Monotone center weights and the averaged odd-quintic theta gate

Scott Schubr

October 24, 2026

Abstract

We prove positive definiteness of the averaged Riemann theta product moment matrix through degree eight for every nonzero nonnegative nondecreasing center weight under its eighth-moment integrability hypothesis, retaining the earlier fifth-, sixth- and seventh-moment conclusions under their respective hypotheses. At these finite degrees, complete-theta upper-tail matrices are positive at every finite cutoff, and separate certificates prove positive two-column local Schur-residual means. All finite physical parameters are included, with local positivity on fixed configurations of at most nine points. Beyond these finite-order averaged results, we give an explicit degree-dependent theorem: for every finite degree N, the local moment Gram has a 99/100 reserve in normalized Hermite coordinates whenever $X \geq 1000N^2$. A complementary exact parity-block algorithm has a computable termination bound. General positive-prefix regression rigidity isolates variance strictness without assuming positivity of a negative residual mean. The generic tools are classical; the complete-theta signs and quantitative domains are the contribution. The stronger all-order monotone cone is false: every sufficiently large cutoff has a negative complete-theta Fourier value at an explicit frequency choice, hence a negative finite rational polynomial moment square. A rigorous explicit Fourier point and interval are certified separately.

1 Definitions and the weighted theorem

The complete theta density is

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

Jacobi symmetry makes it smooth, positive and even on $\mathbb{R}$; for $s \geq 0$ all summands are positive. Put $X = \pi e^{4s}$ and

$$p_s(t) = \frac{\Phi(s + t/2)\Phi(s - t/2)}{\Phi(s)^2}, \qquad h_{2j}(s) = (-1)^j p_s^{(2j)}(0), \qquad h_{2j+1} = 0.$$

The polynomial variable z is distinct from X. In the raw odd basis set

$$G_s = \begin{pmatrix} h_2 & h_4 \\ h_4 & h_6 \end{pmatrix}, \quad r_s = \begin{pmatrix} h_6 \\ h_8 \end{pmatrix}, \quad S_s = h_{10}, \quad \beta_s = G_s^{-1} r_s, \quad \delta_s = S_s - r_s^T G_s^{-1} r_s. \tag{2}$$

Global invertibility is proved below. Write $dm_0 = s^2 \Phi(s)^2 \, ds$. An admissible weight is a finite-valued measurable nonnegative nondecreasing function W on $[0, \infty)$, not zero almost everywhere, such that

$$\int_0^\infty W(s)(1 + X(s)^5) \, dm_0(s) < \infty. \tag{3}$$

Let $Z_W = \int W \, dm_0$, $d\mu_W = W \, dm_0 / Z_W$, and use $\mathbb{E}_W$ for expectation. Define $\mathcal{L}_W(z^{2j}) = \mathbb{E}_W h_{2j}$ through $j = 5$, with odd moments zero.

Theorem 1. *For every admissible W, the matrix of $\mathcal{L}_W$ on $\{z, z^3, z^5\}$ is positive definite. More precisely, with*

$$\bar{G} = \mathbb{E}_W G_s, \quad \bar{r} = \mathbb{E}_W r_s, \quad \bar{\beta} = \bar{G}^{-1}\bar{r},$$

one has

$$\begin{aligned} \Delta_W &:= \mathbb{E}_W S_s - \bar{r}^T \bar{G}^{-1} \bar{r} = \mathbb{E}_W \delta_s + V_W, \\ V_W &:= \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})] > 0, \qquad \mathbb{E}_W \delta_s > 0. \end{aligned} \tag{4}$$

Let $a = \frac{1}{4}\log(a_1/\pi)$, where

$$3.64888758188562 < a_1 < 3.64888758188563$$

is the first fixed-center quintic transition of [1]. If $W(a) > 0$, then

$$\mathbb{E}_W \delta_s > \frac{2\,000\,000\, W(a)}{Z_W}. \tag{5}$$

Corollary 2. *For $H_y(s) = \sinh(2ys)/(2ys)$, continuously defined as 1 when $ys = 0$, Theorem 1 holds for every finite $y \geq 0$. Moreover*

$$\mathbb{E}_0 \delta_s > 17\,000\,000\,000, \qquad V_y > 1\,000\,000\,000 \quad (0 \leq y \leq 1).$$

Here $Z_y = \int H_y \, dm_0 > 0$ defines normalization also at $y = 0$.

For $y > 0$ the customary unnormalized physical kernel has central value $4yZ_y$; its value vanishes at $y = 0$ and is never used as a denominator there. The normalized law has the displayed continuous endpoint.

The earlier bounded-parameter odd-quintic certificate [2] gave a stronger numerical estimate for Δ_y on $[0, 1]$. The advance here is the full parameter half-line, the monotone-weight cone, and the separation of two strictly positive mechanisms.

The moment hypotheses will remain distinct. The fifth-moment assumption (3) suffices for the full degree-five conclusion in Corollary 8. The degree-six theorem in Section 7 assumes $\int W(1 + X^6)\, dm_0 < \infty$ and adds the even polynomial z^6. Its new local sign, complete matrix tail and positive baseline are proved separately; positivity of one finite moment level is not extrapolated to the next. Section 8 adds degree seven under its own seventh-moment condition, with a different upper-tail matrix argument.

2 Classical algebra and the new theta inputs

The Schur identity (4) is completion of squares. Indeed, integrate

$$S_s - 2b^T r_s + b^T G_s b = \delta_s + (b - \beta_s)^T G_s (b - \beta_s)$$

and minimize over b. This is classical matrix-fractional geometry [4]; its positive variance alone does not determine the sign of the mean local innovation. For example, two equally weighted scalar prefixes $G_1 = G_2 = 1$, columns $r_1 = 2, r_2 = -2$, and $S_1 = S_2 = 3$ have local innovations -1 but averaged innovation 3.

The following elementary single-crossing statement is also classical; stronger integral versions occur in [7].

Lemma 3. *Suppose $f \le 0$ on $[0,a]$, $f \ge 0$ on $[a,\infty)$ and $\int f\,dm > 0$. For a nonnegative nondecreasing W with the necessary absolute integrability,*

$$\int W f\,dm \ge W(a)\int f\,dm.$$

If $W(a) = 0$, W is eventually bounded below by a positive constant, and f is strictly positive on a final half-line of positive measure, then $\int W f\,dm > 0$ as well.

Proof. The function $(W(s)-W(a))f(s)$ is nonnegative on both sides of a. If $W(a) = 0$, monotonicity makes W vanish before a, and its eventual positivity makes a strictly positive final contribution. □

We use one authenticated published result rather than reprove the entire compact classification.

Proposition 4 (Compact fixed-center premise from [1])**.** *For the exact density* (1) *and the exact signed derivatives used here, the full fixed-center moment matrix through degree four is positive definite on $3 \le X \le 12$. The minimum negative-square degree is five before a_1, at least six after a_1 through* 12*, and the degree-five matrix is positive semidefinite at a_1.*

This is the relevant portion of the published classification theorem, including its endpoint statement. Because parity separates odd and even blocks and $G_s > 0$, it implies

$$\delta_s < 0 \quad (\pi \le X < a_1), \quad \delta_s = 0 \quad (X = a_1), \quad \delta_s \ge 0 \quad (a_1 < X \le 12). \tag{6}$$

The new deductions that complete the argument are a global lower prefix, a positive full-theta tail beyond 12, and the independently integrated positive baseline. We provide their proofs and finite-arithmetic specifications below. Associated theta kernels and generalized Laguerre inequalities have a broader classical literature [3].

3 The complete-theta positive tail

Proposition 5. *For $X \ge 12$, the full odd matrix $M_o = (h_{2i+2j+2})_{i,j=0}^{2}$ satisfies*

$$M_o \succeq \frac{10}{63001}\,\mathrm{diag}(16X, 16384X^3, 67108864X^5). \tag{7}$$

Consequently $\delta_s \ge (671088640/63001)X^5 > 0$ there.

Proof. The common e^{5s} factor cancels in the paired derivative. Put

$$R_0(z) = 2z - 3, \quad R_{j+1}(z) = 2z(R_j'(z) - R_j(z)), \quad Q_k(X) = \sum_{j=0}^{2k}(-1)^{k+j}\binom{2k}{j}R_j(X)R_{2k-j}(X).$$

For the first atom, $h_{2k} = Q_k/(2X-3)^2$, and exact cancellation gives $\deg Q_k \le k+2$ for $k \le 5$. Its odd matrix determinant is

$$\frac{6039797760X^6P_9(X)}{(2X-3)^6},$$

where

$$\begin{aligned}P_9(X) = {} & 1024X^9 - 12544X^8 + 81216X^7 - 331200X^6 + 902736X^5 \\ & - 1668960X^4 + 2026620X^3 - 1519020X^2 + 637875X - 127575.\end{aligned}$$

The first two leading-minor numerator polynomials, and $P_9 - 256X^9$, have strictly positive coefficients after $X = 12 + v$. Thus the head matrix is positive definite with determinant at least $24159191040X^9$. For $d = (4, 128, 8192)$ and $D = \mathrm{diag}(d_i X^{i+1/2})_{i=0}^{2}$, its congruence by D^{-1} has determinant at least $45/32768$. Its trace is at most

$$\sum_{i=0}^{2} \sum_{j} \frac{|[X^j]Q_{2i+1}|12^{j-(2i+1)-2}}{(2-3/12)^2 d_i^2} = \frac{256401048845}{99883155456} < \frac{13}{5}.$$

Hence its least eigenvalue exceeds $1/5000$ by $\lambda_{\min} \geq \det / \operatorname{tr}^2$.

We now bound every omitted atom. If $r_{jl} = [z^l]R_j$, put

$$C_j = \sum_{l} |r_{jl}|12^{l-j-1}, \qquad B_j = 2^{2j+5}.$$

For $X \geq 12$, $n^2|R_j(n^2X)| \leq C_j X^{j+1} n^{2j+4}$. For $n \geq 2$ the successive majorant ratio is at most $(3/2)^{24}e^{-60} < 1/2$ for $j \leq 10$. Therefore the derivative tail after removing the common head exponential is at most $B_j C_j X^{j+1} e^{-3X}$. Both cross terms and the tail-tail term in the paired product are retained. Set $\varepsilon = 20^{-12} > e^{-36}$. The error in the kth moment after division by X^k, before the full theta denominator correction, is bounded by

$$E_k = \frac{12^k \varepsilon}{(2-3/12)^2} \sum_{j=0}^{2k} \binom{2k}{j} C_j C_{2k-j}(2B_{2k-j} + B_j B_{2k-j}\varepsilon).$$

Here $X^k e^{-3X}$ and $(2-3/X)^{-2}$ decrease for $X \geq 12$, $k \leq 5$. Symmetry in j accounts for both cross terms. The symmetric matrix perturbation norm is at most

$$3 \max_{i,j} \frac{E_{i+j+1}}{d_i d_j} = \frac{5312409501805455073001503693}{14863564800000000000000000000000000} < \frac{1}{25000}.$$

Writing $\Phi = \Phi_1(1+R)$, the denominator correction is exactly $(1+R)^2$ and $0 < R < 1/250$ (Appendix A). Thus

$$(1+R)^2 D^{-1} M_o D^{-1} \succeq (1/5000 - 1/25000)I = I/6250.$$

Since $(1+R)^2 < (251/250)^2$, this proves (7). Minimizing the quadratic form with its monic fifth-degree coefficient fixed to one proves the Schur bound. □

Combining (6) with Proposition 5 gives one global negative-to-nonnegative crossing at $s = a$. Later higher-degree transitions in the compact classification do not change this fixed quintic conclusion.

