Natural permutation classes with arbitrarily large growth oscillations

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Luminary Oddities

Abstract

Pantone and Vatter recently showed that permutation classes need not have proper growth rates, and asked whether every proper atomic class has one. We answer this question negatively, even for natural classes: the sets of finite patterns of a single permutation of the positive integers. For every $R > 1$ we construct a natural class whose upper growth rate is finite and exceeds R times its lower growth rate. The representing permutation and its inverse are computable, and membership in the class is decidable. The permutation is an infinite direct sum of finite blocks. Each high block lists every member of one length from a high-girth grid class of Pantone and Vatter, which we extend to arbitrary regular degree; these blocks keep the upper growth rate large. Between consecutive high blocks we choose a valley length, after all earlier blocks are fixed. At that length, every pattern drawn from later blocks lies in a slowly growing container of forest-shaped grid classes that is closed under direct sums, so the lower growth rate stays small. Low blocks listing this container make membership decidable.

1 Introduction

A permutation class is a set of finite permutations closed under taking patterns. Its speed c_n counts its members of length n. The lower and upper growth rates are

$$\underline{\mathrm{gr}}(\mathcal{C}) = \liminf_{n\to\infty} c_n^{1/n}, \qquad \overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} c_n^{1/n}.$$

When they agree, their common value is the proper growth rate. Pantone and Vatter [10] recently constructed a proper permutation class without a proper growth rate, refuting a conjecture recorded in Vatter's survey [11]. Their class is not atomic [10, Proposition 6.6], and they asked: "Does every proper atomic permutation class have a proper growth rate?" [10, Question 2.6].

We answer this question negatively, already within the family of natural classes, by a modification of the Pantone–Vatter construction. Following Atkinson, Murphy and Ruškuc [3], a natural class has the form $\mathrm{Sub}(\pi)$, the set of finite patterns of a bijection $\pi : \mathbb{N} \to \mathbb{N}$. Every such class is atomic, that is, it is not the union of two strictly smaller subclasses: any two finite patterns occur together in the pattern formed by the union of their occurrence sets.

Our result also gives arbitrarily large multiplicative separation. For every real $R > 1$, there exists a decidable natural class $\mathcal{C}$ satisfying

$$1 \le \underline{\mathrm{gr}}(\mathcal{C}) \le L < H \le \overline{\mathrm{gr}}(\mathcal{C}) \le U < \infty, \qquad \frac{H}{L} > R.$$

Since its upper growth rate is finite, the class is proper. The representing permutation and its inverse can both be chosen computable. For each R the construction gives its own class. Consequently the ratio of upper to lower growth rate is unbounded over proper natural classes. Both rates are located by the explicit envelopes L, H and U below.

1.1 A sum-closed container and sparse finite bursts

The construction starts with the high-girth colored grid method of [10]. For a fixed integer $d \geq 3$, finite d-regular bipartite graphs of arbitrarily large girth can be ordered so that each of their d matching color classes represents a separable permutation. Fill each cell corresponding to an edge with the class $\mathcal{W}$ of permutations having at most $w-1$ descents. The weighted grid-growth theorem of Albert and Vatter [1] makes the resulting grid classes have proper growth rate $H = d^2 w$.

A pattern shorter than the graph girth occupies a forest of cells. We use the container $\mathcal{F}_{d,w}$ consisting of all such forest grids with the same degree and coloring conditions, rather than only patterns common to a chosen sequence of grids. This larger container still has a uniform counting bound. Moreover, it is closed under direct sums: block-diagonal griddings preserve the forest condition, and the color matching permutations are direct sums of separable permutations. The fillings remain in separate cells, so the argument applies to the cell class $\mathcal{W}$ exactly as given.

This closure controls patterns crossing many future bursts. At stage i, we choose a large finite family from one high-growth grid and place each member once in a finite direct-sum block. After fixing the whole preceding prefix, we choose a valley length much larger than that prefix. Every later grid is then required to have girth greater than this valley length. Any pattern of that length selected entirely from the tail is a direct sum of forest-container patterns and hence belongs to the container itself.

Patterns meeting the prefix add at most a factor $2^{S_{i-1}}$, which is subexponential in the valley length because that length is at least iS_{i-1}. The high blocks give an unbounded sequence of peak lengths. The infinite direct sum of all stages is a permutation of $\mathbb{N}$, so its age is a natural class.

The estimates allow the regular degree to amplify the separation. Writing

$$a_d = 8 \cdot 4^d, \qquad L = 4(d-1)(w+1+a_d), \qquad H = d^2 w, \qquad U = d^2(w+1+a_d),$$

we choose d and then w so that $H > RL$. These parameters are fixed before the length limits are taken. The exponential cost of counting ordered colorings is absorbed by the choice of w; it is fixed independently of the peak and valley lengths.

1.2 An exact finite membership procedure

The construction also inserts low bursts listing every member of $\mathcal{F}_{d,w}$ up to successively increasing lengths. These bursts stay inside the low-growth container and are included in the finite prefix cost. For each fixed n, the tail after the first n complete stages has exactly the $\mathcal{F}_{d,w}$ patterns at every length at most n. Containment follows from the girth and sum-closure argument; the next low burst supplies the reverse inclusion.

Consequently the length-n members of the full age form an explicitly computable finite union of direct sums of prefix patterns and container patterns. Finite graph and gridding searches decide container membership and construct the high-growth grids. The peak lengths can be found by finite counting and exact rational comparisons. This proves decidability of the resulting class, as well as computability of the representing permutation and its inverse.

1.3 Relation to the predecessor construction

The high-girth graphs, separable matching orders, cell estimates and spectral enumeration are inherited from [10] and the earlier weighted grid-growth framework [1]. We include the arguments needed for their general-degree use and credit these inputs explicitly. The new step is to combine a sum-closed low-growth container with finite, nonrecurrent high-count bursts in one infinite permutation. The resulting class has upper growth rate at least H, above the low-growth limit class

of [10] for the same cubic parameter w; it is therefore a new class rather than a subclass of their truncation union, consistently with [10, Proposition 6.6], which places every infinite atomic subclass of that union inside its low-growth limit class.

