Near-touching holes in a Dirichlet screen: the retained-segment heat profile, repulsive contact and the exact static Casimir constant

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Abstract

As two holes in a Dirichlet screen approach each other, their connected scalar Casimir interaction is governed by a single planar observable: the heat integral $J(t)$ of Brownian bridges that hit both exterior rays of a retained Dirichlet segment of unit length while avoiding the segment. We determine this observable completely and use it to describe the near-touching interaction with and without surface contact. At short time $J(t) \sim t^{5/2}e^{-1/t}/(32\pi^{3/2})$, with its first correction, by a half-line factorization of the relative determinant; at long time $J(t) \to 1/8$ with an inverse-logarithmic expansion. A translation argument expresses J through the gaps in the screen trace of the bridge and gives exact small- and large-gap counting laws. For two interval apertures we compute the static (zero-mode) heat integral in closed form,

$$Q_g^{\mathrm{stat}} = -2\sum_{n\geq 1} \log\tanh\Big(\pi n \frac{K(k')}{K(k)}\Big) = -\tfrac{1}{4}\log(1-q) - \log\frac{2K(k)}{\pi},$$

by mapping the loops that visit both apertures to the two seams of a flat cylinder. This fixes the finite static constant $\log\pi - \frac{1}{4}\log(\ell_-^{-1}+\ell_+^{-1})$ and attributes the entire double logarithm to loops of nonzero winding. We then add a common repulsive contact of strength ρ_g on both apertures: bounded $g\rho_g$ gives a weighted bridge profile; strong contact gives explicit vacuum constants and a thermal sum of squared boundary kernels for curved and finite-type tangencies in every dimension; and for intervals an unbounded contact with $g\rho_g \to 0$ cuts off the static logarithm at length $1/\rho_g$ while retaining its double-logarithmic correction. The gap-scale free energy and both endpoints of its nonzero Matsubara profile are determined.

Keywords. Scalar Casimir effect, Dirichlet screen, near-contact asymptotics, Brownian loop measure, elliptic integrals, Robin contact, Matsubara sum.

Overview

Sections 1–4 determine the retained-segment profile and the gap-scale free energy of transparent intervals. Sections 5–9 add repulsive contact and prove the exact static formula. The near-contact law for transparent windows, of which J is the local kernel, and the separated many-hole interaction are proved in the companion paper [2]; the underlying connected heat trace is that of [1].

1 The planar observable

Let H_- and H_+ be the two open rays in the horizontal axis outside the retained segment $[-h/2, h/2]$. A planar bridge of duration t, generator Δ, and base point (S, X) contributes when it hits the axis in both rays and avoids the retained segment. Write $J_h(t)$ for the integral of this probability against $(4\pi t)^{-1} dS\, dX$, and write $J(t) = J_1(t)$. This is the local two-aperture kernel in the near-contact theorem of [2]. Brownian scaling gives $J_h(t) = J(t/h^2)$. At $h = 0$ the two slit-edge defects give $J_0(t) = 1/8$ [2, 9, 10]. In particular $0 \le J(t) \le 1/8$. Bridges that hit both rays have lateral range at least h, giving a Gaussian small-time bound up to a constant by one-dimensional bridge range and reflection estimates.

The interaction of two Dirichlet obstacles occupying the exterior rays involves bridges hitting both rays, as in the whisker geometry of [7, Section 3]. The event defining J_h also excludes paths hitting the central segment. The closest classical comparison for finite apertures is the collinear *Neumann* needle calculation [3, Eq. (15)]. Reflection across the axis in the aperture problem leaves an unchanged odd Dirichlet sector and gives N on the openings and D elsewhere in the even sector. Neumann needles occupying the same intervals leave the even Neumann sector unchanged and yield the same mixed sector in the odd component. Thus the connected classical observable is the same in the two models, by cancellation of the unchanged sectors [5]. The known equal-needle force already gives the first two static logarithmic terms. For infinite rays the complementary Neumann half-plane scattering representation of [4, Eqs. (6)–(7)] contains both displayed positive-real-frequency coefficients of Q in its lowest odd channel. We determine the complex-frequency expansion needed for time inversion, the bridge gap law, and the nonzero-frequency thermal response, and match the arbitrary-length finite-interval expansion at the heat level.

2 Half-line determinant and exact short-time equivalent

Theorem 1. *With the principal root $w = \sqrt{z}$,*

$$Q(z) := \int_0^\infty e^{-zt} J(t)\, \frac{dt}{t} = \frac{e^{-2w}}{32\pi w^3} \left(1 - \frac{15}{4w} + O(|w|^{-2})\right)$$

uniformly for $\operatorname{Re} z > 0$*, as* $|z| \to \infty$*. Consequently*

$$J(t) = \frac{t^{5/2}}{32\pi^{3/2}} e^{-1/t} \left(1 - \frac{21}{4} t + O(t^2)\right) \qquad (t \downarrow 0).$$

Proof. For two bounded transparent apertures the relative determinant identity of [2] identifies their connected heat transform with minus the connected log determinant of the compressed response $2\sqrt{D^2 + z}$, where $D = -i\partial_u$ on $\mathbb{R}$. The factor two cancels. Put $r = |w|$ and $\mu = w/r$, and dilate the tangential coordinates by r. The isolated operators become, after removing the common scale factor, the form compression A_μ of $\sqrt{D^2 + \mu^2}$ to the positive half-line. The form domain is the subspace of $H^{1/2}(\mathbb{R})$ supported there. In coordinates $u, v > 0$ that increase into either ray, the equal off-diagonal blocks B have kernel

$$-\frac{\mu}{\pi(r + u + v)} K_1\big(\mu(r + u + v)\big).$$

This follows by square-root subordination, with the identity term supported on the diagonal, or the Bessel formula in [2]. Uniform accretivity of A_μ in the form norm follows from $\operatorname{Re} \sqrt{\xi^2 + \mu^2} \ge$

$c\sqrt{1+\xi^2}$ for $|\arg\mu| \le \pi/4$. Indeed the modulus of $\xi^2+\mu^2$ is at least $(1+\xi^2)/\sqrt{2}$ and the root argument has magnitude at most $\pi/4$.

On the positive axis, pass from bounded intervals to rays by increasing their lengths. The heat side converges by monotone convergence. For the determinant, variational approximation gives strong convergence of the extended inverses, since compact cutoffs approximate in the half-line form space and the restricted forms are uniformly coercive. The cross kernels converge in Hilbert–Schmidt norm by truncation. Uniform boundedness and strong convergence of the inverses and their adjoints preserve this norm convergence under multiplication, by finite-rank approximation. The small norm of B at large r permits the block determinant expansion

$$Q(z) = \sum_{j\ge 1} j^{-1} \operatorname{Tr}[(A_\mu^{-1} B A_\mu^{-1} B)^j].$$

For complex z, form the same product in the original unscaled coordinates. The inverse depends holomorphically on z by local coercive form inversion. The separated Bessel kernel is Hilbert–Schmidt holomorphic on the right half-plane by local exponential domination. The scaled norm bounds below make the series holomorphic throughout its connected large-$|z|$ region. The identity theorem extends the equality from its positive real segment.