4 Positive baseline and independent variance reserve

Proposition 6. *For the full theta density,*

$$\int_0^1 \delta_s \, dm_0 > 2\,000\,000, \qquad I_0 := \int_0^\infty \delta_s \, dm_0 > 2\,000\,000, \qquad I_0/Z_0 > 17\,000\,000\,000.$$

Proof. The first assertion is a finite interval certificate with the full inverse penalty, described in Appendix B. It gives the stronger raw lower bound 2300747. The second follows by dropping the strictly positive tail $s \geq 1$, since $\pi e^4 > 12$. The third uses the independently enclosed full normalizer. This argument does not infer a positive local mean from a known positive averaged Schur complement. □

For two disjoint regions A, B, let $Q_A = \mathbb{E}_W[1_A G_s]$ and $c_A = Q_A^{-1}\mathbb{E}_W[1_A r_s]$, and similarly for B. Completion of squares within the regions and minimization over the global coefficient give the classical parallel-sum lower bound [5, 6]

$$V_W \geq (c_A - c_B)^T (Q_A^{-1} + Q_B^{-1})^{-1}(c_A - c_B) \geq \frac{[u^T(c_A - c_B)]^2}{u^T(Q_A^{-1} + Q_B^{-1})u}. \tag{8}$$

The final inequality is matrix Cauchy–Schwarz, valid without commutativity. For the numerical estimate, use the fixed shifted basis of Appendix B, which preserves V_y; in this paragraph c_A, c_B are the corresponding coefficient vectors. For $A = [0, 3/16]$, $B = [3/16, 1]$, $u = (0, 1)^T$, the certificate encloses the matrices and columns directly over eight closed y bins and proves

$$V_y > 1916442774 > 10^9 \qquad (0 \leq y \leq 1). \tag{9}$$

The complement has nonnegative variance and is discarded. This is a physical-region computation, not the difference of two separately computed total innovations.

5 Proof of the global weighted theorem

The prefix proof in Appendix A gives $G_s > 0$, $h_2 \geq 21$, and $\det G_s \geq 4 \cdot 10^6$ for every s. The paired polynomial formulas in Section 3 give

$$\frac{h_{2j}(s)}{X^j} \longrightarrow (2j-1)!!\, 8^j \quad (1 \leq j \leq 5). \tag{10}$$

Indeed the head numerator has degree $j + 2$, and the full higher-atom correction after division by X^j is $O(X^j e^{-3X})$. Put $v = 8X$. Then $\det G_s / v^4 \to 6$ and the explicit inverse gives

$$\frac{\beta_1}{v^2} \to \frac{225 - 315}{6} = -15, \qquad \frac{\beta_2}{v} \to \frac{105 - 45}{6} = 10.$$

Thus $\beta_1/X^2 \to -960$ and $\beta_2/X \to 80$. In particular $r_s^T G_s^{-1} r_s = O(X^5)$. Smoothness on compact intervals and (3) make all the expectations in (4) absolutely convergent. Positive averaging makes $\bar{G} > 0$.

The global single crossing and Proposition 6 allow Lemma 3 with $f = \delta_s$, proving $\mathbb{E}_W \delta_s > 0$ and (5). If $V_W = 0$, positive definiteness forces $\beta_s = \bar{\beta}$ almost everywhere for μ_W. Every admissible nonzero nondecreasing weight is bounded below by a positive constant on a half-line. Continuity then forces β_s constant on that entire tail, contradicting $\beta_2/X \to 80$. Hence $V_W > 0$. Identity (4) and the Schur criterion prove Theorem 1, also for complex polynomial coefficient vectors.

Finally H_y is nondecreasing because $z\cosh z - \sinh z$ vanishes at zero and has derivative $z \sinh z \geq 0$. Its growth is at most e^{2ys}; the theta density has superexponential decay. Thus (3) holds for every fixed finite y. Propositions 6 and (9) finish Corollary 2.

6 The even block and full degree-five consequence

Proposition 7. *The complete even matrix $M_e = (h_{2i+2j})_{i,j=0}^{2}$ is positive definite for every center. On $X \geq 12$,*

$$M_e \succeq \frac{1}{4000}\operatorname{diag}(1, 1024X^2, 1048576X^4).$$

Proof. Use the recurrence of Section 3, adding $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is $393216X^3\widehat{P}_9(X)$, where

$$\begin{aligned}\widehat{P}_9 = {} & 2048X^9 - 22016X^8 + 120960X^7 - 407232X^6 + 875808X^5 \\ & - 1201680X^4 + 986904X^3 - 424980X^2 + 80190X - 8505.\end{aligned}$$

The first two leading-minor numerators have positive coefficients after $X = 12 + v$, as does

$$393216X^3\widehat{P}_9 - 3145728X^6(2X-3)^6.$$

With $D_e = \operatorname{diag}(1, 32X, 1024X^2)$, the head congruence has determinant at least $3/1024$ and trace at most $1420942925/520224768$. Its least eigenvalue is therefore at least

$$\frac{792872488009728}{2019078796107555625} > 1/3000.$$

The complete head-tail calculation of Section 3, now for $k = 0, \ldots, 4$ with $d = (1, 32, 1024)$, gives perturbation norm at most

$$\frac{5415087591445761969320387}{619315200000000000000000000000000} < 1/1000000.$$

The zeroth entry is included: before dividing by $(1+R)^2$, it changes from 1 to $(1+R)^2$. Thus the tail bound follows from $(1/3000 - 1/1000000)(250/251)^2 > 1/4000$. For $\pi \leq X \leq 12$, the even block is a principal submatrix of the positive M_4 in Proposition 4. Even reflection covers negative centers. □

Corollary 8. *For every admissible W and every nonzero complex polynomial P of degree at most five, $\mathcal{L}_W(|P|^2) > 0$. In particular this holds for every physical weight H_y, $0 \leq y < \infty$.*

Proof. Positive averaging preserves the pointwise strictly positive even block. Odd moments vanish, so a permutation of the full degree-five matrix is the direct sum of that even block and the odd block in Theorem 1. □

The even block on $\{1, z^2, z^4\}$ alone permits arbitrary nonnegative center weights with finite required moments; this does not extend the odd or even-sextic conclusions to arbitrary weights.

7 The even sextic and full degree-six theorem

Suppose, in addition to the stated measurability, sign, monotonicity and nonzero conditions on W, that

$$\int_0^\infty W(s)(1 + X(s)^6)\, dm_0(s) < \infty. \tag{11}$$

Extend $\mathcal{L}_W$ by $\mathcal{L}_W(z^{12}) = \mathbb{E}_W h_{12}$, retaining zero odd moments. Use a distinct notation for the even prefix and its next column:

$$A_s = (h_{2i+2j})_{i,j=0}^{2}, \qquad c_s = (h_6, h_8, h_{10})^T, \qquad b_s = A_s^{-1}c_s, \qquad \eta_s = h_{12} - c_s^T A_s^{-1} c_s.$$

Proposition 7 proves $A_s > 0$ at every center.

Theorem 9. *For every W satisfying* (11)*, the full moment matrix $H_6 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^{6}$ is positive definite. With $\bar{b} = (\mathbb{E}_W A_s)^{-1}\mathbb{E}_W c_s$, its even Schur complement satisfies*

$$\Delta_{6,W} = \mathbb{E}_W \eta_s + \mathbb{E}_W[(b_s - \bar{b})^T A_s (b_s - \bar{b})], \tag{12}$$

and both terms on the right are strictly positive. In particular the theorem holds for every fixed finite physical parameter $y \geq 0$, including the normalized endpoint $y = 0$. Moreover

$$\int_0^\infty \eta_s \, dm_0 > 250000000, \qquad \mathbb{E}_0 \eta_s > 2000000000000. \tag{13}$$

Let $a = \frac{1}{4}\log(a_2/\pi)$, where

$$4.11304745500471102656 < a_2 < 4.11304745500471102658.$$

If $W(a) > 0$, then

$$\mathbb{E}_W \eta_s > \frac{250000000\, W(a)}{Z_W}. \tag{14}$$

7.1 The local sign and the initial obstruction

The compact classification in [1], with the normalization of Proposition 4, has minimum negative-square degree six on (a_1, a_2), at least seven on $(a_2, 12]$, and a positive-semidefinite degree-six matrix at a_2. The degree-six crossing has zero even Schur complement. Before a_1, minimum negative degree five alone does not settle the sign of the even sextic. We supply that missing interval directly.

Proposition 10. *For the actual fixed-center functional, the monic polynomial*

$$P(z) = z^6 - 167z^4 - 7243z^2 + 303062$$

satisfies $\mathcal{L}_X(P^2) < 0$ for every $157/50 \leq X \leq 73/20$.

Proof. The exact whole-cell construction in Appendix D encloses this contraction on 256 closed cells, retaining all theta atoms, and gives a strictly negative upper endpoint on each cell. No point-sample or first-atom approximation is used. □

Since $\pi > 157/50$ and $a_1 < 73/20$, Schur minimization with $A_s > 0$ gives $\eta_s \leq \mathcal{L}_X(P^2) < 0$ on $[\pi, a_1]$. On (a_1, a_2), every square of degree at most five is nonnegative by the compact classification. Thus its negative degree-six square must have a negative even component: the odd component has degree at most five, and cross parity vanishes. The positive even prefix then implies $\eta_s < 0$. At a_2 one has $\eta_s = 0$, and after a_2 through 12 the degree-six matrix is positive semidefinite, so $\eta_s \geq 0$.

Proposition 11. *For every $X \geq 12$, the complete even matrix $E_6 = (h_{2i+2j})_{i,j=0}^{3}$ satisfies*

$$E_6 \succeq \frac{1}{25000}\,\mathrm{diag}(1, 1024X^2, 1048576X^4, 1073741824X^6). \tag{15}$$

In particular $\eta_s \geq (1073741824/25000)X^6 > 0$ there.