The natural-class representation and its connection to atomicity are classical [3, Theorem 1.2]. A proper natural class whose tail classes stabilize, in particular a finitely based or well-quasi-ordered one, has a proper growth rate (Proposition 2.2). Natural counterexamples are therefore infinitely based and contain infinite antichains, as ours do, and within natural classes Questions 2.4 and 2.5 of [10] have positive answers while Question 2.6 has a negative one (Corollary 2.3). The counterexample and its quantitative amplification are proved below through explicit counting and an explicit infinite permutation.

2 Definitions and the principal conclusion

For finite permutations σ and τ, write $\sigma \le \tau$ if a subsequence of τ has the same relative order as σ. A permutation class is a downset for this relation. Write $\mathcal{C}_n = \mathcal{C} \cap S_n$ and $c_n = |\mathcal{C}_n|$. Define

$$\underline{\mathrm{gr}}(\mathcal{C}) = \liminf_{n\to\infty} c_n^{1/n}, \qquad \overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} c_n^{1/n}.$$

The class has a proper growth rate when these numbers coincide and are finite. A proper class is a class different from the class of all finite permutations; every proper class has finite upper growth rate by the Marcus–Tardos theorem [9]. This sense of "proper" is unrelated to proper growth rates, and we write "strictly smaller subclass" for a subclass different from the class itself. A proper edge-coloring has the usual meaning: adjacent edges receive different colors.

A class is *atomic* if it is not the union of two strictly smaller subclasses [3, Theorem 1.2]. Equivalently, it has the joint embedding property: every two members occur in some common member. Indeed, joint embedding precludes such a union by choosing one pattern outside each subclass. Conversely, if σ, τ have no common extension, the subclasses avoiding σ and avoiding τ cover the class.

For a bijection $\pi : \mathbb{N} \to \mathbb{N}$, let $\mathrm{Sub}(\pi)$ be the class of all standardized finite subsequences of π, also called the *age* of π [5]. Following Atkinson, Murphy and Ruškuc [3], a class of this form is called *natural*. Every natural class is atomic: the union of two finite occurrence sets in π gives a finite common extension. We construct an explicit bijection of $\mathbb{N}$; atomic classes in general are the classes $\mathrm{Sub}(\varphi\colon A \to B)$ for linear orders A, B [3, Theorem 1.2].

The direct sum $\alpha \oplus \beta$, for $|\alpha| = r$, places α before β and increases every entry of β by r. Thus all points in the first summand lie below and to the left of all points in the second. A class is sum closed if it is closed under this operation, including finite iterated sums. The *skew sum* reverses the vertical order of the two summands. A permutation is separable if it is obtained from the singleton by repeated direct and skew sums; we include the empty permutation. Separability is hereditary: delete the unused leaves from a direct/skew expression tree and suppress its empty branches. It is decidable by enumerating the finitely many such expression trees of a given size.

Theorem 2.1. *For every real $R > 1$ there is a proper infinite natural permutation class $\mathcal{C}$ such that*

$$1 \le \underline{\mathrm{gr}}(\mathcal{C}) < \overline{\mathrm{gr}}(\mathcal{C}) < \infty, \qquad \frac{\overline{\mathrm{gr}}(\mathcal{C})}{\underline{\mathrm{gr}}(\mathcal{C})} > R.$$

In particular, proper atomic permutation classes need not have proper growth rates. The bijection $\pi : \mathbb{N} \to \mathbb{N}$ representing $\mathcal{C}$ can be chosen computable, with computable inverse. Membership in $\mathcal{C}$ is decidable, and every count c_n is computable.

With the cubic parameters $d = 3$ and $w = 4105$, the construction gives a natural class with the explicit separation

$$\underline{\mathrm{gr}}(\mathcal{C}) \leq 36944 < 36945 \leq \overline{\mathrm{gr}}(\mathcal{C}) \leq 41562. \tag{1}$$

These are exactly the bounds that Pantone and Vatter [10] display for their non-atomic class. The same cubic envelopes now bound a natural class.

Proposition 2.2 (Stable tails)**.** *Let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection with $\mathcal{C} = \mathrm{Sub}(\pi)$ proper, and let $\mathcal{T}_i$ be the age of π restricted to positions $\geq i$. If the chain $\mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \cdots$ is eventually constant, then $\mathcal{C}$ has a proper growth rate. This holds in particular when $\mathcal{C}$ is finitely based [3, Lemma 3.2] and when $\mathcal{C}$ is well-quasi-ordered.*

Proof. Suppose $\mathcal{T}_i = \mathcal{T}$ for all $i \geq i_0$. First, $\mathcal{T}$ is sum closed. Given $\alpha, \beta \in \mathcal{T}$, fix an occurrence of α in π at positions at least i_0, with last position p and largest value v. Only finitely many positions carry values at most v; choose $i > p$ larger than all of them. Since $\beta \in \mathcal{T}_i$, it occurs at positions at least i, to the right of and above the occurrence of α, so $\alpha \oplus \beta \in \mathcal{T}$. This is the argument of [3, Lemma 3.2].

Write $c_n = |\mathcal{C}_n|$ and $t_n = |\mathcal{T} \cap S_n|$. The class $\mathcal{T}$ is infinite, proper and sum closed, so $t_{m+n} \geq t_m t_n$ and $t_n^{1/n}$ converges by Fekete's lemma, to a finite limit by the Marcus–Tardos theorem [9]; this is the argument of Arratia [2]. Also $t_n \leq t_{n+1}$, through $\sigma \mapsto \sigma \oplus 1$. An occurrence of a member of $\mathcal{C}_n$ uses some $r \leq \min\{n, i_0 - 1\}$ of the first $i_0 - 1$ positions, and these are its first r entries. Deleting them leaves a member of $\mathcal{T}$ of length $n - r$, from which the original pattern is recovered by the values of the deleted entries among $1, \ldots, n$ and their order. Hence

$$t_n \leq c_n \leq \sum_{r \leq \min\{n, i_0 - 1\}} \binom{n}{r} r!\, t_{n-r} \leq i_0\, n^{i_0} t_n \qquad (n \geq 1),$$

and $\mathcal{C}$ has the proper growth rate of $\mathcal{T}$.