Set $f_\mu(u) = e^{-\mu u}$. Use the bilinear rank-one kernel $f_\mu(u) f_\mu(v)$. The fixed-order Bessel asymptotic [12, Eq. 10.40.2] gives

$$B = -\frac{\sqrt{\mu}\, e^{-w}}{\sqrt{2\pi}\, r^{3/2}} f_\mu \otimes f_\mu + E, \qquad \|E\|_2 \le C e^{-\operatorname{Re} w} r^{-5/2}.$$

Indeed for $s = u+v$ the change in the algebraic factor $(1+s/r)^{-3/2}$ is $O(s/r)$, the Bessel remainder is $O((r+s)^{-1})$, and $|e^{-\mu s}| \le e^{-s/\sqrt{2}}$. Terms with $j \ge 2$ are exponentially smaller. For the first trace it remains to compute $q(\mu) = \int_0^\infty f_\mu(u)(A_\mu^{-1} f_\mu)(u)\, du$.

For positive real μ, the multiplier $L = (\mu + \partial_u)^{1/2}$ factors the full energy as $\|Lf\|_2^2$. Its inverse is convolution by $\mathbf{1}_{s>0} e^{-\mu s}/\sqrt{\pi s}$, which maps half-line L^2 into the supported form space. Conversely $L = (\mu+\partial_u) L^{-1}$ preserves half-line support in distributions. Thus L is a bijective form isometry onto half-line L^2. The adjoint of the causal inverse maps f_μ to $f_\mu/\sqrt{2\mu}$. Taking its squared norm gives $q(\mu) = 1/(4\mu^2)$. Local form inversion and the holomorphic exponential extend the bilinear pairing to a sector containing the unit-circle arc $|\arg\mu| \le \pi/4$, and the identity theorem gives the same formula there. The leading trace is $\mu e^{-2w} r^{-3} q(\mu)^2/(2\pi)$. The Hilbert–Schmidt error and the uniform inverse norm give a relative $O(r^{-1})$ contribution since the leading coefficient is nonzero with constant modulus on the arc.

To one further order, the rank-one kernel after removal of its scalar prefactor is

$$f_\mu \otimes f_\mu + r^{-1}\left[\frac{3}{8\mu} f_\mu \otimes f_\mu - \frac{3}{2}(u f_\mu \otimes f_\mu + f_\mu \otimes u f_\mu)\right] + O_2(r^{-2}),$$

where O_2 is in Hilbert–Schmidt norm. The next Bessel error and the Taylor remainder of the distance factor are bounded after integration by the same exponential majorant times $C(1+(u+v)^2)$. The adjoint of the causal inverse applied to $u f_\mu$ gives $(u + 1/(4\mu)) f_\mu/\sqrt{2\mu}$, first for positive real μ. Consequently the pairing of $u f_\mu$ with $A_\mu^{-1} f_\mu$ is $3q(\mu)/(4\mu)$, and again extends holomorphically. The cross term in the first trace gives relative correction $r^{-1}(3/(4\mu) - 6 \cdot 3/(4\mu)) = -15/(4w)$.

The correction in the transform can be inverted term by term. The inverse of $z^{-p} e^{-2\sqrt{z}}$ for $p > 0$ is

$$\frac{1}{\sqrt{\pi}\Gamma(p)} \int_0^t u^{p-1}(t-u)^{-3/2} e^{-1/(t-u)}\, du = \frac{t^{2p-3/2} e^{-1/t}}{\sqrt{\pi}} (1 + p(1/2-p)t + O(t^2)).$$

The integral is the convolution of the inverse of z^{-p} with that of $e^{-2\sqrt{z}}$. To expand it, put $u = t^2 v$. The first-order factor multiplying $v^{p-1}e^{-v}$ is $1 + t(3v/2 - v^2)$. Taylor expansion on $v \le t^{-1/4}$ bounds the error by t^2 times an integrable polynomial against e^{-v}. On the omitted tail the exponential $e^{-v/(1-tv)}$ absorbs both the factor $(1-tv)^{-3/2}$ and any fixed power of $1/t$.

For the transform remainder invert on $z = t^{-2} + iy$. Its vertical tail is integrable, and continuity of J, also given by the gap identity below, gives pointwise inversion. The exponent is bounded in modulus after taking out $e^{-1/t}$ by $\exp[-c(yt^2)^2/t]$ for $|yt^2| \le 1$ and by $\exp[-c|yt^2|^{1/2}/t]$ outside. This follows by taking the real part of $\sqrt{1+iyt^2} - 1$. The factor $|z|^{-5/2} \le t^5$ and integration in y give $O(t^{7/2}e^{-1/t})$ for the inverse remainder. Taking $p = 3/2$ and $p = 2$ in the two leading inverse terms and multiplying by t proves the asserted expansion. □

3 Large times and the screen-trace gap law

Theorem 2. *Put $a = \gamma - \log 64$, where γ is Euler's constant. At large time,*

$$J(t) = \frac{1}{8} - \frac{1}{\log t} - \frac{a}{(\log t)^2} - \frac{a^2 - \pi^2/6}{(\log t)^3} + O((\log t)^{-4}).$$

The difference $1/8 - J(t)$ has the full inverse-logarithmic expansion of the planar isolated-segment heat defect.

Proof. At duration one, $J_0(1) - J_h(1)$ counts loops hitting the short segment $[-h/2, h/2]$ and both zero-gap rays, up to polar endpoint events. Its difference from the isolated-segment heat defect counts loops hitting the segment while missing at least one zero-gap ray. Missing a zero-gap ray implies avoiding the corresponding exterior ray at gap h. The relevant loops therefore hit the disk of radius h about the endpoint of that exterior ray while avoiding the ray. Time reversal and the first-half bridge density bound $p_{1/2}(z,q)/p_1(q,q) \le 2$ let us use the Brownian-motion probability to hit that disk before the slit boundary. In the slit exterior wedge of opening 2π, for radius $r \ge 2h$ this probability is $O((h/r)^{1/2})$. Separation of variables gives terms $\sin(n\theta/2)(h/r)^{n/2}$ with bounded coefficients from the constant-one data on the circular arc. Independently, the free bridge reaching the disk has a Gaussian bound for r above a fixed constant. The geometric mean of the two estimates is integrable over starting points and gives $O(h^{1/4})$. The inner disk has area $O(h^2)$. The other ray gives the same bound.