Proof. Appendix C proves the full half-line bound by a first-atom determinant reserve, exact second-atom paired cancellation and a geometric enclosure of every remaining atom. Minimizing (15) with the coefficient of z^6 equal to one gives the innovation bound. □

Consequently η_s has the negative-to-nonnegative crossing required by Lemma 3, at $s = a$, and is strictly positive on a final half-line. The fixed-center obstruction of Proposition 10 is not a counterexample to the averaged theorem.

7.2 Integrability, baseline and proof of Theorem 9

For each fixed $k \le 6$, paired differentiation gives $h_{2k} = O(X^k)$ as $X \to \infty$. One direct justification is to factor the leading atom's paired product as

$$e^{-2X(\cosh(2t)-1)} \frac{(2Xe^{2t}-3)(2Xe^{-2t}-3)}{(2X-3)^2}.$$

Its $2k$th coefficient is a polynomial of degree at most k in X, times bounded amplitude derivatives. The complete remaining paired atoms are $O(X^{2k}e^{-3X})$ after head-square normalization; their differentiated series is uniformly summable for $X \ge \pi$. Thus Proposition 7 gives

$$c_s^T A_s^{-1} c_s \le 4000 \left(h_6^2 + \frac{h_8^2}{1024X^2} + \frac{h_{10}^2}{1048576X^4} \right) = O(X^6).$$

Compact continuity and positive definiteness give a bounded inverse on every bounded center interval. Therefore all terms of (12) are absolutely integrable under (11).

The direct local-innovation integration in Appendix D gives a raw lower bound greater than 262225959 on $0 \le s \le 1$. The remaining innovation is positive by Proposition 11; the complete normalizer upper bound then gives (13). This computation retains the whole inverse penalty; it does not obtain the local mean by subtracting a variance from an averaged innovation.

Apply Lemma 3 to η_s. If $W(a) > 0$, it gives (14). If $W(a) = 0$, monotonicity makes W vanish before a and bounds it below by a positive constant on a final half-line, which supplies a strictly positive integral. Completion of squares proves (12), and its positive mean proves the averaged even block positive definite. The sixth-moment hypothesis implies (3); Theorem 1 supplies the odd block. Parity therefore proves $H_6 > 0$, also for complex coefficient vectors.

For strict variance, the leading moments satisfy $h_{2k}/X^k \to (2k-1)!!8^k$. In scaled coordinates, the regression problem converges to the positive Gaussian moment problem of variance 8. Its monic sixth Hermite polynomial gives

$$b_0/X^3 \to 7680, \qquad b_1/X^2 \to -2880, \qquad b_2/X \to 120.$$

If the variance vanished, positive definiteness of A_s would give $b_s = \bar{b}$ almost everywhere on the final half-line where W is bounded below. Continuity would make b_s constant there, contradicting the last limit. Finally, $H_y \le e^{2ys}$ and the superexponential theta decay imply (11) for every fixed finite y. This completes the theorem.

8 The odd septic and full degree-seven theorem

Assume now

$$\int_0^\infty W(s)(1 + X(s)^7)\, dm_0(s) < \infty, \tag{16}$$

with the same sign, measurability, monotonicity and nonzero conditions as before. Extend $\mathcal{L}_W$ through h_{14}. Put

$$M_7(s) = (h_{2i+2j+2}(s))_{i,j=0}^3, \qquad T_7(a) = \int_a^\infty M_7(s)\, dm_0(s).$$

Let C have columns equal to the coefficient vectors of

$$q_0(z) = z/8, \qquad q_1(z) = (z^3 - 139z)/1024,$$
$$q_2(z) = (z^5 - 461z^3 + 32200z)/262144,$$
$$q_3(z) = (z^7 - 1015z^5 + 242645z^3 - 11576832z)/67108864$$

in the monomial basis (z, z^3, z^5, z^7). Thus $q(z) = C^T(z, z^3, z^5, z^7)^T$ and C is invertible.

Theorem 12. *Every finite $a \geq 0$ satisfies $T_7(a) > 0$. More quantitatively,*

$$C^T \left(\int_a^1 M_7(s)\, dm_0(s) \right) C \succeq \frac{1}{50000} I_4 \qquad (0 \leq a \leq 7/100). \tag{17}$$

For every W satisfying (16), the full moment matrix $H_7 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^7$ is positive definite. This includes every fixed finite physical $y \geq 0$, with the normalized endpoint $y = 0$.

The new ingredient is the odd block on four monomials. The corresponding pointwise three-monomial odd prefix is indefinite near the initial center, so its inverse cannot serve as a positive local metric. The proof uses integrated upper tails instead.

Proposition 13. *For $X \geq 12$ the complete theta matrix satisfies*

$$M_7(s) \succeq \frac{1}{150000} \operatorname{diag}(16X, 16384X^3, 67108864X^5, 1099511627776X^7). \tag{18}$$

Appendix E proves this whole-halfline estimate and (17), with all atoms and derivatives through order fourteen. The compact fixed-center classification [1] has minimum negative-square degree at least nine on $[a_3, 12]$, including its third transition, where

$$a_3 < 414157658489452707553/10^{20}.$$

This implies $M_7 \succeq 0$ there. The exact bridge

$$\pi e^{28/100} > 414157658489452707553/10^{20}$$

is checked by the compact script. Consequently $M_7(s) \succeq 0$ for every $s \geq 7/100$, and it is strictly positive definite on a terminal halfline.

Proof of Theorem 12. For $a \leq 7/100$, add the nonnegative integral over $s \geq 1$ to (17). For $a \geq 7/100$, the entire tail is nonnegative, and (18) makes its quadratic form strictly positive on a terminal interval of positive dm_0 measure for every nonzero vector. Thus $T_7(a) > 0$ for every finite cutoff.

For every fixed k, the paired first-atom argument used in Section 7 gives $h_{2k} = O(X^k)$: its amplitude derivatives remain bounded after division by $(2X-3)^2$, and its opposite-shift exponential has coefficient of degree at most k. Every remaining paired atom has polynomial growth of fixed order and a relative factor at most e^{-3X}; its differentiated all-atom series is uniformly summable for $X \geq \pi$. Hence all entries through h_{14} are absolutely integrable under (16).

For a nonzero real vector u, use the classical layer-cake identity

$$W(s) = \int_0^\infty \mathbf{1}_{\{W(s) > t\}}\, dt.$$

A nonempty superlevel set of a nondecreasing function is a terminal interval, up to an irrelevant endpoint. Its quadratic-form integral equals $u^T T_7(a) u > 0$ at a finite cutoff. Absolute integrability justifies signed Fubini. Since W is not zero almost everywhere, a positive-measure set of levels has such a nonempty tail. Therefore $\int W u^T M_7 u\, dm_0 > 0$. This is the usual directional upper-tail criterion for the positive-semidefinite matrix cone. Theorem 9 supplies the even block under the stronger seventh-moment assumption. Parity proves $H_7 > 0$, also for complex coefficient vectors. Finally $H_y \leq e^{2ys}$ and the theta decay pay (16) for every fixed finite y. □

8.1 A stronger two-column local-residual theorem

Retain the genuinely positive pointwise prefix G_s on $U = (z, z^3)$, and add $V = (z^5, z^7)$. Write

$$M_7 = \begin{pmatrix} G_s & B_s \\ B_s^T & D_s \end{pmatrix}, \quad B_s = \begin{pmatrix} h_6 & h_8 \\ h_8 & h_{10} \end{pmatrix}, \quad D_s = \begin{pmatrix} h_{10} & h_{12} \\ h_{12} & h_{14} \end{pmatrix}, \quad R_s = D_s - B_s^T G_s^{-1} B_s.$$

Proposition 14. *Every finite cutoff satisfies $\int_a^\infty R_s \, dm_0 > 0$. Consequently $\mathbb{E}_W R_s > 0$ for every weight in Theorem 12.*

Proof. Modulo the prefix U, the last two q columns have output transformation

$$S = \begin{pmatrix} 1/262144 & -1015/67108864 \\ 0 & 1/67108864 \end{pmatrix}.$$

The Schur residual in the q basis is exactly $S^T R_s S$: invertible prefix changes and addition of prefix columns cancel in a residual. The direct whole-cell checker in Appendix E encloses, uniformly for $0 \le a \le 7/100$, the raw matrix

$$S^T \left(\int_a^1 R_s \, dm_0 \right) S.$$

Its first diagonal exceeds $3/100000$ and its determinant exceeds $2399/10^{12}$, so it is positive definite. For $s \ge 7/100$, the pointwise PSD classification and (18), together with $G_s > 0$, imply $R_s \succeq 0$ and strict positivity on a terminal halfline. This joins the compact certificate to every finite tail. The growth $B_s^T G_s^{-1} B_s = O(X^7)$ entrywise follows from the scaled Gaussian limit below; it also pays absolute integrability of the residual. The same layer-cake argument completes the claim. □

The integrated odd Schur complement therefore has two positive mechanisms. With $\beta_s = G_s^{-1} B_s$ and $\bar{\beta} = (\mathbb{E}_W G_s)^{-1} \mathbb{E}_W B_s$, completion of squares gives

$$\mathbb{E}_W D_s - (\mathbb{E}_W B_s)^T (\mathbb{E}_W G_s)^{-1} \mathbb{E}_W B_s = \mathbb{E}_W R_s + \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})]. \tag{19}$$

This variable-metric least-squares identity is classical; see [4] and the explicit averaging formulation in [9]. Positivity of the old scalar diagonal alone would not control the new coupling. Proposition 14 pays the entire residual matrix independently.

8.2 Strict variance for any finite higher odd output list

The next statement concerns variance only and does not require a monotone density weight.

Proposition 15. *Fix distinct integers $2 \le k_1 < \cdots < k_m$, take outputs z^{2k_j+1} against the prefix $U = (z, z^3)$, and form B_s and $\beta_s = G_s^{-1} B_s$ from their actual theta moments. Let W be any nonnegative measurable density weight, positive on a set of positive dm_0 measure, with*

$$\int W(s)(1 + X(s)^{2k_m+1}) \, dm_0(s) < \infty.$$

Then the m by m regression variance

$$\mathcal{V}_W = \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})]$$

is strictly positive definite.

Proof. For $\tau = 8X$, the complete paired first-atom expression gives $h_{2k}/\tau^k \to (2k-1)!!$ for every fixed k. Equivalently, after $t = u/\sqrt{8X}$ the normalized paired expression tends locally uniformly on complex compact u sets to $e^{-u^2/2}$. The remaining atoms and their fixed derivatives are exponentially small; Cauchy's formula gives the moment limit. This is a fixed-order asymptotic.