If $\mathcal{C}$ is finitely based, the tail classes are eventually constant by [3, Lemma 3.2]. If the chain does not stabilize, choose $i_1 < i_2 < \cdots$ with $\mathcal{T}_{i_j} \supsetneq \mathcal{T}_{i_{j+1}}$ and $\sigma_j \in \mathcal{T}_{i_j} \setminus \mathcal{T}_{i_{j+1}}$. For $j < k$, a containment $\sigma_j \leq \sigma_k$ would give $\sigma_j \in \mathcal{T}_{i_k} \subseteq \mathcal{T}_{i_{j+1}}$, which is impossible. Thus (σ_j) is a bad sequence, and $\mathcal{C}$ is not well-quasi-ordered. □

Corollary 2.3. *Every proper natural class without a proper growth rate is infinitely based and contains an infinite antichain; in particular so does every class of Theorem 2.1. Thus [10, Questions 2.4 and 2.5] have positive answers within natural classes, while Question 2.6 has a negative one.*

Proof. A permutation class is well-founded under containment, so it is well-quasi-ordered exactly when it contains no infinite antichain. Proposition 2.2 therefore gives the first assertion. For Question 2.5, every well-quasi-ordered class is proper, since the class of all permutations contains infinite antichains. For the classes constructed in the proof of Theorem 2.1, an explicit infinite antichain is supplied by the permutations β_n of [3, Section 4]: they form part of the basis of a single class, hence are pairwise incomparable. Each lies in the container $\mathcal{F}_{d,w}$ of Section 6, through an increasing staircase gridding whose cell graph is a path with identity color matching permutations, and $\mathcal{F}_{d,w} \subseteq \mathcal{C}$ by Lemma 3.1. □

The proof has two interfaces. A sparse-burst lemma converts a sum-closed low-count envelope and high-count finite families into one natural class. Ordered forest grids provide such an envelope, while grids over regular high-girth graphs provide the high-count families. The latter ingredients extend the graph machinery of Pantone and Vatter [10], with attribution at each step.

3 Sparse bursts in a natural permutation class

The grid families used in the application come from the construction of Pantone and Vatter [10], together with the generalized grid-growth theorem of Albert and Vatter [1]. The following completion lemma uses exactly the structural and enumerative consequences recorded in its hypotheses.

Lemma 3.1 (Sparse-burst completion)**.** *Fix constants $1 \le L < H < \infty$, $A \ge 1$, and an integer $q \ge 0$. Let $\mathcal{F}$ be a sum-closed permutation class containing the empty permutation, with*

$$f_0 = 1, \qquad f_n = |\mathcal{F}_n| \le A(n+1)^q L^n \quad (n \ge 0).$$

For each sufficiently large integer h, let $\mathcal{H}_h$ be a permutation class with a proper growth rate at least H, and suppose every pattern of length at most h in $\mathcal{H}_h$ belongs to $\mathcal{F}$. All these classes and constants are fixed before the length limits below.

Then there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ whose age $\mathcal{C}$ contains $\mathcal{F}$, is proper, natural and atomic, and satisfies

$$\underline{\mathrm{gr}}(\mathcal{C}) \le L < H \le \overline{\mathrm{gr}}(\mathcal{C}). \tag{2}$$

Proof. We choose finite stages recursively. Set $S_0 = n_0 = 0$. Once the first $i-1$ stages have been chosen, with total length S_{i-1}, choose an integer

$$m_i > \max\{n_{i-1}, i^2\}, \qquad m_i \ge iS_{i-1}.$$

Choose an index $h_i > m_i$ for which $\mathcal{H}_{h_i}$ is defined. Its high class has growth rate at least H, so there is $n_i > m_i$ with

$$|(\mathcal{H}_{h_i})_{n_i}|^{1/n_i} \ge H - 1/i.$$

List this finite nonempty set in any fixed order, and let B_i be the direct sum of its members, each used once. Let E_i be the direct sum of all members of $\mathcal{F}_0, \dots, \mathcal{F}_i$ in a fixed order, ignoring the empty summand. Sum closure gives $E_i \in \mathcal{F}$. The complete stage and its accumulated size are

$$T_i = E_i \oplus B_i, \qquad S_i = S_{i-1} + |T_i|.$$

Every stage is finite before the next valley length m_{i+1} is chosen.

The direct sum

$$\pi = T_1 \oplus T_2 \oplus \cdots$$

is a literal permutation of $\mathbb{N}$. More precisely, the intervals $\{S_{i-1}+1, \dots, S_i\}$ partition $\mathbb{N}$, and

$$\pi(S_{i-1} + j) = S_{i-1} + T_i(j), \qquad 1 \le j \le |T_i|,$$

is a bijection on each interval. Both the position and value orders therefore have order type ω.

Let $\mathcal{C}$ be its finite pattern age and write $c_n = |\mathcal{C}_n|$. This is an infinite downset, and every member of $\mathcal{F}_k$ occurs in each E_i with $i \ge k$. Thus $\mathcal{F} \subseteq \mathcal{C}$. Any two members have occurrence sets in π whose finite union witnesses a common superpattern. Hence $\mathcal{C}$ has the joint embedding property and is atomic. Explicitly, if it were the union of two strictly smaller subclasses, choose a member missing from each subclass. A common superpattern would belong to neither, contradicting that union.