By scaling, the isolated-segment defect at this duration is the unit-segment defect at time h^{-2}, and the missing-ray error becomes $O(t^{-1/8})$. McGillivray's pinned-sausage expansion [6, Theorem 1.2 and Eq. (8.1)], divided by $4\pi t$, is the free-minus-killed heat trace. In its convention $C(K) = 2\log \mathrm{cap}(K)$. The conformal map of the unit segment complement to the exterior unit disk has derivative 4 at infinity in the sense that its leading term is $4z$, hence the capacity is $1/4$. The second coefficient is $C(K) + \gamma - \log 4 = a$, and the third is $a^2 - \pi^2/6$. The arbitrary-order expansion gives the stated remainder and the full transfer. □

Let B_1, B_2 be independent unit-duration one-dimensional bridges of generator Δ from zero to zero. For a fixed initial normal coordinate X, define

$$Z_X = \{t : X + B_2(t) = 0\}, \qquad Y_X = \{B_1(t) : t \in Z_X\}.$$

Let $\mathcal{G}_X$ be the bounded complementary intervals of the compact set Y_X. Empty and singleton traces have no such gaps. Vary the initial lateral coordinate S. The translated trace $S + Y_X$ avoids the retained segment and meets both exterior rays exactly when the segment lies in a translated

bounded gap, apart from zero-measure endpoint placements. A gap of length ℓ permits lateral shifts of measure $(\ell - h)_+$. Thus

$$4\pi J_h(1) = \int_{\mathbb{R}} \mathbb{E} \sum_{I \in \mathcal{G}_X} (|I| - h)_+ \, dX = \int_h^\infty M(u) \, du, \qquad M(u) = \int_{\mathbb{R}} \mathbb{E}\#\{I \in \mathcal{G}_X : |I| > u\} \, dX. \tag{1}$$

Tonelli applies. For a normal bridge that reaches zero, the total gap length is bounded by the lateral bridge range, which is independent of the normal event. The latter has integrable probability e^{-X^2}. This proves continuity at $h > 0$ by dominated convergence. Planar point polarity and Fubini also give $|Y_X| = 0$ almost surely for every $X \neq 0$. At $h = 0$, the integrated expected range of Y_X therefore equals $4\pi J_0(1) = \pi/2$.

Theorem 3. *The integrated gap count satisfies*

$$M(h) \sim \frac{2\pi}{h \log^2(1/h)} \quad (h \downarrow 0), \qquad M(h) \sim \frac{h^{-4}}{4\sqrt{\pi}} e^{-h^2} \quad (h \to \infty).$$

For the dimensional coefficient $D_d = (4\pi)^{-(d-1)/2}(4\sqrt{\pi})^{-1} \int_0^\infty t^{-(d+2)/2} J(t) \, dt$ *of [2],*

$$D_d = \frac{(4\pi)^{-(d+1)/2}}{2\sqrt{\pi}\, d(d+1)} \int_{\mathbb{R}} \mathbb{E} \sum_{I \in \mathcal{G}_X} |I|^{d+1} \, dX, \qquad \int_0^\infty t^{-(d+2)/2} J(t) \, dt \sim \frac{\Gamma(d/2 - 5/2)}{32\pi^{3/2}} \quad (d \to \infty).$$

Proof. By (1), the two secants of the decreasing function $4\pi J_h(1)$ across h bound $M(h)$, since M is decreasing. At small h use endpoints $h/q, h, qh$, with $q > 1$ fixed. In Theorem 2 and $J_h(1) = J(h^{-2})$, the second inverse-log term changes by $O(\log^{-3}(1/h))$; the leading difference across each secant is $\log q/(2\log^2(1/h))$ for $J_h(1)$. Send $q \downarrow 1$ after $h \downarrow 0$. At large h, Theorem 1 gives $J_h(1) \sim h^{-5} e^{-h^2}/(32\pi^{3/2})$. With endpoints $h - b/h, h, h + b/h$ for fixed $b > 0$, the ratios of the shifted values to the central value tend to e^{2b} and e^{-2b}. Send $b \downarrow 0$ after $h \to \infty$.

For the moment identity, set $h = t^{-1/2}$ and integrate (1), using $2\int_0^\ell h^{d-1}(\ell - h) \, dh = 2\ell^{d+1}/[d(d+1)]$. For large d, instead set $v = 1/t$; the large-v integrand is $v^{d/2-7/2} e^{-v}/(32\pi^{3/2})$ up to arbitrarily small relative error, by Theorem 1. The integral on a fixed bounded v-interval is negligible at the gamma scale. This proves both claims. □

4 Gap-scale free energy of finite intervals

Let the two transparent intervals have fixed lengths $\ell_-, \ell_+ > 0$, with gap g centered at zero. Let $\mathcal{I}_g(t)$ denote their connected all-hit heat gain. At $g = 0$ retain the interval labels and disregard the polar common endpoint. Fix a common unit of length in logarithms. Define

$$\begin{aligned} C_* &= \lim_{L \to \infty} \left[\int_0^{L^2} J(t) \frac{dt}{t} - \frac{1}{4} \log L + \log \log L \right], \\ C_{\ell_-, \ell_+} &= C_* + \int_0^1 (\mathcal{I}_0(t) - 1/8) \frac{dt}{t} + \int_1^\infty \mathcal{I}_0(t) \frac{dt}{t}, \\ N_\lambda &= 2 \sum_{n \geq 1} Q((2\pi n/\lambda)^2). \end{aligned}$$

Theorem 4. *At inverse temperature* $\beta = \lambda g$*, locally uniformly for* $\lambda > 0$*, the connected free energy is*

$$\mathcal{F}_{g, \lambda g} = -\frac{\log(1/g)}{8\lambda g} + \frac{\log\log(1/g)}{2\lambda g} - \frac{C_{\ell_-, \ell_+} + N_\lambda}{2\lambda g} + o(g^{-1}).$$

The nonzero Matsubara function obeys

$$N_\lambda \sim \frac{\lambda^3}{128\pi^4} e^{-4\pi/\lambda} \qquad (\lambda \downarrow 0),$$

$$N_\lambda = 2\lambda D_1 - \tfrac{1}{4}\log\lambda + \log\log\lambda - C_* + \tfrac{\gamma}{8} - \frac{a+\gamma}{2\log\lambda} + O((\log\lambda)^{-2}) \qquad (\lambda \to \infty).$$

Proof. The bounded intervals are contained in the infinite rays of gap g. For small physical times,

$$0 \le J(t/g^2) - \mathcal{I}_g(t) \le Ce^{-c/t}$$

uniformly in sufficiently small g. A bridge in the difference visits a ray beyond an outer endpoint and also visits the opposite ray. Its contact-axis range in unit-duration coordinates is at least $\min(\ell_-, \ell_+)/\sqrt{t}$. Integrate over contact-axis starts using the lateral range to bound their measure, over the normal starts by the reflection probability, and then apply the Gaussian range tail. For each fixed $t > 0$, the all-hit event converges as $g \downarrow 0$, outside the zero-probability event of hitting one of the limiting endpoints. Gaussian localization gives convergence of the spatial integral. At large time use the enclosing-window $O(t^{-1})$ bound [2, 1].

Split the zero-mode integral as

$$\int_0^\infty \mathcal{I}_g(t)\frac{dt}{t} = \int_0^{g^{-2}} J(T)\frac{dT}{T} + \int_0^1 [\mathcal{I}_g(t) - J(t/g^2)]\frac{dt}{t} + \int_1^\infty \mathcal{I}_g(t)\frac{dt}{t}.$$

The last two terms converge by the preceding bounds. The first is $\frac{1}{4}\log(1/g) - \log\log(1/g) + C_* + o(1)$ by Theorem 2. In the nonzero modes the substitution $t = g^2T$ gives $\mathcal{I}_g(g^2T) \uparrow J(T)$. The Gaussian bounds on $J(T)$ at zero and on the nonzero-mode sum at infinity give dominated convergence locally uniformly in $\lambda > 0$. The free-energy prefactor is $-1/(2\lambda g)$ [2]. The first nonzero Matsubara pair dominates as $\lambda \downarrow 0$ by Theorem 1, which also bounds the higher-mode tail.