The leading Gaussian prefix is

$$G_0 = \begin{pmatrix} \tau & 3\tau^2 \\ 3\tau^2 & 15\tau^3 \end{pmatrix}.$$

For $c_k = (2k+1)!!$, the cross column is $(c_k\tau^{k+1}, (2k+3)c_k\tau^{k+2})^T$. Inversion gives the actual leading limits

$$\frac{\beta_{1,k}}{\tau^k} \longrightarrow (1-k)c_k, \qquad \frac{\beta_{2,k}}{\tau^{k-1}} \longrightarrow kc_k/3. \tag{20}$$

Indeed congruence by $\operatorname{diag}(\tau^{1/2}, \tau^{3/2})$ makes the prefix converge to $\left(\begin{smallmatrix} 1 & 3 \\ 3 & 15 \end{smallmatrix}\right) > 0$, so inversion is justified. The inverse-energy entry for outputs i, j is $O(X^{k_i+k_j+1})$; the highest displayed moment pays every entry and the square identity. Compact continuity handles bounded centers.

The complete theta sum is real analytic locally around each positive real center, by normal convergence on a sufficiently small complex neighborhood. Positivity of Φ and $\det G_s$ makes β_s real analytic on $(0, \infty)$. If $x^T \mathcal{V}_W x = 0$, the positive metric forces $\beta_s x = \bar{\beta} x$ almost everywhere where $W > 0$. Some finite interior interval contains a positive-measure subset of this support. The real-analytic identity theorem extends this equality to every $s > 0$. Divide its second row by $\tau^{k_m - 1}$ and pass to infinity in (20); the nonzero leading coefficient gives $x_m = 0$. Descending through the distinct powers gives every $x_j = 0$. Thus $\mathcal{V}_W > 0$. □

For a single atomic center the variance is zero; a density-support hypothesis cannot be omitted. The proposition supplies no uniform-in-order eigenvalue bound and does not dominate a negative residual mean at higher degree. For the two outputs z^5, z^7, the Gaussian leading regression is

$$\beta_0(\tau) = \begin{pmatrix} -15\tau^2 & -210\tau^3 \\ 10\tau & 105\tau^2 \end{pmatrix}, \quad \det(\beta_0(\tau_2) - \beta_0(\tau_1)) = 525(\tau_2 - \tau_1)^4.$$

Thus actual full-rank contrasts occur at sufficiently large separated centers. A useful quantitative regional bound still requires paid regional masses and errors. In particular a nonmonotone compactly supported density need not charge such distant regions; its strictness above follows from analyticity instead. Applied to (19), Proposition 15 makes the second term strictly positive as well as the first.

8.3 All associated indices at a fixed finite moment degree

For an integer $n \geq 1$ define the associated kernel in the normalization of [3] by

$$K_n(t) = \int_{\mathbb{R}} \Phi(s+t)\Phi(s-t)s^{2n}\, ds, \qquad J_n(t) = \int_0^\infty s^{2n}\Phi(s+t/2)\Phi(s-t/2)\, ds.$$

Corollary 16. *Every $n \geq 1$ has a positive-definite normalized derivative-moment matrix of degree seven at the origin for K_n. If $O_n(J_n)$ denotes the raw odd-four moment block of J_n, then*

$$C^T O_n(J_n) C \succeq \frac{(7/100)^{2n-2}}{50000} I_4. \tag{21}$$

Proof. Take $W(s) = s^{2n-2}$, with $W \equiv 1$ for $n = 1$. Every such weight is admissible under (16). Evenness gives exactly $J_n(t) = K_n(t/2)/2$. Normalized moments differ by the positive diagonal

congruence $\operatorname{diag}(2^{-j})_{j=0}^{7}$, so Theorem 12 proves the jet claim. For $n = 1$, (21) is the cutoff-zero reserve. For $p = 2n - 2 > 0$, signed Fubini and the positive Stieltjes representation of s^p give

$$O_n(J_n) = \int_0^\infty p a^{p-1} T_7(a)\, da.$$

Insert the congruent reserve for $0 \le a \le 7/100$ and drop the PSD remaining tails. Their early scalar mass is $(7/100)^p$. □

Here n ranges over associated indices, not derivative orders. The theorem uses associated indices $n \ge 1$. The case $n = 0$ would require $W = s^{-2}$; its classical autocorrelation positivity has a different proof. Csordas's Theorem 4.5 gives admissibility of the associated theta kernels, while Theorem 4.6 and Open Problem 4.7 address global positive definiteness and Fourier/Laguerre inequalities. The positive-K_1, negative-K_2 example there uses a different Gaussian-polynomial density.

9 The even octic and full degree-eight theorem

Assume the same sign, monotonicity, measurability and nonzero conditions on W, now with

$$\int_0^\infty W(s)(1 + X(s)^8)\, dm_0(s) < \infty. \tag{22}$$

Extend $\mathcal{L}_W$ through h_{16} and put

$$E_8(s) = (h_{2i+2j}(s))_{i,j=0}^{4}, \qquad T_8(a) = \int_a^\infty E_8(s)\, dm_0(s).$$

Let C_8 have columns the coefficient vectors, in $(1, z^2, z^4, z^6, z^8)$, of

$$\begin{aligned}
&q_0 = 1, && q_1 = (z^2 - 47)/64,\\
&q_2 = (z^4 - 275z^2 + 6419)/8192,\\
&q_3 = (z^6 - 703z^4 + 98976z^2 - 1557845)/2097152,\\
&q_4 = (z^8 - 1399z^6 + 512772z^4 - 49595891z^2 + 598408111)/1073741824.
\end{aligned}$$

Theorem 17. *For every finite $a \ge 0$, $T_8(a) > 0$. Uniformly on $0 \le a \le 7/100$,*

$$C_8^T \left(\int_a^1 E_8(s)\, dm_0(s) \right) C_8 \succeq I_5/100000. \tag{23}$$

On that interval one also has the raw monomial bound $T_8(a) \succeq I_5/400000$. For every W satisfying (22), the full degree-eight moment matrix $H_8 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^{8}$ is positive definite. Every fixed finite physical $y \ge 0$ is included, with the normalized endpoint $y = 0$.

Proposition 18. *For $X \ge 12$ the complete even-five matrix satisfies*

$$E_8(s) \succeq \frac{1}{2500000} \operatorname{diag}(1, 1024X^2, 1048576X^4, 4294967296X^6, 70368744177664X^8). \tag{24}$$

Proof of Theorem 17. Appendix F proves (23) by whole-cell integration of all fifteen symmetric entries, retaining every theta atom through derivative sixteen. Proposition 18 is proved there by a new strict head determinant reserve and complete error budget. The compact classification [1] gives $E_8 \succeq 0$ on $a_3 \le X \le 12$, because the minimum negative-square degree is at least nine. The exact

bridge past a_3 in Section 8 therefore gives $E_8(s) \succeq 0$ for every $s \geq 7/100$, with strict positivity on a terminal halfline. For $a \leq 7/100$, add the nonnegative $s \geq 1$ tail to (23); for larger a, integrate the PSD matrix with its strictly positive terminal part. This proves every finite upper tail PD.

The exact basis has

$$\det C_8 = 2^{-70}, \qquad \|C_8\|_F^2 = \frac{870866245054201579}{288230376151711744} < 4.$$

From $C_8^T T C_8 \succeq \eta I$ one obtains $T \succeq \eta (C_8 C_8^T)^{-1} \succeq \eta I / \|C_8\|_{\mathrm{op}}^2$, proving the conservative raw $1/400000$ floor. A normalized weighted floor would require division by Z_W.

The fixed-order bound $h_{2k} = O(X^k)$ also holds for $k = 8$. Thus (22) pays entrywise absolute integrability through h_{16}. The signed layer-cake argument in Theorem 12, applied now to T_8, gives a strictly positive weighted even-five matrix for every admitted W. Its odd-four block is already PD by Theorem 12, since the eighth-moment hypothesis implies the seventh. Parity proves full $H_8 > 0$, also for complex coefficients. The superexponential theta tail pays (22) for every finite physical parameter. □

9.1 Two added even columns and a positive residual mean

Let A_s be the globally PD even-three prefix on $U = (1, z^2, z^4)$ from Proposition 7, and take $V = (z^6, z^8)$. Write

$$E_8 = \begin{pmatrix} A_s & B_s \\ B_s^T & D_s \end{pmatrix}, \qquad R_{8,s} = D_s - B_s^T A_s^{-1} B_s.$$

Proposition 19. *Every finite $a \geq 0$ satisfies $\int_a^\infty R_{8,s}\, dm_0(s) > 0$. Hence $\mathbb{E}_W R_{8,s} > 0$ for every weight in Theorem 17.*

Proof. Modulo U, the last two q columns have output transformation

$$S_8 = \begin{pmatrix} 1/2097152 & -1399/1073741824 \\ 0 & 1/1073741824 \end{pmatrix}.$$

The transformed local Schur residual is $S_8^T R_{8,s} S_8$. The separate complete-theta whole-cell certificate in Appendix F directly bounds its integral from a to 1 for every $0 \leq a \leq 7/100$. Its first diagonal exceeds $1/10000$ and its determinant exceeds $1/100000000$. It retains all three-by-three inverse-prefix terms and the output cross coupling. There is no subtraction of an estimated variance to infer this mean. For $s \geq 7/100$, the full matrix PSD statement and $A_s > 0$ imply $R_{8,s} \succeq 0$, strict on a terminal halfline. Add this tail and apply layer cake. Gaussian scaling gives inverse-energy entries $O(X^6), O(X^7), O(X^8)$; compact inverse continuity, including $s = 0$, and (22) pay absolute integrability. □

With $\beta_s = A_s^{-1} B_s$ and $\bar{\beta} = (\mathbb{E}_W A_s)^{-1} \mathbb{E}_W B_s$, the integrated even Schur complement is

$$\mathbb{E}_W R_{8,s} + \mathbb{E}_W[(\beta_s - \bar{\beta})^T A_s (\beta_s - \bar{\beta})].$$

The first term is PD by Proposition 19. The second is PD by the following common rigidity mechanism. It does not by itself settle the sign of a mean residual at higher degree.

Proposition 20 (Parity-prefix regression rigidity)**.** *Let $\epsilon \in \{0, 1\}$ and an integer $p \geq 0$, and suppose the actual pointwise moment matrix G_s on $U = (z^\epsilon, z^{2+\epsilon}, \ldots, z^{2p+\epsilon})$ is PD for every $s \geq 0$, including*

the endpoint. For distinct integers $k_j > p$, take outputs $z^{2k_j+\epsilon}$ and their regression $\beta_s = G_s^{-1}B_s$. If $W \geq 0$ is any measurable density weight positive on a set of positive dm_0 measure, and

$$\int W(s)(1 + X(s)^{2k_{\max}+\epsilon})\, dm_0(s) < \infty,$$

then its integrated regression variance is PD. Monotonicity is not required for this variance statement.