Every member of $(\mathcal{H}_{h_i})_{n_i}$ occurs as one summand of B_i, so

$$c_{n_i}^{1/n_i} \ge H - 1/i. \tag{3}$$

Since $m_i < n_i < m_{i+1}$, these lengths tend to infinity and give the upper-growth lower bound in (2).

For the low subsequence, fix i, and put $m = m_i$ and $S = S_{i-1}$. Any length-m pattern uses $r \le \min\{S, m\}$ points from the earlier prefix and $m - r$ points from the remaining tail. The former precede and lie below all the latter, so the pattern is $\alpha \oplus \beta$. There are at most $\binom{S}{r}$ choices for α, by counting the actual prefix subsets.

The occurrence of β meets finitely many low blocks E_j or underlying $\mathcal{H}_{h_j}$ summands with $j \ge i$. A selected portion of E_j belongs to $\mathcal{F}$ by heredity. A selected high-block portion has length at most $m_i < h_j$, because $m_j \ge m_i$, and also belongs to $\mathcal{F}$. Their finite direct sum is in $\mathcal{F}_{m-r}$. This accounts for all mixed tail occurrences, regardless of which future stages or summands they use. Consequently

$$\begin{aligned} c_m &\le \sum_{r=0}^{\min\{S,m\}} \binom{S}{r} f_{m-r} \\ &\le 2^S A(m+1)^q L^m. \end{aligned}$$

The empty-tail case is included by $f_0 = 1$; the second inequality uses $L \ge 1$. Taking m_ith roots gives

$$c_{m_i}^{1/m_i} \le L\, 2^{1/i} \big(A(m_i+1)^q\big)^{1/m_i}.$$

Here $S_{i-1}/m_i \le 1/i$ and $m_i \to \infty$, so the two factors after L tend to one. Thus $\underline{\mathrm{gr}}(\mathcal{C}) \le L$, proving the strict separation together with (3).

Finally $\mathcal{C}$ is proper. Otherwise $c_{m_i} = m_i!$, whose m_ith roots tend to infinity, contrary to the finite valley bound. This completes the construction. □

Remark 3.2 (Prefix-anchored patterns). If $\sigma \in \mathcal{C} \setminus \mathcal{F}$ has length k, every occurrence of σ meets the first k stages. Indeed $m_j > j^2$ gives $h_j > m_j > k$ for $j > k$, so every length-k pattern wholly in the later tail lies in $\mathcal{F}$ by the short-pattern and sum-closure argument. Hence at most S_k occurrences of σ are pairwise disjoint, and joint embedding is realized by unions of occurrence sets that may share prefix points. In particular $\mathcal{C}$ is not sum closed, although π is an infinite direct sum: an iterated sum of $S_k + 1$ copies of σ lies outside $\mathcal{C}$. Such σ exists because $\overline{\mathrm{gr}}(\mathcal{C}) \ge H > L \ge \overline{\mathrm{gr}}(\mathcal{F})$. This is as it must be: a proper sum-closed class has a proper growth rate by Arratia's theorem [2]; see [10, Proposition 2.1].

4 Ordered graphs and filled grid classes

We use the graph and grid machinery of Pantone–Vatter [10], retaining a variable regular degree. Their ordered cubic construction and counting argument provide the model for the lemmas below. The additional feature needed here is a sum-closed forest container for short patterns, together with an exponential container for all patterns of the eventual direct-sum construction.

Fix integers $d \ge 3$ and $w \ge 1$. Let $\mathcal{W} = \mathcal{W}_w$ be the class of permutations with at most $w - 1$ descents. An *ordered bipartite graph* has a separate linear order on each part X and Y. We use X for columns, ordered left to right, and Y for rows, ordered bottom to top. A matching induces a permutation between its covered vertices in their inherited orders; uncovered vertices are ignored. Deleting edges of a matching therefore gives a pattern of its matching permutation.

Separable permutations were defined in Section 2. They form a hereditary class that is closed under direct sums. The number of separable permutations of length k is at most 8^k: for $k \ge 1$, a binary separating tree has k leaves, at most 4^{k-1} possible shapes and 2^{k-1} choices of direct/skew

labels at its internal vertices. The assertion at $k = 0$ is immediate. This is the crude bound of [10, Proposition 3.1].

Given an ordered bipartite graph F, put $\mathcal{W}$ in every cell corresponding to an edge and leave the other cells empty. Write $\mathrm{Grid}_w(F)$ for the resulting grid class. Thus a gridding partitions positions into consecutive column intervals and values into consecutive row intervals, with the points in each occupied cell forming a member of $\mathcal{W}$. Grid cells may contain no entries. Every such grid class is hereditary.

Lemma 4.1 (The cell class)**.** *For $k \geq 1$,*

$$\frac{w^k}{\binom{k+w-1}{w-1}} \leq |\mathcal{W}_k| \leq w^k. \tag{4}$$

Thus $\mathcal{W}_w$ has proper growth rate w. We also have the uniform occupied-cell bound

$$|\mathcal{W}_k| \leq (w+1)^{k-1} \qquad (k \geq 1). \tag{5}$$

Proof. A word of length k on w letters, indexed by the values $1, \ldots, k$, produces a permutation by listing the values assigned to its first letter increasingly, then those assigned to its second letter, and so on. Its image is exactly $\mathcal{W}_k$, since these are the permutations divisible into w consecutive possibly empty increasing segments. For a fixed image, the segment lengths determine the word. There are at most $\binom{k+w-1}{w-1}$ such weak compositions. Counting the words therefore gives (4), and its polynomial denominator gives the proper growth rate.

