho_g}(t)\,\frac{dt}{t}, \qquad N_\lambda = 2\sum_{n\geq 1} Q_0((2\pi n/\lambda)^2).$$

At $g = 0$ retain the two interval labels, with the common endpoint omitted as a polar set. In fixed units of length, set

$$\begin{aligned} C_* &= \lim_{L\to\infty}\left[\int_0^{L^2} J(T)\frac{dT}{T} - \frac{1}{4}\log L + \log\log L\right], \\ C_R &= \int_0^1 [J_0^1(s) - 1/8]\frac{ds}{s} + \int_1^\infty J_0^1(s)\frac{ds}{s}, \\ C_{\ell_-,\ell_+} &= \log\pi - \tfrac{1}{4}\log(\ell_-^{-1} + \ell_+^{-1}). \end{aligned}$$

The constant $C_*$ exists by the retained-segment heat expansion Sections 1–4; $C_R$ is finite by (4).

**Theorem 7.** *Suppose $d = 1$ and $c_g \to 0$. If $\rho_g \to \rho < \infty$, then*

$$\begin{aligned} Z_g = \tfrac{1}{4}\log(1/g) - \log\log(1/g) + C_{\ell_-,\ell_+} \\ + \int_0^\infty [\mathcal{I}_{0,\rho}(t) - \mathcal{I}_{0,0}(t)]\,\frac{dt}{t} + o(1). \end{aligned}$$

*If $\rho_g \to \infty$, then*

$$Z_g = \tfrac{1}{4}\log(1/c_g) - \log\log(1/c_g) + C_* + C_R + o(1).$$

*In both cases*

$$\mathcal{F}_{g,\lambda g} = -\frac{Z_g + N_\lambda + o(1)}{2\lambda g}$$

*locally uniformly for $\lambda > 0$. In particular $\rho_g = g^{-\sigma}$ with $0 < \sigma < 1$ reduces the coefficient of $\log(1/g)$ in $Z_g$ to $(1-\sigma)/4$.*

*Proof.* If the physical coefficients are bounded, subtract the transparent finite-interval integral. At fixed $t > 0$ the gain difference converges by polarity at the limiting endpoints and continuity of the weight in $\rho_g$. The small-time bound in (4) is $O(\sqrt{t})$ for this difference. At large time use the fixed enclosing transparent window's $O(t^{-1})$ heat gain [1, 2]. Dominated convergence gives the stated finite shift. The transparent constant follows from Theorem 8 below.

If $\rho_g \to \infty$, truncate the infinite rays to the actual intervals. The unweighted heat loss at small physical time is bounded by $Ce^{-b/t}$, uniformly for small $g$, since a lost all-hit bridge reaches beyond an outer endpoint and must have lateral range at least $\min(\ell_-, \ell_+)$. The weighting can only decrease this loss, and (4) also bounds it by $C/(\rho_g\sqrt{t})$. Use dominated convergence below $t = 1$ and integrate directly above $t = 1$ to get $Z_g = Q_{c_g}(0) + o(1)$.

For $c > 0$ split that ray integral at $T = c^{-2}$. Insert the transparent ray integral below the cutoff and rescale the resulting difference by $s = c^2 T$. This gives the exact identity

$$Q_c(0) = \int_0^{c^{-2}} J(T)\frac{dT}{T} + \int_0^1 [J_c^1(s) - J_c(s)]\frac{ds}{s} + \int_1^\infty J_c^1(s)\frac{ds}{s},$$

where the subscripts $c$ on $J_c^1, J_c$ denote the retained segment lengths. By point polarity, as $c \downarrow 0$ the last two integrands converge to the corresponding zero-gap ones; (4) supplies integrable majorants. The first integral equals the two logarithms plus $C_* + o(1)$ by its definition.

Finally at each fixed $T > 0$, the expanding intervals in scaled coordinates and $c_g \to 0$ give $\mathcal{I}_{g,\rho_g}(g^2T) \to J(T)$. Insert this in the nonzero Matsubara modes. The unweighted small-time Gaussian bound and the exponential large-$T$ Matsubara bound give a majorant locally uniformly in $\lambda$. This proves the last formula. □

# 9 Exact transparent static calibration

Let the transparent interval endpoints be $a < b < c < d_0$, and put

$$q = \frac{(b-a)(d_0-c)}{(c-a)(d_0-b)}, \qquad k = \sqrt{q}, \quad k' = \sqrt{1-q}.$$

Section 4 compared the finite interval constant to a boundary four-point expression from [11, Appendix A]. The following argument fixes the normalization in the aperture heat observable. The classical Neumann needle geometry has the same connected static observable by reflection parity (Section 1) [3]. In that geometry [3, Eq. (15)] already gives the equal-length leading and double logarithms. Dubédat's crosscut transfer formula [20, Section 2.2] also gives the cylinder product below, since the transfer between the seams has eigenvalues $\operatorname{sech}(\pi n W/H_0)$. We derive it directly from the connected heat trace to track its winding classes.

**Theorem 8.** *For transparent intervals the zero-mode heat integral is exactly*

$$Q_g^{\mathrm{stat}} = -2\sum_{n\ge1} \log\tanh\!\left(\pi n\,\frac{K(k')}{K(k)}\right) = -\frac{1}{4}\log(1-q) - \log\frac{2K(k)}{\pi}.$$

*Thus its finite constant as $g \downarrow 0$ is $C_{\ell_-,\ell_+}$ above. In the decomposition by cylinder winding, the nonzero class of signed winding $m$ has mass $1/[|m|(e^{s|m|}-1)]$ with $s = 2\pi K(k')/K(k)$. The zero-winding all-hit mass is $2\sum_{n\ge1}\log(1+e^{-sn})$. The nonzero classes give the entire double logarithm.*

*Proof.* Let $\Omega$ be the plane cut along the screen retained outside both intervals. By the killed heat diagonal and its conditional bridge, the measure $dt/t$ times that diagonal and bridge law is the Brownian loop measure of $\Omega$ after forgetting the root. The all-hit static integral is its mass of loops visiting both apertures. The generator normalization agrees with [21] because changing from $\Delta/2$ to $\Delta$ simply changes the integration variable in $dt/t$. The event is independent of the time parametrization, so the conformal invariance of the unrooted loop measure applies [21, Proposition 6], including for the cylindrical surface [22, Theorem 2.2].

A real Möbius map sends $a, b, c, d_0$ cyclically to $0, q, 1, \infty$. The Schwarz–Christoffel integral in the upper half-plane has derivative proportional to $[z(z-q)(z-1)]^{-1/2}$. Its rectangle has its opposite aperture sides of height $H_0 = 2K(k)$ and its gap side of width $W = 2K(k')$ [12, Section 19.32]. These lengths follow by substituting $z = q\sin^2\theta$ on $(0, q)$ and $z = 1-(1-q)\sin^2\theta$ on $(q, 1)$. Schwarz reflection across either open aperture maps the lower half-plane to a reflected rectangle. The two reflections differ by the translation $2W$ across the pair of parallel aperture sides. Taking this horizontal coordinate modulo $2W$ therefore gives one conformal map from $\Omega$ to the flat cylinder $(\mathbb{R}/2W\mathbb{Z}) \times (0, H_0)$, with two seams $x = 0, W$. The two half-planes map to the two half-cylinders. The remaining sides give the two Dirichlet circles, and the four endpoints lie on their boundary.

Cutting the cylinder at one seam gives a rectangle of width $2W$; cutting both gives two rectangles of width $W$. Inclusion–exclusion on the loop event separates the common vertical

Dirichlet heat trace. Horizontally, with $P_L$ the period-$L$ and $D_L$ the Dirichlet-interval length-$L$ traces, the factor is

$$P_{2W}(t) - 2D_{2W}(t) + 2D_W(t) = \frac{4W}{\sqrt{4\pi t}} \sum_{m\in\mathbb{Z},\ m\ \text{odd}} e^{-m^2W^2/t}.$$

This is the image sum for the circle and the Poisson formula $D_L(t) = L(4\pi t)^{-1/2}\sum_m e^{-m^2L^2/t} - 1/2$. Multiply by the vertical sum $\sum_{n\ge1} e^{-(\pi n/H_0)^2 t}$. Integration using $\int_0^\infty t^{-3/2}e^{-A^2/t-B^2t}dt = \sqrt{\pi}e^{-2AB}/A$ gives $4\sum_{n\ge1}\sum_{m\ge1,\ m\ \text{odd}} e^{-snm}/m$. The series converges for $s>0$ and summing in $m$ yields the first formula. With nome $p = e^{-s/2}$, Jacobi's products [12, Sections 20.4 and 20.9] give

$$\prod_{n\ge1}\frac{1+p^{2n}}{1-p^{2n}} = [\vartheta_3(0,p)\vartheta_4(0,p)]^{-1/2} = (1-q)^{-1/8}[2K(k)/\pi]^{-1/2}.$$

The first equality follows by collecting the even and odd factors; the second uses $2K(k)/\pi = \vartheta_3^2$ and $k' = \vartheta_4^2/\vartheta_3^2$. This proves the elliptic expression.

Every loop of horizontal winding $m \neq 0$ visits both seams. Select just the $m$-th periodic image in the uncut cylinder before integrating. Its coefficient is $2W$ instead of $4W$, and summing the vertical factor gives $1/[|m|(e^{s|m|}-1)]$, in agreement with the general annular formula [22, Eq. (4.3)]. Summing the nonzero classes gives $-2\log P(s)$ with $P(s) = \prod_{n\ge1}(1-e^{-sn})$. Subtraction from the first formula leaves $2\log[P(2s)/P(s)]$ for the zero-winding loops. Put $L = \log(16/(1-q))$. Then $s = 2\pi^2/L + O((1-q)/L)$ by the elliptic endpoint expansion. The eta transformation [12, Chapter 23] gives

$$\log P(s) = -\frac{\pi^2}{6s} + \tfrac{1}{2}\log(2\pi/s) + \frac{s}{24} + O(e^{-4\pi^2/s}).$$

Consequently the contributions near contact are

$$Q_{\text{wound}} = \frac{L}{6} - \log L + \log\pi - \frac{s}{12} + o(L^{-1}), \qquad Q_{\text{zero}} = \frac{L}{12} - \log 2 + \frac{s}{12} + o(L^{-1}).$$

This gives the winding claim. Finally $1-q = g(\ell_-^{-1}+\ell_+^{-1}) + O(g^2)$ and $2K(k) = \log(16/(1-q)) + O((1-q)|\log(1-q)|)$. Inserting these in the exact heat integral gives $C_{\ell_-,\ell_+}$. □

For clarity, the transparent cylinder result concerns the static heat integral. Positive contact would pull back to a spatially varying coefficient along the seams under the same conformal map. Under zero contact, put $A = \ell_-^{-1}+\ell_+^{-1}$ and $L_g = \log(1/g)$. The elliptic expression refines the static expansion by the next term $-\log(16/A)/L_g + O(L_g^{-2})$ in $Q_g^{\text{stat}}$. The nonzero-mode discrepancy with $N_\lambda$ is exponentially small in $1/g$ for $\lambda$ in compact positive sets. Indeed at small physical time $t$ it is bounded in the heat integral by $Ce^{-b/t}\sum_{n\neq0} e^{-t(2\pi n/(\lambda g))^2}$, and at $t \ge 1$ the sum itself is exponentially small. Splitting the small-time exponential in half and using $b/t + t(2\pi n/(\lambda g))^2 \ge c|n|/g$ gives the claimed bound after integration with $dt/t$. Thus the next term in $\mathcal{F}_{g,\lambda g}$ beyond that of Sections 1–4 is $+\log(16/A)/(2\lambda g L_g)$, with remainder $O(1/(gL_g^2))$ locally uniformly in $\lambda > 0$.

## 10 Independent contact strengths: geometry and observables

We take the near-contact geometry from [2] and Sections 5–9. Two bounded convex apertures lie in opposite closed half-spaces of a $d$-dimensional screen, with unique contact of their closures

at the origin before translation. Their facing graphs are $s = -a_-(y)$ and $s = a_+(y)$, where $s$ is the gap direction and $y \in \mathbb{R}^{d-1}$ is tangential to the contact. The interiors lie on the outer sides of the respective graphs. Translate the apertures by $\mp g/2$ along $s$. For a quadratic contact the two graphs are $C^2$ and $B_- + B_+$ is positive definite, with $B_i$ their Hessians at zero. Put $\alpha = (d-1)/2$ and $H(w) = 1 + \frac{1}{2} w^{\mathsf{T}}(B_- + B_+)w$. For separable finite type assume $a_i(y) = \sum_j b_{ij}|y_j|^{p_j} + o(\sum_j |y_j|^{p_j})$ in orthonormal coordinates, where $p_j > 1$, $b_{ij} \geq 0$, and $b_{-j} + b_{+j} > 0$. Then $\alpha = \sum_j p_j^{-1}$ and $H(w) = 1 + \sum_j (b_{-j} + b_{+j})|w_j|^{p_j}$. For $d = 1$ use intervals of fixed lengths $\ell_-, \ell_+ > 0$, with $\alpha = 0$, $H = 1$, and zero-dimensional integration interpreted as evaluation.

Write $\rho_{i,g} \geq 0$ for the constant repulsive coefficient added to the aperture form on side $i$, and $c_{i,g} = g\rho_{i,g}$. The connected heat gain $\mathcal{I}_g(t)$ is the two-opening relative heat trace minus the two one-opening relative heat traces, each relative to the cut half-spaces. It is the all-hit integral of bridges of generator $\Delta$ that avoid the retained screen and visit both apertures, with weight $\exp[-\sum_i \rho_{i,g} L_t^i]$ [1]. Here $L_t^i$ is the normal occupation local time accumulated at hits in aperture $i$. Define the gap-scaled positive transform

$$q_g(\kappa) = \int_0^\infty e^{-\kappa^2 T} \mathcal{I}_g(g^2 T)\, \frac{dT}{T}, \qquad \kappa \geq 0.$$

The vacuum energy is $-(2\pi g)^{-1} \int_0^\infty q_g(\kappa) d\kappa$, and the free energy at inverse temperature $\beta$ is $-(2\beta)^{-1} \sum_{n \in \mathbb{Z}} q_g(2\pi|n|g/\beta)$ [2]. In gap coordinates the diagonal boundary response is the form compression of $2\sqrt{|D|^2 + \kappa^2}$ plus $c_{i,g}$, and the off-diagonal kernel is that of the two-arm square root. The normalized pair determinant equals the positive transform $q_g$ after taking minus the logarithm.

The common-strength theory Sections 5–9 uses a trace of two separated kernels for large contact. With unequal strengths there is a regime in which only one diagonal inverse tends to a scalar. Mixed-boundary half-plane Green functions [23, 10] and the diffraction basis for the massive half-line square root [19, Appendix A.1] are antecedents for the flat profile. Explicit Dirichlet Green functions for fractional half-space powers [24] describe the massless local inverse, while small Robin patches on a fixed boundary [25] give a different perturbation of a local potential. The problem here is a connected trace on varying apertures, including an unbounded inverse on the remaining transparent side. We estimate its cross-kernel sandwich, track the collapsing geometry, and evaluate the resulting flat coefficient when that side is transparent.

## 11 One strong contact

Let $A_\kappa$ be the positive-ray compression of $2\sqrt{D^2 + \kappa^2}$ in one tangential dimension, and let $B_{\kappa,h}$ on two inward half-lines have symmetric kernel

$$-\frac{2\kappa K_1(\kappa(h + x + y))}{\pi(h + x + y)}, \qquad x, y > 0, \quad h > 0.$$

At $\kappa = 0$ the kernel is $-2/[\pi(h + x + y)^2]$. Put

$$W_{1,b}(\kappa, h) = \mathrm{Tr}\big[B_{\kappa,h}(A_\kappa + b)^{-1} B_{\kappa,h}^*\big], \tag{7}$$

$$W_{d,b}(\kappa, h) = (2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} W_{1,b}\big(\sqrt{\kappa^2 + |k|^2}, h\big)\, dk. \tag{8}$$

At $\kappa = b = 0$, the inverse in (7) is understood through the form-dual variational supremum. Its trace is finite, as shown below.

**Theorem 9.** *Suppose $c_{-,g} \to \infty$ and $c_{+,g} \to b \in [0,\infty)$. Then, with $\gamma_* = d-1-\alpha$,*

$$g^{\gamma_*} c_{-,g} q_g(\kappa) \longrightarrow \int_{\mathbb{R}^{d-1}} W_{d,b}(\kappa, H(w))\, dw.$$

*The convergence is uniform on compact subsets of $[0,\infty)$, with a common exponential upper bound at infinity. Hence the vacuum limit after multiplication by $g^{d-\alpha} c_{-,g}$ is minus the integral of the displayed limit against $d\kappa/(2\pi)$. At $\beta = \lambda g$ the same normalization gives minus its Matsubara sum with prefactor $1/(2\lambda)$, locally uniformly for $\lambda > 0$. For fixed $\beta > 0$ the free energy has the vacuum equivalent.*

*For $b = 0$ the planar expression and the static value in all dimensions are*

$$W_{1,0}(\kappa, h) = \frac{1}{2\pi} \int_h^\infty (r-h) r^{-3} e^{-2\kappa r}\, dr,$$
$$W_{d,0}(0,h) = \frac{\Gamma(d/2)}{2^d \pi^{(d+2)/2} d(d+1)}\, h^{-d}.$$

*The vacuum limit is then $-A_d^{(1)} \int H(w)^{-d-1} dw$, where*

$$A_d^{(1)} = \frac{\Gamma((d+1)/2)}{2^{d+2}\pi^{(d+3)/2}(d+1)(d+2)}.$$

### 11.1 The half-line sandwich

For $\kappa > 0$ the form of $A_\kappa$ is $2\|Lf\|_2^2$ with $L = (\kappa + \partial_x)^{1/2}$. Its causal inverse is convolution by $\mathbf{1}_{s>0} e^{-\kappa s}/\sqrt{\pi s}$. The inverse maps half-line $L^2$ into the supported form domain, and $L = (\kappa + \partial_x) L^{-1}$ preserves support. Thus $L$ maps the form space bijectively onto half-line $L^2$ as in Sections 1–4. The zero-contact sandwich equals $T_{\kappa,h}^* T_{\kappa,h}/2$ with $T_{\kappa,h} = L^{-*} B_{\kappa,h}^*$. Evaluate the convolution by using

$$\frac{\kappa K_1(\kappa r)}{r} = \int_\kappa^\infty e^{-sr} \sqrt{s^2 - \kappa^2}\, ds.$$

The backward convolution on an exponential of rate $s$ gives $(s+\kappa)^{-1/2}$, leaving $\sqrt{s-\kappa}$ in the integral. Hence $T_{\kappa,h}$ has kernel $-e^{-\kappa r}/(\sqrt{\pi}\, r^{3/2})$ at $r = h + x + y$. This is Hilbert–Schmidt and gives the first explicit expression in the theorem.

At $\kappa = 0$ regularize the diagonal form by $2\sqrt{D^2 + m^2}$ while keeping the massless cross block. As $m \downarrow 0$, the forms decrease to $2|D|$ with discrepancy at most $2m\|f\|_2^2$. The inverse quadratic form converges to the variational supremum by taking the limit on each compactly supported test. The backward convolutions are dominated in absolute value by the positive convolution with $\mathbf{1}_{s>0}/\sqrt{\pi s}$, so their sandwiched kernels converge in Hilbert–Schmidt norm to $T_{0,h}$. This identifies the variational limit as the trace-class sandwich. The static planar value $W_{1,0}(0,h) = 1/(4\pi h)$ also follows from the classical mixed Laplace Green function [23, Eq. (4.12)]. By variational comparison the sandwich for $b \geq 0$ is bounded by that for $b = 0$, including its exponentially decaying trace bound.

### 11.2 Bounded intervals in dimension one

Let $P_{i,g}$ restrict the corresponding ray to its scaled interval. The diagonal contractions $R_{-,g} = c_{-,g}(A_{-,g} + c_{-,g})^{-1}$, extended by zero, tend strongly to the identity. On a smooth compactly supported interior test this follows by the resolvent identity, since the compressed square root of that test is the restriction of its full-line image; then use contractivity and density. With

$C_g = B_g(A_{+,g} + c_{+,g})^{-1}B_g^*$ and $B_g = P_{-,g}B_{\kappa,1}P_{+,g}$, the variational comparison gives $0 \le C_g \le P_{-,g}C_{\kappa,0}P_{-,g}$, where $C_{\kappa,0} = T_{\kappa,1}^* T_{\kappa,1}/2$.

At any fixed left coordinate the diagonal form on the right cross column converges to the full right-ray value with contact $b$. The lower limit follows by eventual admissibility of every compactly supported form test. For the upper limit use the full-ray contact $b - \varepsilon > 0$ if $b > 0$ and then send $\varepsilon \downarrow 0$ using its gap. If $b = 0$, use the massless-compatible zero-contact variational bound. The dominating trace density from the transparent half-line sandwich is integrable, so the traces converge to $W_{1,b}(\kappa, 1)$. The first determinant term times $c_{-,g}$ is $\operatorname{Tr}(R_{-,g}C_g)$. The difference from $\operatorname{Tr} C_g$ is at most $\operatorname{Tr}[(P_{-,g} - R_{-,g})C_{\kappa,0}]$, which tends to zero by trace class and strong convergence. The normalized cross product before multiplication has operator norm and trace at most $W_{1,0}(\kappa, 1)/c_{-,g}$. The higher trace terms therefore tend to zero after multiplication by $c_{-,g}$, with the same uniform exponential bound. This proves the planar transform limit.