Proof. Put $\tau = 8X$. The scaled prefix converges to the positive Gaussian parity Gram, so its inverse converges. Gaussian integration by parts gives

$$\mathbb{E}[x^d \mathrm{He}_r(x)] = \frac{d!}{2^{(d-r)/2}((d-r)/2)!}, \qquad \mathbb{E}[\mathrm{He}_r^2] = r!$$

when $d \geq r$ has the same parity. Only the highest retained Hermite polynomial contributes to the highest monomial coefficient. Therefore the actual last regression row satisfies

$$\frac{\beta_{p,k}}{\tau^{k-p}} \longrightarrow \frac{(2k+\epsilon)!}{(2p+\epsilon)!\, 2^{k-p}(k-p)!} > 0.$$

Its inverse-energy entry for columns i, j is $O(X^{k_i+k_j+\epsilon})$. The displayed moment pays integrability at infinity. The assumed PD at $s = 0$, as well as all positive centers, bounds the inverse on each closed compact interval. Positivity merely on the open halfline would not by itself supply that endpoint bound.

As in Proposition 15, a variance-null vector makes $\beta_s x$ constant on a positive-measure set and hence everywhere on $(0, \infty)$ by real analyticity. The last-row asymptotics, divided successively by the largest remaining distinct power, eliminate every output coefficient. This proves strictness. The theorem applies to the averaged weights specified in its hypotheses. □

The verified even-three prefix gives last-row limits $15\tau, 210\tau^2$ for outputs z^6, z^8. The odd-two special case gives $10\tau, 105\tau^2$ for z^5, z^7. No globally positive longer prefix is inferred from an averaged theorem or from eventual positivity alone.

Corollary 21. *For every integer $n \geq 1$, the associated kernel K_n in Corollary 16 has a positive-definite normalized degree-eight derivative-moment matrix at the origin. Its raw even-five block for $J_n(t) = K_n(t/2)/2$ obeys*

$$C_8^T O_{\mathrm{even},n}(J_n) C_8 \succeq \frac{(7/100)^{2n-2}}{100000} I_5.$$

Proof. Use $W = s^{2n-2}$, with $W = 1$ for $n = 1$, exactly as in Corollary 16. The eighth-moment integrability follows from theta decay. The early Stieltjes mass on $0 \leq a \leq 7/100$ is $(7/100)^{2n-2}$; the remaining tails are PSD. The identity $J_n(t) = K_n(t/2)/2$ pays both the half-line factor and the positive diagonal dilation of derivative jets. □

This covers every associated index at one finite derivative degree. The global positive-definiteness criterion requires the stated all-derivative-order premise.

10 An explicit degree-dependent theorem at every finite order

Define the pointwise signed functional $\ell_s(z^j) = h_j(s)$. No positive measure representing all its moments is assumed. Write He_j for the monic probabilists' Hermite polynomials, generated by $e^{uz-u^2/2} = \sum_j \mathrm{He}_j(z)u^j/j!$.

Theorem 22. *For every integer $N \geq 1$ and every complete-theta center $X \geq 1000N^2$, set*

$$P_j(z) = \frac{\mathrm{He}_j(z/\sqrt{8X})}{\sqrt{j!}}, \qquad 0 \leq j \leq N.$$

Then

$$(\ell_s(P_iP_j))_{i,j=0}^{N} \succeq \frac{99}{100} I. \tag{25}$$

Thus the full local monomial moment matrix through degree N is PD on this explicit final halfline. Every upper-tail integral beginning there is PD. The basis depends on the center and the threshold on the degree; this is not positivity of every order at any one finite center.

The theorem gives a common quantitative formula beyond unspecified fixed-order Gaussian convergence. Hermite approximations in the Jensen-polynomial setting [10, 11] concern a different global coefficient sequence and do not imply the present raw center-moment estimate. We claim no superiority over their degree/shift regions.

Lemma 23 (Hermite coefficient row bound)**.** *Let $M(t) = \sum_j m_j t^j/j!$ be analytic on a neighborhood of $|t| \leq 4\sqrt{N}$, with $m_0 = 1$ and real coefficients. For its signed polynomial functional put $f(t) = M(t)e^{-t^2/2}$ and $B = \max_{|t|=4\sqrt{N}} |f(t) - 1|$. Its normalized Hermite Gram matrix G_N satisfies $\|G_N - I\|_{\mathrm{op}} < B$ if $B > 0$, and $G_N = I$ if $B = 0$.*

Proof. The product generating function is $e^{uv}f(u+v)$. Write $f = 1 + \sum_{m\geq 1} a_m t^m$. For $r + s = m$ and $l = i - r = j - s \geq 0$, the contribution to entry i, j is

$$a_m \binom{m}{r} \frac{\sqrt{i!j!}}{l!} = a_m \binom{m}{r} \sqrt{(i)_r (j)_s}.$$

For a fixed row and m, each admissible r determines at most one j, and the factorial factor is at most $N^{m/2}$. Terms $m > 2N$ cannot enter. Hence the absolute row sum of $G_N - I$ is bounded by

$$\sum_{m=1}^{2N} |a_m| (2\sqrt{N})^m.$$

Cauchy's estimate gives $|a_m| \leq B/(4\sqrt{N})^m$, and the resulting geometric sum is strictly less than B. Real symmetry bounds the operator norm by this row norm. This is coefficient algebra and Cauchy's inequality, with no positivity premise on the signed functional. □

Proof of Theorem 22. Take $M_X(t) = p_s(it/\sqrt{8X})$, whose moments are $h_{2k}/(8X)^k$, and set

$$w = \frac{t}{\sqrt{2X}}, \qquad \delta = \sqrt{8N/X} < 1/10.$$

On $|t| \leq 4\sqrt{N}$, the shifted atom arguments are $Xe^{\pm iw}$, and the elementary exponential series gives

$$|e^{\pm iw} - 1| < 1/9, \quad \Re e^{\pm iw} > 8/9, \quad |e^{\pm iw}| < 10/9.$$

The theta series therefore converges normally on a neighborhood of the closed disc; the first-atom amplitudes have no zero there.

The first-atom generating function is exactly

$$M_1(t) = A(t)e^{2X(1-\cos w)}, \quad A(t) = \frac{4X^2 - 12X\cos w + 9}{(2X-3)^2}.$$

For $f_1 = M_1 e^{-t^2/2} = Ae^E$, one has $E = 2X(1 - \cos w - w^2/2)$. The complex cosine remainder and $\cosh\delta < 10/9$ give

$$|E| \le \frac{|t|^4 \cosh\delta}{48X} \le \frac{4}{675}, \qquad |A-1| \le \frac{160N}{3X^2} < \frac{1}{10000}.$$

Since $e^r \le (1-r)^{-1}$ for $0 \le r < 1$,

$$|f_1 - 1| \le \frac{1627}{268400}, \qquad |f_1| \le \frac{270027}{268400}.$$

To pay every higher atom, put $Z = Xe^z$, $|z| \le \delta$. For $n \ge 2$,

$$\left|\frac{n^2(2n^2Z-3)}{2Z-3}\right| \le 2n^4, \qquad \Re Z > X/2.$$

Thus the complete relative remainder obeys

$$|R(z)| \le 2\sum_{n\ge 2} n^4 e^{-(n^2-1)X/2} \le 64e^{-3X/2} < 1/100000.$$

The consecutive-term ratio is at most $(3/2)^4 e^{-5X/2} < 1/2$; the conservative last bound already follows for $X \ge 20$ from $e > 2$. It applies to both shifted remainders and the real central $R_0 \ge 0$.

Full normalization is $f_X = f_1 Q$, where

$$Q = \frac{(1+R_+)(1+R_-)}{(1+R_0)^2}, \qquad |Q-1| \le 4b + 2b^2 = \frac{200001}{5000000000}, \quad b = 1/100000.$$

Consequently, throughout the disc,

$$|f_X - 1| \le \frac{1627}{268400} + \frac{270027}{268400}\frac{200001}{5000000000} = \frac{8189005670027}{1342000000000000} < \frac{1}{100}.$$

Lemma 23 gives (25). The exact rational constants and a separate finite generic-coefficient reconstruction are reproducible with `verify_hermite_tail.py`. □

For each fixed N, positive averaging also allows any nonnegative density weight supported in $X \ge 1000N^2$, positive on a set of positive measure, with the necessary Nth moment. This qualitative averaged conclusion does not preserve a 99/100 floor in one fixed monomial basis. The order-dependent initial region is not covered. A complementary terminating finite-order certificate algorithm is given in Appendix G; it often supplies much smaller low-order thresholds than the uniform formula.

11 Local positivity for at most nine physical kernel points

Define, only for the physical weights and each fixed finite $y \geq 0$,

$$k_y(t) = \frac{1}{Z_y} \int_0^\infty s^2 H_y(s) \Phi(s + t/2) \Phi(s - t/2)\, ds.$$

For $y > 0$ this is the customary physical kernel divided by its nonzero central value. At $y = 0$ it is $Q_1(t)/Q_1(0)$, where $Q_1(t) = 2 \int s^2 \Phi(s + t/2) \Phi(s - t/2)\, ds$.

Corollary 24. *Fix $1 \leq n \leq 9$ and distinct real numbers $r_0, \dots, r_{n-1}$. For every fixed finite $y \geq 0$ there exists $\epsilon_0 > 0$ such that*

$$[k_y(\epsilon(r_i - r_j))]_{i,j=0}^{n-1} > 0 \qquad (0 < |\epsilon| < \epsilon_0).$$

With $\mathcal{V}(r) = \prod_{i<j} (r_j - r_i)$ and $H_{n-1} = (\mathcal{L}_y(z^{i+j}))_{i,j=0}^{n-1}$,

$$\det[k_y(\epsilon(r_i - r_j))] = \frac{\epsilon^{n(n-1)} \mathcal{V}(r)^2 \det H_{n-1}}{\prod_{j=0}^{n-1} (j!)^2} + o(\epsilon^{n(n-1)}).$$

The denominator is 1194393600 *for $n = 6$ and* 619173642240000 *for $n = 7$.*

Proof. For every fixed derivative order j and $u \geq 1$, the complete derivative-polynomial series gives $|\Phi^{(j)}(u)| \leq C_j e^{(4j+9)u} e^{-\pi e^{4u}}$. For $|t| \leq T$ and $s \geq T/2 + 1$, the differentiated paired product is therefore bounded by $C_{j,T} e^{C_j s} e^{-c_T e^{4s}}$, with $c_T = 2\pi e^{-2T} > 0$. The factor $s^2 H_y(s) \leq s^2 e^{2ys}$ preserves integrability. On the remaining compact interval smooth theta convergence supplies domination. Consequently k_y is smooth, real and even, $k_y(0) = 1$, and $k_y^{(2j)}(0) = (-1)^j \mathcal{L}_y(z^{2j})$.