For (5), take a permutation with $s \leq w-1$ descents and place w dividers in its $k+1$ gaps, allowing repeated dividers. All descent gaps must receive a divider. There are

$$\binom{k+w-s}{k} \geq w+1$$

placements, since $w - s \geq 1$ and $k \geq s+1$. Encode each placement by the word on $w+1$ letters whose ith letter is the segment containing the value i. Increasing order within each segment recovers both the permutation and the dividers, including empty segments. Hence $(w+1)|\mathcal{W}_k| \leq (w+1)^k$, as in [10, Proposition 3.2]. The empty permutation has count one. □

The spectral formula for grid growth was established by Bevan [4, Theorem 3.6] and extended to general cell classes by Albert–Vatter [1, Theorem 1.1]; see also [10, Section 4]. We prove the uniform interleaving bound and, in Proposition 6.4, the regular common-cell case needed here directly. The following interleaving argument is the upper bound of [10, Lemma 4.2], which yields [10, Proposition 4.3]; we include it to fix the all-length constants. For a prescribed matrix $N = (n_{ij})$ of cell sizes, let $n = \sum n_{ij}$, $c_i = \sum_j n_{ij}$ and $r_j = \sum_i n_{ij}$. The number of gridded permutations with those sizes is

$$I(N) \prod_{ij \in E(F)} |\mathcal{W}_{n_{ij}}|, \qquad I(N) = \prod_i \binom{c_i}{n_{i1}, \ldots, n_{iu}} \prod_j \binom{r_j}{n_{1j}, \ldots, n_{tj}}. \tag{6}$$

The two multinomial products choose the interleavings within columns and rows. Together with the cell permutations, these choices determine the gridded permutation.

Lemma 4.2 (Uniform interleaving bound)**.** *For every admissible N of total size $n \geq 1$,*

$$I(N) \leq \rho(F)^{2n}. \tag{7}$$

Proof. First give every possible edge of the complete bipartite graph a positive weight b_{ij}. Let ρ be the spectral radius of its weighted adjacency matrix, and let $x_i, y_j > 0$ be a Perron eigenvector, so that

$$\rho x_i = \sum_j b_{ij} y_j, \qquad \rho y_j = \sum_i b_{ij} x_i.$$

Retaining one term in the multinomial expansion gives

$$(\rho x_i)^{c_i} \geq \binom{c_i}{n_{i1}, \ldots, n_{iu}} \prod_j (b_{ij} y_j)^{n_{ij}},$$

and similarly for every row. Multiplying all these inequalities and cancelling the positive vertex coordinates yields

$$I(N) \prod_{ij} b_{ij}^{2n_{ij}} \leq \rho^{2n}.$$

Now let the weights tend to 1 on $E(F)$ and 0 off it. All sizes off $E(F)$ are zero. Continuity of the spectral radius gives (7). In particular this limiting argument covers disconnected graphs and isolated vertices. □

5 Separable matching colors at arbitrary degree

We first extend the high-girth construction of [10, Appendix A] from three colors to d colors. The classical group input is that a finitely generated free group is residually a finite 2-group: for every nonidentity element there is a homomorphism to a finite 2-group on which that element remains nonidentity. This is the Iwasawa–Hall residual-p theorem [8], [7, Section 5]; it is also recorded in [10, Appendix A.1].

Lemma 5.1 (Ordered regular graphs)**.** *For every $d \geq 3$ and $h \geq 4$, there exists a finite simple ordered bipartite d-regular graph $G^{(h)}$ with girth greater than h and a proper d-edge-coloring whose color matching permutations are separable.*

Proof. In the free group on $a_1, \ldots, a_d$, list the finitely many nonempty freely reduced words of length at most h. For each listed word v, choose a homomorphism ϕ_v to a finite 2-group P_v with $\phi_v(v) \neq 1$. Put

$$P = \prod_v P_v, \qquad s_c = (\phi_v(a_c))_v \quad (1 \leq c \leq d).$$

Then P is a finite 2-group and no listed word vanishes at the s_c. In particular the s_c are pairwise distinct.

Choose a subgroup chain

$$P = K_0 > K_1 > \cdots > K_k = \{1\}, \qquad [K_j : K_{j+1}] = 2.$$

Such a chain exists by induction: a nontrivial finite 2-group has a central subgroup N of order 2; lift a chain from P/N and append $\{1\}$ below N. The right cosets $K_j g$, ordered by containment, form a complete rooted binary tree. Ordering the two children at every node orders its leaves, hence orders P.

Right multiplication by any $s \in P$ sends $K_j g$ to $K_j g s$ and preserves containment. It is therefore an automorphism of this binary tree, and it carries the subtree below each coset $K_j g$ isomorphically onto the subtree below $K_j g s$. By induction on depth, each such isomorphism between subtrees

induces a separable map of their ordered leaves: at the top node it is the direct sum of the maps on the two children, or their skew sum when the children are exchanged. Hence the leaf permutation of the right multiplication is separable. This is the binary-coset ordering argument of [10, Appendix A.2], and the same order works for all d chosen elements.

Take two copies $X = \{x_g : g \in P\}$ and $Y = \{y_g : g \in P\}$, each with that order, and join x_g to y_{gs_c} with color c. Distinct s_c prevent parallel edges. Each color is a perfect matching, since right multiplication is bijective. Thus the graph is simple and d-regular, and the coloring is proper. Its color matching permutations are separable by the chosen order. This is the Haar-graph construction of [10, Appendix A.3], with d generators in place of three.

A cycle has even length $2r$. Reading its colors in order gives a relation

$$s_{c_1} s_{c_2}^{-1} s_{c_3} s_{c_4}^{-1} \cdots s_{c_{2r-1}} s_{c_{2r}}^{-1} = 1.$$

Consecutive cycle edges have different colors, so the corresponding word in the a_c is freely reduced and nonempty. The construction of P forces $2r > h$. This proves the girth assertion. □

A graph with such a coloring will be called *admissibly colored*. The term refers to both properness and separability of its color matching permutations. A graph is admissibly colored as soon as one such coloring exists. The next count extends [10, Proposition 6.1] to d colors.