For large λ, Poisson summation as in [8] gives

$$2\lambda D_1 - N_\lambda = \int_0^\infty J(t)W(t/\lambda^2)\frac{dt}{t}, \qquad W(x) = (4\pi x)^{-1/2} - 2\sum_{n\ge1} e^{-4\pi^2 n^2 x}.$$

Equivalently $W(x) = 1 - 2(4\pi x)^{-1/2}\sum_{n\ge1} e^{-n^2/(4x)}$. Thus $k(x) = W(x) - \mathbf{1}_{x<1}$ has all finite absolute logarithmic moments against dx/x. Its zeroth moment is $-\gamma$, by

$$\int_0^\infty x^{s-1}W(x)\,dx = -2\Gamma(s)(2\pi)^{-2s}\zeta(2s) \qquad (0 < \operatorname{Re} s < 1/2).$$

Indeed the positive n-sum has its termwise Mellin transform for $\operatorname{Re} s > 1/2$. Subtract $(4\pi x)^{-1/2}$ on $(0,1)$ and continue its integral $1/(\sqrt{4\pi}(s-1/2))$. In the indicated strip this term equals minus the integral of the same power over $(1,\infty)$. The constant term at $s = 0$ of the displayed expression is $-\gamma$, since $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. The pole $1/s$ is exactly the Mellin transform of the indicator.

The sharp cutoff contribution is

$$\int_0^{\lambda^2} J(t)\frac{dt}{t} = \tfrac{1}{4}\log\lambda - \log\log\lambda + C_* + \frac{a}{2\log\lambda} + O((\log\lambda)^{-2}).$$

Set $t = \lambda^2 x$ in the difference. For $\lambda^{-1} < x < \lambda$, Theorem 2 gives

$$J(\lambda^2 x) = \frac{1}{8} - \frac{1}{2\log\lambda} + O\Big(\frac{1 + |\log x|}{(\log\lambda)^2}\Big).$$

The bounds on k and J make the omitted tails $O(\lambda^{-1/2})$. Thus the difference is $-\gamma/8 + \gamma/(2\log\lambda) + O((\log\lambda)^{-2})$, giving the stated large-period law. □

For comparison, the noncompact Dirichlet–Neumann boundary four-point function [11, Appendix A] gives the normalized expression

$$(1-q)^{-1/8}[2K(\sqrt{q})/\pi]^{-1/2}, \qquad q = \frac{\ell_-\ell_+}{(\ell_-+g)(\ell_++g)},$$

where K is the complete elliptic integral of the first kind. Identifying twice its logarithm with the static heat integral would give $C_{\ell_-,\ell_+} = \log\pi - \frac{1}{4}\log(\ell_-^{-1}+\ell_+^{-1})$. Theorem 4 gives a heat-level definition and proof of the finite limit. Theorem 8 below proves this identification and the stated value of C_{ℓ_-,ℓ_+}.

5 Contact and local geometry

Let two bounded apertures in the plane of a Dirichlet screen in $\mathbb{R}^{d+1}$ have gap $g > 0$. In each aperture impose a common repulsive surface form $\rho_g \int |f|^2$, $\rho_g \geq 0$. Across the screen the openings allow transmission. Reflection leaves an odd full-Dirichlet half-space problem and an even half-space problem with Robin coefficient $\rho_g/2$ on the openings and Dirichlet on the rest. The factor $1/2$ follows from the unitary restriction of an even full-space function to the half-space. It also appears as the ratio of the contact term to the two-arm response $2\sqrt{|D|^2+z}$.

Karabali and Nair [13, Section 3] develop the square-root boundary action for general boundary conditions and apply it to a plate and a parallel half-plate. Fosco and Mazzitelli [14] study variable delta interactions between parallel thin plates. The derivative expansion for facing imperfect mirrors [15] and the curved delta arc in a wedge [16, Section 6] treat different geometries. Frank and Geisinger [17] and Sher [18] analyze large Robin parameters at smooth boundaries, including the scale $\rho\sqrt{t}$ in the heat kernel. We use the connected interaction of apertures in one screen. Inclusion–exclusion cancels the individual aperture terms, and the relevant $\rho_g\sqrt{t}$ scale becomes $g\rho_g$ at the gap diffusion time. The previous transparent near-contact laws [2] answer the approach question of [19, Section V] in several geometries. Here we vary the contact through the competing gap scale and obtain a separate squared-kernel limit when $g\rho_g$ diverges.

Take $s \in \mathbb{R}$ along the gap and $y \in \mathbb{R}^{d-1}$ tangential to it in the screen, with x normal to the screen. Before translating the apertures by $\mp g/2$ in s, suppose their closures have unique contact at zero and lie on opposite sides of $s = 0$. For $d \geq 2$, assume bounded convex apertures with facing graphs $s = -a_-(y)$ and $s = a_+(y)$ near zero. The interiors are respectively on the left and right. We use either of the following hypotheses from [2].

- The graphs are C^2 with positive definite sum $B_- + B_+$ of their Hessians at zero. Put $\alpha = (d-1)/2$ and $H(w) = 1 + \frac{1}{2}w^{\mathsf{T}}(B_- + B_+)w$.
- For $p_j > 1$, $a_i(y) = \sum_j b_{ij}|y_j|^{p_j} + o(\sum_j |y_j|^{p_j})$ in orthonormal coordinates, where $b_{ij} \geq 0$ and $b_{-j} + b_{+j} > 0$. Set $\alpha = \sum_j p_j^{-1}$ and $H(w) = 1 + \sum_j (b_{-j}+b_{+j})|w_j|^{p_j}$.

In $d = 1$ take fixed interval lengths $\ell_-, \ell_+ > 0$, put $\alpha = 0$ and $H = 1$, and interpret the zero-dimensional integral as evaluation.

Let $\mathcal{I}_{g,\rho}(t)$ be the connected heat gain obtained by the two-opening relative heat trace minus the two one-opening relative traces, each relative to the cut half-spaces. Denote by L_t the normal local time at the plane, in the occupation-density convention for generator Δ. The connected gain is the free diagonal integral of bridges that avoid the retained screen and hit both openings,

with weight $e^{-\rho L_t}$ [1]. The weight is common to all surviving subsets, so the same inclusion–exclusion applies. The scalar vacuum and thermal observables in units $\hbar = c = k_{\mathrm{B}} = 1$ are [2]

$$\mathcal{E}_g = -\frac{1}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \mathcal{I}_{g,\rho_g}(t)\, dt, \tag{2}$$

$$\mathcal{F}_{g,\beta} = -\frac{1}{2\beta} \int_0^\infty \mathcal{I}_{g,\rho_g}(t) \sum_{n\in\mathbb{Z}} e^{-t(2\pi n/\beta)^2}\, \frac{dt}{t}. \tag{3}$$

6 Planar estimates and bounded scaled contact

Let $J_h^b(t)$ be the planar all-hit gain for the exterior rays of a retained segment of length h, with bridge weight e^{-bL_t}. For $b = 0$, write $J_h(t)$ and $J(t) = J_1(t)$. The zero-gap rays have $J_0(t) = 1/8$ [2] and Sections 1–4. Brownian scaling gives

$$J_h^b(t) = J_{h/\sqrt{t}}^{b\sqrt{t}}(1) = J_1^{bh}(t/h^2) \quad (h > 0).$$

For later use set $Q_b(z) = \int_0^\infty J_1^b(t) e^{-zt}\, dt/t$ for $z > 0$, also for $z = 0$ when $b > 0$.