### 11.3 Localization of the weak inverse in dimension $d \ge 2$

Let $K_{i,g}$ be the aperture domains dilated by $1/g$, $A_{i,g}$ the compressed two-arm square root at $\kappa$, and $B_g$ its kernel $n_{\kappa^2}(X - Y)$ between the two domains. At a left point $Y$, the diagonal of $C_g = B_g(A_{+,g} + c_{+,g})^{-1}B_g^*$ is the quadratic form of the right inverse on

$$f_{g,Y}(X) = \mathbf{1}_{K_{+,g}}(X) n_{\kappa^2}(X - Y).$$

For $p = 2d/(d+1)$, the homogeneous Sobolev estimate $\|v\|_{2d/(d-1)} \le C\|v\|_{\dot{H}^{1/2}(\mathbb{R}^d)}$ [26, Theorem 6.5] gives the column-form bound $C\|f_{g,Y}\|_p^2$. Indeed dualize and take the supremum of $2\operatorname{Re}\langle f_{g,Y}, v\rangle$ minus the response form. No positive spectral gap is needed. The kernel is bounded for $|Z| \ge 1$ by $C|Z|^{-d-1}e^{-a\kappa|Z|}$ by Sections 5–9. If $L$ is the distance from $Y$ to the right aperture, integration over $|Z| \ge L \ge 1$ gives $Ce^{-a\kappa}L^{-d-1}$ for the column bound, after reducing $a$ if needed. The same decay power holds for derivatives in $Y$ at fixed $\kappa$, by differentiating the kernel.

In the left chart use $y_j = g^{1/p_j} w_j$ with $p_j = 2$ in the quadratic case, and physical inward depth $gu$. The pair-distance estimate of Sections 5–9 gives $L \gtrsim 1 + u + \sum_j |w_j|^{p_j}$ against points of the right chart. For a left point in a smaller chart, any point outside the right chart is at positive physical distance by uniqueness of contact. Since $u + \sum_j |w_j|^{p_j} \lesssim g^{-1}$ in the left chart, the bound holds against the entire right aperture. For a left point outside its smaller chart the distance to every right point is at least $c/g$, and the total off-chart column estimate is $O(ge^{-a\kappa})$, using the scaled volume $O(g^{-d})$. The scaled left volume element in the chart is $g^{-\gamma_*} dw\, du$. The localized majorant is integrable because the effective dimension in these variables is $1 + \alpha < d + 1$.

Fix $(w, u)$ with $u > 0$, and translate $Y$ to the origin. At any bounded tangential offset the right graph converges uniformly to the half-space boundary $s = H(w) + u$. This is the graph limit in Sections 5–9, where the local Lipschitz constants tend to zero on shrinking neighborhoods of contact. Other parts of the right boundary leave each compact set. The translated columns converge strongly in $L^p$ by dominated convergence, since the support is at distance at least one and the kernel has the stated radial decay.

The inverse forms on these columns converge by the variational formula. For the lower bound, every smooth test compactly supported inside the limiting half-space is admissible for all small $g$. Such tests are dense in the supported energy space. In the massless homogeneous case, translate a function inward, apply a compact cutoff, and smooth without crossing the boundary. To justify removing the cutoff, let $\chi_R(X) = \chi(X/R)$ with smooth $\chi$ equal to one near zero. The difference-quotient norm of $\chi_R v$ is bounded by the norm of $v$ plus a term $C\int |v(Y)|^2 h_R(Y) dY$, where

$$h_R(Y) = \int_{\mathbb{R}^d} |\chi_R(X) - \chi_R(Y)|^2 |X - Y|^{-d-1}\, dX.$$

This term is bounded by $C\|v\|^2_{\dot H^{1/2}}$ by Hölder and the Sobolev estimate, since $h_R$ has uniformly bounded $L^d$ norm. Indeed $h_R(Y) = R^{-1}h_1(Y/R)$ and $h_1$ is bounded and decays as $|Y|^{-d-1}$ at infinity. Whole-space test density and this uniform cutoff bound give convergence. For the inhomogeneous energy also use convergence in $L^2$.

For the upper bound take near-maximizing tests. They have bounded homogeneous energy by the dual bound on the columns and the quadratic cost. If $\kappa > 0$ or $b > 0$, they are also bounded in $L^2$ by coercivity. Take a weak subsequential limit in the relevant space. It vanishes strictly below the limiting plane by the convergence of the domains on compact sets. The pairing converges by strong $L^p$ convergence, and energy is lower semicontinuous. When $b > 0$ use the eventual form bound with contact $b - \varepsilon$, followed by $\varepsilon \downarrow 0$; when $b = 0$ drop the nonnegative contact part. This proves the variational upper bound, including $\kappa = b = 0$.

The column majorant now permits integration, giving the limit of $g^{\gamma_*} \operatorname{Tr} C_g$. Tangential Plancherel on the limiting half-space separates the right inverse into the half-line forms of (7) with momentum $\sqrt{\kappa^2 + |k|^2}$. Integration of the left depth at each $w$ gives $W_{d,b}(\kappa, H(w))$. This separation also holds in the massless form-dual sense by positive diagonal mass regularization followed by monotone convergence. The planar fiber trace is finite at zero, and its trace decays exponentially in the fiber momentum for $h \geq 1$.

### 11.4 The strong inverse and the trace remainder

Put $T_g = (A_{+,g} + c_{+,g})^{-1/2} B_g^*$. The difference between $\operatorname{Tr} C_g$ and $\operatorname{Tr}(R_{-,g} C_g)$ equals $\|(I - R_{-,g})^{1/2} T_g^*\|_2^2$. Cut off $T_g^*$ near the left boundary using the multiplier $\min(1, \operatorname{dist}(Y, K^c_{-,g})/\delta)$. The squared normalized Hilbert–Schmidt norm removed by the cutoff is at most the left column-form integral times the squared removed multiplier. It vanishes as $g \downarrow 0$ and then $\delta \downarrow 0$, by the majorant above and the uniform scaled boundary distance at fixed positive depth in the Lipschitz chart. For fixed $\delta$ the cutoff kernel has zero extensions with normalized integrated $H^1$ norms bounded. To see this, differentiate the cross column in $Y$ under the bounded finite-domain weak inverse square root and use the derivative column estimate and the Lipschitz cutoff. The cutoff $A_{-,g}$ form is thus bounded after normalization by $g^{\gamma_*}$. The bound $I - R_{-,g} \leq A_{-,g}/c_{-,g}$ shows that the trace error is $o(g^{-\gamma_*})$.

For the higher traces we also need an operator norm bound for $C_g$ independent of its growing trace. Both apertures are contained in facing supporting half-spaces at separation one in scaled coordinates. By the weak variational formula, the finite-domain $C_g$ is dominated by the compression of the zero-contact facing-half-space sandwich. Its tangential Fourier fibers are the positive planar operators with norm at most their traces $W_{1,0}(\sqrt{\kappa^2 + |k|^2}, 1) \leq W_{1,0}(0, 1)$. This is a uniform operator norm bound even at zero. The positive normalized cross product has norm $O(c_{-,g}^{-1})$ and trace $O(c_{-,g}^{-1} g^{-\gamma_*} e^{-a\kappa})$. Thus the remainder after multiplication by $g^{\gamma_*} c_{-,g}$ is $O(c_{-,g}^{-1} e^{-a\kappa})$. This finishes the pointwise transform limit in $d \geq 2$.

In every dimension the limiting transform is continuous, from the explicit $b = 0$ formula and from the positive inverse gap and kernel continuity if $b > 0$, followed by dominated convergence in the tangent variables. The normalized transforms decrease with $\kappa$ by their heat definition, so a finite mesh gives compact uniformity. The exponential bound permits vacuum integration and Matsubara summation. For fixed physical temperature the scaled mesh is $2\pi g/\beta$. The error of its half-weighted sum, after multiplication by the mesh step, is at most the step times the value at zero, by monotone comparison with the integral. That value is uniformly bounded in the normalization of the theorem. This proves the fixed-temperature assertion.

To compute the coefficients, insert $W_{1,0}$ into (8) and integrate radially. The static tangential integral gives a gamma factor from $e^{-2|k|r}$ and then $\int_h^\infty (r - h) r^{-d-2} dr = h^{-d}/[d(d+1)]$. For the

vacuum integrate over $\kappa \geq 0$ and the $d-1$ momenta. At $h=1$ this gives

$$\frac{|S^{d-1}|}{2(2\pi)^d}\int_0^\infty q^{d-1}W_{1,0}(q,1)\,dq.$$

Substitute the planar formula, use $\int_0^\infty q^{d-1}e^{-2qr}dq = \Gamma(d)/(2r)^d$, and then $\int_1^\infty (r-1)r^{-d-3}dr = 1/[(d+1)(d+2)]$. Gamma duplication yields $A_d^{(1)}$; the same duplication simplifies the static expression. Dilation gives the height factor $h^{-d-1}$ in the vacuum integral.

### 11.5 Time representation for the transparent limit

When $b=0$, the positive transform limit is the Laplace transform with measure $j(T)dT/T$, where

$$j(T) = \frac{T^{-d/2}}{2^d\pi^{(d+2)/2}}\int_{\mathbb{R}^{d-1}}\int_{H(w)}^\infty (r-H(w))r^{-2}e^{-r^2/T}\,dr\,dw.$$

Indeed the inverse Laplace transform of $e^{-2r\sqrt{z}}$ is $r\,e^{-r^2/T}/(\sqrt{\pi}T^{3/2})$. The tangential momentum integral contributes $(4\pi T)^{-(d-1)/2}$ by the Gaussian integral after the shift $z+|k|^2$. Integrating the positive formula yields the expression. Convergence of the finite-measure Laplace transforms including total mass at $\kappa=0$ proves weak convergence of $g^{\gamma_*}c_{-,g}\mathcal{I}_g(g^2T)dT/T$ to this measure.

Write $v_H = |\{w : H(w)-1<1\}|$, with $v_H=1$ in $d=1$. The sublevel volumes of $H-1$ are $v_H s^\alpha$ at level $s>0$. Thus the double integral may be replaced by

$$\frac{v_H}{\alpha+1}\int_1^\infty (r-1)^{\alpha+1}r^{-2}e^{-r^2/T}\,dr.$$

At small $T$, substitute $r=1+Ts$ and apply dominated convergence to get the equivalent $\frac{v_H}{\alpha+1}\Gamma(\alpha+2)2^{-\alpha-2}T^{\alpha+2}e^{-1/T}$ for this integral. At large $T$ and $\alpha>0$, substituting $r=\sqrt{T}s$ gives $\frac{v_H}{2(\alpha+1)}\Gamma(\alpha/2)T^{\alpha/2}$. If $d=1$ and $\alpha=0$, the integral instead has leading term $\frac{1}{2}\log T$, since its integrand differs from $r^{-1}e^{-r^2/T}$ by an integrable function. In that case $j(T)\sim T^{3/2}e^{-1/T}/(8\pi^{3/2})$ at zero and $j(T)\sim \log(T)/(4\pi^{3/2}\sqrt{T})$ at infinity. The interval vacuum equivalent is $-1/(48\pi^2 g c_{-,g})$; at gap-scale temperature the interval equivalent is

$$-\frac{1}{4\pi\lambda g c_{-,g}}\int_1^\infty (r-1)r^{-3}\coth(2\pi r/\lambda)\,dr,$$

by summing the elementary exponentials in $W_{1,0}$.

## 12 Independent bounded and divergent strengths

Let $J_h^{a,b}(T)$ be the planar all-hit relative heat gain for the two rays outside a retained segment of length $h$, with weight $\exp(-aL_T^- - bL_T^+)$ as in Section 1. Its unweighted version is $J_h(T)$; write $J=J_1$. At the gap-scale short-time limit the weights act on the fixed normal bridge local time. We first estimate the large-time effect of a single positive strength.

**Theorem 10.** *For $a,b,h\geq 0$ and $T>0$, writing $m=\max(a,b)$, one has*

$$J_h^{a,b}(T) \leq C(1+m\sqrt{T})^{-1/4}, \qquad 0\leq J_h(T)-J_h^{a,b}(T)\leq C\min(1,m\sqrt{T}).$$

*If $c_{-,g} \to a < \infty$ and $c_{+,g} \to b < \infty$, the limits after multiplication by $g^{d-\alpha}$ are*

$$-\frac{(4\pi)^{-(d-1)/2}}{4\sqrt{\pi}} \int_{\mathbb{R}^{d-1}} \int_0^\infty T^{-(d+2)/2} J^{a,b}_{H(w)}(T)\, dT\, dw \quad \text{for the vacuum,}$$

$$-\frac{(4\pi)^{-(d-1)/2}}{2\lambda} \int_{\mathbb{R}^{d-1}} \int_0^\infty T^{-(d+1)/2} J^{a,b}_{H(w)}(T) \sum_{n\in\mathbb{Z}} e^{-T(2\pi n/\lambda)^2}\, dT\, dw \quad \text{at } \beta = \lambda g.$$

*The thermal limit holds locally uniformly for $\lambda > 0$ in $d \geq 2$ and also in $d = 1$ if $m > 0$. At fixed positive physical inverse temperature the leading law is the vacuum law.*

*If both $c_{i,g} \to \infty$, the strong common-strength vacuum and thermal laws of Sections 5–9 hold with the product $c_{-,g}c_{+,g}$ replacing the squared common strength.*

*Proof.* For the first inequality scale to $T = 1$ and take $a = m \geq 2$ by symmetry. At a fixed normal starting coordinate $X$ the normal zero times give the compact lateral trace of the lateral increments. In the all-hit event the minimum of this trace plus the lateral base point lies in the left ray. The placements for which that minimum is within distance $\delta$ of the left endpoint have lateral measure at most $\delta$. Integrating the normal hit probability $e^{-X^2}$ bounds this part by $C\delta$.

All other contributing bridges have a contact at least $\delta$ inside the left ray. By time reversal it suffices to handle such a hit by $1/2$, at a cost of two. The bridge law up to time $3/4$ is bounded by four times planar Brownian motion from the same base point, by the Gaussian transition density. Stop the motion at its first such contact before $1/2$. During the following $U \leq 1/4$ time units it can exit the left ray in lateral coordinate only by a displacement of at least $\delta$, whose probability is at most $Ce^{-\delta^2/(4U)}$ by reflection. Otherwise all normal local time during that time stays in the left ray. Starting normally at zero, its occupation local time has the law of a constant times $\sqrt{U}$ times a half-normal variable by the standard reflected Brownian local-time identity. Hence the strong Markov property bounds the weighted expectation by $C[(1 + a\sqrt{U})^{-1} + e^{-\delta^2/(4U)}]$, uniformly in the base point. Independently, the all-hit event has Gaussian bound $Ce^{-c(S^2+X^2)}$ from the two independent bridge coordinates. Use the geometric mean of the bounds and integrate in $(S, X)$. With $\delta = a^{-1/4}$ and $U = a^{-3/4}/4$ the deep term is $O(a^{-5/16})$ and the shallow term $O(a^{-1/4})$. Scaling the retained gap and local time proves the assertion. The second inequality follows from $1 - e^{-aL^- - bL^+} \leq m(L^- + L^+)$ and the integrated unit normal local-time moment in Sections 5–9, with the same range bound over lateral placements.

For finite scaled limits, $T = t/g^2$ gives the two limiting local-time weights in the near-contact bridge proof [2]. Off the polar limiting endpoints the compact trace at normal-zero times has positive distance from the endpoints. The tangential freeze converges uniformly along each path, so aperture labels agree on the whole normal-zero set for all small $g$. Integration against its local-time measure therefore gives convergence of both weights. The transparent majorants from [2] apply to the weighted integrals. In $d = 1$ the first estimate of the theorem adds $O(T^{-1/8})$ at infinity when $m > 0$, which makes the thermal zero mode integrable. This proves the limits including local uniformity in $\lambda$. At fixed physical temperature the same monotone mesh comparison as in Theorem 9 applies. For $d \geq 2$ the appropriately normalized zero mode is uniformly bounded by the transparent Matsubara sum at $\lambda = 1$. For $d = 1$ it is at most $O(\log(1/g))$, again sufficient against the $O(g)$ step.

If both strengths diverge, the leading trace term becomes $\mathrm{Tr}(R_{-,g}B_gR_{+,g}B_g^*)/(c_{-,g}c_{+,g})$, with $R_{i,g} = c_{i,g}(A_{i,g} + c_{i,g})^{-1}$. The strong kernel localization and two boundary-cutoff estimates in Sections 5–9 apply to each factor with its own $c_{i,g}$, both tending to infinity. The higher traces have an extra operator norm factor $O((c_{-,g}c_{+,g})^{-1})$ by the separated kernel Schur bound. Thus the same limits and temperature estimates follow with the product normalization. □

## 13 The marginal interval regime

Suppose $d = 1$ and both scaled strengths vanish. Write $Z_g = q_g(0)$ and put $r_g = \max_i \rho_{i,g}$, $\epsilon_g = g r_g$. As in Sections 1–4 and Sections 5–9, define

$$\begin{aligned} C_* &= \lim_{L\to\infty} \left[ \int_0^{L^2} J(T)\frac{dT}{T} - \tfrac{1}{4}\log L + \log\log L \right], \\ C_{\ell_-,\ell_+} &= \log\pi - \tfrac{1}{4}\log(\ell_-^{-1} + \ell_+^{-1}), \\ N_\lambda &= 2\sum_{n\ge 1} \int_0^\infty e^{-(2\pi n/\lambda)^2 T} J(T)\frac{dT}{T}. \end{aligned}$$

The time limit in $C_*$ exists by the retained-segment inverse-logarithmic law Sections 1–4. The finite geometric constant is the heat-normalized elliptic result of Sections 5–9.

**Theorem 11.** *If the physical pair $(\rho_{-,g}, \rho_{+,g})$ tends to a finite $(\rho_-, \rho_+)$, then*

$$Z_g = \tfrac{1}{4}\log(1/g) - \log\log(1/g) + C_{\ell_-,\ell_+} + \int_0^\infty [\mathcal{I}_{0;\rho_-,\rho_+}(t) - \mathcal{I}_{0;0,0}(t)]\frac{dt}{t} + o(1).$$

*At gap zero the intervals retain their two labels. If $r_g \to \infty$ while $\epsilon_g \to 0$, set $\theta_{i,g} = \rho_{i,g}/r_g$. Then*

$$Z_g = \tfrac{1}{4}\log(1/\epsilon_g) - \log\log(1/\epsilon_g) + C_* + C_R(\theta_g) + o(1),$$

*where*

$$C_R(\theta) = \int_0^1 [J_0^{\theta_-,\theta_+}(s) - 1/8]\frac{ds}{s} + \int_1^\infty J_0^{\theta_-,\theta_+}(s)\frac{ds}{s}.$$

*This is uniform in the contact ratios $\theta_g$ on $\max_i \theta_i = 1$, including an exactly transparent side. In both cases the free energy at gap-scale temperature is $-(Z_g + N_\lambda + o(1))/(2\lambda g)$, locally uniformly in $\lambda > 0$.*

*Proof.* For a finite limiting physical pair, take the difference from the transparent finite-interval heat integral. It converges at fixed physical time by polarity of the limiting endpoints and by local-time label convergence. The small-time difference is $O(\sqrt{t})$ by Theorem 10; the large-time bound is the fixed enclosing window's $O(t^{-1})$ heat gain [1, 2]. Dominated convergence and the transparent expansion prove the first formula.

When $r_g \to \infty$, replace the finite apertures by their infinite rays of gap $g$. At small physical times the truncation changes the heat gain by at most $Ce^{-c/t}$, because a removed all-hit bridge must cross past an outer endpoint and have lateral range at least the shorter interval length. Independent of this bound, its difference is bounded by $C(1 + r_g\sqrt{t})^{-1/4}$ by Theorem 10. Dominated convergence below time one and direct integration above one show the integrated truncation error is $o(1)$.

In gap units split the ray integral at $T = \epsilon_g^{-2}$, and use $s = \epsilon_g^2 T$ in the difference from its transparent sharp-cutoff part. The resulting static integral is

$$\int_0^{\epsilon_g^{-2}} J(T)\frac{dT}{T} + \int_0^1 [J_{\epsilon_g}^{\theta_-,g,\theta_+,g}(s) - J_{\epsilon_g}(s)]\frac{ds}{s} + \int_1^\infty J_{\epsilon_g}^{\theta_-,g,\theta_+,g}(s)\frac{ds}{s}.$$

The transparent cutoff term equals the stated logarithms plus $C_* + o(1)$. The two matching terms converge to those defining $C_R(\theta_g)$, uniformly in $\theta_g$. At each fixed $s$ the origin is polar, so the all-hit indicators and aperture labels agree eventually with their zero-gap limits on each

eligible path, independently of the contact strengths. The weights are bounded by one, and the all-hit probability has an integrable spatial majorant. The time majorants are $O(\sqrt{s})$ for the difference at zero and $O(s^{-1/8})$ at infinity, uniformly by Theorem 10. This also shows continuity of $C_R$.

For every fixed scaled time the weights tend to one because both $c_{i,g} \to 0$, and the scaled finite intervals exhaust their rays. Thus the nonzero Matsubara sum converges to $N_\lambda$, by the small-time Gaussian crossing bound and exponential large-time suppression from the nonzero Matsubara frequencies. The domination is uniform on compact positive $\lambda$-intervals. □

## 14 Variable and degenerate contact: problem and normalization

We use the connected scalar heat trace and boundary determinant of [1], Sections 5–9 and Sections 10–13. The apertures approach across a gap $g$; $u, v \geq 0$ denote inward aperture depths in units of $g$, and $\kappa \geq 0$ is the gap-scaled mass. The normalized transform $q_g(\kappa)$ is the minus-log determinant of $I$ minus the positive cross product obtained by inserting the two diagonal form inverses. The diagonal operator before a contact is the supported compression of $2\sqrt{|D|^2 + \kappa^2}$. A contact is a nonnegative multiplication term in the form, and the gap-scaled contact is $g$ times the physical coefficient. The connected vacuum energy is $-(2\pi g)^{-1} \int_0^\infty q_g(\kappa)\, d\kappa$. At inverse temperature $\beta$ the free energy is $-(2\beta)^{-1} \sum_{j\in\mathbb{Z}} q_g(2\pi|j|g/\beta)$.