Put $m = n - 1 \leq 8$. We give the classical Taylor-confluence argument using the leading moment matrix H_m. Let $V_{ia} = r_i^a$ for $0 \leq i, a \leq m$ and $C_\epsilon = \operatorname{diag}(1, \epsilon^{-1}, \dots, \epsilon^{-m}) V^{-1}$. Taylor expansion through order $2m$ has remainder $o(\epsilon^{2m})$ and bivariate terms

$$\epsilon^{p+q} r_i^p r_j^q \frac{(-1)^q k_y^{(p+q)}(0)}{p! q!}, \qquad p + q \leq 2m.$$

Multiplication by V^{-1} maps the evaluation vector of degree $p \leq m$ to its coordinate vector. For $p > m$, its coordinate $a \leq m$ acquires a factor ϵ^{p-a} tending to zero; p, q cannot both exceed m. The scaled remainder is $o(1)$ since $a + b \leq 2m$. Hence the real congruence of the kernel matrix tends entrywise to

$$\left(\frac{(-1)^b k_y^{(a+b)}(0)}{a! b!} \right)_{a,b=0}^{m} = J H_m J^*, \qquad J = \operatorname{diag}(i^a / a!)_{a=0}^m.$$

This is positive definite by Theorem 17; the complex phases express a real matrix because cross-parity moments vanish. Openness of positive definiteness and invertibility of C_ϵ prove the claim. Finally $\det C_\epsilon = \epsilon^{-m(m+1)/2} / \mathcal{V}(r)$, proving the determinant asymptotic. □

This classical consequence concerns the stated local point configuration. It is asserted only for the physical family, whose locally uniform shifted-derivative domination through order sixteen was just proved.

12 Why generic log-concavity is insufficient

Arbitrary nonnegative weights are excluded: the known negative fixed-center square near $X = 13/4$ remains negative under a smooth bump supported in a sufficiently small neighborhood. More fundamentally, even smooth positive even log-concavity of a density does not imply the averaged odd-quintic gate.

Proposition 25. *The density $\varphi(s) = e^{-s^2}(1+\cos(4s)/10)$ is positive, even, Schwartz and uniformly strictly log-concave. At $y = 0$ its actual normalized averaged paired-moment functional has a positive odd two-by-two prefix but gives a negative square to*

$$P(z) = z^5 - 45z^3 + 59z.$$

Proof. For $c = \cos(4s)$ and $\epsilon = 1/10$,

$$(\log \varphi)'' = -2 - 16\epsilon \frac{c+\epsilon}{(1+\epsilon c)^2} \le -2/9.$$

If $c + \epsilon \ge 0$ this is immediate; otherwise the positive correction is at most $16\epsilon/(1-\epsilon) = 16/9$. Let $J(t) = \int_{\mathbb{R}} s^2 \varphi(s+t/2)\varphi(s-t/2)\, ds$ and $F(\xi) = \int \varphi(s) e^{i\xi s}\, ds$. The substitution $a = s+t/2, b = s-t/2$ has Jacobian one and gives

$$\widehat{J}(\xi) = \tfrac{1}{2}(F'(\xi)^2 - F(\xi)F''(\xi)), \qquad F(\xi) = \sqrt{\pi}\left(e^{-\xi^2/4} + \frac{e^{-(\xi-4)^2/4} + e^{-(\xi+4)^2/4}}{20}\right).$$

All operations are justified for Schwartz functions. Up to a common positive factor the signed Fourier density is

$$\begin{aligned}\nu(\xi) = (\tfrac{1}{2} - \tfrac{3}{80}e^{-8})e^{-\xi^2/2} + \tfrac{1}{800}\{e^{-(\xi-4)^2/2} + e^{-(\xi+4)^2/2}\} \\ - \tfrac{3}{20}e^{-2}\{e^{-(\xi-2)^2/2} + e^{-(\xi+2)^2/2}\}.\end{aligned}$$

For $J/J(0)$ the polynomial moment functional equals $\int P^2\nu / \int \nu$. Evenness makes this the same normalized half-line $y = 0$ functional. Variance-one Gaussians with means $0, 2, 4$ give respectively

$$\mathbb{E}P^2 = 11191, \quad 334171, \quad 2544391.$$

Thus the numerator in Gaussian units, with $q = e^{-2}$, is

$$N(q) = \frac{4782591}{400} - \frac{1002513}{10}q - \frac{33573}{80}q^4 < -1600.$$

The positive Taylor series for e^2 through degree 20, with omitted-term ratio at most $2/22$, proves $1353/10000 < q < 677/5000$. Substitution of these rational endpoints proves the strict displayed inequality. The same exact Gaussian moment calculation gives mass $> 2/5$, unnormalized second moment $> 1/3$, and odd prefix determinant $> 5/2$. Division by the positive mass preserves its definiteness and the negative square. These explicit checks are included in the companion arithmetic script. □

This example does not refute the actual theta result or stronger curvature criteria for three-point positivity. The associated-kernel Fourier identities are classical [3]; the example serves to distinguish their hypotheses from generic log-concavity.

A Global lower prefix: finite and unbounded enclosures

We give the analytic bounds behind the lower-prefix certificate. All derivatives in this appendix are physical s derivatives unless polynomial arguments are displayed. Let $b = 3/2$, $\tau_n = n^2 - 1$, $\Phi = \Phi_1(1 + R)$, and $\ell = \log \Phi$. Algebra gives the positive Laplace representation

$$R = \int_3^\infty e^{-vX}\, d\sigma(v), \quad d\sigma = \sum_{n\ge 2} \left(n^4 \delta_{\tau_n} + bn^2 \tau_n e^{b(v-\tau_n)} 1_{v\ge\tau_n}\, dv \right).$$

Define $P_0 = 1$, $P_{j+1}(z) = z(P_j - P_j')$. The physical derivative of e^{-vX} is $(-4)^j P_j(vX)e^{-vX}$. For $j \le 6$, both $P_j(9 + w)$ and $[(z^j - P_j(z))]_{z=9+w}$ have nonnegative coefficients; so do the shifted ordinary derivatives of P_4, P_6. Hence the derivatives R_j alternate in sign and are bounded by the highest Laplace moments.

For every $n \ge 2$ their majorants are

$$M_{n,j}(X) = 4^j X^j e^{-\tau_n X} \left[n^4 \tau_n^j + bn^2 \tau_n \sum_{l=0}^{j} \binom{j}{l} \frac{\tau_n^{j-l} l!}{(X-b)^{l+1}} \right].$$

They decrease on $X \ge 3$ for $j \le 6$. Their complete $n \ge 3$ tails are bounded by

$$\mathcal{T}_j = \frac{C_j 3^{4+2j}}{20^8(1-q_j)}, \quad C_j = 12^j \left(1 + \sum_{l=0}^{j} \binom{j}{l} l!(2/27)^l \right), \quad q_j = (4/3)^{4+2j}/20^7 < 1.$$

The ratio bound retains every atom. In particular

$$\begin{aligned} |R_1| &\le 1104/8000 + (100/99)24 \cdot 3^6/2^{24} < 1/4, \\ 0 < R_2 &\le 44736/8000 + (100/99)288 \cdot 3^8/2^{24} < 6, \\ |R_3| &\le M_{2,3}(3)e^9/8000 + \mathcal{T}_3 = 34736311628179317/147622497952000 < 256, \\ R &\le 28/8000 + 2 \cdot 3^4/[20^8(1 - (4/3)^4/20^7)] < 1/250. \end{aligned}$$

Here $e^3 > 20$ follows from its Taylor polynomial through degree eight. The first two weaker tails use $e^{-24} < 2^{-24}$ and ratio $< 1/100$.

Set $U = 1 + R$. For the leading atom

$$\ell_1'' = -16[X + T_2(b/X)], \quad \ell_1'''' = -256[X + T_4(b/X)], \quad \ell_1^{(6)} = -4096[X + T_6(b/X)],$$

where

$$T_2(r) = \frac{r}{(1-r)^2}, \quad T_4(r) = \frac{r(1 + 4r + r^2)}{(1-r)^4}, \quad T_6(r) = \frac{r(1 + 26r + 66r^2 + 26r^3 + r^4)}{(1-r)^6}.$$

The fourth logarithmic correction is

$$(\log U)'''' = R_4/U - (4R_1R_3 + 3R_2^2)/U^2 + 12R_1^2R_2/U^3 - 6R_1^4/U^4.$$

It is at most $R_4 + 9/2$. The qualitative sign $\ell'''' < 0$ is also proved in Umihara [8]; we include the quantitative independent bounds needed here. The positive decreasing $n = 2$ fourth contribution is $C_4(X)e^{-3X}$, where

$$C_j(X) = 4^j \left[16P_j(3X) + \frac{18}{X-b} \sum_{l=0}^{j} P_j^{(l)}(3X) \left(\frac{X}{X-b} \right)^l \right] \quad (j = 4, 6).$$

Indeed $(P_4-P_4')(9+w) = 1162+921w+241w^2+26w^3+w^4 > 0$, so its integrated fourth contribution decreases. Its $n \geq 3$ fourth tail is less than

$$\frac{100}{99}\frac{62208 \cdot 3^{12}}{20^8} = \frac{14348907}{11000000} < 2.$$

The exact endpoint coefficients are $C_4(3) = 43140096$, $C_4(16/5) = 42636828818522112/887410625$, $C_4(7/2) = 59421432$ and $C_4(4) = 278436163584/3125$. Direct substitution yields

X interval	upper for $C_4(X)e^{-3X}$	lower for $256[X + T_4(b/X)]$
$[3, 16/5]$	5393	5430
$[16/5, 7/2]$	3337	3801
$[7/2, 4]$	1748	2557
$[4, \infty)$	557	1024

The exponential lower bounds are respectively $8000, 14400, 34000, 160000$. They follow from $e^3 > 20$, $e^{3/5} > 9/5$, $e^{3/2} > 17/4$. Each final column exceeds the middle column plus $2 + 9/2$.