At duration one, vary the lateral base point of the ray bridge while keeping the lateral increments and normal path fixed. The available lateral length is bounded by the full lateral bridge range Sections 1–4, which has finite expectation and is independent of the normal bridge. The occupation local time of a generator-Δ normal unit bridge from X to X satisfies

$$\Pr_X(L_1 > u) = e^{-(|X|+u)^2} \quad (u \geq 0).$$

Indeed the reflected joint bridge density for variance-one Brownian motion with local time above l is $p_1(|x|+|y|+l)$ before conditioning on the endpoint y. Divide by $p_1(y-x)$ at $x = y = X/\sqrt{2}$ and use that variance-two semimartingale local time is twice its occupation local time. Integration over X gives $2e^{-u^2}\, du$ for the positive part of the local-time density. Hitting the line and acquiring positive local time are almost surely the same event for these bridges. The bound on the lateral range and $1 - e^{-s} \leq s$ give

$$J_h^b(t) \leq C \min(1, (b\sqrt{t})^{-1}), \qquad 0 \leq J_h(t) - J_h^b(t) \leq C \min(1, b\sqrt{t}). \tag{4}$$

These estimates are uniform in $h \geq 0$ and apply to the respective interval observables by event inclusion. The separate Gaussian estimate $J_1(T) \leq Ce^{-\eta/T}$ at $T \leq 1$ follows from the lateral range needed to meet both rays.

Set $c_g = g\rho_g$. For $b \geq 0$ define

$$D_d(b) = \frac{(4\pi)^{-(d-1)/2}}{4\sqrt{\pi}} \int_0^\infty T^{-(d+2)/2} J_1^b(T)\, dT.$$

Theorem 5. *If $c_g \to c < \infty$, then*

$$g^{d-\alpha} \mathcal{E}_g \longrightarrow -\int_{\mathbb{R}^{d-1}} H(w)^{-d} D_d(cH(w))\, dw.$$

For $d \geq 2$, and also $d = 1$ when $c > 0$, one has locally uniformly for $\lambda > 0$

$$g^{d-\alpha} \mathcal{F}_{g,\lambda g} \longrightarrow -\frac{(4\pi)^{-(d-1)/2}}{2\lambda} \int_{\mathbb{R}^{d-1}} \int_0^\infty T^{-(d+1)/2} J_{H(w)}^c(T) \sum_{n\in\mathbb{Z}} e^{-T(2\pi n/\lambda)^2}\, dT\, dw. \tag{5}$$

Proof. For $t = g^2T$, dilation in the two fast bridge coordinates changes the weight to $e^{-c_g L_T}$. The slow tangential freezing in [2, proof of the near-contact theorem] uses the uniform continuity of a scaled bridge and point polarity at the two limiting endpoints. Multiply the converging all-hit indicators by this bounded weight, which converges pointwise for fixed T. Every spatial and time majorant of the transparent proof still applies, since the weight is at most one. For the additional $d = 1, c > 0$ zero mode use (4) at large T with c_g bounded below. The small-time Gaussian bound and the Matsubara sum control the other terms uniformly for λ in compact positive sets. For the vacuum integral use $J_h^c(T) = J_1^{ch}(T/h^2)$ and substitute $T = h^2u$. This gives the factor h^{-d} and the asserted result. The limiting integrals are strictly positive, by a positive-probability bridge tube that forces two visits and by strict positivity of the local-time factor. □

7 Strong scaled contact

Write n_{κ^2} for the off-diagonal kernel of $2\sqrt{|D|^2 + \kappa^2}$ in dimension d, in the convention of [2]. Define, for $h > 0$,

$$V_d(\kappa, h) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{d-1}} |n_{\kappa^2}(h + u + v, x)|^2 \, dx \, du \, dv, \qquad A_d = \frac{2\Gamma(d/2+1)^2\Gamma(d/2+2)}{\pi^{d/2+2}\Gamma(d+4)}.$$

Theorem 6. *If $c_g \to \infty$, then*

$$g^{d-\alpha} c_g^2 \mathcal{E}_g \longrightarrow -A_d \int_{\mathbb{R}^{d-1}} H(w)^{-d-2} \, dw,$$
$$g^{d-\alpha} c_g^2 \mathcal{F}_{g,\lambda g} \longrightarrow -\frac{1}{2\lambda} \int_{\mathbb{R}^{d-1}} \sum_{n \in \mathbb{Z}} V_d(2\pi|n|/\lambda, H(w)) \, dw.$$

The second limit is locally uniform for $\lambda > 0$. The static kernel integral is

$$V_d(0, h) = \frac{4\Gamma((d+1)/2)^2\Gamma((d+3)/2)}{\pi^{(d+3)/2}\Gamma(d+3)} \, h^{-d-1}.$$

In particular $A_1 = 1/(64\pi)$ and $V_1(0, 1) = 2/(3\pi^2)$.

Proof. Let $\gamma_* = d - 1 - \alpha$ and let $K_{i,g}$ be the apertures dilated by $1/g$ in the screen. At scaled momentum κ the relative response, after removing a common factor, has isolated diagonal forms $A_{i,g} + c_g$, with $A_{i,g}$ the compression of $2\sqrt{|D|^2 + \kappa^2}$. The cross block B_g acts by n_{κ^2} between the two scaled domains. Their separation is at least one. Subordination gives

$$|n_{\kappa^2}(x)| \le C|x|^{-d-1} e^{-b\kappa|x|} \quad (|x| \ge 1).$$

Differentiating its integral gives $|\nabla n_{\kappa^2}(x)| \le C_\kappa |x|^{-d-1}$ there for fixed κ. The Schur test on $|x - y| \ge 1$ also gives $\|B_g\| \le Ce^{-b\kappa}$ uniformly in g.

In the facing charts use the physical tangential coordinates $y_j = g^{1/p_j} w_j$ and $y' = y + gx$, with $p_j = 2$ in the quadratic case. The two inward depths are gu, gv. The pair separation divided by g is $(1 + (a_-(y) + a_+(y'))/g + u + v, x)$, which tends to $(H(w) + u + v, x)$. In the finite-type case a_+ is convex and $a_+(y) = o(|y|)$. Its Lipschitz constants on shrinking centered neighborhoods tend to zero: bound its slope in any direction between points in a small ball by its oscillation on a concentric ball of twice the radius divided by that radius. This gives

$(a_+(y') - a_+(y))/g \to 0$ at fixed chart variables. For a fixed local chart, bounded Lipschitz constants and $a_-(y) + a_+(y) \gtrsim \sum_j |y_j|^{p_j}$ give a lower bound for the scaled pair distance by

$$C^{-1}\left(1 + u + v + |x| + \sum_j |w_j|^{p_j}\right).$$

In fact the nonnegative graph heights give the bound in u, v, the lateral difference gives the bound in $|x|$, and replacing $a_+(y')$ by $a_+(y)$ costs at most $Cg|x|$.