In the curved and separable finite-type setting, write the facing depths before translation as $a_-(y)$ and $a_+(y)$. In smooth quadratic contact let $B_\pm$ be their Hessians at the tangency, with $B_- + B_+$ positive definite, and put $\alpha = (d-1)/2$ and $H(w) = 1 + w^{\mathsf{T}}(B_- + B_+)w/2$. For separable finite type the leading profiles are $\sum_j b_{\pm,j}|y_j|^{p_j}$, $p_j > 1$, $b_{-,j} + b_{+,j} > 0$; put $\alpha = \sum_j p_j^{-1}$ and $H(w) = 1 + \sum_j (b_{-,j} + b_{+,j})|w_j|^{p_j}$. In the quadratic case take $p_j = 2$ for the depth estimate. The geometry and full hypotheses are those in Sections 5–9 and Sections 10–13. In dimension $d = 1$ take two intervals, $\alpha = 0, H = 1$.

Inhomogeneous repulsive boundary strengths and their determinant representation are used in [14, 27] for thin mirrors. The interface here is a Dirichlet screen with two apertures and the connected subtraction is essential. The constant-contact near-tangency results in Sections 5–9 and Sections 10–13 supply the geometric estimates and the transparent separated column. The new limits address varying depths, zeros that make the reciprocal nonintegrable, and two strong profiles with independent rates. The finer zero-layer laws below apply to flat geometries and to physical intervals.

## 15 Depth-dependent contact on moving apertures

Use the geometry in Sections 10–13 and Sections 5–9, $\gamma_* = d - 1 - \alpha$, height $H(w)$, and positive transform $q_g$. In the facing charts let $y_j = g^{1/p_j} w_j$ and write inward depths as $gu$ on the left and $gv$ on the right. In dimension one omit $w$, and use $H = 1$, $\gamma_* = 0$. Take bounded continuous nonnegative profiles $a(w, u)$, $b(w, v)$ on $\mathbb{R}^{d-1} \times [0, \infty)$. Give the left aperture scaled strength $M_g[a(w, u) + \epsilon_g]$ in its chart and $M_g$ outside, with

$$M_g \to \infty, \qquad \epsilon_g \downarrow 0, \qquad M_g \epsilon_g \to \infty.$$

On the right prescribe the scaled strength $b(w, v)$ in the chart and any uniformly bounded nonnegative measurable strength outside it. These are scaled strengths, so the physical coefficients are divided by $g$. We use one fixed facing chart per side, and only its local representation enters the limit.

For $q \ge 0, \ell > 0$, let

$$f_{q,\ell}(v) = -2qK_1(q(\ell+v))/[\pi(\ell+v)]$$

with $f_{0,\ell}(v) = -2/[\pi(\ell+v)^2]$. Let $A_q$ be the zero-exterior right-ray compression of $2\sqrt{|D|^2+q^2}$. Set

$$C_b(\kappa, w, u) = (2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} \langle f_{q,\ell}, (A_q + b(w,\cdot))^{-1} f_{q,\ell} \rangle \, dk.$$

Here $q = \sqrt{\kappa^2 + |k|^2}$ and $\ell = H(w)+u$. The zero-energy inverse denotes the form-dual variational expression.

**Theorem 12.** *The transform limit below holds under* (9)*, and at any fixed $\kappa \ge 0$ where its right side is infinite:*

$$g^{\gamma_*} M_g q_g(\kappa) \longrightarrow \Phi_{a,b}(\kappa) := \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{C_b(\kappa, w, u)}{a(w,u)} \, du \, dw.$$

*A sufficient hypothesis for uniform convergence on compact $\kappa$-intervals, an exponential upper bound, and vacuum and thermal limits is*

$$\int \frac{(1 + u + \sum_j |w_j|^{p_j})^{-(d+1)}}{a(w,u)} \, du \, dw < \infty. \tag{9}$$

*Under* (9)*, multiply the energy and free energy by $g^{d-\alpha} M_g$. The vacuum limit and the free-energy limit at $\beta = \lambda g$ are respectively*

$$-\frac{1}{2\pi} \int_0^\infty \Phi_{a,b}(\kappa) \, d\kappa, \qquad -\frac{1}{2\lambda} \sum_{n \in \mathbb{Z}} \Phi_{a,b}(2\pi|n|/\lambda).$$

*The latter convergence is locally uniform in $\lambda > 0$. At fixed positive physical $\beta$, the normalized free energy has the vacuum limit. The case $b \equiv 0$ is included.*

### 15.1 Weak inverse and separated columns

At each fixed left point in the scaled local chart, translate tangentially to its tangency coordinate. The right domain converges locally to the half-space at distance $H(w) + u$. Every fixed local tangential offset changes $w_j$ by $O(g^{1-1/p_j})$, while the scaled facing graph becomes flat on bounded sets. Thus the right potential converges locally uniformly in the limiting interior to $b(w, v)$. At $d \ge 2$ use the homogeneous Sobolev dual bound $C\|f_{g,Y}\|^2_{2d/(d+1)}$ for each separated column and its spatial derivatives. Compactly supported tests give the variational lower limit. Near-maximizers have bounded homogeneous energy by this dual bound. Their zero extensions are bounded in local $H^{1/2}$ after smooth cutoff, using Sobolev for local $L^2$ bounds and the difference-quotient multiplier estimate in Sections 10–13. Rellich compactness gives strong $L^2_{\mathrm{loc}}$ and weak energy convergence along a subsequence. The weak limit vanishes off the limiting right half-space. On compact subsets of that half-space the potential integrals converge by the local $L^2$ convergence and boundedness. Exhausting the interior and dropping the nonnegative potential cost elsewhere proves lower semicontinuity of the potential term, including profiles with zeros and no positive gap. For $\kappa > 0$, there is in addition a global $L^2$ bound from the mass. The pairing passes by the strong $L^{2d/(d+1)}$ column convergence. The limit energy-plus-potential domain has compactly supported test density: first cut off at infinity using the homogeneous multiplier argument and dominated convergence in the potential norm, then translate inward and smooth on bounded sets where the potential is bounded.

For $d = 1$, the translated weak domains are intervals $(0, \ell_+/g)$, and the massless energy completion of the supported ray has norm $2\|L_0 v\|_2^2$ where $L_0 = \partial_x^{1/2}$. Its inverse on the completion is the causal half-integral

$$(I_0 h)(x) = \int_0^x \frac{h(y)}{\sqrt{\pi(x-y)}}\, dy.$$

Here is one justification of surjectivity in the completion. For smooth $h$ compactly supported in the open ray, put $I_m h = \mathbf{1}_{x>0} e^{-mx}/\sqrt{\pi x} * h$ after zero extension. These are supported form functions for $m > 0$. Their Fourier transforms are $(m + i\xi)^{-1/2}\widehat{h}(\xi)$, so they converge in homogeneous energy to $I_0 h$ by dominated convergence against $|\xi|\, d\xi$. Hardy control on the ray identifies this limit in local $L^2$. Inversion by $L_0$ follows by the Fourier multiplier identity and hence extends by density of such $h$ in $L^2$. The restriction of $I_0$ to any bounded interval is compact on $L^2$: the part of its kernel with $x - y < \delta$ has $L^1$ convolution norm $O(\sqrt{\delta})$, and the remainder is Hilbert–Schmidt there. Thus the same local compactness, potential lower semicontinuity, and weak form argument apply even at $\kappa = 0$. For density with a bounded potential, the support condition gives the elementary Hardy bound $\int_0^\infty |v(x)|^2 dx/x \le C\mathfrak{a}_0[v]$ by retaining interactions with the negative ray in the homogeneous difference-quotient form. Smooth cutoffs $\chi(x/R)$ on the line act uniformly in energy since their difference-quotient multiplier $h_R(x)$ satisfies $h_R(x) \le C/x$ for $x > 0$. This follows by scaling from boundedness near zero and $O(x^{-2})$ decay at infinity. Density in the completed energy then gives energy cutoff convergence, while the potential norm converges by domination. The compactly supported restriction can be translated inward and mollified, as in higher dimension. For $\kappa > 0$, the same cutoff convergence includes the $L^2$ norm. The full-ray column pairing is bounded by the causal adjoint $I_0^* f$ in $L^2$. Restricted and moving columns converge in this dual norm by dominated convergence, applied first pointwise to $I_0^*$ and then in $L^2$ using the massless column at separation bounded below. This also applies to their depth derivatives, dominated by a constant times the massless column at separations at least one.

Consequently the diagonal column forms of

$$C_g = B_g (A_{+,g} + c_{+,g})^{-1} B_g^*$$

have pointwise limit $C_b$. Fiber decomposition follows by tangential Plancherel since the frozen potential only depends on depth. At mass zero it follows by monotone removal of a positive diagonal mass using the variational expression. In every dimension, these columns obey

$$C_g(Y, Y) \le C e^{-c\kappa} (1 + u + \sum_j |w_j|^{p_j})^{-(d+1)}$$

in the left chart by the column estimates in Sections 10–13, and the off-chart contribution after $g^{\gamma_*}$ normalization is $O(g e^{-c\kappa})$. In dimension one use the pointwise $I_0^*$ factorization, which gives a column-square bound $C(H + u)^{-2}$. For fixed $\kappa$, the same normalized integrated bound holds for the squared norms of the spatial derivatives of the weak-inverted cross columns. The operator bound $\|C_g\| \le C$ at all $\kappa \ge 0$ is the transparent supporting-half-space comparison in Sections 10–13, which uses only nonnegativity of $c_{+,g}$.

### 15.2 The weighted strong inverse

Let $a_g = a + \epsilon_g$ in the left chart and $a_g = 1$ off it. Then $R_g = M_g(A_{-,g} + M_g a_g)^{-1} = (M_g^{-1} A_{-,g} + a_g)^{-1}$, and $R_g \le a_g^{-1}$ in form order. For $F = (A_{+,g} + c_{+,g})^{-1/2} B_g^*$, the difference between

$\mathrm{Tr}(a_g^{-1}C_g)$ and $\mathrm{Tr}(R_gC_g)$ is nonnegative. The variational formula bounds it above, for any admissible Hilbert–Schmidt trial kernel $V$ on the left, by

$$\|a_g^{1/2}(V - a_g^{-1}F^*)\|_2^2 + M_g^{-1}\|A_{-,g}^{1/2}V\|_2^2. \tag{10}$$

Choose $V = m(w,u)F^*$, where a real smooth multiplier $m$ compactly supported in $u > 0, a > 0$ approximates $1/a$ on an inner compact set. This suffices to make the limsup of the first term after normalization by $g^{\gamma_*}$ arbitrarily small. Indeed use an exhaustion of $u > 0, a > 0$, a smooth cutoff $0 \le \eta \le 1$ equal to one on the selected compact set, and uniform approximation of $1/a$ on the cutoff support. The lost part is dominated by the integrable weighted column bound from (9). The off-chart contribution is negligible. The normalization of the energy term in (10) is bounded for each chosen multiplier, since the scaled Lipschitz constants of the slow coordinates and inward depth are uniformly bounded, and the multiplier vanishes at the actual left edge. The derivative-column estimate controls the zero-extended $H^1$ norm of $V$. This proves that the weighted multiplication upper bound is asymptotically attained. Dominated convergence evaluates the weighted trace.

The relative determinant and all-hit identity hold for these bounded multiplication contacts at each fixed $g$. In the Galerkin/Poisson proof of [1] and Sections 10–13 the contact is independent of the resolvent variable, and it adds a nonnegative trace-form term to the energy. All differentiated Poisson factors are unchanged, and strong Galerkin inverse convergence uses the same form density. The corresponding bridge weight is the exponential of the integral of the bounded coefficient against normal occupation local time. Thus it preserves positivity, monotonicity in $\kappa$, and comparison under increasing a coefficient. The positive normalized cross product has operator norm $O((M_g\epsilon_g)^{-1})$ and trace $M_g^{-1}\,\mathrm{Tr}(R_gC_g)$. Its higher trace terms are negligible relative to the first by the determinant power series. This proves the limit under (9), including exponential domination. Limiting spectral continuity follows from variational convergence as the mass varies: on $\kappa > 0$ use the mass gap and strong dual convergence of cross columns. At zero use compact tests for the lower bound, and for the upper bound drop the nonnegative mass increment and use dual convergence in the massless space, with the common column majorant. Tangential and depth integrations preserve continuity by (9). The monotone mesh and sum bounds of Sections 10–13 now give the stated energy limits.

For a divergent pointwise profile without (9), the determinant first term and $R_g \le a_g^{-1}$ yield domination if the weighted column envelope is integrable, but this sufficient criterion need not be necessary at positive $\kappa$ or for large weak profiles. The divergence can be shown by comparison with a raised profile. Since $b$ is globally bounded, the transparent column form and the bounded-contact column form have comparable local positive lower bounds on every compact $(w,u)$-set. Thus if $\int C_b/a = \infty$, regardless of where divergence occurs, increase the left profile to $a + \delta$ in the chart. The proved limit gives $\liminf g^{\gamma_*}M_g q_g(\kappa) \ge \int C_b/(a+\delta)$, and monotone convergence sends this to infinity. When (9) fails but $\int C_b/a < \infty$, no limit is claimed here because upper tightness of the finite-gap weighted columns has not been established. In particular the extended pointwise assertion above applies under (9) and in the divergent case, with this intermediate case unasserted.

For example if $a(w,u) \asymp u^\nu$ for $u$ near zero and is bounded below away from the edge, then $\nu < 1$ gives (9), and $\nu \ge 1$ gives divergence for every $\kappa$ by a single compact tangential patch. Likewise a zero set of codimension $m$ with local strength comparable to distance to the power $\nu$ has threshold $\nu = m$ wherever the corresponding tubular volume is comparable to $r^m$. When $b \equiv 0$, the massless unweighted column obtained from the causal factorization in Sections 10–13 is exactly a positive constant times $(H(w) + u)^{-(d+1)}$. Thus (9) gives the precise threshold for

a finite normalized static limit in that case. These conclusions concern the $M_g$ normalization. Section 16 gives the finer laws in the flat and interval settings.

### 15.3 Strong profiles with a zero floor

All the preceding claims under (9) still hold with $\epsilon_g = 0$ if $M_g g^{\gamma_*} \to \infty$, or if the strong profile has the uniform lower bound

$$a(w,u) \ \geq c \min(1, u^\nu), \qquad 0 \leq \nu < 1. \tag{11}$$

In the first case the normalized cross product has norm at most its trace, bounded by $C M_g^{-1} g^{-\gamma_*}$ from (9) and the weighted form inequality. This includes arbitrary integrable reciprocal profiles in $d = 1$, with no extra condition on $M_g$. A vanishing floor does not affect any other part of the variational trace proof, since $a > 0$ almost everywhere under (9) and weighted multiplication is interpreted through its quadratic form. For each finite $g$ the bounded-domain compressed square-root response has a strictly positive lower bound even where contact vanishes.

For (11), weighted Schur gives a uniform operator bound on $a_g^{-1/2} C_g a_g^{-1/2}$, which handles every $M_g \to \infty$. First, the kernel of $C_g$ is nonnegative and is bounded pointwise by the facing-half-space transparent sandwich $C_\kappa^{\mathrm{H}}$. Indeed the cross columns have one sign and the compressed square-root inverses preserve positivity. The enlarged-domain inverse comparison on nonnegative forcing follows by subtracting the form equations on the smaller domain and testing with the positive part of the difference. This is an admissible supported test since the larger-domain solution is nonnegative and the difference's positive part vanishes off the smaller domain. The square-root form has the jump difference representation from subordination plus a nonnegative mass term, so this test gives the order as in the standard disjoint-sign argument. The same comparison drops a nonnegative contact. Form-dual limits at mass zero follow by positive mass regularization. The kernel $C_\kappa^{\mathrm{H}}(x, x', t)$ uses flat depths $x, x' \geq 0$ and tangential difference $t$. It decreases in each depth, since each positive cross column does by radial subordination and the inverse preserves order.

The integral in $t$ is the zero-tangential-momentum planar sandwich $C_\kappa^{(1)}(x, x')$. To justify integrability before taking Fourier value zero, first integrate the nonnegative kernel against $e^{-|t|^2/R^2}$. On the Fourier side this is a positive approximate identity acting on the continuous planar fibers. Continuity and boundedness follow from their explicit causal columns, and monotone convergence in $R$ now proves the asserted integral. The factorization in Sections 10–13 gives

$$0 \leq C_\kappa^{(1)}(x, x') \leq C_0^{(1)}(0, x') = \frac{1}{2\pi} \int_0^\infty (1+z)^{-3/2} (1 + x' + z)^{-3/2} \, dz.$$

Its integral against $1 + \mathbf{1}_{x'<1}(x')^{-\nu}$ is bounded for $\nu < 1$: near zero the kernel is bounded, and for the unweighted tail one can integrate in $x'$ first. Inside the left aperture chart, the flat depth $x'$ is at least the actual inward depth $u'$, so monotonicity allows replacing $x'$ by $u'$ in the kernel. The shear between the flat and actual depth coordinates has Jacobian one. Outside the chart the multiplier is one and the entire flat half-space bound suffices. We conclude

$$\sup_Y \int C_g(Y, Y') a_g(Y')^{-1} \, dY' \leq C.$$

The weighted Schur test with positive testing function $a_g^{-1/2}$ proves the desired operator norm bound, by approximation if the multiplier is unbounded. Equivalently it bounds $C_g^{1/2} a_g^{-1} C_g^{1/2}$.

The form inequality therefore bounds the normalized cross product's norm by $C/M_g$. Its logarithmic higher powers are negligible as before.

Finally the divergence conclusion when $\int C_b/a = \infty$ also needs no floor. Increasing the coefficient by a fixed $\delta$ in the chart gives the already proved lower estimate for the original gain by positivity of the bridge comparison. In particular the edge threshold $\nu = 1$ for the $M_g$ normalization holds at actual zero edge strength with no regularization and arbitrary $M_g \to \infty$.

## 16 A power-law zero of the strong contact

Use depth $x \geq 0$ on a half-line or on $(0, L_M)$, $L_M \to \infty$, with zero extension in the kinetic form. Let $a$ be a continuous nonnegative profile with one zero $x_0 \geq 0$. Assume $a(x)$ is bounded below by a positive constant outside each neighborhood of $x_0$, and

$$a(x) = c|x - x_0|^p(1 + o(1)), \qquad c > 0, \quad p > 0.$$

Limits at $x_0 = 0$ are from the right. Set $M \to \infty$, floor $\varepsilon_M \to 0$, $\varepsilon_M \geq 0$, and take the compressed form

$$H_M = 2\sqrt{D^2 + \kappa_M^2} + M(a + \varepsilon_M)$$

with $\kappa_M \to \kappa \geq 0$. If $f$ is an $L^2$ source continuous near $x_0$, put $Q_M(f) = \langle f, H_M^{-1} f\rangle$. The proof also works for Hilbert-valued sources. In the logarithmic and power regimes it covers varying sources uniformly bounded in $L^2$ and locally uniformly continuous near $x_0$, with a limit of their squared norms at $x_0$. In the integrable regime the varying-source version additionally uses pointwise norm convergence, uniform $L^2$ tails, and uniform local $H^1$ bounds away from the zero, all available for the columns below.

**Theorem 13.** *Let $s_0 = 2$ for an interior zero and $s_0 = 1$ for a boundary zero, and let $D_0 = \mathbb{R}$ in the interior and $D_0 = \mathbb{R}_+$ at the boundary. All operators on $\mathbb{R}_+$ below use zero extension in the full-line Fourier form.*

- *If $0 < p < 1$, $MQ_M(f) \to \int_0^\infty \|f(x)\|^2/a(x)\,dx$.*
- *If $p = 1$, put $\rho_M = \max(M^{-1/2}, \varepsilon_M)$. Then*

$$\frac{M}{\log(1/\rho_M)} Q_M(f) \longrightarrow \frac{s_0}{c}\|f(x_0)\|^2.$$

  *For zero floor this is $Q_M = (s_0/(2c))M^{-1}\log M\,\|f(x_0)\|^2 + o(M^{-1}\log M)$.*

- *If $p > 1$, put $\ell = M^{-1/(p+1)}$, $\lambda_M = M\ell\,\varepsilon_M$. If $\lambda_M \to \lambda < \infty$, then*

$$\ell^{-2} Q_M(f) \longrightarrow \|f(x_0)\|^2\, \mathcal{C}_{p,c}^{D_0}(\lambda),$$

  *where*

$$\mathcal{C}_{p,c}^{D_0}(\lambda) = \sup_v \left(2\,\mathrm{Re}\int_{D_0} v - \langle v, (2|D| + c|y|^p + \lambda)v\rangle\right),$$

  *with the supremum over smooth compactly supported tests in $D_0$, equivalently its weighted form completion. The constant is finite and positive. If $\lambda_M \to \infty$, then*

$$M\varepsilon_M^{1-1/p} Q_M(f) \longrightarrow \|f(x_0)\|^2 \int_{D_0} \frac{dy}{1 + c|y|^p} = \|f(x_0)\|^2\, \frac{s_0\pi}{pc^{1/p}\sin(\pi/p)}.$$

*The assertions concern normalized limits, including $f(x_0) = 0$. As $\lambda \to \infty$ the critical profile satisfies $\lambda^{1-1/p}\mathcal{C}_{p,c}^{D_0}(\lambda) \to \int_{D_0}(1 + c|y|^p)^{-1}dy$, matching the last regime.*

## 16.1 Proof of the three regimes

The quadratic inverse is the form supremum

$$Q_M(f) = \sup_v (2\,\mathrm{Re}\langle f, v\rangle - \langle v, H_M v\rangle).$$

It is bounded by the quadratic inverse of the positive multiplication part wherever the latter is finite. Also uniformly in $\kappa_M \geq 0$,

$$H_M \succeq c_1\, \ell^{-1} I, \qquad \ell = M^{-1/(p+1)}.$$

Indeed set $x = x_0 + \ell y$. The scaled potential $\ell M a(x_0 + \ell y)$ is bounded below by $c_2 \min(|y|^p, 1)$ on the scaled domain for large $M$, by the assumed local behavior and the positive lower bound away from the zero. The operator $2|D| + c_2 \min(|y|^p, 1)$ has a positive $L^2$ lower bound: split into frequencies $|\xi| \leq r$ and $|\xi| > r$. The former part has mass in $[-1, 1]$ at most $Cr\|v\|_2^2$ by Fourier Cauchy–Schwarz, and the latter has total squared norm bounded by $\langle v, |D|v\rangle / r$. Choose $r$ fixed small. The potential bounds mass outside $[-1, 1]$, so absorb the low-frequency term. Zero extension gives the same inequality on the half-line or on expanding bounded intervals. The floor additionally gives $H_M \succeq M\varepsilon_M I$.