For the determinant, put $A = -\ell''$, $d = -\ell'''' > 0$, $e = \ell^{(6)}$. Direct paired differentiation gives

$$h_2 = -\ell''/2, \quad h_4 = (\ell'''' + 6\ell''^2)/8, \quad h_6 = -(\ell^{(6)} + 30\ell''\ell'''' + 60\ell''^3)/32,$$

$$64 \det G = 24A^4 - 18A^2 d - d^2 - Ae. \tag{26}$$

The fourth logarithmic error budget is $46595/128$. Expanding $\log(1 + R)$ through order six and dropping its negative groups gives

$$(\log U)^{(6)} \leq R_6 + (15/8)R_4 + 841095/16.$$

The constants follow respectively from $4(1/4)256 + 3 \cdot 6^2 + 6(1/4)^4$ and $120(1/4)6 \cdot 256 + 30 \cdot 6^3 + 360(1/4)^4 6$. For a cell $[L, H]$, bound R_4 below by $C_4(H)e^{-3H}$ and above by $C_4(L)e^{-3L} + \mathcal{T}_4$. Bound R_6 above by $C_6^+(L, H)e^{-3L} + \mathcal{T}_6$, where C_6^+ replaces polynomial arguments by $3H$, denominators by $L - b$, and $X/(X - b)$ by $L/(L - b)$. Positive shifted derivative coefficients justify the enclosure. With these bounds $r_{4,-}, r_{4,+}, r_{6,+}$ define

$$A_- = 16[L + T_2(b/H)] - 6, \qquad A_+ = 16[H + T_2(b/L)] + 1/16,$$
$$d_+ = 256[H + T_4(b/L)] - r_{4,-}/(251/250) + 46595/128,$$
$$e_+ = -4096[L + T_6(b/H)] + r_{6,+} + (15/8)r_{4,+} + 841095/16.$$

On the 96 closed cells $L = 3 + j/32$, $H = L + 1/32$, rational arithmetic proves $A_- > 0$, $d_+ > 0$, $A_-^2 \geq 3d_+/8$ and

$$24A_-^4 - 18A_-^2 d_+ - d_+^2 - (A_+ \text{ if } e_+ \geq 0 \text{ else } A_-)e_+ > 256000000.$$

The minimum lower floor is 264806350, and the minimum guard floor is 3928. The guard justifies the monotonic substitution in A; the other signs justify every remaining substitution. Exponential enclosures here use $\sum_{j=0}^{80} q^j/j!$ and its remainder $(q^{81}/81!)/(1 - q/82)$ for $0 \leq q \leq 18$. For $X \geq 6$, the sixth envelope at 6 gives $e < 150000$; also $16X - 6 \leq A \leq 16X + 33$ and $d \leq 256X + 782$. The resulting lower polynomial in $v = X - 6$ has coefficients

$$(1211952476, 958667264, 274580480, 34209792, 1572864),$$

and its monotonicity guard has coefficients $(28923/4, 2784, 256)$. These are positive. Equation (26) proves globally $\det G \geq 4 \cdot 10^6$ and $h_2 \geq 21$. Thus both the finite interval and the remaining unbounded domain are covered.

B Whole-cell baseline and regional certificate

The separate baseline uses 1024 closed cells $[j/1024, (j+1)/1024]$, not point evaluations. Work in the unit-triangular basis

$$U = z, \quad V = z^3 - 139z, \quad R = z^5 - 461z^3 + 32200z.$$

The scalar monic innovation and matrix-weighted residual variance are invariant under these fixed lower-degree basis shifts. Let $b(s) = \pi e^{5s-X}$ and use tildes for moment numerators multiplied by Φ^2/b^2. Then

$$s^2\Phi^2\delta_s = s^2b^2(\widetilde{S} - \widetilde{r}^T\widetilde{G}^{-1}\widetilde{r}). \tag{27}$$

This exact cancellation avoids division by a small density.

To specify the complete derivative enclosures, use $q_0(z) = 2z - 3$, $q_{j+1}(z) = (5 + 4z\partial_z - 4z)q_j(z)$ through $j = 10$. The head derivative is

$$d_j = \sum_{n=1}^{3} n^2 q_j(n^2X)e^{-(n^2-1)X}.$$

For a whole cell with $X \in [X_-, X_+]$, the omitted derivative tail is bounded by the sum over monomials $q_{jl}z^l$ of

$$|q_{jl}|X_+^l 4^{2l+2}e^{-15X_-} \Big/ \left[1 - (5/4)^{2l+2}e^{-9X_-}\right].$$

Every denominator is positive. The paired head moment polynomial is formed before interval evaluation:

$$\widetilde{m}_{2k}^{(n,m)} = \frac{(-1)^k}{4^k}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}[n^2q_j(n^2X)][m^2q_{2k-j}(m^2X)]e^{-(n^2+m^2-2)X},$$

summed for $1 \le n \le m \le 3$ with multiplicity two off the diagonal. With head and tail derivative bounds, the absolute moment error is

$$4^{-k}\sum_{j=0}^{2k}\binom{2k}{j}(|d_j|t_{2k-j} + t_j|d_{2k-j}| + t_jt_{2k-j}).$$

All cross terms are retained. The six fields are the linear combinations

$$\begin{aligned}
a &= m_2,\\
b &= m_4 - 139m_2,\\
c &= m_6 - 278m_4 + 19321m_2,\\
r_1 &= m_6 - 461m_4 + 32200m_2,\\
r_2 &= m_8 - 600m_6 + 96279m_4 - 4475800m_2,\\
S &= m_{10} - 922m_8 + 276921m_6 - 29688400m_4 + 1036840000m_2.
\end{aligned}$$

All are formed at numerator level before enclosure. Absolute errors are combined using absolute coefficients. On every cell the lower bounds for a and $ac - b^2$ are positive. The inverse penalty is at most the upper bound of $(cr_1^2 - 2br_1r_2 + ar_2^2)/(ac - b^2)$, and also at most $2(cr_1^2 + ar_2^2)/(ac - b^2)$ with absolute upper bounds. Use the smaller of these valid upper bounds, and subtract it from the

lower endpoint for S in (27). Integrating the resulting interval times s^2b^2 proves the baseline. The recorded rational lower exceeds 2300747; no adaptive subdivision is needed.

For normalization, $B_0 = 81/40$ and the positive theta series give, for $s \geq 1$,

$$\Phi(s) \leq 2B_0\pi X e^{5s-X}, \qquad H_y(s) \leq e^{2s} \quad (0 \leq y \leq 1).$$

Indeed $|2n^2X - 3| \leq B_0 n^2 X$ for $X > 120$, and $\sum n^4 e^{-(n^2-1)X} < 2$ by the successive ratio $16e^{-3X} < 1/2$. Therefore

$$s^2 H_y \Phi^2 \leq 4B_0^2\, 4^4 s^2 e^{20s-6e^{4s}}.$$

Since $e^4 > 40$ and $e^3 > 20$, $20s - 6e^{4s} \leq -220 - 940(s-1)$ and $e^{-220} < 20^{-73}$. The full normalizer tail is at most

$$\frac{4B_0^2\, 4^4}{20^{73}} \left(\frac{1}{940} + \frac{2}{940^2} + \frac{2}{940^3} \right). \tag{28}$$

The compact denominator interval for the whole bin $0 \leq y \leq 1/8$ contains Z_0 after adding this tail. Interval division of the baseline by this full denominator proves the 17 billion bound.

For the variance, integrate the five fields of G, r separately on $A = [0, 3/16]$ and $B = [3/16, 1]$, with the common factor $s^2b^2H_y$. Use the eight closed bins $j/8 \leq y \leq (j+1)/8$, $j = 0, \ldots, 7$. Regional interval inverses have positive determinant. Enclose the second-coordinate contrast in (8), square its minimum absolute value, and divide by upper bounds for the inverse sum and full Z_y. The binwise gain floors exceed, respectively,

$$1966012227,\ 1955643969,\ 1946391148,\ 1938242732,$$
$$1931186825,\ 1925210688,\ 1920300777,\ 1916442774.$$

This includes the $1/Z_y$ scaling of raw regional matrices. It certifies (9) independently of a total averaged innovation.

The interval script uses exact rational operations with outward rounding at 80 decimal places; the other finite checks use exact rational polynomial arithmetic. Pi is enclosed by the Machin alternating series; exponentials use a positive Taylor series after range reduction, with an explicit geometric remainder; the hyperbolic quotient uses its positive series. They reconstruct polynomials and tails instead of reading stored numerical enclosures. Explicit exception gates, active also under optimized Python, require raw baseline $> 2 \cdot 10^6$, normalized baseline $> 17 \cdot 10^9$, and every variance bound $> 10^9$. The public scripts are `verify_prefix.py` (lower-order controls and the 96-cell prefix), `verify_tail.py` (odd complete tail), `verify_even.py` (even complete tail), `verify_separated.py` (baseline and regional variance), and `verify_counterexample.py` (signed Gaussian control). The accompanying README specifies exact commands. Arithmetic checking supports the displayed analytic proof; the compact classification theorem is a cited input. These finite checks support the analytic argument.

C Complete even-sextic tail certificate

Use R_j, Q_k from Section 3, now through $j = 12$ and $k = 6$, and $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is

$$\det(Q_{i+j})_{i,j=0}^{3} = 4638564679680 X^6 P_{14}(X),$$

where

$$\begin{aligned}P_{14} = {} & 131072X^{14} - 2293760X^{13} + 20570112X^{12} - 120004608X^{11} \\ & + 493764096X^{10} - 1486503936X^{9} + 3312458496X^{8} - 5440255488X^{7} \\ & + 6483101760X^{6} - 5448021120X^{5} + 3089651040X^{4} - 1112123520X^{3} \\ & + 247531950X^{2} - 39803400X + 3444525.\end{aligned}$$

After $X = 12 + v$, each of its first three leading-minor numerators has nonnegative coefficients with a positive constant coefficient. The reserve polynomial

$$\det(Q_{i+j})_{i,j=0}^{3} - 2^{50}X^{12}(2X-3)^{8}$$

also has nonnegative shifted coefficients. These exact coefficient checks prove the head matrix positive definite and its determinant at least $2^{50}X^{12}$ on the whole half-line. They do not infer definiteness from a determinant alone.

Let $D = \operatorname{diag}(1, 32X, 1024X^2, 32768X^3)$, with constant factors $d = (1, 32, 1024, 32768)$. For the preconditioned head, the determinant is at least $1/1024$ and the trace is at most

$$T = \sum_{i=0}^{3}\sum_{l}\frac{|[X^l]Q_{2i}|12^{l-2i-2}}{(2-3/12)^2 d_i^2} < 8.093.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$. Hence its least eigenvalue is at least

$$\frac{27}{1024T^3} = \frac{2905790377975168091802036781455507456}{58394607561635613985907354403273651574229} > \frac{1}{21000}.$$

Here and below T denotes the displayed exact rational sum, before its decimal upper bound.