The local Jacobian is $g^{d+1+\alpha}$ for the two physical variables together, hence $g^{-\gamma_*}$ in the scaled aperture spaces. The aperture indicators tend to one at fixed positive u, v by the local interior graph description. The squared radial decay exponent $2(d+1)$ exceeds the effective volume exponent $d + 1 + \alpha$ of the chart variables. Outside the local charts compactness and unique contact give a positive physical distance lower bound. This part gives $O(g^2 e^{-b\kappa})$ in the squared scaled kernel integral, using the bounded physical volumes. Dominated convergence gives

$$g^{\gamma_*}\|B_g\|_2^2 \longrightarrow \int_{\mathbb{R}^{d-1}} V_d(\kappa, H(w))\, dw, \qquad g^{\gamma_*}\|B_g\|_2^2 \le Ce^{-b\kappa}. \tag{6}$$

It remains to control the diagonal inverse in this same limit. Put $R_{i,g} = c_g(A_{i,g} + c_g)^{-1}$, a positive contraction. For either output side of the cross block, multiply its kernel in the output variable by

$$\chi_{i,g,\delta}(x) = \min(1, \operatorname{dist}(x, K_{i,g}^c)/\delta), \qquad \delta > 0.$$

At a fixed chart tuple with positive output depth, the scaled boundary distance is bounded below by a positive constant times that depth for small g, by the graph's bounded Lipschitz constant. The majorant for (6) therefore shows the removed kernel has negligible normalized squared Hilbert–Schmidt norm when $g \downarrow 0$ and then $\delta \downarrow 0$. The cutoff columns have zero extensions in H^1. Their normalized integrated squared gradients are bounded by $C_{\kappa,\delta}$ by differentiating the kernel in the output variable and using the bound on the cutoff gradient. The normalized integrated $A_{i,g}$ form of these columns is bounded by the same type of constant, since $2\sqrt{|\xi|^2 + \kappa^2} \le C_\kappa(1 + |\xi|^2)$. The spectral bound

$$(I - R_{i,g})^2 \le c_g^{-1} A_{i,g}$$

then shows that the action of $I - R_{i,g}$ on the cutoff columns is negligible in the normalized Hilbert–Schmidt norm. First send $g \downarrow 0$ and then $\delta \downarrow 0$, using contractivity for the removed part.

Define $q_g(\kappa) = \int_0^\infty e^{-\kappa^2 T}\mathcal{I}_{g,\rho_g}(g^2T)\, dT/T$. The pair relative determinant identity from [2] expands this as a positive trace logarithm. Its first term times c_g^2 is $\operatorname{Tr}(R_{-,g}B_gR_{+,g}B_g^*)$. The preceding cutoff bound and Hilbert–Schmidt Cauchy–Schwarz show its difference from $\|B_g\|_2^2$ is $o(g^{-\gamma_*})$. After multiplication by c_g^2, the remaining trace series is bounded by $Cc_g^{-2}\|B_g\|_2^2\|B_g\|^2$, by bounding all but the first normalized cross-pair factor in operator norm. Thus $g^{\gamma_*}c_g^2 q_g(\kappa)$ has the pointwise limit in (6) and is bounded by $Ce^{-b\kappa}$. It is a decreasing function of κ by the positive heat representation. Continuity of the limit and a finite mesh give uniform convergence on compact κ-intervals. The exponential bound controls the upper tails when integrating in κ or summing Matsubara frequencies. In (2) the corresponding momentum formula is $-(2\pi g)^{-1}\int_0^\infty q_g(\kappa)d\kappa$; in (3) the prefactor at $\beta = \lambda g$ is $-(2\lambda g)^{-1}$. Both have scaling $g^{\alpha-d}$ once $g^{-\gamma_*}$ is included.

Tangential Plancherel gives

$$V_d(\kappa, h) = (2\pi)^{-(d-1)}\int_{\mathbb{R}^{d-1}} V_1(\sqrt{\kappa^2 + |k|^2}, h)\, dk, \qquad V_1(\kappa, h) = h^{-2}V_1(h\kappa, 1).$$

The planar kernel is $-2\kappa K_1(\kappa r)/(\pi r)$ at separation r. Radially integrating the κ, k variables gives the vacuum factor h^{-d-2} times

$$\frac{|S^{d-1}|}{2(2\pi)^d}\frac{4}{\pi^2}\int_0^\infty \frac{s\,ds}{(1+s)^{d+4}}\int_0^\infty q^{d+1}K_1(q)^2\,dq.$$

The first integral is $1/[(d+2)(d+3)]$. The second is $2^{d-1}\Gamma(d/2+1)^2\Gamma(d/2)\Gamma(d/2+2)/\Gamma(d+2)$. To see this, use $K_1(q)=\frac{q}{4}\int_0^\infty x^{-2}e^{-x-q^2/(4x)}dx$ twice and integrate in q first. In the remaining two variables x, y, substitute $x+y$ and $x/(x+y)$. Their gamma and beta integrals give the stated moment. This yields A_d. At $\kappa=0$ the kernel is $-2\Gamma((d+1)/2)\pi^{-(d+1)/2}|x|^{-d-1}$. Integrate its square over the $d-1$ tangential coordinates first, by the radial beta integral, and then integrate $\int_0^\infty s(h+s)^{-d-3}ds = h^{-d-1}/[(d+1)(d+2)]$. This proves the static formula. □

In Theorems 5 and 6, the leading law at fixed $\beta>0$ is the vacuum law. Indeed the scaled momentum step is $\Delta=2\pi g/\beta$. For a positive decreasing function, the half-weighted mesh sum after multiplication by Δ differs from its integral by at most Δ times its value at zero. Use $g^{\gamma_*}c_g^2 q_g$ in the strong case, bounded at zero by (6). In the bounded case use $g^{\gamma_*}q_g$. Its value at zero is bounded by the transparent Matsubara sum at the scaled period one, which is $O(1)$ in $d\geq 2$ and $O(\log(1/g))$ in $d=1$ by [2]. Multiplication by Δ makes the error negligible in both cases. Positivity of the limiting vacuum constants permits comparison of equivalents.

8 Interval zero mode with weak scaled contact

For $d=1$, write

$$Z_g=\int_0^\infty \mathcal{I}_{g,\rho_g}(t)\,\frac{dt}{t},\qquad N_\lambda = 2\sum_{n\geq 1}Q_0((2\pi n/\lambda)^2).$$

At $g=0$ retain the two interval labels, with the common endpoint omitted as a polar set. In fixed units of length, set

$$\begin{aligned}
C_* &= \lim_{L\to\infty}\left[\int_0^{L^2} J(T)\frac{dT}{T}-\frac{1}{4}\log L+\log\log L\right],\\
C_R &= \int_0^1 [J_0^1(s)-1/8]\frac{ds}{s}+\int_1^\infty J_0^1(s)\frac{ds}{s},\\
C_{\ell_-,\ell_+} &= \log\pi-\tfrac{1}{4}\log(\ell_-^{-1}+\ell_+^{-1}).
\end{aligned}$$

The constant C_* exists by the retained-segment heat expansion Sections 1–4; C_R is finite by (4).