Lemma 5.2 (The graph count)**.** *Set $a_d = 8 \cdot 4^d$. The number $q_{d,m}$ of ordered bipartite graphs with $m \geq 1$ edges, no isolated vertices, and an admissible d-coloring satisfies*

$$q_{d,m} \leq m^2 \binom{m+d-1}{d-1} a_d^m. \tag{8}$$

Proof. It is enough to count graphs together with a chosen admissible coloring. The two part sizes t, u lie in $[1, m]$, giving at most m^2 choices. The color sizes $m_1, \ldots, m_d \geq 0$ sum to m, giving $\binom{m+d-1}{d-1}$ choices. For color c, choose its covered vertices in both parts and its matching permutation. The number is at most

$$\binom{t}{m_c}\binom{u}{m_c} 8^{m_c} \leq 4^m 8^{m_c}.$$

Multiplication over the d colors gives $4^{dm} 8^m = a_d^m$. The data determine the whole ordered colored graph; invalid choices only overcount. Vertices are named by their ranks in the two given orders, so an additional labeling factor is not needed. □

6 A sum-closed forest container

Let $\mathcal{F}_{d,w}$ be the union of $\mathrm{Grid}_w(F)$ over all finite admissibly d-colored ordered bipartite forests of maximum degree at most d, together with the empty permutation. Let $\mathcal{D}_{d,w}$ be the analogous union over all finite admissibly colored ordered bipartite graphs of maximum degree at most d, allowing cycles. The graphs in these unions may be disconnected.

Proposition 6.1 (Closure and short patterns)**.** *Both $\mathcal{F}_{d,w}$ and $\mathcal{D}_{d,w}$ are hereditary and sum closed, and $1 \in \mathcal{F}_{d,w}$. If $G^{(h)}$ is the graph of Lemma 5.1, or any admissibly colored d-regular ordered bipartite graph of girth greater than h, then*

$$\mathrm{Grid}_w(G^{(h)})_{\leq h} \subseteq \mathcal{F}_{d,w}. \tag{9}$$

Proof. Hereditary closure follows from that of each grid class. To take a direct sum, place the two witnessing matrices in diagonal blocks, with the first block's columns to the left and its rows below those of the second. The new graph is their disjoint union. Degrees and the forest condition, when imposed, are preserved. Use the original color names on each component. For every color, its matching permutation is the direct sum of the old matching permutations and is separable. Cell fillings remain separate, so this argument does not require $\mathcal{W}$ itself to be sum closed. A one-cell gridding shows that $1 \in \mathcal{F}_{d,w}$.

For (9), take a gridded permutation of length $n \le h$ in the indicated high-girth grid. Keep only cells containing entries and delete isolated row and column vertices. The resulting ordered graph has at most n edges and inherits its proper coloring. It is a forest, since a cycle would use at most $n \le h$ edges, contrary to the girth. Each new color matching is a submatching of the old one, and its matching permutation is a pattern of a separable permutation. Thus it is an allowed forest gridding. □

Every positive-length member of either container can similarly be normalized to its *occupied graph*, with no isolated vertices and $1 \le m \le n$ edges, all corresponding cells occupied. Deleting unused edges preserves the coloring and its separable matching permutations. This normalization is important because the original union allows arbitrarily many unused rows and columns.

Write

$$f_n = |(\mathcal{F}_{d,w})_n|, \qquad b_d = 4(d-1).$$

Proposition 6.2 (The forest envelope)**.** *For every $n \ge 1$,*

$$f_n \le a_d b_d^n n^2 \binom{n+d-1}{d-1} (w+1+a_d)^{n-1}. \tag{10}$$

Consequently $\overline{\mathrm{gr}}(\mathcal{F}_{d,w}) \le L$, where $L = b_d(w+1+a_d)$. Also $f_0 = 1$ and $f_n \le f_{n+1}$.

Proof. The classical forest bound gives $\rho(F) < 2\sqrt{d-1}$ for every finite forest of maximum degree at most d. This is a special case of a result of Godsil [6, Theorem 3(b)]; see also [10, Proposition 5.3]. A short proof also fixes its applicability to disconnected forests. Spectral radius is the maximum over components, so take a tree with an edge, root it at a leaf r, and let x be a positive unit Perron vector. Each vertex has at most $d-1$ children. With $p(v)$ the parent of $v \ne r$,

$$\rho(F) = 2\sum_{v \ne r} x_v x_{p(v)} \le 2\left(1 - x_r^2\right)^{1/2} \left(\sum_{v \ne r} x_{p(v)}^2\right)^{1/2} < 2\sqrt{d-1}.$$

Here $\sum_{v \ne r} x_{p(v)}^2 = \sum_u \mathrm{ch}(u) x_u^2 \le d-1$, where $\mathrm{ch}(u) \le d-1$ is the number of children of u; the root has one child because it is a leaf. The last inequality is strict because $x_r > 0$. Isolated components have spectral radius zero.

For a normalized occupied forest with m edges, fix positive cell sizes summing to n. Equations (6) and (7), together with (5), bound its gridded fillings by

$$b_d^n (w+1)^{n-m}.$$

The cell sizes have $\binom{n-1}{m-1}$ positive compositions. Lemma 5.2 counts the normalized graphs, giving

$$\begin{aligned} f_n &\le b_d^n \sum_{m=1}^{n} m^2 \binom{m+d-1}{d-1} a_d^m \binom{n-1}{m-1} (w+1)^{n-m} \\ &\le b_d^n n^2 \binom{n+d-1}{d-1} \sum_{m=1}^{n} \binom{n-1}{m-1} a_d^m (w+1)^{n-m}. \end{aligned}$$

The last sum is $a_d(w+1+a_d)^{n-1}$, proving (10). Taking roots gives the growth bound for fixed d, w. The empty permutation gives $f_0 = 1$, and the injection $\pi \mapsto \pi \oplus 1$ proves monotonicity. □

Proposition 6.3 (The global envelope)**.** *For every $n \geq 1$,*

$$|(\mathcal{D}_{d,w})_n| \leq a_d(d^2)^n n^2 \binom{n+d-1}{d-1}(w+1+a_d)^{n-1}. \tag{11}$$

In particular its upper growth rate is at most $U = d^2(w+1+a_d)$.