For $p < 1$, the multiplication inverse upper bound is integrable by local continuity of $f$ and the bound on $a$ away from zero. For the lower bound use the form of $M^{-1}H_M$ against fixed compact smooth tests and let $M \to \infty$. The limiting multiplication variational supremum is the integral, by cutoffs avoiding the zero and density on the truncated sets. Varying Hilbert-valued sources for the trace application below can also be treated without assuming strong global convergence: truncate away from the zero and the far edge, use smooth scalar cutoffs times the sources divided by $a$ approximated by a continuous bounded function, and use uniform $H^1$ bounds of the sources on compact intervals. This permits convergence to the pointwise norm integral.

For $p = 1$, split $f$ at $|x - x_0| = \rho_M$. The inner inverse quadratic form is $O(M^{-1})$, using $\|f\|^2_{\mathrm{inner}} = O(\rho_M)$ and $\|H_M^{-1}\| \leq CM^{-1/2}$ when $\rho_M = M^{-1/2}$, or $\|H_M^{-1}\| \leq (M\varepsilon_M)^{-1}$ otherwise. The outer inverse is bounded by $M^{-1} \int_{\mathrm{outer}} \|f\|^2/(a + \varepsilon_M)$. Its leading logarithm is at most $(s_0/c)M^{-1} \log(1/\rho_M)\|f(x_0)\|^2$, with the coefficient verified by integrating $1/r$ on each side between $\rho_M$ and fixed small depth and then shrinking the upper depth. The inner-outer cross term is of smaller order by Cauchy–Schwarz for the positive inverse. For a lower test, choose

$$v(x) = \frac{f(x_0)}{Mc|x - x_0|}$$

with smooth cutoffs to depths between $\rho_M$ and a small fixed upper depth. Its potential and linear terms produce the same logarithm, up to an arbitrarily small relative error by continuity of $f$ and the asymptotic of $a$. The additional floor energy is $O(\varepsilon_M/(M\rho_M)) = O(M^{-1})$. The compressed kinetic cost satisfies $\langle v, |D|v\rangle \leq \|v\|_2\|v'\|_2 = O(M^{-2}\rho_M^{-2}) = O(M^{-1})$, and the fixed bounded mass adds at most $2\kappa_M\|v\|^2 = o(M^{-1})$. This proves the logarithmic coefficient including its one-sided boundary factor.

For $p > 1$ and finite $\lambda$, the inverse quadratic form divided by $\ell^2$ is that of the rescaled operator

$$2\sqrt{D_y^2 + (\ell\kappa_M)^2} + \ell M a(x_0 + \ell y) + \lambda_M$$

on the source $f(x_0 + \ell y)$, by unitary dilation. On every bounded scaled interval the potential converges uniformly to $c|y|^p + \lambda$. The form inverses converge on compactly supported scalar sources: for the lower limit use compact smooth tests, which eventually belong to the scaled

domains. For the upper limit their maximizing vectors have bounded energy and $L^2$ norm by uniform coercivity, hence weak $H^{1/2}$ and strong local $L^2$ subsequences. The supported limit belongs to the full-line or half-line form space, and restricting potentials to compact sets then growing the sets gives the lower semicontinuity of the limiting energy. The mass change is bounded in form by $2\ell\kappa_M\|v\|^2$. Smooth compact tests are dense in the limit form: cut off at infinity with a smooth scaled cutoff in $H^{1/2}$ and weighted $L^2$; for a supported ray translate the cut-off function slightly inward and mollify without crossing the boundary. Translation and approximation are valid in full-line $H^{1/2}$, and on the compact support the potential is bounded. This proves equality of the variational limits.

For the rescaled inverse quadratic form, the part of the source outside $|y| \le R$ has squared dual norm at most $CR^{1-p} + C\ell^{p-1}\|f\|_2^2$, by the multiplication bound. In the local region use $\ell Ma(x_0+\ell y) \ge c_3|y|^p$, and on the remaining region use the lower bound $c_4\ell^{-p}$. Cross terms tend uniformly to zero as $R \to \infty$. On each fixed scaled compact set the source differs in $L^2$ by $o(1)$ from the constant vector $f(x_0)$, so vector-valued sources reduce there to the scalar variational comparison. The same tail estimate proves that the constant source is in the dual form space of the limit, and the constant is strictly positive by a nonzero nonnegative test. For the large-$\lambda$ limit of the limiting constant, rescale $y = \lambda^{1/p}z$; the kinetic coefficient relative to $\lambda$ becomes $\lambda^{-1-1/p}$, giving the multiplication limit by the same upper bound and compact tests.

When $\lambda_M \to \infty$, rescale instead $x = x_0 + \varepsilon_M^{1/p}y$ and divide the kinetic plus potential by $M\varepsilon_M$. The kinetic coefficient $1/(M\varepsilon_M^{1+1/p}) = \lambda_M^{-(p+1)/p}$ tends to zero, and the mass coefficient $\kappa_M/(M\varepsilon_M)$ tends to zero. The potential tends to $1+c|y|^p$ uniformly on compacts, and the exact multiplication upper bound and compact smooth lower tests give its inverse integral. The omitted portion outside scaled radius $R$ is again bounded in squared dual norm by $CR^{1-p}+C\varepsilon_M^{1-1/p}\|f\|^2$, using the fixed positive bound away from $x_0$. This proves the floor-dominated formula.

## 16.2 The connected trace and energy

Between two inward rays at separation $h > 0$, let $B_{\kappa,h}(x,y) = f_{\kappa,h+x}(y)$ be the separated column kernel defined in Section 15. The diagonal response $A_\kappa$ is the supported compression of $2\sqrt{D^2+\kappa^2}$. For weak constant $b \ge 0$ let

$$F_{\kappa,h,b}(x) = (A_\kappa + b)^{-1/2}B_{\kappa,h}^*\delta_x, \qquad w_{\kappa,h,b}(x) = \|F_{\kappa,h,b}(x)\|^2.$$

Use the form-dual interpretation at $\kappa = b = 0$. Then the first connected determinant trace with strong profile $M(a+\varepsilon_M)$ is $Q_M(F)$, acting on the left depth with a Hilbert-valued source. The transparent half-line factorization in Sections 10–13 gives

$$w_{\kappa,h,b}(x) \le w_{\kappa,h,0}(x) = \frac{1}{2\pi}\int_{h+x}^{\infty} e^{-2\kappa r}r^{-3}\,dr.$$

It also bounds the source derivative norm in $x$ by differentiating the explicit backward-convolved separated kernel, and hence gives uniform local equicontinuity and global $L^2$ bounds with exponential mass decay for fixed $h$. The same statements hold for finite right interval compressions, by the variational upper bound on each column and every difference or derivative column. The compressed column norms converge pointwise when the right interval expands and its constant contact tends to $b$, by the variational comparison in Sections 10–13. This remains true along $\kappa_M \to \kappa$: for a positive limiting mass or contact use fixed test convergence and lower comparison with the nearby positive mass or contact; at zero mass and contact use the transparent massless upper bound and fixed compact tests for the lower bound. Cross columns converge in

the transparent dual norm by the same factorization or dominated backward convolution. For the $p < 1$ lower bound with a varying source, the column derivative estimates make compact cutoff tests $v = \chi(x)F_M(x)/a_{\mathrm{appr}}(x)$ uniformly bounded in $H^1$, where $a_{\mathrm{appr}}$ is a smooth positive approximation to $a$ on the support away from its zero. This validates the source condition used above.

The normalized connected cross product has norm at most $C\ell$ times the bound on the weak-side sandwich, by the strong spectral gap, uniformly for finite left/right intervals. Its higher trace terms are at most $O(\ell)$ times its first trace, so all source limits apply to the full minus-log determinant. The three normalized strong-side profiles are therefore respectively a weighted integral of $w$, a logarithmic coefficient times $w(x_0)$, and a fractional-oscillator coefficient times $w(x_0)$, with the additional floor crossover as above. Exponential mass domination follows from the column bound and the upper estimates in the source proof, with uniform constants.

For fixed gap-scaled separation $h$, the flat density per unit tangent area in a screen of dimension $d$ follows by tangential Fourier integration of these formulas with mass $\sqrt{\kappa^2 + |k|^2}$, as in Sections 10–13. For transparent weak contact the static point coefficient is

$$w_{d,0}(0, h; x_0) = \frac{\Gamma(d/2)}{2^d \pi^{(d+2)/2}(d+1)}(h + x_0)^{-d-1},$$

and its integral against $d\kappa/(2\pi)$ is

$$\frac{\Gamma((d+1)/2)}{2^{d+2}\pi^{(d+3)/2}(d+2)}(h + x_0)^{-d-2}.$$

These are minus the separation derivatives of the static and vacuum transparent constants, since its full trace integrates the depth density from zero to infinity.

In the physical $d = 1$ interval model at gap $g$, use $h = 1$ and prescribe the strong gap-scaled contact $M_g(a(x) + \varepsilon_g)$ in inward depth $x$ measured in units of $g$, with $M_g \to \infty$. The opposing contact converges to finite $b$. Let $S_g$ be the normalizing factor from the corresponding source regime ($M_g$, $M_g/\log(1/\rho_g)$, $\ell_g^{-2}$, or $M_g\varepsilon_g^{1-1/p}$). Then $S_g q_g(\kappa) \to P(\kappa)$ from the preceding trace formulas, along compact $\kappa$-converging sequences and with a common exponential tail. The vacuum limit is $gS_gE_g \to -(2\pi)^{-1}\int_0^\infty P(\kappa)\,d\kappa$. At inverse temperature $\beta = \lambda g$, the same normalization gives $-(2\lambda)^{-1}\sum_{m\in\mathbb{Z}} P(2\pi|m|/\lambda)$. At fixed positive $\beta$, monotone mesh comparison for the positive transformation of the all-hit heat formula gives the same vacuum equivalent: the mesh is $2\pi g/\beta$ and its error is bounded by the mesh times the uniformly bounded normalized value at $\kappa = 0$. The nonnegative variable contact enters the all-hit formula through the exponential of its local-time integral. Curved higher-dimensional apertures with a shrinking strong-side zero layer require a separate simultaneous geometry limit; the flat-tangent density above does not claim that extension.

### 16.3 The Airy coefficient for a quadratic interior zero

For $p = 2$ and an interior zero, the full-line constant $\mathcal{C}^{\mathbb{R}}_{2,c}(\lambda)$ admits an exact Airy expression. The form solution $v$ to

$$(2|D| + cx^2 + \lambda)v = 1$$

is $L^1$ by its $L^2$ and $xv \in L^2$ bounds. With Fourier convention $\hat{v}(\xi) = \int e^{-ix\xi}v(x)\,dx$ it satisfies

$$-c\hat{v}'' + (2|\xi| + \lambda)\hat{v} = 2\pi\delta_0.$$

For $\xi > 0$ the decaying solution is a multiple of $\mathrm{Ai}(a\xi + b)$, $a = (2/c)^{1/3}$, $b = \lambda/(ca^2)$. The solution is even by uniqueness of the form solution. The derivative jump $-2c\hat{v}'(0+) = 2\pi$ gives

$$\mathcal{C}^{\mathbb{R}}_{2,c}(\lambda) = \hat{v}(0) = -\frac{\pi}{2^{1/3}c^{2/3}} \frac{\mathrm{Ai}(\lambda/(2^{2/3}c^{1/3}))}{\mathrm{Ai}'(\lambda/(2^{2/3}c^{1/3}))}.$$

This directly gives

$$\mathcal{C}^{\mathbb{R}}_{2,c}(0) = \frac{\pi\,\Gamma(1/3)}{6^{1/3}c^{2/3}\Gamma(2/3)}, \qquad \mathcal{C}^{\mathbb{R}}_{2,c}(\lambda) \sim \pi/\sqrt{c\lambda}.$$

The Airy spectral problem for the full-line massless relativistic oscillator is treated by [28]. Here the Fourier source is a point mass and its forced resolvent value is the aperture coefficient. The values at zero and the asymptotics used here are given in [12, Sections 9.2 and 9.7]. The boundary-zero coefficient corresponds to a different supported half-line form.

## 17 Two independently strong degenerate contacts

On each inward ray (or each expanding finite interval in scaled depth), take a profile $a_i$ with one zero $x_i$, exponent $p_i > 0$, leading coefficient $c_i > 0$, and floor $\epsilon_i \to 0$ as in the source theorem. Let $M_i \to \infty$ independently. In any of the allowed regimes of that theorem let $S_i$ denote the normalization of Theorem 13, let $G_i = (A_\kappa + M_i(a_i + \epsilon_i))^{-1}$, and define a positive measure $\sigma_i$ on the closed inward ray $[0, \infty)$ as follows.

- For $p_i < 1$, $\sigma_i(dx) = a_i(x)^{-1}\,dx$.
- For $p_i = 1$, $\sigma_i = (s_{0,i}/c_i)\delta_{x_i}$, with the logarithmic normalization $M_i/\log(1/\rho_i)$.
- For $p_i > 1$ and a finite scaled floor, $\sigma_i = \mathcal{C}^{D_i}_{p_i,c_i}(\lambda_i)\delta_{x_i}$.
- In the floor-dominated power regime, $\sigma_i = (\int_{D_i}(1 + c_i|y|^{p_i})^{-1}\,dy)\delta_{x_i}$.

**Theorem 14.** *The first connected trace has the joint limit*

$$S_-S_+\ \mathrm{Tr}(G_-B_{\kappa,h}G_+B^*_{\kappa,h}) \longrightarrow \iint |B_{\kappa,h}(x,y)|^2\,\sigma_-(dx)\sigma_+(dy). \tag{12}$$

*The rates $M_-, M_+$ need no coupling. The same holds for the full minus-log determinant. This gives the flat density per tangential area by Fourier integration in all screen dimensions, and physical interval vacuum and thermal laws in dimension one with the energy factors of Section 16, with $S_g = S_{-,g}S_{+,g}$.*

We use a positive-kernel argument that includes point evaluations at the boundary. Fix $w(x) = (1+x)^{-1}$, and define the finite measure $d\mu_i^{(M)}(x,x') = S_i w(x)G_i(x,x')w(x')\,dx\,dx'$. The kernel is nonnegative. Indeed the restricted square-root form is the zero-exterior part of the subordinate Brownian jump form with a nonnegative mass term. Its semigroup with additional nonnegative multiplication is positive and dominated on nonnegative functions by the full-line translation-invariant semigroup. The domination can also be obtained by testing resolvent differences with positive parts as in the depth-profile proof, followed by the semigroup product limit. The full-line semigroup has a density for each positive time, by the usual square-root subordination of the heat kernel. Thus the dominated semigroup and its positive resolvent have nonnegative kernels, with no pointwise bound at their diagonal needed for the weighted finite measure. The total mass of $\mu_i^{(M)}$ is $S_iQ_{M_i}(w)$, uniformly bounded by the source theorem.

For compact continuous tests $\phi, \psi$ of the two variables, polarization of the source theorem shows that $\int \phi(x)\psi(x')\, d\mu_i^{(M)} \to \int \phi(x)\psi(x)w(x)^2\, d\sigma_i(x)$. The measures are tight: for $R > x_i + 1$, the first marginal on $x > R$ is at most

$$[S_i Q_{M_i}(w 1_{x>R})]^{1/2} [S_i Q_{M_i}(w)]^{1/2},$$

and the first term under the square root is bounded by $(S_i/M_i) \int_{x>R} w^2/a_i = O(\int_{x>R} w^2)$. Here $S_i/M_i$ stays bounded in all regimes. The second marginal is the same by symmetry. The product tests and tightness give weak convergence of the measures to the weighted diagonal measure, including its possible point concentration.

The trace is the tensor-product quadratic form of the real $L^2$ cross kernel for $G_- \otimes G_+$. Its kernel integrand uses the two positive Green kernels times $B(x', y)\overline{B(x, y')}$. This product is nonnegative since the cross kernel has a single real sign; the formula follows first for rectangle step kernels and then by monotone nonnegative approximations to $|B|$ and convergence in $L^2$. Divide each $B$ by its two depth weights to express the normalized trace as an integral against $\mu_-^{(M)} \otimes \mu_+^{(M)}$. The new test

$$\frac{B_{\kappa,h}(x', y)\overline{B_{\kappa,h}(x, y')}}{w(x)w(x')w(y)w(y')}$$

is bounded and continuous. Indeed for $r = h + x + y$, the Bessel Laplace representation gives

$$|B_{\kappa,h}(x, y)| \le C e^{-\kappa r/2} r^{-2}$$

uniformly in $\kappa \ge 0$, by bounding $\sqrt{s^2 - \kappa^2} \le s$ and integrating. Also $(1 + x)(1 + y) \le C_h r^2$. This test is continuous as $\kappa$ varies, uniformly on compact depth sets, and retains a common exponential bound in mass. Thus weak product convergence proves (12) along compact $\kappa$-converging sequences, with exponential domination by the uniformly bounded measure masses.

The norm of the positive connected product is at most $C_h \ell_- \ell_+$, where $\ell_i = M_i^{-1/(p_i+1)}$, by the two strong inverse spectral gaps and bounded $B$. Hence the determinant remainder is relatively $O(\ell_- \ell_+)$ compared to the first trace. Tangential Fourier integration for the flat model and the interval vacuum and Matsubara limits follow from the exponential bound. At fixed physical temperature use the same monotone mesh comparison of the all-hit heat transform as in the source theorem.

When both limiting measures are point masses with weights $m_-, m_+$, the static coefficient in a one-dimensional screen is $m_- m_+ \cdot 4/(\pi^2 r_0^4)$, $r_0 = h + x_- + x_+$. The vacuum integral of the positive trace coefficient is

$$m_- m_+ \frac{3}{16\pi r_0^5}.$$

To check the latter, $\int_0^\infty t^2 K_1(t)^2\, dt = 3\pi^2/32$. Indeed insert $K_1(t) = (t/4) \int_0^\infty u^{-2} e^{-u - t^2/(4u)} du$, integrate the resulting $t^4$ Gaussian, then use the sum and proportion of the two Schwinger variables. The remaining product is $\Gamma(5/2)\Gamma(1/2)\mathrm{B}(3/2, 3/2) = 3\pi^2/32$. Multiplying by $4/(\pi^2 r_0^5)$ and $1/(2\pi)$ gives the expression. The physical vacuum energy carries the minus sign of the connected determinant convention.

## 18 Weak contact at interval apertures: setup and main limits

Use the connected scalar Dirichlet-screen model of Section 10. The two interval apertures have inward depth variables in $(0, L_-)$, $(0, L_+)$, in units of a length $g$; their separation is $h > 0$ in those

units. In the physical fixed-length example with actual gap $g$, $h = 1$ and $L_\pm = \ell_\pm/g$. At mass $\kappa$, $A_\pm$ are the compressed operators $2\sqrt{D^2+\kappa^2}$, with zero extension, and the cross operator $B$ has kernel

$$-\frac{2\kappa K_1(\kappa(h+u+v))}{\pi(h+u+v)}$$

for $\kappa > 0$, and $-2/[\pi(h+u+v)^2]$ at zero. The contact strengths here are scaled by $g$. For left strength $M$ and right strength $b$, set

$$Q_{M,b} = (A_- + M)^{-1/2} B (A_+ + b)^{-1} B^* (A_- + M)^{-1/2}, \qquad q_{M,b}(\kappa) = -\log\det(I - Q_{M,b}).$$

The vacuum energy is $-1/(2\pi g)$ times the integral of $q$ over positive mass. At physical inverse temperature $\beta$ the free energy is $-(2\beta)^{-1}\sum_{j\in\mathbb{Z}} q_{M,b}(2\pi|j|g/\beta)$.

**Theorem 15.** *Let $L_\pm \to \infty$, $\eta \downarrow 0$, and $M \to \infty$, at independent rates. Put*

$$T = \min\{L_+, \eta^{-1}\}, \qquad l = \log(T/h), \qquad m = \log((h+L_-)/h), \qquad D_* = l^2 - (l-m)_+^2.$$

*For the first trace $W(\eta) = \mathrm{Tr}[B(A_+ + \eta)^{-1}B^*]$ at $\kappa = 0$,*

$$W(0) - W(\eta) \sim \eta D_*/(2\pi^2). \tag{13}$$

*For the exact transform,*

$$M[q_{M,0}(0) - q_{M,\eta}(0)] \sim \eta D_*/(2\pi^2). \tag{14}$$

*The joint vacuum energy difference satisfies*

$$\frac{gM}{\eta}(E_{M,\eta} - E_{M,0}) \longrightarrow \frac{1}{8\pi^3 h}.$$

*With $\lambda_g = \beta/g$, if $\lambda_g \to \infty$, the free-energy difference satisfies*

$$\frac{gM}{\eta}(\mathcal{F}_{M,\eta} - \mathcal{F}_{M,0}) = \frac{1}{8\pi^3 h} + \frac{D_*}{4\pi^2\lambda_g} + o(1 + D_*/\lambda_g).$$

*If $\lambda_g$ is bounded above, the second term alone is the leading equivalent for the normalized difference.*

For comparable scaled lengths $D_* \sim \log^2(T/h)$. The static formula and exact determinant comparison remain valid if the left length instead stays bounded below by a positive constant. The short interval can then limit the leading logarithm through $m$. The vacuum and temperature formulas above use both lengths tending to infinity.