Theorem 7. *Suppose $d=1$ and $c_g\to 0$. If $\rho_g\to\rho<\infty$, then*

$$\begin{aligned}
Z_g = \tfrac{1}{4}\log(1/g)-\log\log(1/g)+C_{\ell_-,\ell_+}\\
+\int_0^\infty [\mathcal{I}_{0,\rho}(t)-\mathcal{I}_{0,0}(t)]\,\frac{dt}{t}+o(1).
\end{aligned}$$

If $\rho_g\to\infty$, then

$$Z_g=\tfrac{1}{4}\log(1/c_g)-\log\log(1/c_g)+C_*+C_R+o(1).$$

In both cases

$$\mathcal{F}_{g,\lambda g}=-\frac{Z_g+N_\lambda+o(1)}{2\lambda g}$$

locally uniformly for $\lambda>0$. In particular $\rho_g=g^{-\sigma}$ with $0<\sigma<1$ reduces the coefficient of $\log(1/g)$ in Z_g to $(1-\sigma)/4$.

Proof. If the physical coefficients are bounded, subtract the transparent finite-interval integral. At fixed $t > 0$ the gain difference converges by polarity at the limiting endpoints and continuity of the weight in ρ_g. The small-time bound in (4) is $O(\sqrt{t})$ for this difference. At large time use the fixed enclosing transparent window's $O(t^{-1})$ heat gain [1, 2]. Dominated convergence gives the stated finite shift. The transparent constant follows from Theorem 8 below.

If $\rho_g \to \infty$, truncate the infinite rays to the actual intervals. The unweighted heat loss at small physical time is bounded by $Ce^{-b/t}$, uniformly for small g, since a lost all-hit bridge reaches beyond an outer endpoint and must have lateral range at least $\min(\ell_-, \ell_+)$. The weighting can only decrease this loss, and (4) also bounds it by $C/(\rho_g\sqrt{t})$. Use dominated convergence below $t = 1$ and integrate directly above $t = 1$ to get $Z_g = Q_{c_g}(0) + o(1)$.

For $c > 0$ split that ray integral at $T = c^{-2}$. Insert the transparent ray integral below the cutoff and rescale the resulting difference by $s = c^2T$. This gives the exact identity

$$Q_c(0) = \int_0^{c^{-2}} J(T)\frac{dT}{T} + \int_0^1 [J_c^1(s) - J_c(s)]\frac{ds}{s} + \int_1^\infty J_c^1(s)\frac{ds}{s},$$

where the subscripts c on J_c^1, J_c denote the retained segment lengths. By point polarity, as $c \downarrow 0$ the last two integrands converge to the corresponding zero-gap ones; (4) supplies integrable majorants. The first integral equals the two logarithms plus $C_* + o(1)$ by its definition.

Finally at each fixed $T > 0$, the expanding intervals in scaled coordinates and $c_g \to 0$ give $\mathcal{I}_{g,\rho_g}(g^2T) \to J(T)$. Insert this in the nonzero Matsubara modes. The unweighted small-time Gaussian bound and the exponential large-T Matsubara bound give a majorant locally uniformly in λ. This proves the last formula. □

9 Exact transparent static calibration

Let the transparent interval endpoints be $a < b < c < d_0$, and put

$$q = \frac{(b-a)(d_0-c)}{(c-a)(d_0-b)}, \qquad k = \sqrt{q}, \quad k' = \sqrt{1-q}.$$

Section 4 compared the finite interval constant to a boundary four-point expression from [11, Appendix A]. The following argument fixes the normalization in the aperture heat observable. The classical Neumann needle geometry has the same connected static observable by reflection parity (Section 1) [3]. In that geometry [3, Eq. (15)] already gives the equal-length leading and double logarithms. Dubédat's crosscut transfer formula [20, Section 2.2] also gives the cylinder product below, since the transfer between the seams has eigenvalues $\operatorname{sech}(\pi nW/H_0)$. We derive it directly from the connected heat trace to track its winding classes.

Theorem 8. *For transparent intervals the zero-mode heat integral is exactly*

$$Q_g^{\mathrm{stat}} = -2\sum_{n\ge1} \log\tanh\left(\pi n\,\frac{K(k')}{K(k)}\right) = -\frac{1}{4}\log(1-q) - \log\frac{2K(k)}{\pi}.$$

Thus its finite constant as $g \downarrow 0$ is C_{ℓ_-,ℓ_+} above. In the decomposition by cylinder winding, the nonzero class of signed winding m has mass $1/[|m|(e^{s|m|}-1)]$ with $s = 2\pi K(k')/K(k)$. The zero-winding all-hit mass is $2\sum_{n\ge1}\log(1+e^{-sn})$. The nonzero classes give the entire double logarithm.

Proof. Let Ω be the plane cut along the screen retained outside both intervals. By the killed heat diagonal and its conditional bridge, the measure dt/t times that diagonal and bridge law is the Brownian loop measure of Ω after forgetting the root. The all-hit static integral is its mass of loops visiting both apertures. The generator normalization agrees with [21] because changing from $\Delta/2$ to Δ simply changes the integration variable in dt/t. The event is independent of the time parametrization, so the conformal invariance of the unrooted loop measure applies [21, Proposition 6], including for the cylindrical surface [22, Theorem 2.2].

A real Möbius map sends a, b, c, d_0 cyclically to $0, q, 1, \infty$. The Schwarz–Christoffel integral in the upper half-plane has derivative proportional to $[z(z-q)(z-1)]^{-1/2}$. Its rectangle has its opposite aperture sides of height $H_0 = 2K(k)$ and its gap side of width $W = 2K(k')$ [12, Section 19.32]. These lengths follow by substituting $z = q\sin^2\theta$ on $(0, q)$ and $z = 1-(1-q)\sin^2\theta$ on $(q, 1)$. Schwarz reflection across either open aperture maps the lower half-plane to a reflected rectangle. The two reflections differ by the translation $2W$ across the pair of parallel aperture sides. Taking this horizontal coordinate modulo $2W$ therefore gives one conformal map from Ω to the flat cylinder $(\mathbb{R}/2W\mathbb{Z}) \times (0, H_0)$, with two seams $x = 0, W$. The two half-planes map to the two half-cylinders. The remaining sides give the two Dirichlet circles, and the four endpoints lie on their boundary.

Cutting the cylinder at one seam gives a rectangle of width $2W$; cutting both gives two rectangles of width W. Inclusion–exclusion on the loop event separates the common vertical Dirichlet heat trace. Horizontally, with P_L the period-L and D_L the Dirichlet-interval length-L traces, the factor is

$$P_{2W}(t) - 2D_{2W}(t) + 2D_W(t) = \frac{4W}{\sqrt{4\pi t}} \sum_{m\in\mathbb{Z},\ m\ \text{odd}} e^{-m^2W^2/t}.$$

This is the image sum for the circle and the Poisson formula $D_L(t) = L(4\pi t)^{-1/2}\sum_m e^{-m^2L^2/t} - 1/2$. Multiply by the vertical sum $\sum_{n\ge1} e^{-(\pi n/H_0)^2 t}$. Integration using $\int_0^\infty t^{-3/2}e^{-A^2/t-B^2t}dt = \sqrt{\pi}e^{-2AB}/A$ gives $4\sum_{n\ge1}\sum_{m\ge1,\ m\ \text{odd}} e^{-snm}/m$. The series converges for $s > 0$ and summing in m yields the first formula. With nome $p = e^{-s/2}$, Jacobi's products [12, Sections 20.4 and 20.9] give

$$\prod_{n\ge1}\frac{1+p^{2n}}{1-p^{2n}} = [\vartheta_3(0,p)\vartheta_4(0,p)]^{-1/2} = (1-q)^{-1/8}[2K(k)/\pi]^{-1/2}.$$

The first equality follows by collecting the even and odd factors; the second uses $2K(k)/\pi = \vartheta_3^2$ and $k' = \vartheta_4^2/\vartheta_3^2$. This proves the elliptic expression.