Proof. An occupied graph in this container has maximum degree at most d, so $\rho(F) \leq d$ by the maximum row-sum bound for its adjacency matrix. The same count as in Proposition 6.2, now using d^{2n} in place of b_d^n, gives (11). Components and their order are already encoded in the ordered graph count. □

Proposition 6.4 (The high-growth grids)**.** *For every $h \geq 4$, with $G^{(h)}$ as in Proposition 6.1, the class $\mathcal{H}_h = \mathrm{Grid}_w(G^{(h)})$ is contained in $\mathcal{D}_{d,w}$, has proper growth rate $H = d^2w$, and satisfies $(\mathcal{H}_h)_{\leq h} \subseteq \mathcal{F}_{d,w}$.*

Proof. The first inclusion follows from the admissible coloring and regularity; the short-pattern assertion is Proposition 6.1. For the rate, fix $G^{(h)}$. Let each part have t vertices and let $e = dt$ be its number of edges. Its spectral radius is d: the all-ones vector has eigenvalue d, and the maximum row sum is d. For every cell-size matrix of total n, equations (6), (7) and (4) give at most $(d^2w)^n$ gridded permutations. There are at most $(n+1)^e$ matrices, so this proves the upper root limit at most d^2w.

For a lower bound at every sufficiently large length, write $n = ke + r$ with $0 \leq r < e$, and put $k+1$ entries in any r allowed cells and k entries in the others. Every row and column sum lies in $[dk, dk+d]$. Stirling's formula applied to (6) gives

$$\begin{aligned}\log I(N) &= \sum_i c_i \log c_i + \sum_j r_j \log r_j - 2\sum_{ij\in E} n_{ij}\log n_{ij} + O(\log n)\\ &= 2n\log(n/t) - 2n\log(n/e) + O(\log n) = 2n\log d + O(\log n).\end{aligned}$$

Here the fixed number of factorial remainders is $O(\log n)$; row, column and cell sizes differ from $n/t, n/t, n/e$ by bounded amounts. The lower cell bound in (4) gives

$$\prod_{ij\in E} |\mathcal{W}_{n_{ij}}| \geq \frac{w^n}{\prod_{ij\in E}\binom{n_{ij}+w-1}{w-1}} = w^n \exp(-O(\log n)),$$

since e and w are fixed. Thus this single size matrix has at least $(d^2w)^n \exp(-O(\log n))$ gridded permutations. A permutation of length n has at most $(n+1)^{2t}$ griddings on this fixed matrix, by choosing its row and column division positions. The same lower root limit therefore holds for ungridded permutations. This proves proper growth rate d^2w by a direct count; all constants here may depend on the fixed graph and w. □

Thus, with fixed d, w, the three rates used in the sparse construction are

$$L = 4(d-1)(w+1+a_d), \qquad H = d^2w, \qquad U = d^2(w+1+a_d), \qquad a_d = 8\cdot 4^d. \tag{12}$$

The polynomial factors in (10) and (11) are subexponential because the degree d is fixed before taking the length limits.

7 Effective bursts and a decidable age

The low bursts E_i in Lemma 3.1 give an exact finite description of the age at each length. This also makes the completion effective for the grid classes used here.

Proposition 7.1 (Effective sparse completion)**.** *In Lemma 3.1, suppose that H is rational, membership in $\mathcal{F}$ is decidable, and membership in $\mathcal{H}_h$ is decidable uniformly in h. Then the stages can be chosen so that $\pi : \mathbb{N} \to \mathbb{N}$ and its inverse are computable and its natural atomic age $\mathcal{C}$ is decidable. The exact counts $|\mathcal{C}_n|$ are computable.*

Proof. Fix an index h_0 from which the high families are defined. Make the recursive choices deterministic by taking

$$m_i = 1 + \max\{n_{i-1}, i^2, iS_{i-1}\}, \qquad h_i = \max\{h_0, m_i + 1\}.$$

The finite low burst E_i is computable by testing all permutations of lengths at most i. For a fixed h_i, enumerate lengths $n = m_i + 1, m_i + 2, \ldots$, counting $(\mathcal{H}_{h_i})_n$ by its membership algorithm, until

$$|(\mathcal{H}_{h_i})_n|^{1/n} \geq H - 1/i.$$

This is an exact integer test: if $H = a/b$ with positive integers $a > b$, it is equivalent to

$$|(\mathcal{H}_{h_i})_n|(bi)^n \geq (ai - b)^n.$$

The proper growth rate of $\mathcal{H}_{h_i}$ is at least H, so the search terminates. Choose the first passing length and list its high family lexicographically. All stages $T_i = E_i \oplus B_i$, and their sizes S_i, are therefore computable. To compute $\pi(k)$, compute stages until $S_i \geq k$ and use the finite interval formula in Lemma 3.1. The same interval search followed by inversion of the finite T_i computes $\pi^{-1}(k)$.

It remains to decide age membership. A computable permutation yields a semi-decision procedure by occurrence search; the low bursts upgrade it to a decision procedure. Fix $n \geq 1$ and let $P_n = T_1 \oplus \cdots \oplus T_n$. Every later stage $i > n$ satisfies $h_i > m_i > i^2 > n$. A pattern of length at most n wholly in the tail after P_n is a finite direct sum of low-envelope patterns, by the proof of Lemma 3.1. Conversely E_{n+1}, which lies in this tail, contains every member of $\mathcal{F}_t$ for $t \leq n$. Thus

$$\mathrm{Age}_t(\pi^{>n}) = \mathcal{F}_t \qquad (0 \leq t \leq n), \tag{13}$$

where $\pi^{>n}$ denotes the tail after the first n stages.