General delta-profile boundary determinants occur for inhomogeneous parallel mirrors [14], and perturbations about a constant background are studied for a single inhomogeneous mirror [27]. The finite-interval inverse below uses the fractional Poisson kernel [29]. In the present separated connected trace it cuts off a static long-depth response. The relative error from the strong side is estimated after taking the weak-contact difference. This is what permits independent rates.

# 19 Weak-contact response on the half-line

The depth profiles in this section are defined on the full rays. For the limiting static half-line model, write $W_{\eta p}(0,h)$ for the trace profile with weak multiplication contact $\eta p(v)$, with $p$ bounded, measurable and nonnegative. The parameter $\eta \downarrow 0$ is the weak amplitude.

If $p(v) \to p_\infty$ as $v \to \infty$, then

$$\frac{W_0(0,h) - W_{\eta p}(0,h)}{\eta \log^2(1/(\eta h))} \longrightarrow \frac{p_\infty}{2\pi^2}. \tag{15}$$

If instead $p$ is integrable against the weight in (16), the correct first variation is finite and exact:

$$\lim_{\eta \downarrow 0} \frac{W_0(0,h) - W_{\eta p}(0,h)}{\eta} = \frac{1}{\pi^2} \int_0^\infty \frac{p(v)}{v} \left[ \log(1 + v/h) - \frac{v}{v+h} \right] dv. \tag{16}$$

In particular this holds for compactly supported $p$. For general bounded $p \geq 0$ the limit in (16) holds in the extended positive reals.

## 19.1 Exact column and variational deficit

Let $A_q$ be the zero-extension ray compression of $2\sqrt{D^2+q^2}$ and $\mathfrak{a}_q$ its quadratic form. Write $f_{q,\ell}$ for the cross column at separation $\ell = h + u$, with $u$ the left depth. Its transparent inverse is

$$V_{q,\ell}(v) := A_q^{-1} f_{q,\ell}(v) = -\frac{\sqrt{v}}{\pi\sqrt{\ell}(\ell+v)} e^{-q(\ell+v)}. \tag{17}$$

In the half-line factorization of Section 11, $L_q^{-*} f_{q,\ell}(v) = -e^{-q(\ell+v)}/[\sqrt{\pi}(\ell+v)^{3/2}]$, and $A_q^{-1} f = \frac{1}{2} L_q^{-1} L_q^{-*} f$. Multiply out the two exponentials and use

$$\int_0^v (v-s)^{-1/2} (\ell+s)^{-3/2}\, ds = \frac{2\sqrt{v}}{\sqrt{\ell}(\ell+v)}.$$

This also directly yields the supported massless energy representative. The per-column loss $D_{\eta p}(q,\ell)$ is exactly

$$D_{\eta p}(q,\ell) = \inf_z \left\{ \mathfrak{a}_q[z - V_{q,\ell}] + \eta \int_0^\infty p|z|^2 \right\}, \tag{18}$$

from the variational supremum by completing the energy square. For each column $D_{\eta p}/\eta$ increases as $\eta \downarrow 0$, by taking the infimum of $\mathfrak{a}_q/\eta + \int p|z|^2$. If $\int p|V_{q,\ell}|^2 < \infty$, testing with $V_{q,\ell}$ gives the upper bound. For a finite bounded subsequential ratio, near-minimizing $z$ converges in energy to $V_{q,\ell}$, hence in local $L^2$ by Hardy or the mass gap. Fatou gives the lower bound even if the response integral is infinite. Therefore the limit is the integral of $p|V_{q,\ell}|^2$, finite or infinite. Monotone convergence in left depth proves the integrated response, and integration of the explicit squared massless column uses

$$\int_h^\infty \frac{d\ell}{\ell(\ell+v)^2} = v^{-2} \left[ \log(1 + v/h) - \frac{v}{v+h} \right].$$

### 19.2 Truncation bounds

Write

$$S_p(R) = \frac{1}{\pi^2}\int_0^R \frac{p(v)}{v}\left[\log(1+v/h) - \frac{v}{v+h}\right] dv, \qquad L = \log(1/\eta).$$

For $\eta$ sufficiently small the integrated loss satisfies

$$\frac{\eta S_p(\eta^{-1}L^{-2})}{1+CL^{-2}} \le W_0(0,h) - W_{\eta p}(0,h) \le \eta S_p(2\eta^{-1}L^2) + O(\eta/L). \tag{19}$$

The constants may depend on $h$ and the profile bound. This estimates nonconstant and vanishing tails directly.

For the lower bound in (19), keep only $(0,R)$ in the weighted contact term for each column. Write $B_\ell = \|p^{1/2}V_{0,\ell}\|_{L^2(0,R)}$, $y = \mathfrak{a}_0[z - V_{0,\ell}]^{1/2}$ and $c_R = (C\|p\|_\infty R)^{1/2}$. Hardy and the reverse triangle inequality bound the objective in (18) below by $y^2 + \eta(B_\ell - c_R y)_+^2$, whose minimum for $y \ge 0$ is $\eta B_\ell^2/(1+\eta c_R^2)$. Integrate over $\ell$, then set $R = \eta^{-1}L^{-2}$.

For the upper bound take $z = \chi(v/R)V_{0,\ell}$ at $h \le \ell \le R$ and $z = 0$ for $\ell > R$, with $0 \le \chi \le 1$ equal to one up to one and zero past two. The lost energy for $\ell \le R$ is $O(1/(R\ell))$. Indeed after $v = Ry$ the omitted tail has amplitude factor $(R\ell)^{-1/2}$ times a function zero below one, bounded by $Cy^{-1/2}$, with derivative bounded by $Cy^{-3/2}$ at infinity, uniformly for $\ell/R \in [0,1]$. Its difference-quotient seminorm is uniformly bounded: for $|x-y| \le x/2$ use the derivative bound and integrate $O(x^{-2})$ in $x$ away from zero. On the complementary positive quadrant $|x-y| \gtrsim x, y$, so the squared values integrate against $dx/x$. Interactions with the negative half-line also integrate against $dx/x$. Homogeneous energy at this order is invariant under scaling the argument, leaving the squared amplitude $1/(R\ell)$. For $\ell > R$ the whole-column loss is at most $1/(4\pi\ell^2)$ by the causal column trace of Sections 10–13. Thus the integrated energy error is $O((1+\log R)/R)$. The potential error is at most $\eta S_p(2R)$. Take $R = \eta^{-1}L^2$ to finish (19).

If $p(v) \to p_\infty$, then $S_p(R)/\log^2 R \to p_\infty/(2\pi^2)$ by the elementary logarithmic integral and tail comparison. Since both cutoff logarithms in (19) equal $L + O(\log L)$, this gives (15), including $p_\infty = 0$. More generally if $p(v) \sim p_0(\log v)^{-\sigma}$ at infinity with $p_0 > 0$, then the deficit has equivalents

$$\frac{p_0}{\pi^2(2-\sigma)}\eta L^{2-\sigma} \quad (0 \le \sigma < 2), \qquad \frac{p_0}{\pi^2}\eta \log L \quad (\sigma = 2).$$

For $\sigma > 2$, (16) supplies a finite linear derivative. These coefficients come from $S_p$, whose tail integrand is asymptotic to $p_0(\log v)^{1-\sigma}/(\pi^2 v)$.

## 20 The joint static limit

### 20.1 The finite-interval inverse column

Write $\ell = h + u$, $R = L_+$. The right forcing is $f_\ell(v) = -2/[\pi(\ell+v)^2]$ for $0 < v < R$. Its transparent inverse has zero exterior extension and the explicit formula

$$V_\ell^R(v) = (A_+)^{-1}f_\ell(v) = -\frac{\sqrt{v}}{\pi\sqrt{\ell}(\ell+v)}\sqrt{\frac{R-v}{R+\ell}}, \qquad 0 < v < R. \tag{20}$$

Indeed the positive of (20) is the interval Poisson kernel at the exterior point $-\ell$. The ball formula with $n = 1, s = 1/2$, recentered at $R/2$, gives the square-root factor and $1/\pi$. For rigor, take a smooth unit-mass exterior source supported in a shrinking neighborhood of $-\ell$,

solve the homogeneous real-line half-Laplacian problem with that exterior value, and subtract the source to leave the zero extension. The zero-interior source enters the inner equation with kernel $-2/(\pi|v-y|^2)$, so the remaining interior solution solves the compressed problem with its opposite sign. The forcing converges in $L^2(0,R)$ and the compressed inverse is bounded for each finite $R$, by the exterior part of the energy. The Poisson formula converges pointwise to its value at $-\ell$. Hence it gives (20) with the sign shown. The Poisson formula and its exterior-data theorem are [29, Eq. (1.14) and Theorem 2.10].

The limiting half-line column is $V_\ell^\infty(v) = -\sqrt{v}/[\pi\sqrt{\ell}(\ell+v)]$, also given directly by the two causal integrals in Section 19. For every $b > 0$, the weak-inverted finite-interval column $v_{b,\ell} = (A_+ + b)^{-1} f_\ell$ is nonpositive and

$$|v_{b,\ell}| \le \left|(A_{\mathbb{R}_+} + b)^{-1} f_\ell^\infty\right|$$

on the finite interval. To see the order use positive forcing and test the difference of the form equations by the positive part of the finite-interval solution minus the half-line solution. This test is supported in the finite interval by nonnegativity of the half-line solution. The cross energy of positive and negative parts is nonpositive for the jump form, and adding mass $b$ preserves the inequality. Inverses preserve positivity by energy contraction under absolute values. The same argument also applies at $b = 0$ in the supported homogeneous space, or follows here from (20).

### 20.2 Matching cutoffs for the first trace

Let

$$S_L(X) = \int_h^{h+L} \int_0^X |V_\ell^\infty(v)|^2 \, dv \, d\ell.$$

By explicit integration in $v$,

$$\begin{aligned} S_L(X) &= \frac{1}{\pi^2} \int_h^{h+L} \left[\log(1 + X/\ell) - \frac{X}{X+\ell}\right] \frac{d\ell}{\ell} \\ &= \frac{l_X^2 - (l_X - m_L)_+^2}{2\pi^2} + O(1 + \min\{l_X, m_L\}), \end{aligned} \tag{21}$$

Here $l_X = \log(X/h)$ and $m_L = \log((h+L)/h)$. The estimate is uniform for $X > h$. For $\ell \le X$ the bracket is $\log(X/\ell) + O(1)$, and its integral for $\ell > X$ is $O(1)$ using the bound $O((X/\ell)^2)$. Evaluating (21) at $X = T\log^{\pm 2} T$, or at twice that value, changes its leading term by relative $o(1)$. In fact the main expression is of order $l_X \min(l_X, m_L)$, its slope in $l_X$ is $2\min(l_X, m_L)$ before the factor $1/(2\pi^2)$, and the change of $l_X$ is $O(\log\log T)$.

For the lower bound on the per-column loss, use the finite-interval identity

$$D_\eta^R(\ell) = \inf_z \{\mathfrak{a}[z - V_\ell^R] + \eta\|z\|^2\},$$

where $\mathfrak{a}$ is the zero-extension form of $2|D|$. The ray Hardy bound, obtained by retaining interactions against the negative ray, gives $\|z - V_\ell^R\|_{L^2(0,U)} \le (CU\mathfrak{a}[z - V_\ell^R])^{1/2}$ if $U < R$. Minimize $y^2 + \eta(\|V_\ell^R\|_{L^2(0,U)} - \sqrt{CU}\, y)_+^2$ over $y \ge 0$. This gives

$$D_\eta^R(\ell) \ge \frac{\eta}{1 + C\eta U} \int_0^U |V_\ell^R(v)|^2 \, dv.$$

Set $U = T/\log^2 T$ and restrict to $\ell \le U$. The squared finite-to-infinite column ratio on this region is $1 + O(U/R) = 1 + O(1/\log^2 T)$. In $S_{L_-}(U)$ the part with $\ell > U$ is bounded by a

constant, because its integrand is at most $v/(\pi^2\ell^3)$. Thus the finite loss integrated in $\ell$ has lower bound $\eta(1 - O(1/\log^2 T))(S_{L_-}(U) - O(1))$.

For the upper bound when $T = R$, use the trial $z = V_\ell^R$, giving total loss at most $\eta S_{L_-}(R)$. When $T = \eta^{-1}$, comparison of the positive weak inverses gives $D_\eta^R(\ell) \le D_\eta^\infty(\ell)$. In detail, for $b > 0$ the derivative of the per-column quadratic inverse is minus the squared column norm. The cumulative loss integrates these norms over $0 < b < \eta$, by the resolvent identity and continuity at zero in the form-dual expression. The finite-domain norm is bounded by the full-ray norm by positivity. Now take the half-line cutoff used in (19), but integrate only over $h \le \ell \le h + L_-$. With $X = \eta^{-1}\log^2(1/\eta)$ this gives the bound $\eta S_{L_-}(2X) + O(\eta/\log(1/\eta))$, since the integrated cutoff energy is $O((1+\log X)/X)$ even over the whole ray. The lower and upper bounds with (21) prove (13). The same argument permits $L_-$ to stay bounded away from zero without tending to infinity, using the displayed two-log expression.

### 20.3 The exact determinant

Let $D_- = A_- + M$ and $Q_b = D_-^{-1/2}B(A_+ + b)^{-1}B^*D_-^{-1/2}$. The positive transparent full-ray sandwich bounds the compressed sandwich in operator norm by the transparent trace $1/(4\pi h)$ of Theorem 9, even when the right interval grows. Thus $0 \le Q_b \le (C_h/M)I$. For $M$ large one can differentiate the Fredholm logarithm on $b \in [0, \eta]$, at fixed finite intervals. The identity

$$-\partial_b q_{M,b}(0) = \operatorname{Tr}\left[(I - Q_b)^{-1}D_-^{-1/2}B(A_+ + b)^{-2}B^*D_-^{-1/2}\right]$$

has positive trace factors. The bound $I \le (I - Q_b)^{-1} \le (1 - C_h/M)^{-1}I$ therefore gives a relative $1 + O(M^{-1})$ error between $M[q_{M,0}(0) - q_{M,\eta}(0)]$ and

$$\operatorname{Tr}(R_-\Delta_\eta), \quad R_- = M(A_- + M)^{-1}, \quad \Delta_\eta = \int_0^\eta B(A_+ + b)^{-2}B^*\,db.$$

Here $\operatorname{Tr}\Delta_\eta = W_{L_-,L_+}(0) - W_{L_-,L_+}(\eta)$. This comparative estimate scales with the deficit itself.

To control $I - R_-$, use the Hilbert–Schmidt columns of $\Delta_\eta = VV^*$, with right input labels $v, b$, $0 < b < \eta$. The kernel of $V$ is $v_{b,\ell}(v)$, with left coordinate $\ell = h + u$. By positivity and the pointwise bound $|\partial_\ell f_\ell| \le (2/h)|f_\ell|$,

$$|\partial_\ell v_{b,\ell}(v)| \le (2/h)|v_{b,\ell}(v)|.$$

The derivative statement can first be taken in $L^2(0, R)$, since $f_\ell$ is smooth there in $\ell$ and the finite inverse is bounded. The pointwise inequality then follows from applying the positive inverse to the two forcing inequalities. In particular the integrated squared left-derivative norm is at most $(2/h)^2 \operatorname{Tr}\Delta_\eta$.

The diagonal loss density satisfies, uniformly for $\ell \ge h$,

$$\int_0^\eta \|v_{b,\ell}\|_2^2\,db \ \le C_h\eta(1 + \log T).$$

When $T = R$, bound it by the transparent norm integrated only to $R$, using (20) and positivity in $b$. When $T = \eta^{-1}$, compare to the half-line column at $\ell = h$ by forcing monotonicity and the domain comparison. For this single column the half-line energy cutoff at $X = \eta^{-1}$ has lost energy $O_h(\eta)$ and potential norm cost $O_h(\eta(1 + \log(1/\eta)))$, by the same tail bound as in (19).

Choose $0 \le \chi \le 1$ on the left interval, zero on its endpoints, and equal to one outside depth $\delta$ of both endpoints, with derivative bounded by $C/\delta$. For $L_- \to \infty$ take $\delta = 1$ eventually. The removed squared norm is $O_h(\delta\eta(1 + \log T))$, which is $o(\operatorname{Tr}\Delta_\eta)$ for fixed $\delta$ since $\min(l, m) \to \infty$.

The retained columns extend by zero to $H^1$, with squared form cost at most $C_h(1+\delta^{-2})\operatorname{Tr}\Delta_\eta$ using the derivative estimate and $2|\xi| \le 1+|\xi|^2$. Finally

$$\operatorname{Tr}[(I-R_-)\Delta_\eta] = \|(I-R_-)^{1/2}V\|_{\mathrm{HS}}^2 \le 2\|(1-\chi)V\|_{\mathrm{HS}}^2 + 2M^{-1}\|A_-^{1/2}\chi V\|_{\mathrm{HS}}^2.$$

Both terms are relatively negligible. If $L_-$ is merely bounded below, take $\delta = M^{-1/4}$ eventually and use that $\operatorname{Tr}\Delta_\eta \gtrsim_h \eta\log T$ with a constant depending on the lower left length. The two terms are still relatively negligible. This proves (14).

# 21 Vacuum and temperature crossover

## 21.1 Positive mass and vacuum

For the current parameter sequence put

$$F(\kappa) = \frac{M}{\eta}[q_{M,0}(\kappa) - q_{M,\eta}(\kappa)], \qquad G_- = (A_- + M)^{-1}.$$

The forcing $f_{\kappa,\ell}(v)$ is the cross column at left depth $\ell - h$.

For $\kappa > 0$, the half-line column at separation $\ell = h + u$ is

$$V_{\kappa,\ell}(v) = -\frac{\sqrt{v}}{\pi\sqrt{\ell}(\ell+v)}e^{-\kappa(\ell+v)}.$$

Along converging positive masses and any geometric/contact rates as above,

$$F(\kappa) \longrightarrow J(\kappa) = \int_h^\infty d\ell \int_0^\infty |V_{\kappa,\ell}(v)|^2\, dv. \tag{22}$$

For proof, differentiate the exact connected determinant in right strength $b$, as in Section 20. The derivative is the trace of $(I-Q_b)^{-1}$ against the positive form $G_-^{1/2}B(A_+ + b)^{-2}B^*G_-^{1/2}$. The norm bound $\|Q_b\| \le C_h/M$, uniform in mass, removes the first factor at relative error $O(M^{-1})$. The scaled derivative is then the trace of $R_- = M(A_- + M)^{-1}$ against the Gram operator of $v^g_{b,\kappa,\ell} = (A_+ + b)^{-1}f_{\kappa,\ell}$ on the finite right interval, $\ell$ restricted to the finite left interval.

For each fixed positive $\kappa$, expanding right supports give strong $L^2$ convergence of the zero-contact columns by form exhaustion and the mass gap. The same holds along $\kappa_g \to \kappa > 0$, since the square-root form difference has $L^2$ operator bound $2|\kappa_g - \kappa|$, and the forcing kernels converge in $L^2$. Right strength $0 \le b \le \eta$ changes the column in $L^2$ by at most $b\|v^g_{0,\kappa,\ell}\|/(2\kappa)$, by the resolvent identity. These convergences give the convergence in Hilbert–Schmidt norm of the full cross-column kernels on the expanding supports, by dominated convergence in left depth using the positive inverse comparison with the full-ray column at any fixed smaller positive mass. The contractions $R_-$, extended by zero, converge strongly to the identity. For compact interior tests this follows from the resolvent identity and uniform boundedness of the applied square root for bounded $\kappa$; extend to all $L^2$ by density. Combining both convergences gives the stated trace limit uniformly for $b \in [0,\eta]$ along such mass sequences, and then (22) after averaging over $b$.

The same positivity comparison and determinant bound give the exact majorant

$$0 \le F(\kappa) \le (1 - C_h/M)^{-1}J(\kappa) \qquad (\kappa > 0).$$

The positive function $J$ decreases in $\kappa$ and is integrable. Indeed integration of the exponential and then $\int_0^\infty v/(\ell+v)^3\, dv = 1/(2\ell)$ gives

$$\int_0^\infty J(\kappa)\, d\kappa = \frac{1}{4\pi^2 h}.$$

The joint vacuum difference is therefore linear:

$$\frac{gM}{\eta}(E_{M,\eta} - E_{M,0}) \longrightarrow \frac{1}{8\pi^3 h},$$

where $g$ is the length used in the dimensionless gap scaling. For physical fixed-length intervals at gap $g$ use $h = 1$, $L_\pm = \ell_\pm/g$. The vacuum change is positive because the weak contact reduces the magnitude of the attraction in this scalar convention.

### 21.2 The zero and nonzero thermal modes

Let $\lambda_g = \beta/g$, where $\beta$ is physical inverse temperature. The exact scaled free-energy difference is

$$\frac{gM}{\eta}(\mathcal{F}_{M,\eta} - \mathcal{F}_{M,0}) = \frac{F(0)}{2\lambda_g} + \frac{1}{\lambda_g}\sum_{j\geq 1} F(2\pi j/\lambda_g).$$

If $\lambda_g \to \infty$, the mesh $a_g = 2\pi/\lambda_g$ tends to zero. Equation (22) is uniform on compact positive mass intervals by the converging-sequence assertion. The nonzero sum therefore converges to the vacuum constant after multiplication by $1/\lambda_g$, including at the moving endpoints: the positive integrable decreasing majorant $J$ bounds $a_g \sum_{0<ja_g<d} F(ja_g)$ by a fixed multiple of $\int_0^d J$, and the analogous upper tail by the integral above a nearby cutoff. Thus

$$\frac{gM}{\eta}(\mathcal{F}_{M,\eta} - \mathcal{F}_{M,0}) = \frac{1}{8\pi^3 h} + \frac{D_*}{4\pi^2\lambda_g} + o(1 + D_*/\lambda_g).$$

The nonzero modes and the zero mode are comparable at $\lambda_g$ of order $D_*$. For comparable scaled interval lengths, $D_* \sim \log^2(\min(L_+, \eta^{-1})/h)$. Fixed positive physical $\beta$ in the fixed-length physical intervals gives $D_*/\lambda_g \to 0$, since $D_* = O(\log^2(1/g))$ and $\lambda_g = \beta/g$.