Every loop of horizontal winding $m \neq 0$ visits both seams. Select just the m-th periodic image in the uncut cylinder before integrating. Its coefficient is $2W$ instead of $4W$, and summing the vertical factor gives $1/[|m|(e^{s|m|}-1)]$, in agreement with the general annular formula [22, Eq. (4.3)]. Summing the nonzero classes gives $-2\log P(s)$ with $P(s) = \prod_{n\ge1}(1-e^{-sn})$. Subtraction from the first formula leaves $2\log[P(2s)/P(s)]$ for the zero-winding loops. Put $L = \log(16/(1-q))$. Then $s = 2\pi^2/L + O((1-q)/L)$ by the elliptic endpoint expansion. The eta transformation [12, Chapter 23] gives

$$\log P(s) = -\frac{\pi^2}{6s} + \tfrac{1}{2}\log(2\pi/s) + \frac{s}{24} + O(e^{-4\pi^2/s}).$$

Consequently the contributions near contact are

$$Q_{\text{wound}} = \frac{L}{6} - \log L + \log\pi - \frac{s}{12} + o(L^{-1}), \qquad Q_{\text{zero}} = \frac{L}{12} - \log 2 + \frac{s}{12} + o(L^{-1}).$$

This gives the winding claim. Finally $1-q = g(\ell_-^{-1}+\ell_+^{-1}) + O(g^2)$ and $2K(k) = \log(16/(1-q)) + O((1-q)|\log(1-q)|)$. Inserting these in the exact heat integral gives C_{ℓ_-,ℓ_+}. □

For clarity, the transparent cylinder result concerns the static heat integral. Positive contact would pull back to a spatially varying coefficient along the seams under the same conformal map. Under zero contact, put $A = \ell_-^{-1}+\ell_+^{-1}$ and $L_g = \log(1/g)$. The elliptic expression refines the static expansion by the next term $-\log(16/A)/L_g + O(L_g^{-2})$ in Q_g^{stat}. The nonzero-mode discrepancy with N_λ is exponentially small in $1/g$ for λ in compact positive sets. Indeed at small physical time t it is bounded in the heat integral by $Ce^{-b/t}\sum_{n\neq 0} e^{-t(2\pi n/(\lambda g))^2}$, and at $t \geq 1$ the sum itself is exponentially small. Splitting the small-time exponential in half and using $b/t + t(2\pi n/(\lambda g))^2 \geq c|n|/g$ gives the claimed bound after integration with dt/t. Thus the next term in $\mathcal{F}_{g,\lambda g}$ beyond that of Sections 1–4 is $+\log(16/A)/(2\lambda g L_g)$, with remainder $O(1/(gL_g^2))$ locally uniformly in $\lambda > 0$.

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References

[1] S. Schubr, *Small Dirichlet apertures and the screen limit: edge laws, sharp rates and many-window interactions*, Luminary Research Archive, 2026, `https://research.luminaryoddities.org/small-dirichlet-apertures-screen-limit`.

[2] S. Schubr, *Scalar Casimir interactions of holes in a Dirichlet screen: many-hole terms, monotone attraction and near-contact laws*, Luminary Research Archive, 2026, `https://research.luminaryoddities.org/scalar-casimir-interactions-holes-dirichlet-screen`.

[3] G. Bimonte, T. Emig, and M. Kardar, *Conformal field theory of critical Casimir interactions in 2D*, EPL **104** (2013), 21001. arXiv:1307.3993.

[4] M. F. Maghrebi and N. Graham, *Electromagnetic Casimir energies of semi-infinite planes*, EPL **95** (2011), 14001. arXiv:1102.1486.

[5] M. F. Maghrebi, R. Abravanel, and R. L. Jaffe, *Implications of the Babinet principle for Casimir interactions*, Phys. Rev. D **84** (2011), 061701. arXiv:1103.5395.

[6] I. McGillivray, *The spectral shift function for planar obstacle scattering at low energy*, arXiv:1111.4377, 2011.

[7] D. Kabat, D. Karabali, and V. P. Nair, *Edges and diffractive effects in Casimir energies*, Phys. Rev. D **81** (2010), 125013. arXiv:1002.3575.

[8] D. Kabat and D. Karabali, *Thermal diffractive corrections to Casimir energies*, Phys. Rev. D **84** (2011), 065029. arXiv:1107.0952v2.

[9] I. G. Avramidi, *Heat kernel asymptotics of Zaremba boundary value problem*, Math. Phys. Anal. Geom. **7** (2004), 9–46. arXiv:math-ph/0110020.

[10] M. Nursultanov, J. Rowlett, and D. A. Sher, *The heat kernel on curvilinear polygonal domains in surfaces*, Ann. Math. Qué. **49** (2025), 1–61. doi:10.1007/s40316-024-00237-4.

[11] T. Erler and C. Maccaferri, *String field theory solution for any open string background*, J. High Energy Phys. **10** (2014), 029. arXiv:1406.3021v2.

[12] F. W. J. Olver et al., eds., *NIST Digital Library of Mathematical Functions*, §10.40.

[13] D. Karabali and V. P. Nair, *Diffractive effects and general boundary conditions in Casimir energy*, Phys. Rev. D **87** (2013), 105021. arXiv:1304.0511.

[14] C. D. Fosco and F. D. Mazzitelli, *Casimir energy due to inhomogeneous thin plates*, Phys. Rev. D **101** (2020), 045012. arXiv:1912.12651.

[15] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, *Derivative expansion for the electromagnetic and Neumann Casimir effects in* $2+1$ *dimensions with imperfect mirrors*, Phys. Rev. D **91** (2015), 105019. arXiv:1503.02913.

[16] C. D. Fosco and B. Cardin Guntsche, *Static and dynamical Casimir effects in a wedge within the worldline formulation*, arXiv:2611.05768, 2026.

[17] R. L. Frank and L. Geisinger, *Semi-classical analysis of the Laplace operator with Robin boundary conditions*, Bull. Math. Sci. **2** (2012), 281–319. arXiv:1208.2327.

[18] D. A. Sher, *Heat trace and Riesz means with a large Robin parameter*, arXiv:2610.40804, 2026.

[19] D. Kabat, D. Karabali, and V. P. Nair, *Casimir interaction between holes in a plate*, Phys. Rev. D **82** (2010), 025014. arXiv:1005.3352.

[20] J. Dubédat, *SLE and the free field: Partition functions and couplings*, J. Amer. Math. Soc. **22** (2009), 995–1054. arXiv:0712.3018.

[21] G. F. Lawler and W. Werner, *The Brownian loop soup*, Probab. Theory Relat. Fields **128** (2004), 565–588. arXiv:math/0304419.

[22] Y. Wang and Y. Xue, *The Brownian loop measure on Riemann surfaces and applications to length spectra*, doi:10.1002/cpa.70003.