For a finite permutation P, let $\mathrm{Sub}_r(P)$ be its set of length-r patterns, with the unique empty pattern at $r = 0$. Equation (13) gives the exact identity

$$\mathcal{C}_n = \bigcup_{r=0}^{\min\{n, S_n\}} \{\alpha \oplus \beta : \alpha \in \mathrm{Sub}_r(P_n),\ \beta \in \mathcal{F}_{n-r}\}. \tag{14}$$

Indeed every occurrence splits at this prefix–tail cut as a direct sum. For the converse, combine an occurrence of α in P_n with an occurrence of β in E_{n+1}; both orders put the former points first. Every set on the right is finite and computable. Enumerating and deduplicating this union decides membership and gives the exact count $|\mathcal{C}_n|$. The empty pattern is handled by $|\mathcal{C}_0| = 1$. □

Lemma 7.2 (Effective forest and grid inputs)**.** *The forest envelopes and high-grid families used here satisfy the effectiveness assumptions of Proposition 7.1.*

Proof. A length-n gridding has at most n occupied cells. After deleting empty rows and columns, it has at most n vertices on each side of its occupied-cell graph. To test membership in $\mathcal{F}_{d,w}$, enumerate these finite ordered graphs and their proper d-edge colorings, and enumerate the cuts of the position and value orders of the input permutation into the corresponding strips. Acyclicity, maximum degree, emptiness of the cells not corresponding to edges, and the cell condition of at most $w-1$ descents are finite tests. Each color matching is also tested for separability: for its length r, enumerate all binary expression trees with r leaves and internal labels $\oplus, \ominus$, evaluate them, and compare with the matching permutation. This finite list is exactly the separable permutations of that length by their definition; the empty matching is included separately. Thus both acceptance and rejection in the forest envelope are decidable.

For each h, enumerate finite ordered d-regular bipartite graphs with proper d-edge colorings. Check girth greater than h and separability of all color matchings by the same finite tests. The finite graph existence theorem supplies a witness, so this search halts. Choose its first returned witness as $G^{(h)}$; these choices are made uniformly in h. Its finite grid specifies $\mathcal{H}_h$. Membership in that fixed grid class is decided by the finitely many possible strip cuts, with empty strips allowed, followed by the $\mathcal{W}$-cell tests. Hence membership in $\mathcal{H}_h$ and its finite counts are computable uniformly in h and length. Its proved growth rate supplies termination of the high-length search in Proposition 7.1. □

8 Arbitrarily large oscillation

Proof of Theorem 2.1. Fix $R > 1$. Choose an integer $d \geq 3$ such that

$$d^2 > 4R(d-1),$$

and then choose an integer $w \geq 1$ such that

$$w > \frac{4R(d-1)(1+a_d)}{d^2 - 4R(d-1)}, \qquad a_d = 8 \cdot 4^d. \tag{15}$$

Both choices are finite and are made before any length limit. Set

$$L = 4(d-1)(w+1+a_d), \qquad H = d^2 w, \qquad U = d^2(w+1+a_d).$$

Equation (15) gives $H > RL$.

Take the sum-closed forest container constructed above. For each h, choose $G^{(h)}$ to be the first ordered colored graph returned by the finite witness search in Lemma 7.2, and use its grid class as $\mathcal{H}_h$. Proposition 6.2 gives $f_n \leq (n+1)^{d+1} L^n$, since $a_d < w+1+a_d$ and $\binom{n+d-1}{d-1} \leq (n+1)^{d-1}$, so Lemma 3.1 applies with $\mathcal{F} = \mathcal{F}_{d,w}$, $A = 1$ and $q = d+1$. Every member of $\mathrm{Grid}_w(G^{(h)})$ of length at most h belongs to $\mathcal{F}_{d,w}$ (Proposition 6.1), while each fixed high-girth grid family has proper growth rate H. Lemma 3.1, applied with $\mathcal{H}_h = \mathrm{Grid}_w(G^{(h)})$, therefore gives a bijection $\pi : \mathbb{N} \to \mathbb{N}$ whose natural class $\mathcal{C} = \mathrm{Sub}(\pi)$ satisfies

$$\underline{\mathrm{gr}}(\mathcal{C}) \leq L, \qquad \overline{\mathrm{gr}}(\mathcal{C}) \geq H.$$

Every low burst belongs to $\mathcal{F}_{d,w} \subseteq \mathcal{D}_{d,w}$, and every high summand belongs to $\mathcal{D}_{d,w}$. Both containers are sum closed. Every finite pattern of π is a pattern of a finite direct sum $T_1 \oplus \cdots \oplus T_k$ whose summands lie in $\mathcal{D}_{d,w}$, so sum closure and heredity place it in $\mathcal{D}_{d,w}$. Its count gives $\overline{\mathrm{gr}}(\mathcal{C}) \leq U < \infty$. In particular the class is proper. It is infinite, so $c_n \geq 1$ for every n and $\underline{\mathrm{gr}}(\mathcal{C}) \geq 1$. Consequently

$$\frac{\overline{\mathrm{gr}}(\mathcal{C})}{\underline{\mathrm{gr}}(\mathcal{C})} \geq \frac{H}{L} > R.$$

The class is atomic because it is natural. For the effectiveness assertions, $H = d^2 w$ is an integer, Lemma 7.2 supplies the decidable input classes, and Proposition 7.1 gives a computable representing permutation, computable inverse, and decidable age with computable counts. This completes the proof. □

For the explicit separation (1), take $d = 3$ and $w = 4105$. Then $a_d = 512$ and the same construction gives

$$L = 8(4105 + 513) = 36944, \qquad H = 9 \cdot 4105 = 36945, \qquad U = 9(4105 + 513) = 41562.$$

These agree with the cubic bounds of [10], now for a natural class. For this instance the ratio of the envelopes is $H/L = 36945/36944$; the general choice of d and w above gives arbitrarily large ratios.

Assistance and data availability

AI-assisted tools were used in mathematical exploration and manuscript preparation. All arguments supporting the conclusions are theoretical and complete within this article.

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