For $\lambda_g$ bounded above, including $\lambda_g \downarrow 0$, the same $J$ majorant with its exponential tail at positive mass bounds the whole nonzero contribution after multiplication by $\lambda_g$ by $O(1)$. Since $D_* \to \infty$, the zero mode alone gives the leading difference $\eta D_*/(4\pi^2\lambda_g gM)$. The result concerns constant strong and weak contacts. A strong profile with a vanishing layer would require control of the weak-loss columns on that layer, which is not supplied by the separate fixed-contact estimates.

## 22 Weak response at a degenerate strong contact

Let $A_\pm$ be the zero-extension compressions of $2\sqrt{D^2+\kappa^2}$ on the gap-scaled interval depths. Their separated cross operator $B$ has kernel $-2\kappa K_1(\kappa(h+x+v))/[\pi(h+x+v)]$, with value $-2/[\pi(h+x+v)^2]$ at $\kappa = 0$. We use the connected scalar determinant and the vacuum and Matsubara normalization of Sections 10–13. The dimensionless separation is $h > 0$, with $g$ the length unit. For fixed physical interval lengths and actual gap $g$, one has $h = 1$ and $L_\pm = \ell_\pm/g$.

The previous weak-contact result Sections 18–21 treats an independently diverging constant strong contact, while Sections 14–17 treats a strong contact with a zero and a fixed limiting weak contact. The present calculation requires a further split in depth: the static weak-loss source has one logarithm at any fixed depth, and a second appears in its total squared norm over the long depths. Applying the single-source strong limit to a globally normalized source would discard one of these contributions. The fractional localization identity and the positivity of the separated inverse permit a joint limit. Related boundary-diffraction and thermal logarithms in scalar Casimir models are discussed in [7, 8].

### 22.1 Main limits

Use two finite gap-scaled intervals of lengths $L_-, L_+ \to \infty$, fixed separation $h > 0$, and the scalar connected transform $q_b(\kappa)$ with weak constant $b \geq 0$ on the right. On the left use $M[a(x) + \varepsilon_M]$, where $M \to \infty$, $\varepsilon_M \geq 0$ tends to zero, and the continuous strong profile has exactly one zero $x_0$, satisfies $a(x) \sim c|x - x_0|^p$ there for $c > 0, p \geq 1$, and is bounded below away from every neighborhood of the zero. Assume $a(x) \to a_\infty > 0$ as $x \to \infty$. The weak amplitude is $\eta \downarrow 0$, with no relative rate imposed on $M, \eta, L_\pm$. Let $S$ be the strong normalization and $m_0 > 0$ the corresponding weight from Theorem 13. Write $s_0 = 2$ for an interior zero and $s_0 = 1$ for a boundary zero.

- If $p = 1$, use $\rho_M = \max(M^{-1/2}, \varepsilon_M)$, $S = M/\log(1/\rho_M)$, and $m_0 = s_0/c$.
- If $p > 1$, set $\ell_{\mathrm{sc}} = M^{-1/(p+1)}$ and $\lambda_M = M\ell_{\mathrm{sc}}\varepsilon_M$. For $\lambda_M \to \lambda_0 < \infty$, use $S = \ell_{\mathrm{sc}}^{-2}$ and the weight $m_0 = \langle 1, (2|D| + c|y|^p + \lambda_0)^{-1}1\rangle$ in the dual form sense. Its domain is the full line for an interior zero and the zero-extension half-line for a boundary zero. For $\lambda_M \to \infty$, use $S = M\varepsilon_M^{1-1/p}$ and $m_0 = s_0\pi/[pc^{1/p}\sin(\pi/p)]$.

In all cases $S/M \to 0$.

Set

$$T = \min(L_+, \eta^{-1}), \quad l = \log(T/h), \quad m = \log((h + L_-)/h), \quad D = l^2 - (l - m)_+^2, \quad \ell_0 = h + x_0.$$

**Theorem 16.** *Then the exact static deficit has the additive asymptotic*

$$q_0(0) - q_\eta(0) = \frac{\eta\, m_0 l}{\pi^2 \ell_0 S} + \frac{\eta D}{2\pi^2 a_\infty M} + o\Big(\frac{\eta l}{S} + \frac{\eta D}{M}\Big). \tag{23}$$

*Both terms are positive. The first term comes from the contact zero. The second comes from the far depths. The dimensionless balance is $(S/M)D/l$. For comparable scaled lengths the two terms have the same order when $l$ has order $M/S$, which equals $\log(1/\rho_M)$ at a linear zero and $M^{(p-1)/(p+1)}$ at a power zero with finite scaled floor.*

*The joint vacuum difference is*

$$\frac{gS}{\eta}(E_\eta - E_0) \longrightarrow \frac{m_0}{8\pi^3 \ell_0^2}. \tag{24}$$

*Write $\mathcal{F}_b$ for the free energy at right strength $b$. At $\lambda = \beta/g \to \infty$, the difference $(gS/\eta)(\mathcal{F}_\eta - \mathcal{F}_0)$ is the same vacuum constant plus*

$$\frac{1}{2\lambda}\left[\frac{m_0 l}{\pi^2 \ell_0} + \frac{SD}{2\pi^2 a_\infty M}\right] + o\Big(1 + \frac{l + (S/M)D}{\lambda}\Big). \tag{25}$$

*For $\lambda$ bounded above the displayed zero-mode term is the leading equivalent. In the physical fixed-length interval setting $h = 1, L_\pm = \ell_\pm/g$, fixed positive physical $\beta$ gives the vacuum equivalent.*

## 23 From determinant difference to a positive source

Let $G_- = (A_- + M[a + \varepsilon_M])^{-1}$ for the left supported square-root form and $Q_b = G_-^{1/2} B(A_+ + b)^{-1} B^* G_-^{1/2}$ for the connected cross product. At any mass $\kappa$, differentiate the right strength $b$ in the exact determinant at fixed finite intervals, as in Sections 18–21. The negative derivative of

$q_b$ is the trace of $(I-Q_b)^{-1}$ against the positive operator $G_-^{1/2}B(A_+ + b)^{-2}B^*G_-^{1/2}$. The strong form gap gives $\|G_-\| \le C\ell_{\mathrm{sc}}$, $\ell_{\mathrm{sc}} = M^{-1/(p+1)}$, uniformly in mass and floor. The transparent sandwich bounds $\|Q_b\| \le C\ell_{\mathrm{sc}}$. Therefore the determinant deficit is the first-trace deficit to a relative $1 + O(\ell_{\mathrm{sc}})$, independently of $\eta$:

$$q_0(\kappa) - q_\eta(\kappa) = (1+O(\ell_{\mathrm{sc}}))\langle F_\kappa, (G_-\otimes I)F_\kappa\rangle, \qquad F_\kappa(x)(b,v) = (A_+ + b)^{-1} f_{\kappa,h+x}(v), \quad 0 < b < \eta. \tag{26}$$

The Hilbert space in the source labels is $L^2((0,\eta)\times(0,L_+))$. Its kernel has one sign by positivity of the compressed inverse. Its spatial derivative satisfies the relative bound uniformly at static mass,

$$\|\partial_x F_0(x)\| \le (2/h)\|F_0(x)\|,$$

by $|\partial_x f_{0,h+x}| \le (2/h)|f_{0,h+x}|$ and applying the positive weak inverse. The norm squares integrate in $x$ to the unweighted static first-trace deficit of Sections 18–21, which is $\eta D/(2\pi^2)(1+o(1))$.

## 24 Local weak-loss coefficient

Uniformly on every compact left depth set,

$$\frac{\|F_0(x)\|^2}{\eta l} \longrightarrow \frac{1}{\pi^2(h+x)}. \tag{27}$$

For $R = L_+$ the transparent interval inverse column and the ray column are

$$V_\ell^R(v) = V_\ell^\infty(v)\sqrt{(R-v)/(R+\ell)}, \qquad V_\ell^\infty(v) = -\frac{\sqrt{v}}{\pi\sqrt{\ell}(\ell+v)}.$$

For fixed $\ell$ in a compact positive interval, $\int_0^X |V_\ell^\infty|^2 = \log(X/h)/(\pi^2\ell) + O(1)$. The finite Hardy lower test from Sections 18–21, truncated at $U = T/\log^2 T$, gives the same integral times $\eta(1 + o(1))$ because $\eta U = o(1)$ and the finite-column ratio is $1 + o(1)$ on this test. For the upper bound when $T = R$, trial with $V_\ell^R$ gives $\eta\int_0^R |V_\ell^\infty|^2$. When $T = \eta^{-1}$, use the half-line domain comparison for the integrated derivative and the half-line cutoff at $X = \eta^{-1}\log^2(1/\eta)$, with energy cost $O_h(1/(X\ell))$. These bounds match (27). The same upper comparison at $\ell = h$ and forcing monotonicity give $\|F_0(x)\|^2 \le C_h\eta(1+l)$ for all left depths.

## 25 Adding the zero layer and the bulk

We use two static bounds that retain constant accuracy across arbitrary ratios of $\eta$ and $M$. First, for any fixed smooth $0 \le \chi_0 \le 1$ compactly supported in left depth and equal to one in a neighborhood of $x_0$, apply the source theorem Theorem 13 to $\chi_0 F_0/\sqrt{\eta l}$. These sources are uniformly bounded in $L^2$ since the left support is fixed, and locally uniformly continuous at $x_0$ by the relative derivative estimate and local bound above. Their value norm squares tend to $1/(\pi^2\ell_0)$ by (27). This applies also to an edge zero with $\chi_0(0) = 1$, since the source need not be a form test there. It gives

$$Q_H(\chi_0 F_0) \sim \eta l\, m_0/(\pi^2\ell_0 S),$$

where $Q_H$ denotes the inverse quadratic form with $G_-\otimes I$.

Second, for sources supported away from the zero, multiplication bounds and a far-depth test give the bulk term. Fix any cutoff $\chi_1$ zero near the zero and equal to one beyond a fixed sufficiently large depth. The upper bound

$$Q_H(\chi_1 F_0) \le M^{-1} \int \frac{\|\chi_1 F_0\|^2}{a + \varepsilon_M}$$

has limsup coefficient $1/(2\pi^2 a_\infty)$ under normalization $M/(\eta D)$. To see it, all fixed bounded depths away from the zero cost $O(\eta l)$, smaller than $\eta D$ since $D/l \to \infty$ when both lengths grow. On the remaining far depths use $a(x) \to a_\infty$ and the full unweighted trace norm. For the lower bound take a nonnegative scalar cutoff $\psi$ which vanishes up to fixed depth $R_0$ beyond the zero and at the far endpoint, and equals one between $R_0 + 1$ and the last unit of the left interval. Choose $R_0$ large enough that $|a(x) - a_\infty| \le \delta$ there. For the source restricted to those far depths, the trial

$$v = \frac{\psi F_0}{M(a_\infty + 2\delta)}$$

is in the zero-extension $H^1$ form domain. For sufficiently small floor its potential and linear terms give at least $1/[M(a_\infty + 2\delta)]$ times the integrated source norm except in the two cutoff layers. The discarded layers cost $O_{R_0}(\eta l)$ by the pointwise bound. The kinetic energy is $O_{R_0}(\eta D/M^2)$, by $2|\xi| \le 1 + |\xi|^2$, the derivative bound for $F_0$ and the fixed cutoff slopes. Then send $\delta \downarrow 0$. Thus the far source gives the same bulk term up to relative $o(1)$.

For the sum upper bound take $\chi_0^2 + \chi_1^2 = 1$ with the above support arrangement, using cosine and sine of a smooth cutoff angle. The fractional localization identity (cf. [30]) gives

$$\sum_j \langle \chi_j v, H \chi_j v \rangle \le \langle v, Hv \rangle + C\|v\|^2 \le (1 + C\ell_{\mathrm{sc}}) \langle v, Hv \rangle.$$

Indeed the potential terms sum exactly, and the difference-quotient energy error is bounded by the integral of $\|v(x)\| \|v(y)\| \min(1, |x-y|^2)/|x-y|^2$, using the squared cutoff differences and convolution with an $L^1$ kernel. This holds for Hilbert-valued tests and zero extensions. In the variational supremum write the linear term using the two cutoffs and use the inequality to obtain $Q_H(F_0) \le (1 + C\ell_{\mathrm{sc}}) \sum Q_H(\chi_j F_0)$. For the lower bound, keep disjoint near and far sources with cutoffs bounded by one and drop the remaining middle part. All pairings between retained pieces and the discarded piece are nonnegative: $G_-$ has positive kernel and $F_0(x)(b, v) \le 0$. Thus $Q_H(F_0)$ is at least the sum of the near quadratic form and the far trial lower bound. Their limits match the upper bound to an $o(1)$ relative error in the sum, giving (23) via (26).

## 26 Massive fibers and the energy limits

For $\kappa > 0$ the weak-loss source $F_\kappa/\sqrt{\eta}$ is uniformly $L^2$ in both depths, with squared norm bounded by $J(\kappa)$ from Sections 18–21. This follows from positivity comparison with the transparent half-line inverse, independently of $b$ and the interval lengths. The same comparison bounds its value norm at fixed left depth by the corresponding half-line norms and gives local equicontinuity. The Bessel Laplace formula expresses the forcing magnitude at distance $r$ as a constant times $\int_\kappa^\infty e^{-sr} \sqrt{s^2 - \kappa^2}\, ds$. After $s = \kappa + t$, integration by parts gives the ratio of its absolute derivative to itself at most $\kappa + 2/r$, since $t(\sqrt{t(t+2\kappa)})'/\sqrt{t(t+2\kappa)} \le 1$. Applying the positive inverse gives the same relative bound for the inverted columns. Along converging positive masses its squared value norm at $x_0$ tends to $\|V_{\kappa,\ell_0}\|_2^2$ by strong compressed inverse convergence and the positive

mass gap, averaging $0 < b < \eta$. The source theorem of Sections 14–17 applies globally without a spatial cutoff and gives

$$\frac{S}{\eta}[q_0(\kappa) - q_\eta(\kappa)] \longrightarrow m_0 \|V_{\kappa,\ell_0}\|_2^2. \tag{28}$$

This convergence is uniform on compact positive mass intervals by the converging-sequence statement.

For a common integrable majorant use the inverse comparison $H_\kappa \geq H_0$, followed by the same two fixed cutoffs and fractional localization at zero mass. The far part after normalization by $S/\eta$ is bounded by $C(S/M)J(\kappa) \leq CJ(\kappa)$. The near part is bounded by $C \sup_{x \leq R} \|F_\kappa(x)\|^2/\eta$ using the upper source estimates for compactly supported sources; this is at most $C\|V_{\kappa,h}\|_2^2$ by monotone positive forcing. Both functions are decreasing and integrable in $\kappa$, and decay exponentially at infinity. For the latter one also directly obtains $\int_0^\infty \|V_{\kappa,\ell_0}\|_2^2 \, d\kappa = 1/(4\pi^2\ell_0^2)$, by integrating the exponential and then $v/(\ell_0 + v)^3$. The factor $1/(2\pi)$ gives (24).

With $\lambda \to \infty$, the nonzero Matsubara modes converge to the same vacuum integral by compact convergence and decreasing integrable domination of the Riemann sum near zero and infinity, as in Sections 18–21. The zero mode retains exactly (23), with the Matsubara factor $1/(2\lambda)$ giving (25). If $\lambda$ is bounded above, the nonzero sum multiplied by $\lambda$ is $O(1)$ under the same normalized bound, whereas the normalized zero-mode deficit grows at least as $l$. Finally for fixed physical lengths and fixed positive physical inverse temperature, $l = O(\log(1/g))$, $D = O(\log^2(1/g))$, and $S/M \leq C$, so the normalized zero-mode correction vanishes. This proves the stated temperature comparisons.

# 27 Degenerate strong contacts at curved tangencies: model and limiting response

Let two bounded convex apertures in a $d$-dimensional screen, $d \geq 2$, lie on opposite sides of a supporting hyperplane before they are translated by $\mp g/2$ along the separation direction $s$. Their closures meet only at the origin before translation. Use the facing graphs of Sections 10–13: in physical coordinates they are $s = -h_-(y)$ and $s = h_+(y)$ before translation. At a smooth quadratic tangency $h_\pm$ are $C^2$ near zero and their Hessian sum is positive definite. In the separable finite-type case

$$h_i(y) = \sum_j b_{ij}|y_j|^{r_j} + o\left(\sum_j |y_j|^{r_j}\right), \qquad r_j > 1, \quad b_{ij} \geq 0, \quad b_{-j} + b_{+j} > 0.$$

Set $r_j = 2$ in the quadratic case. In gap units the Cartesian coordinates are $(s/g, Y)$, where $Y = y/g$. We use $w_j = g^{1-1/r_j}Y_j$, $\alpha = \sum_j 1/r_j$, $\gamma = d - 1 - \alpha > 0$, and the gap height $H(w) = 1 + \sum_j (b_{-j} + b_{+j})|w_j|^{r_j}$ in the finite-type case. In the quadratic case the added term is half the quadratic form of the Hessian sum. The variable $u$ denotes the inward vertical depth in units of $g$ from the relevant facing graph. The volume element is $g^{-\gamma}\, dw\, du$.

The aperture response is the zero-exterior compression of $2\sqrt{|D|^2 + \kappa^2}$, with gap-scaled mass $\kappa$ and an added nonnegative multiplication contact. A gap-scaled coefficient is $g$ times the physical coefficient. Let $G_i$ be the inverses of the two compressed responses. Let $B$ be the separated cross operator from the right to the left aperture with kernel $n_\kappa$, the off-diagonal kernel of $2\sqrt{|D|^2 + \kappa^2}$. The connected transform $q_g = -\log\det(I - G_-^{1/2} B G_+ B^* G_-^{1/2})$ has vacuum and

thermal normalizations

$$E_g = -\frac{1}{2\pi g}\int_0^\infty q_g(\kappa)\,d\kappa, \qquad \mathcal{F}_g(\beta) = -\frac{1}{2\beta}\sum_{j\in\mathbb{Z}} q_g(2\pi|j|g/\beta).$$

We use the connected heat and determinant identities of [1], Sections 10–13 and Sections 14–17, including positivity and monotonicity of $q_g$ in $\kappa$. The occupation weight for a bounded multiplication contact is the exponential of its negative integral against the normal local time. It preserves those properties at each fixed $g$.

For a strong side prescribe $M[a(w,u)+\epsilon]$ in a fixed facing chart and $M$ off the chart, with $M\to\infty$ and $\epsilon\to 0$, $\epsilon\ge 0$. We use bounded continuous nonnegative $a$ with a single zero $u_0(w)$ in each depth ray and

$$a(w,u)\sim c(w)|u-u_0(w)|^p, \qquad a(w,u)\ge c_*\min(1,|u-u_0(w)|^p). \tag{29}$$

Here $p>0$, $c_*>0$, $c$ is positive continuous, the expansion is locally uniform in $w$, and $u_0$ is bounded and locally Lipschitz. Either $u_0\equiv 0$ or $\inf u_0>0$. The notation $\mathcal{D}$ denotes respectively the open half-line or the line, and $\nu=1$ or 2, in these cases.

The normalization $S$ and limiting depth measure $\sigma_w$ are as follows. For $p<1$ take $S=M$ and $\sigma_w = du/a(w,u)$. For $p=1$ take $S=M/\log(1/\rho)$ with $\rho=\max(M^{-1/2},\epsilon)$, and put $\sigma_w=(\nu/c(w))\delta_{u_0(w)}$. For $p>1$ write $\ell=M^{-1/(p+1)}$. If $M\ell\epsilon\to L<\infty$ take $S=\ell^{-2}$ and $\sigma_w=m(w)\delta_{u_0(w)}$, with

$$m(w)=\sup_{\psi\in C_c^\infty(\mathcal{D};\mathbb{R})}\left\{2\int_{\mathcal{D}}\psi-\langle\psi,2|D|\psi\rangle-\int_{\mathcal{D}}(c(w)|z|^p+L)|\psi(z)|^2\,dz\right\}. \tag{30}$$

Functions in the kinetic form are extended by zero. If $M\ell\epsilon\to\infty$, take $S=M\epsilon^{1-1/p}$ and $m(w)=\int_{\mathcal{D}}(1+c(w)|z|^p)^{-1}\,dz$. These normal responses are finite by the gap and the integrability of the reciprocal power at infinity. In particular the quadratic interior oscillator has the Airy evaluation in Sections 14–17; the spectral antecedent is [28].

**Theorem 17** (One strong contact)**.** *Use* (29) *on the left. On the right use bounded continuous nonnegative scaled contact $b(w,v)$ in the chart and any uniformly bounded nonnegative coefficient off it. Then*

$$g^\gamma S q_g(\kappa) \;\longrightarrow\; P_1(\kappa)=\int_{\mathbb{R}^{d-1}}\int_{[0,\infty)} C_b(\kappa,w,u)\,\sigma_w(du)\,dw.$$

*Here $C_b(\kappa,w,u)$ is the frozen weak-inverted squared column. Explicitly, for $h=H(w)+u$, $q=\sqrt{\kappa^2+|k|^2}$, and $f_{q,h}(v)=-2qK_1(q(h+v))/[\pi(h+v)]$, it is*

$$(2\pi)^{-(d-1)}\int_{\mathbb{R}^{d-1}}\langle f_{q,h},(A_q+b(w,\cdot))^{-1}f_{q,h}\rangle\,dk.$$

*The operator $A_q$ is the supported ray compression of $2\sqrt{D^2+q^2}$. At $q=0$ use $f_{0,h}=-2/(\pi(h+v)^2)$ and the inverse form on its energy dual. This theorem includes $b\equiv 0$, and $\epsilon\equiv 0$ at every order $p$.*

**Theorem 18** (Two strong contacts)**.** *On both sides use* (29) *with the respective parameters, and with $a_i$ depending only on $u$ over each fixed tangency chart. Use that profile on the entire vertical fiber above each chart point. The two rates and floor regimes are independent. Then*

$$g^\gamma S_- S_+ q_g(\kappa) \;\longrightarrow\; P_2(\kappa)=\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{[0,\infty)^2}|n_\kappa(H(w)+u+v,z)|^2\,\sigma_-(du)\sigma_+(dv)\,dz\,dw. \tag{31}$$

Both limits are uniform on compact mass intervals and their normalized transforms have a common exponential envelope. With $A = S$ in Theorem 17 and $A = S_- S_+$ in Theorem 18, $g^{d-\alpha} A E_g$ converges to $-\int_0^\infty P_i(\kappa)\, d\kappa/(2\pi)$. At $\beta = \lambda g$ the limit is

$$g^{d-\alpha} A \mathcal{F}_g(\lambda g) \;\longrightarrow\; -\frac{1}{2\lambda} \sum_{j \in \mathbb{Z}} P_i(2\pi |j|/\lambda),$$

locally uniformly in $\lambda > 0$. At fixed positive physical $\beta$ the free energy has the vacuum limit.

The diffraction formulation of [7, 19, 8] provides the boundary action and thermal setting. Inhomogeneous thin-mirror contacts appear in [14]. Hard thin films with regional $H^{1/2}$ energy have a tangential variational limit [31], and directional fractional equations on thin domains are studied in [32]. Local Schrödinger confinement with a degenerate hypersurface zero is treated in [33]. In the present problem the zero-exterior first-order kinetic form retains a forced normal response, and the near-contact geometry and the contact layer enter the trace at independent rates. The flat or interval responses needed below are from Sections 14–17.

## 28 Vertical inverses and recovery trials

Denote by $N_i$ the direct integral over the gap tangential coordinate $Y$ of the inverse of the vertical compression of $2\sqrt{D_s^2 + \kappa^2}$ plus the same contact. Convexity makes each nonempty slice an interval. For every compact slow-coordinate set its length on the inward side tends to infinity in gap units uniformly. Indeed near the tangency choose a fixed short inward normal segment with interior endpoint, then use a neighborhood of that endpoint and convexity. The facing graph has vanishing local Lipschitz constant at tangency Sections 10–13. Zero extension gives $0 \le G_i \le N_i$ by form order. For any supported $H^1$ trial $V$, possibly Hilbert-valued, the surcharge from replacing the vertical by the full kinetic form satisfies

$$0 \le \mathfrak{A}_i[V] - \mathfrak{A}_i^{\mathrm{vert}}[V] \le 2\|V\|_2 \|\nabla_Y V\|_2. \tag{32}$$

This follows from the Fourier symbol difference bounded by $2|\xi_Y|$ and Cauchy–Schwarz. The derivative is at fixed Cartesian normal coordinate.

The vertical form has a uniform gap $C^{-1}\ell^{-1}$ for $\ell = M^{-1/(p+1)}$, at every $p > 0$. After centering at the zero and dilating by $\ell$, the rescaled potential is bounded below by $c \min(1, |z|^p)$. Constant off-chart strength also satisfies the bound for small $\ell$. For the whole-line comparison, the low-frequency part $|\xi| \le r$ of a unit vector has squared mass $O(r)$ in $[-1, 1]$, by Fourier Cauchy–Schwarz. The high-frequency norm squared is bounded by $\langle |D| \rangle / r$, and the potential controls the mass outside $[-1, 1]$. Fixing a sufficiently small $r$ proves the gap on the line and on every zero-extended slice. With a nonzero floor the inverse is also bounded by $C/(M\epsilon)$ for small $\epsilon$. All statements hold at zero mass.

We record the two consequences of this comparison used below. The signal $F = F_g(w, u)$ takes values in a Hilbert space that may vary with $g$. Suppose its norm squared converges to a function $J(w, u)$ along compact mass-converging sequences, and its full Cartesian spatial derivatives and values are uniformly bounded on compact $(w, u)$-sets. Evaluation of $F_g$ on a boundary layer is allowed when the field extends to a fixed normal neighborhood there with these bounds.

**Lemma 19** (Slice upper limit and local recovery)**.** *For the slice upper limit suppose also that in the chart*

$$\|F_g(w, u)\|^2 \le C e^{-c\kappa} (1 + u + \Psi(w))^{-d-1}, \qquad \Psi(w) = \sum_j |w_j|^{r_j}.$$

*Then the chart source has vertical inverse quadratic with upper limit, on the scale $g^\gamma S$, at most $\int J(w,u)\,\sigma_w(du)\,dw$. The bound is exponentially dominated in mass. For local recovery, the derivative and pointwise hypotheses alone suffice: the full inverse quadratic has lower limit at least the same integral, by compact exhaustion.*

*Proof.* For $p < 1$ drop the vertical kinetic form. The reciprocal majorant is integrable because $u_0$ is bounded, $p < 1$, and $\alpha < d$. This yields the upper bound by dominated convergence. For $p = 1$, on $|u - u_0| \le \rho$ the gap bounds the inverse quadratic by $O(M^{-1})$ times the local column envelope. Indeed the inverse gap there is $O((M\rho)^{-1})$ and the source squared mass is $O(\rho)$. On its complement drop the kinetic term. Integrating $1/|u - u_0|$ between $\rho$ and a small fixed radius yields the normalized coefficient $\nu J(w,u_0)/c(w)$. The finite-radius error tends to zero with that radius by (29) and the derivative estimate. Fixed outer depths contribute $O(M^{-1})$ on each slice. Cross terms are lower order by Cauchy–Schwarz in the inverse norm.

For $p > 1$ with finite scaled floor, split at $|u-u_0| = R\ell$. The complementary inverse quadratic times $S$ is bounded by the multiplication form and costs at most

$$Ce^{-c\kappa}\left[(1+\Psi(w))^{-d-1}R^{1-p} + (1+\Psi(w))^{-d}\ell^{p-1}\right].$$

In the core the gap and local derivative bound allow replacing the vector by $F_g(w,u_0)$ in the normalized inverse norm. After dilation the scalar forcing is the indicator of the truncated interval or ray. The rescaled forms for this forcing converge to the oscillator form. For the upper limit the gap bounds the $H^{1/2}$ norm of near-maximizers. Local $L^2$ compactness gives convergence in the pairing and in compact potential integrals. Exhaustion gives lower semicontinuity of the potential term, and the limiting functions remain zero-exterior in the half-line case. Compact smooth tests give the reverse bound and are dense by cutoff, inward translation on the ray, and smoothing. Increasing $R$ then gives (30), using the reciprocal-power bound outside $R$ for the inverse norm of the removed forcing. For dominant floor, use the multiplication bound directly with $u - u_0 = \epsilon^{1/p} z$. The near-zero integrand is bounded by the local envelope times $C/(1+|z|^p)$; away from the zero the normalization includes $S/M = \epsilon^{1-1/p} \to 0$. Each upper estimate gives the integrable tangential envelope $Ce^{-c\kappa}(1+\Psi(w))^{-d}$, using $R = 1$ for the oscillator bound before taking limits.

For the full lower bounds use the inverse variational formula with compact slow-coordinate support. Let $s_0(Y)$ be the Cartesian gap coordinate of the profile zero. Its local slope $\theta_g$ tends to zero on each such support because the graph slopes vanish and $\partial_Y w_j = g^{1-1/r_j}$. For a trial

$$V(s,Y) = a_0\psi\left(w, \frac{u - u_0(w)}{t}\right) F_g(s_0(Y), Y)$$

with smooth compact $\psi$ in the allowed limiting domain, the estimates for its squared norm and its squared tangential derivative norm are $O(g^{-\gamma}a_0^2 t)$ and $O(g^{-\gamma}a_0^2(t + \theta_g^2/t))$. These follow from the full derivative bound and the chain rule for Lipschitz centers. The trial has no zero-extension jump. Thus (32) costs at most $Cg^{-\gamma}a_0^2(t + \theta_g)$.

For the oscillator take $a_0 = t = \ell$. On the normalized scale the surcharge is $O(\ell + \theta_g)$, with no coupling of these two quantities. The linear pairing converges with $F_g(w,u)$ replaced by $F_g(w,u_0)$, by its normal derivative bound. At each $w$ the vertical kinetic form dilates to $2\sqrt{D_z^2 + (\kappa\ell)^2}$, and the potential converges to $c(w)|z|^p + L$. The resulting optimized functional is $\int J(w,u_0)m(w)\,dw$. To justify its optimization choose finitely many compact normal tests near-optimal on a compact set of $w$, using continuity of $c$ and relative form comparison. Partition the set by choice of test and approximate the partition by disjoint compact subsets with smooth

cutoffs in disjoint neighborhoods. Their discarded integral can be arbitrarily small, since the normal objective and $J$ are bounded on the compact set. The subsequent compact exhaustion gives the asserted lower bound.

For dominant floor take $t = \epsilon^{1/p}$ and $a_0 = 1/(M\epsilon)$. The normalized vertical kinetic term is $O((M\epsilon t)^{-1})$ on compact mass sets; the tangential cost is bounded by this factor times $C(t+\theta_g)$. Both vanish since $M\ell\epsilon \to \infty$ implies $M\epsilon t \to \infty$. Optimize the remaining multiplication functional as for the oscillator.

For $p = 1$ use trial numerator $\chi(w)F_g(s_0(Y), Y)$ and denominator $Mc(w)|u - u_0|$, cut off smoothly on the annulus from $\rho$ to a small fixed outer radius. Approximate $1/c$ on the support by a smooth factor and subsequently remove the approximation. The squared trial and normal derivative norms are $O(g^{-\gamma}M^{-2}\rho^{-1})$ and $O(g^{-\gamma}M^{-2}\rho^{-3})$. The tangential squared derivative adds terms of order $g^{-\gamma}M^{-2}(\rho^{-1} + \theta_g^2\rho^{-3})$. The total kinetic cost is $O(g^{-\gamma}M^{-2}\rho^{-2})$ and the floor cost is $O(g^{-\gamma}\epsilon/(M\rho))$. Both are $O(g^{-\gamma}/M)$. The leading linear and potential terms give $(2\chi - \chi^2)\nu J(w, u_0)/c(w)$ as the outer radius shrinks. Exhaust with $0 \le \chi \le 1$. For $p < 1$ use $V = M^{-1}m(w, u)F_g(w, u)$ for compact smooth $m$ supported away from the zero and boundary. The kinetic cost after normalization is $O(M^{-1})$; approximate the reciprocal on compacts and exhaust. This proves local recovery. □

## 29 One strong side

Set $F_g(X) = G_+^{1/2}B^*\delta_X$ for a left point $X$. The weak-column limit of Sections 14–17 gives

$$\|F_g(w, u)\|^2 \longrightarrow C_b(\kappa, w, u), \qquad u \ge 0.$$

Translate to the evaluation point. The opposite domain converges locally to its facing half-space, and its contact converges locally uniformly to $b(w, \cdot)$ in the interior. The separated source converges in $L^{2d/(d+1)}$, and the dual homogeneous Sobolev estimate bounds the inverse pairing.

Compactly supported tests give the lower limit. Near-maximizers are bounded in homogeneous energy, and local Rellich compactness and exhaustion of the nonnegative potential term give the upper limit. At $u = 0$ the source still has separation at least one. Tangential Fourier decomposition gives the formula of Theorem 17.

The same Sobolev estimate for the differentiated separated kernel bounds each first Cartesian derivative, even across a short extension of the left facing graph. It gives the envelope of Lemma 19 for the squared chart columns. The normalized off-chart integral is $O(ge^{-c\kappa})$.

The first trace is the inverse quadratic on this Hilbert-valued column. In its upper bound split the source into chart and off-chart pieces and apply the triangle inequality in the inverse norm. The off-chart square costs at most $1/M$ times the off-chart column integral by multiplication order, so it is $O(gS/M)$ after normalization. The chart upper limit and full lower limit now agree by Lemma 19. The positive connected product has norm $O(\ell)$: use the strong inverse gap and the uniformly bounded transparent supporting-half-space sandwich of Sections 10–13. Higher determinant traces have relative size $O(\ell)$ by the positive logarithmic series. This proves the transform limit.

## 30 Two strong sides

Apply the vertical comparison to both inverses. With $\omega(u) = (1+u)^{-1}$, the positive fiber measure

$$S_i\omega(u)N_i(u, u')\omega(u')\,du\,du'$$

has uniformly bounded mass. In a chart its total mass is bounded by the full-ray vertical inverse at zero mass on $\omega$, using domain and form comparison and the one-dimensional source theorem Sections 14–17. Outside the chart use constant $M_i$ and $S_i/M_i \le C$. For any fixed slow $w$, or any fixed gap-coordinate tangential displacement from that point, the fibers are eventually in the chart and their lengths tend to infinity. The one-dimensional weighted-measure convergence in Sections 14–17 gives weak convergence on the closed depth ray squared to the diagonal image of $\omega(u)^2\sigma_i(du)$. It applies along converging compact mass sequences and retains boundary atoms.

Represent the vertical upper trace as the quadratic of $N_- \otimes N_+$ on the cross kernel. Fubini in the tangential coordinates reduces its depth integral to the two positive weighted fiber measures. There are two separated kernels tested against those measures with the four weight factors divided out. This identity follows by positive rectangle approximation on the depth variables. Let $z = Y' - Y$ and put $\Psi(w) = \sum |w_j|^{r_j}$. The near-contact distance estimate Sections 10–13 gives, for two chart points at depths $u, v$, separation

$$|R| \gtrsim 1 + u + v + |z| + \Psi(w).$$

The supports also give $(1+u)(1+v) \le C|R|^2$ globally. The kernel bound is

$$|n_\kappa(R)| \le Ce^{-c\kappa|R|}|R|^{-d-1}.$$

The resulting product envelope is $Ce^{-c\kappa}(1+|z|+\Psi(w))^{-2(d-1)}$. It is integrable in $(w,z)$ since $2(d-1) > d-1+\alpha$. Outside a smaller fixed chart on either side the physical separation is bounded below by uniqueness of contact. The weighted product is then $O(e^{-c\kappa}g^{2(d-1)})$, and the product of both scaled tangential volumes is $O(g^{-2(d-1)})$. The normalized contribution is $O(e^{-c\kappa}g^\gamma)$. For each fixed $(w,z)$ the graph shifts give the gap $H(w)$, and the kernel tests converge uniformly on compact depths. Boundedness and tightness of the weakly convergent weighted measures and the common product envelope justify convergence to (31) for the vertical upper trace.

To replace both vertical inverses by full inverses use

$$\begin{aligned} 0 \le\ & \operatorname{Tr}(N_- B N_+ B^*) - \operatorname{Tr}(G_- B G_+ B^*) \\ \le\ & \operatorname{Tr}[(N_- - G_-) B N_+ B^*] + \operatorname{Tr}[(N_+ - G_+) B^* N_- B]. \end{aligned}$$

This follows by inserting the mixed trace and comparing the other factor in each positive term. For the left defect take $F_g(X) = \sqrt{S_+}N_+^{1/2}B^*\delta_X$. The right depth measure gives the squared norm of this field and its first full Cartesian derivatives. Each column factor divided by the right weight alone costs at most $C(1+u+v+|z|)^{-d}$ on compact mass sets, by the separated kernel bound and its differentiated version. Integrating over $z$ with the bounded right measure masses yields $C(1+u)^{-d-1}$ for the squared field and derivative bounds. The same right-measure convergence as above gives the norm-square limit

$$J(w,u) = \int_{\mathbb{R}^{d-1}} \int_{[0,\infty)} |n_\kappa(H(w)+u+v,z)|^2 \, \sigma_+(dv)\, dz.$$

The bound extends slightly across a boundary zero by separation. The compact recovery part of Lemma 19 gives the entire limit $\int J(w,u)\,\sigma_-(du)\,dw$ as a lower bound for the normalized mixed trace with $G_-$. This equals the vertical upper limit. The analogous recovery on the right removes the second defect. Neither defect estimate involves a ratio of strong scales.

The inverses have norms $O(\ell_-)$ and $O(\ell_+)$, and $B$ is uniformly bounded by the separated kernel Schur estimate of Sections 10–13. The determinant remainder is relatively $O(\ell_-\ell_+)$. This proves Theorem 18.

## 31 Energies and point-response checks

All bounds and convergence arguments above hold along convergent compact mass sequences. The limiting integrals are continuous by the column variational limit or the continuous separated kernel and the same dominating envelopes. The exponential majorants justify vacuum and gap-temperature integration and summation. For fixed positive physical $\beta$, use the monotone mesh bound for the connected transform, with step $2\pi g/\beta$ and endpoint error at most a constant times this step times the bounded normalized static transform.

If the weak side is transparent, the flat one-sided trace $W_{d,0}(\kappa, h)$ of Sections 10–13 satisfies $C_0(\kappa, w, u) = -\partial_h W_{d,0}(\kappa, h)$ at $h = H(w) + u$, since the column integral differs only by the integration over additional inward depth. For two atomic measures the vacuum density in (31) is the product of their two masses times

$$(d+2)(d+3)A_d[H(w) + u_- + u_+]^{-d-4},$$

before applying the overall energy sign. Here

$$A_d = \frac{2\Gamma(d/2+1)^2\Gamma(d/2+2)}{\pi^{d/2+2}\Gamma(d+4)}$$

is the flat two-strong vacuum constant of Sections 5–9. Indeed $A_d h^{-d-2}$ integrates the vacuum squared kernel over both depths, and two derivatives in $h$ remove those integrations. These are coefficients at the assigned gap-scaled profiles.

## 32 Periodic corrugation of a strong contact

A nonnegative surface contact adds a multiplication term to the aperture response of a Dirichlet screen. The kinetic term is the supported compression of twice the free square root. A strong contact confines the part of the response contributing to the Casimir interaction near a zero of the coefficient. For an order-$p$ zero with strength multiplier $M$, the normal confinement width is $\ell = M^{-1/(p+1)}$. A slow tangential modulation can be approximated by separate vertical inverses. When the contact zero varies in depth by order $\ell$ over a tangential distance of order $\ell$, tangential and normal frequencies enter the leading kinetic energy together.

Periodic scattering decompositions and wavelength crossovers were developed for geometrically deformed plates by Emig, Hanke, Golestanian and Kardar [37] and Büscher and Emig [35]. Fosco and Mazzitelli [14] derive a scalar interaction determinant for flat parallel mirrors with spatially varying multiplication contacts. Here the screen geometry is fixed and the contact well moves in depth within a coplanar aperture. We use the separated aperture kernel of the boundary-operator formulation [7] and Sections 10–13. The constant-depth strong limit and its supported half-line oscillator are developed in Sections 14–17.

Floquet trace formulas per unit volume are standard [36, Lemma 2.2]. Large periodic potentials lead to Bloch cells in local second-order homogenization [34]. Homogenization of periodic Lévy-type kinetic coefficients is treated in [39, 40]. In the present model the kinetic symbol is fixed, the confining potential translates within the cell, and the coefficient is a forced inverse. Its zero-quasimomentum value enters the leading trace even though the weak factor is integrated over all physical tangential momenta.

Braides, Pinamonti and Solci [31, Theorem 4.7] obtain a logarithmic tangential correction for the regional $H^{1/2}$ seminorm of a thin product film. A logarithmic term also arises from low-frequency Neumann modes in the Casimir derivative expansion [38]. The edge logarithm

here comes from the normal Fourier multiplier applied to an optimized supported response. The coefficient depends on the tangential derivative of its integrated normal mass. In the interior, translation preserves that mass and the leading term has no logarithm.

## 33 The contact model and main results

Let $Y \in \mathbb{R}^m$, $m \geq 1$, be the common tangential coordinate. The left and right apertures have inward depths $u > 0$ and $v > 0$, respectively, and fixed separation $h > 0$ in gap units. The diagonal boundary response at mass $\kappa \geq 0$ is the compression of $2\sqrt{D_u^2 + |D_Y|^2 + \kappa^2}$, with zero extension across the edge, plus the contact. We use $D = -i\partial$ and the unnormalized Fourier transform $\widehat{f}(\xi) = \int e^{-ix\xi} f(x)\,\mathrm{d}x$. The right contact is the constant $b \geq 0$. The left contact is

$$M\big[c|u - u_0 - \ell\chi(Y/\delta_\ell)|^p + \epsilon_\ell\big], \qquad \ell = M^{-1/(p+1)}, \quad M\ell\epsilon_\ell \longrightarrow \lambda \geq 0. \tag{33}$$

Here $c > 0$, $p > 1$, $\epsilon_\ell \geq 0$, and $\chi$ is a real smooth function on $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$. The physical period in each tangent coordinate is $2\pi\delta_\ell$. We use normalized Haar measure on $\mathbb{T}^m$, denoted by $\mathrm{d}\theta$ or by the averaging brackets $\langle\cdot\rangle_\theta$. The ratio $r_\ell = \ell/\delta_\ell$ tends to $r \in [0, \infty]$. Either $u_0 > 0$ is fixed, or $u_0 = 0$ and $\chi \geq 0$. Write $J = \mathbb{R}$ in the first case and $J = (0, \infty)$ in the second.

For $0 \leq r < \infty$, define

$$\mathfrak{b}_r[U] = \left\langle U, 2\sqrt{D_z^2 + r^2|D_\theta|^2}\,U \right\rangle + \int_{J\times\mathbb{T}^m} (c|z - \chi(\theta)|^p + \lambda)|U|^2\,\mathrm{d}z\,\mathrm{d}\theta, \tag{34}$$

$$C_r(J) = \sup_U \left(2\,\mathrm{Re}\int_{J\times\mathbb{T}^m} U\,\mathrm{d}z\,\mathrm{d}\theta - \mathfrak{b}_r[U]\right). \tag{35}$$

The $z$ variable is extended by zero outside $J$ in the kinetic form. The admissible functions lie in its closed form domain. Define $C_\infty(J)$ by the same supremum restricted to $\theta$-independent functions. Its one-dimensional potential is $c\langle|z - \chi|^p\rangle_\theta + \lambda$.

Write $A_q$ for the right ray compression of $2\sqrt{D_v^2 + q^2}$, and put

$$f_{q,L}(v) = -\frac{2qK_1(q(L+v))}{\pi(L+v)} \quad (q > 0), \qquad f_{0,L}(v) = -\frac{2}{\pi(L+v)^2}, \qquad L > 0.$$

The weak-side factor is

$$\mathcal{W}_b(\kappa, L) = (2\pi)^{-m}\int_{\mathbb{R}^m} \big\langle f_{q,L}, (A_q + b)^{-1} f_{q,L}\big\rangle_{\mathrm{dual}}\,\mathrm{d}k, \qquad q = \sqrt{\kappa^2 + |k|^2}. \tag{36}$$

The form-dual notation includes $q = b = 0$. We define the periodic connected transform $q_\ell^{\mathrm{area}}(\kappa)$ as the trace per unit tangential volume of the negative logarithm of identity minus the positive cross product of the two diagonal inverses and two separated kernels. The Bloch formula and finiteness of this logarithm are proved in Section 34.

**Theorem 20** (Periodic cell law)**.** *Under* (33)*,*

$$\ell^{-2} q_\ell^{\mathrm{area}}(\kappa) \longrightarrow C_r(J)\mathcal{W}_b(\kappa, h + u_0).$$

*The convergence is uniform on compact mass intervals including zero, and the normalized traces have a common exponentially decaying mass bound. In gap units the corresponding vacuum integral and free energy at inverse temperature $\beta_* > 0$, normalized by $\ell^{-2}$, converge to*

$$-\frac{C_r(J)}{2\pi}\int_0^\infty \mathcal{W}_b(\kappa, h + u_0)\,\mathrm{d}\kappa, \qquad -\frac{C_r(J)}{2\beta_*}\sum_{j\in\mathbb{Z}} \mathcal{W}_b(2\pi|j|/\beta_*, h + u_0),$$

*respectively. The second limit is locally uniform for $\beta_* > 0$.*

*If $\chi$ is nonconstant, $C_r(J)$ is strictly decreasing on $[0,\infty]$. At $r = 0$ it is the average of the local vertical inverses. In $J = \mathbb{R}$ this average equals the single full-line coefficient with potential $c|z|^p + \lambda$. At $r = \infty$ it is the inverse response for the averaged potential.*

**Theorem 21** (Small-ratio corrections)**.** *On the line let $v_* = (2|D|+c|z|^p+\lambda)^{-1}1$ in the form-dual sense. Then*

$$\lim_{r\downarrow 0} \frac{C_0(\mathbb{R}) - C_r(\mathbb{R})}{r^2} = \langle |\nabla_\theta \chi|^2 \rangle_\theta \, \langle v_*, |D| v_* \rangle. \tag{37}$$

*On the supported ray let $v_t = (2|D| + c|z-t|^p + \lambda)^{-1}1$ and define the normal response mass $m_0(\theta) = \int_0^\infty v_{\chi(\theta)}(z)\,\mathrm{d}z$. Then*

$$\lim_{r\downarrow 0} \frac{C_0((0,\infty)) - C_r((0,\infty))}{r^2 \log(1/r)} = \frac{\langle |\nabla_\theta m_0|^2 \rangle_\theta}{\pi}. \tag{38}$$

*Both coefficients are strictly positive for a nonconstant $\chi$.*

The two scales in Theorem 21 are asymptotics of the limiting cell coefficients.

## 34 The unfolded trace

Write $G_\ell$ for the strong diagonal inverse on the left. Write $C_\kappa$ for the right sandwich by the separated cross kernel and the right inverse. In the usual Fourier transform in $Y$, the sandwich is diagonal with positive blocks $C_{\kappa,k}$ in normal depth. Its Hilbert-valued column is

$$F_{\kappa,k}(u) = (A_q + b)^{-1/2} f_{q,h+u}.$$

In particular the squared column norm is the inner product in (36) evaluated at $L = h + u$, and at zero mass it has the form-dual interpretation. The donor's causal half-line factorization Section 11 gives

$$\|F_{\kappa,k}(u)\|^2 \le \frac{1}{2\pi} \int_{h+u}^\infty e^{-2qs} s^{-3}\,\mathrm{d}s \le C_h e^{-2hq}. \tag{39}$$

One way to obtain the first bound is to set $L_q = (q + \partial_v)^{1/2}$, so the supported form of $A_q$ is $2\|L_q \cdot\|_2^2$. Its backward inverse applied to $f_{q,h+u}$ is $-e^{-q(h+u+v)}/[\sqrt{\pi}(h+u+v)^{3/2}]$. At $q = 0$ this is in $L^2_v$, and it represents the homogeneous dual norm. Adding $b$ can only decrease the quadratic. The same bound applied to the differentiated separated column controls the first depth derivative on compact $q$-intervals. Column norms depend continuously on $q \ge 0$: the separated columns converge in the massless vertical dual norm, and the inverse quadratic converges by its form supremum, using compact tests for the lower bound and form order at $q = 0$ for the upper bound. Away from zero the spectral gap also gives continuous dependence in $L^2$. The transparent trace bound integrated in $u$ gives $\sup_{\kappa,k} \|C_{\kappa,k}\| \le 1/(4\pi h)$.

Fold the tangent Fourier transform into $k + n/\delta_\ell$ with $k \in [-1/(2\delta_\ell), 1/(2\delta_\ell))^m$ and $n \in \mathbb{Z}^m$. This groups the Fourier channels into a periodic fiber in $\theta = Y/\delta_\ell$, with modes $e^{in\cdot\theta}$. Multiplication by the contact is periodic in $\theta$. The unscaled kinetic symbol in this fiber is $2\sqrt{\xi_u^2 + |k + n/\delta_\ell|^2 + \kappa^2}$. The trace per unit tangent volume is $(2\pi)^{-m} \int_{\mathrm{BZ}} \mathrm{Tr}(\cdot)(k)\,\mathrm{d}k$. Indeed normalized Brillouin-zone measure divided by the real-space cell volume gives this Fourier measure. The localized positive trace formula [36, Lemma 2.2] extends to the additional normal Hilbert space by summing diagonal coefficients in an orthonormal basis and using positivity.

In each folded fiber $C_\kappa$ acts diagonally in the tangent channels. For its $n$-th term, conjugation by multiplication by $e^{in\cdot\theta}$ shifts the strong kinetic quasimomentum from $k$ to $k+n/\delta_\ell$ and leaves the contact unaltered. The source becomes constant in $\theta$. Unfolding the sum and integral gives the exact first trace

$$\mathrm{Tr}_{\mathrm{area}}(G_\ell^{1/2} C_\kappa G_\ell^{1/2}) = (2\pi)^{-m} \int_{\mathbb{R}^m} Q_{\ell,\kappa,k}(F_{\kappa,k})\,\mathrm{d}k. \tag{40}$$

Here $k$ is unfolded physical momentum, and $Q_{\ell,\kappa,k}$ is the left inverse quadratic on the $\theta$-constant source for that momentum. For a Hilbert-valued source, the formula sums the scalar inverse quadratics over the right-depth coefficient basis. All are nonnegative. Thus finite channel and coefficient truncations give the same identity upon taking increasing sums. The integral is finite by the uniform bound proved below.

Under the unitary dilation $u = u_0 + \ell z$, the operator $\ell$ times the strong diagonal response has form

$$2\sqrt{D_z^2 + |r_\ell D_\theta + \ell k|^2 + (\ell\kappa)^2} + c|z - \chi(\theta)|^p + M\ell\epsilon_\ell. \tag{41}$$

The support is $z > -u_0/\ell$. The normal dilation scales the source amplitude by $\ell^{1/2}$ and the inverse scales by $\ell$. Hence $\ell^{-2} Q_{\ell,\kappa,k}(F_{\kappa,k})$ is the inverse quadratic of (41) on $F_{\kappa,k}(u_0 + \ell z)$, constant in $\theta$. This exhibits the small shifted momentum $\ell k$ separately from the period frequency $r_\ell D_\theta$.

## 35 Convergence of the forced inverse

The forms (41) have a uniform positive gap in $L^2$, including at zero floor. Their kinetic symbol dominates $2|\xi_z|$ and the potential is at least $c'|z|^p$ for $|z|$ large, uniformly in $\theta$. For a direct proof of the gap choose one fixed interval containing all the shifted wells and split the $z$-Fourier transform at a small frequency. The low-frequency part has squared $L^2$ mass on that interval bounded by a constant times the frequency cutoff times its global squared norm. The entire high-frequency squared norm is controlled by the vertical kinetic energy divided by the cutoff. The potential controls the squared norm outside the interval. Choosing a sufficiently small fixed cutoff and absorbing the low-frequency term proves the gap. Integrating in $\theta$ gives the same estimate for the cell. It holds after zero extension and with arbitrary quasimomentum and mass. In particular $\|G_\ell\| \le C\ell$.

For a source bounded in pointwise Hilbert norm, the gap controls its form-dual norm on a fixed bounded normal interval. Outside $|z| \le R$, the reciprocal potential controls the omitted form-dual norm by $CR^{(1-p)/2}$ times the supremum of the source norm. These estimates follow by dual Cauchy–Schwarz against the kinetic gap or the potential term, respectively. In particular

$$\ell^{-2} Q_{\ell,\kappa,k}(F_{\kappa,k}) \le C \sup_{u\ge 0} \|F_{\kappa,k}(u)\|^2. \tag{42}$$

On a fixed normal window $|z| \le R$, the two sources $F_{\kappa,k}(u_0 + \ell z)$ and $F_{\kappa,k}(u_0)$ differ in norm by $O(\ell R)$, uniformly for bounded masses and momenta, by the separated column derivative bound. This freezes the auxiliary Hilbert-vector factor. The truncated inverse is then a scalar quadratic on the constant normal and tangential source restricted to the window, times $\|F_{\kappa,k}(u_0)\|^2$. The earlier tail estimate applies also to the limiting unit source. It suffices to prove variational convergence for truncated scalar sources and take $R \to \infty$. In the slow case we will equivalently use smooth tests for the full source after the tail estimate.

For $0 < r < \infty$, the mass and quasimomentum perturbations change the kinetic form in (41) by at most $2\ell(|k| + \kappa)\|U\|_2^2$. The remaining symbols converge on smooth tests as $r_\ell \to r$. A

compactly supported test in $J$ is eventually supported in the scaled domain, giving the variational lower limit. For the upper limit, near-maximizers for the compact scalar source have bounded energy and $L^2$ norm. Since $r_\ell$ is bounded below, they are bounded in $H^{1/2}$ of the full cylinder, after zero extension, and have a strongly convergent subsequence in local $L^2$. The limiting support is $J$. Restricting to compact normal sets in the potential energy and to bounded Fourier frequencies in the kinetic energy gives lower semicontinuity after exhaustion. Smooth periodic tests compactly supported in $J$ form a core: cut off at infinity in $H^{1/2}$ and the potential norm, then translate inward for a supported ray and mollify on bounded sets. Thus the upper and lower limits match.

For $r_\ell \to 0$, dropping the tangent term bounds the inverse from above by the vertical direct integral over the limiting domain $J$, after the source replacement and tail estimate. Its unit-source response is $C_0(J)$. When $J = \mathbb{R}$, translation separately at each $\theta$ reduces it to the well centered at zero. Tests $\varphi(z - \chi(\theta))$ with $\varphi$ smooth and compactly supported approximate its response from below. For the ray, let $V_0(z, \theta) = v_{\chi(\theta)}(z)$ be the vertical optimizer. As shown in Section 37, its angular derivative belongs to $L^2$. It can be cut off at infinity, translated slightly inward and mollified, in both the vertical form and the angular derivative norm. These trials recover the upper bound because the extra kinetic cost on a trial $U$ is at most $2r_\ell \|U\|_2 \|\nabla_\theta U\|_2$ before the small mass and quasimomentum terms. This argument requires no tangent compactness when $r = 0$.

For $r_\ell \to \infty$, a bounded-energy near-maximizer for the compact source has squared $L^2$ norm in nonconstant tangent modes $O(r_\ell^{-1})$ for any fixed $k$, since their kinetic cost is at least $2r_\ell|n| - 2\ell|k|$. The constant mode is locally compact in normal $L^2$ by the vertical energy bound. In the limit the optimizer is independent of $\theta$, so restriction to bounded normal intervals and exhaustion give its potential cost with the averaged potential. The normal kinetic energy of the average has the usual weak lower bound. Conversely a smooth compact test in $J$ independent of $\theta$ supplies the desired lower bound. This proves the response $C_\infty(J)$.

In all cases the limiting normalized fiber value is $C_r(J)\|F_{\kappa,k}(u_0)\|^2$. The arguments apply along sequences with $\kappa$ converging to any finite mass, since only uniform source derivatives and convergence of the squared point-column norms are used. By (39) and (42), the normalized unfolded integrand has an integrable bound $C_h e^{-2h\sqrt{\kappa^2+|k|^2}}$. Its integral decays exponentially in $\kappa$. The same bound applies for all sufficiently small $\ell$ and gives compact-uniform convergence in $\kappa$, since the limit is continuous.

Finally set $Q = G_\ell^{1/2} C_\kappa G_\ell^{1/2}$ in the periodic space. By the uniform weak sandwich bound and the strong gap, $\|Q\| = O(\ell)$. In each fiber $Q$ is positive, and the trace of $\sum_{j\ge2} Q^j/j$ is at most $O(\ell)\operatorname{Tr} Q$. The first trace is integrable by (40), so the per-area negative logarithm is finite and the higher terms are negligible in the $\ell^{-2}$ normalization. This proves the transform limit. The vacuum and Matsubara formulas follow by the common exponential bound, with a uniform majorant for the Matsubara sum on compact inverse-temperature intervals.

## 36 Dependence on the wavelength ratio

The constant source defines a bounded functional on the cell form space because $p > 1$. Coercivity gives a unique maximizing form element at each finite $r$, including zero, as well as in the constant-mode space at infinity. The ordered kinetic multipliers give $C_{r_1}(J) \ge C_{r_2}(J)$ for $0 \le r_1 < r_2$. Equality between two finite ratios forces the $r_2$ optimizer into the constant tangent channel, since the kinetic difference is strictly positive on each other channel. Equality with the infinite-ratio response would make the constant-mode optimizer a maximizer for the full finite-ratio form.

A $\theta$-independent function $U(z)$ maximizing the full finite form would solve the Euler equation with potential $c|z-\chi(\theta)|^p+\lambda$ for every $\theta$. To see this, test in $z$ on compact sets and use continuity in $\theta$ of the resulting potential pairing. Subtract at two distinct shift values $a$ and $b$. The result $U(z)(|z-a|^p-|z-b|^p)=0$ implies that $U$ is supported at their midpoint and hence is zero as an $L^2$ function. This contradicts the unit forcing. Thus the decrease is strict.

# 37 Small-ratio corrections

Let $V_0$ be the vertical optimizer and let $\mathcal{D}_r$ be the positive excess of the $r$-dependent kinetic form over its vertical part. Completing the vertical form square gives

$$C_0(J)-C_r(J)=\inf_V\{\mathfrak{b}_0[V-V_0]+\mathcal{D}_r[V]\}. \tag{43}$$

With Fourier variable $\xi$ in $z$ and mode $n\in\mathbb{Z}^m$ in $\theta$, the multiplier of $\mathcal{D}_r/r^2$ is zero for $n=0$ and otherwise equals

$$\frac{2|n|^2}{\sqrt{\xi^2+r^2|n|^2}+|\xi|}. \tag{44}$$

Fourier integration in $\xi$ uses $\mathrm{d}\xi/(2\pi)$.

## 37.1 Interior zero

On the line, $V_0(z,\theta)=v_*(z-\chi(\theta))$. Its normal Fourier transform is $e^{-i\xi\chi(\theta)}\widehat{v}_*(\xi)$. The multipliers (44) increase to $|n|^2/|\xi|$ as $r\downarrow 0$ for $\xi\neq 0$. Parseval in $\theta$ gives

$$\int_{\mathbb{R}}\sum_n\frac{|n|^2}{|\xi|}|\widehat{V}_0(\xi,n)|^2\frac{\mathrm{d}\xi}{2\pi}=\langle|\nabla_\theta\chi|^2\rangle_\theta\int_{\mathbb{R}}|\xi|\,|\widehat{v}_*(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}.$$

The integral is finite by the vertical form bound, which also makes $V_0$ admissible for the full finite-ratio form. Testing (43) with $V_0$ gives the upper coefficient in (37). For the lower bound, let $V_r$ be the full optimizer. The upper estimate and (43) give $\mathfrak{b}_0[V_r-V_0]=O(r^2)$ and hence $V_r\to V_0$ in $L^2$. Restrict the divided excess to finitely many modes and a compact interval of $\xi$ separated from zero. There the multipliers converge uniformly and the quadratic expressions pass by strong $L^2$ convergence. Exhausting the Fourier region proves the matching lower bound. The coefficient is positive when $\chi$ is nonconstant because $v_*$ is a nonzero $L^2$ function with strictly positive $|D|$ energy.

## 37.2 Zero at the supported edge

For $t$ in the range of $\chi$, the forms defining $v_t$ have common domain supported $H^{1/2}$ with $(1+z^p)$-weighted $L^2$. Their energy norms are uniformly equivalent to this norm by the vertical gap and the potential at large $z$. The shifted potential depends $C^1$ on $t$ as a map from the form space to its dual. In fact

$$\partial_t|z-t|^p=-p\,\operatorname{sgn}(z-t)|z-t|^{p-1}.$$

After division by $1+z^p$, the derivative is continuous in multiplier norm, by continuity on compact intervals and a uniform $O((1+z)^{-1})$ tail bound. Inverting the form operators shows that $t\mapsto v_t$ is $C^1$ in this form space. Consequently $V_0(z,\theta)=v_{\chi(\theta)}(z)$ and its first tangent derivatives belong to $L^1_z(L^2_\theta)$ by weighted Cauchy–Schwarz. In particular $\widehat{V}_0(\xi,\cdot)$ is continuous at zero as an $H^1_\theta$-valued function and equals $m_0$ there.

The mass $\int_0^\infty v_t \,\mathrm{d}z$ is strictly increasing in $t$. Translating to the center of the potential gives the same confining form on the expanding supported interval $(-t, \infty)$. Domain inclusion gives monotonicity. The smaller-domain optimizer is nonzero and nonnegative by absolute-value contraction for the jump form. If the inverse quadratic were unchanged upon expansion, this optimizer extended by zero would also solve the Euler equation on the larger domain. A nonzero nonnegative test compactly supported in the added strip has zero potential pairing and strictly negative kinetic pairing with the smaller-domain optimizer, by the off-diagonal jump kernel. This contradicts the positive pairing with the unit source. Thus $m_0$ is nonconstant when $\chi$ is nonconstant, and its squared tangent gradient has positive mean.

To compute the upper coefficient in (38), evaluate the divided excess of $V_0$ using (44). On $|\xi| \geq d_0 > 0$ it is bounded uniformly in $r$ by $C_{d_0}\|\nabla_\theta V_0\|_2^2$. For $|\xi| \leq d_0$ we have the bound $C|n|^2/(|\xi|+r)$ for every nonzero integer mode. The $H^1_\theta$ continuity therefore permits replacement of $\widehat{V}_0(\xi,\cdot)$ by $m_0$ with error $o_{d_0}(1)\log(1/r) + O_{d_0}(1)$, where the first coefficient tends to zero as $d_0 \downarrow 0$. For each fixed nonzero $n$,

$$\lim_{r\downarrow 0} \frac{1}{\log(1/r)} \int_{-d_0}^{d_0} \frac{2|n|^2}{\sqrt{\xi^2 + r^2|n|^2} + |\xi|} \frac{\mathrm{d}\xi}{2\pi} = \frac{|n|^2}{\pi}.$$

For example the multiplier is asymptotic to $|n|^2/|\xi|$ above the scale $r|n|$. The bound $C|n|^2/(|\xi|+r)$ also gives dominated summation against the $H^1_\theta$ coefficients of $m_0$. Thus $\mathcal{D}_r[V_0]/[r^2\log(1/r)] \to \langle|\nabla_\theta m_0|^2\rangle_\theta/\pi$. The vertical optimizer is an admissible trial for each finite ratio since its tangential derivative lies in $L^2$. Equation (43) gives the upper bound.

For the lower bound (43) first gives $\|V_r - V_0\|_2^2 = O(r^2\log(1/r))$ from the vertical gap. Retain finitely many nonzero modes and restrict to $r \leq |\xi| \leq d_0$. The divided multiplier on this region is bounded by $C/r$ for the fixed modes. Thus the divided excess of $V_r - V_0$ on the region is $O(r\log(1/r))$, whereas that of $V_0$ is $O(\log(1/r))$. The reverse triangle inequality in the weighted Fourier norm shows that replacing $V_r$ by $V_0$ changes the leading logarithmic coefficient by zero. For each retained mode, the removed interval $|\xi| < r$ would contribute only $O(1)$ to the divided integral. Letting $d_0 \downarrow 0$ after the small-$r$ limit and then exhausting the mode sets proves the matching lower coefficient. This completes the proof of Theorem 21.

**Acknowledgement.** OpenAI Codex and Anthropic Claude assisted with mathematical exploration, source comparison, proof checking, numerical verification and manuscript preparation. The author is responsible for the work.

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