To retain enough cancellation at $X = 12$, treat the second atom by its paired polynomials rather than by absolute derivative products. Put

$$\begin{aligned}R_{j,n}(X) &= n^2 R_j(n^2X), \\ C_k(X) &= 2\sum_{j=0}^{2k}(-1)^{k+j}\binom{2k}{j}R_{j,1}(X)R_{2k-j,2}(X), \\ B_k(X) &= 16Q_k(4X).\end{aligned}$$

The additional retained numerator is exactly $C_k(X)e^{-3X} + B_k(X)e^{-6X}$: both ordered cross pairs and the diagonal second-atom pair are included once. Write

$$\varepsilon = (3/8)^{36}, \quad \varepsilon_8 = (3/8)^{96}, \quad c_j = \sum_{l}|[X^l]R_j|12^{l-j-1},$$

$$H_j = \sum_{l}|[X^l]R_j|(1+2^{2l+2})12^{l-j-1}, \qquad b_j = 2\cdot 3^{2j+4}.$$

The elementary inequality $e > 8/3$ gives $e^{-3X} \le \varepsilon$ and $e^{-8X} \le \varepsilon_8$ for $X \ge 12$. The retained two-atom derivative head is bounded by $H_j X^{j+1}$. For the remaining atoms $n \ge 3$, the ratio of successive $n^{2j+4}e^{-(n^2-1)X}$ terms is at most

$$(4/3)^{28}e^{-7X} \le (4/3)^{28}(3/8)^{84} < 1/2 \qquad (j \le 12).$$

Thus their complete derivative tail is at most $b_j c_j X^{j+1} e^{-8X}$.

For $k = 0, \dots, 6$, define the rational numerator-error bound

$$e_k = \frac{1}{(2-3/12)^2}\Bigg\{\varepsilon \sum_l |[X^l]C_k| 12^{l-k-2} + \varepsilon^2 \sum_l |[X^l]B_k| 12^{l-k-2} + 12^k \varepsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \big[H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} + b_j b_{2k-j} c_j c_{2k-j} \varepsilon_8\big]\Bigg\}.$$

This bounds the omitted moment numerator after division by $X^k(2X-3)^2$. The first line retains the exact second-atom cancellation; the remaining lines retain both head-tail terms and the tail-tail term. Every normalized monomial $X^a e^{-3X}$ arising in the first line decreases from 12, since $a \le k \le 6$; the terms with e^{-6X} also decrease. The other terms use the decreasing functions $X^k e^{-8X}$ and $X^k e^{-16X}$. These facts justify the uniform substitution $X = 12$, rather than merely an endpoint test.

The symmetric preconditioned perturbation consequently has norm at most

$$N = 4 \max_{0 \le i,j \le 3} \frac{e_{i+j}}{d_i d_j} < \frac{1}{180000}.$$

The exact value is approximately 0.00000529965257745. The zeroth moment is included in this bound. As in the earlier tails, before the full denominator is restored its zeroth entry is $(1+R)^2$, not one. For the denominator itself,

$$0 < R \le \frac{2X}{2X-3}\, 32 e^{-3X} < 1/250.$$

The geometric ratio here is bounded by $(3/2)^{28} e^{-5X} < 1/2$; the exponent 28 safely covers the zeroth derivative as well. Both rational endpoint ratio checks use $e > 8/3$. Therefore

$$D^{-1} E_6 D^{-1} \succeq \left(\frac{1}{21000} - \frac{1}{180000}\right)\left(\frac{250}{251}\right)^2 I \succ \frac{1}{25000} I,$$

which proves Proposition 11. The standalone checker `verify_sextic_tail.py` reconstructs the polynomials, leading minors, determinant reserve and every rational bound in this appendix.

D Whole-cell sextic sign and mean certificates

The arithmetic uses the interval method and paired-polynomial construction of Appendix B, extended through physical derivative twelve and moments $m_0, \dots, m_{12}$. In particular $q_{j+1} = (5 + 4z\partial_z - 4z)q_j$ is now continued through $j = 12$; the same geometric formula encloses every $n \ge 4$ derivative. Every head-tail and tail-tail product remains in the moment error. The zeroth moment is treated by the same calculation.

For Proposition 10, divide $[157/50, 73/20]$ into 256 equal closed X intervals. The coefficients of the square of the displayed monic polynomial, in ascending even powers, are

$$91846575844,\ -4390156132,\ -48761659,\ 3025286,$$
$$13403,\ -334,\ 1.$$

Combine these with the paired moment polynomials before enclosure. On every whole cell, the sum of its head upper bound and its full absolute tail error is strictly negative. Dividing by the positive complete density square proves the stated fixed-center sign. The script `verify_sextic_initial.py` supplies the exact upper bound and rejects empty partitions.

For the mean, use the fixed scaled basis

$$\begin{aligned} V_0 &= 1, \\ V_1 &= z^2/64 - 1, \\ V_2 &= z^4/4096 - 6z^2/64 + 3, \\ V_3 &= z^6/262144 - 15z^4/4096 + 45z^2/64 - 15. \end{aligned}$$

The residual innovation for V_3 is $\eta_s/262144^2$. Lower-degree shifts do not affect this residual, and the scaling is restored at the end. Work in raw paired numerators as in (27). If their prefix is

$$\widetilde{A} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \qquad \widetilde{c} = (u, v, w)^T,$$

its six adjugate entries are

$$df - e^2, \quad ce - bf, \quad be - cd, \quad af - c^2, \quad bc - ae, \quad ad - b^2.$$

The whole-cell lower endpoints of a, $ad - b^2$ and $\det \widetilde{A}$ are positive. Contract the interval adjugate with (u, v, w) on both sides to obtain a numerator upper bound U. If $D_- > 0$ is the determinant lower bound, then

$$\widetilde{c}^T \widetilde{A}^{-1} \widetilde{c} \le \max(0, U)/D_-.$$

Subtract this entire inverse penalty from the lower top-square endpoint, multiply by the interval for $s^2\pi^2 e^{10s-2X}$ and the cell width, and sum. Exact outward arithmetic on the 1024 closed intervals $[j/1024, (j+1)/1024]$ gives, after multiplication by 262144^2,

$$\int_0^1 \eta_s \, dm_0 > 262225959 > 250000000.$$

The positive remainder for $s \ge 1$ follows from Proposition 11. An independent compact normalizer enclosure plus the full tail (28) gives

$$\frac{\int_0^1 \eta_s \, dm_0}{Z_0} > 2049676440449 > 2000000000000.$$

The checker `verify_sextic_mean.py`, run with argument `1024`, reconstructs these enclosures with the same outward 80-decimal rational arithmetic, Machin bound for pi and Taylor/geometric bounds for exponentials. Explicit exception gates retain all prefix signs and require the conservative raw and normalized thresholds in (13). All three new scripts may be run normally and under optimized Python. The paired recurrences, infinite tails, sign transfer and inverse-energy arguments above are the analytic justification for their finite checks.

E Complete odd-septic and all-cutoff certificates

E.1 The entire local tail beyond twelve

Cancel the e^{5s} factors in the opposite paired shifts. Set

$$R_0(X) = 2X - 3, \quad R_{j+1} = 2X(R_j' - R_j), \quad Q_k = \sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_j R_{2k-j} \quad (0 \le k \le 7).$$

The first-atom matrix has entries $Q_{i+j+1}/(2X-3)^2$, with $\deg Q_k \le k+2$. Exact arithmetic reconstructs

$$\det(Q_{i+j+1}) = 1994953897436774400 X^{10} P_{14}(X),$$

where the ascending coefficients of P_{14} are

$$442047375, -3150592200, 11194876350, -25569432000,$$
$$39943380960, -44288622720, 35956699200, -21886364160,$$
$$10134653184, -3589152768, 967872000, -195207168,$$
$$28237824, -2686976, 131072.$$

After $X = 12+v$, the first three leading-minor numerators have nonnegative coefficients and positive constants. The polynomial

$$\det(Q_{i+j+1}) - 2^{68} X^{16} (2X-3)^8$$

also has nonnegative shifted coefficients. Sylvester's criterion and this reserve give head PD and normalized determinant at least $2^{68} X^{16}$.

Let $d = (4, 128, 8192, 1048576)$ and $D = \operatorname{diag}(d_i X^{i+1/2})_{i=0}^3$. The preconditioned head determinant is at least $1/65536$, and its trace is bounded by

$$T = \sum_{i=0}^{3} \frac{\sum_l |[X^l] Q_{2i+1}| 12^{l-2i-3}}{(2-3/12)^2 d_i^2} = \frac{46528533188040845}{14728370570919936}.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$, so the least eigenvalue exceeds $27/(65536 T^3) > 1/80000$.

Retain the second atom in paired form. Put $R_{j,n} = n^2 R_j(n^2 X)$ and

$$C_k = 2 \sum_j (-1)^{k+j} \binom{2k}{j} R_{j,1} R_{2k-j,2}, \qquad B_k = 16 Q_k(4X).$$

The additional numerator is $C_k e^{-3X} + B_k e^{-6X}$. Its normalized degrees are at most k, so every monomial times the relevant exponential decreases for $X \ge 12$. With $\epsilon = (3/8)^{36}$ and $\epsilon_8 = (3/8)^{96}$, the exact inequality $e > 8/3$ supplies the endpoint majorant

$$A_k = \epsilon \sum_l |[X^l] C_k| 12^{l-k-2} + \epsilon^2 \sum_l |[X^l] B_k| 12^{l-k-2}.$$

For $0 \le j \le 14$, define

$$c_j = \sum_l |[X^l] R_j| 12^{l-j-1},$$
$$H_j = \sum_l |[X^l] R_j| (1 + 2^{2l+2}) 12^{l-j-1}, \qquad b_j = 2\, 3^{2j+4}.$$

The retained derivative head is bounded by $H_j X^{j+1}$. The entire $n \geq 3$ derivative tail is bounded by $b_j c_j X^{j+1} e^{-8X}$: its successive-atom ratio is at most $(4/3)^{32} e^{-7X} < (4/3)^{32}(3/8)^{84} < 1/2$. This pays every atom through derivative fourteen. Both cross terms and the tail-tail product remain in

$$F_k = 12^k \epsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \left(H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} + b_j b_{2k-j} c_j c_{2k-j} \epsilon_8 \right).$$

Set $e_k = (A_k + F_k)/(2 - 3/12)^2$. A symmetric four-by-four entry bound gives the operator error

$$N = 4 \max_{0 \leq i,j \leq 3} \frac{e_{i+j+1}}{d_i d_j} < \frac{1}{200000}.$$

All comparisons are exact rational gates, not rounded decimal tests. Finally, writing $\Phi = \Phi_1(1+R)$, the positive zeroth tail has

$$R \leq \frac{2X}{2X-3}\, 32 e^{-3X} < 1/250.$$

The geometric test $(3/2)^{32}(3/8)^{60} < 1/2$ pays that zeroth sum. Restoring $(1+R)^{-2}$ after the perturbation yields

$$\frac{1/80000 - 1/200000}{(1+1/250)^2} > 1/150000,$$

which proves (18). The standalone checker `verify_septic_tail.py` reconstructs these polynomials, shifted minors and rational budgets.

E.2 Every real short cutoff, rather than sampled cutoffs

For the compact integrators use the physical derivative recurrence $q_{j+1} = (5 + 4X\partial_X - 4X)q_j$ through $j = 14$, beginning at $q_0 = 2X - 3$, as in Appendix B. Retain atoms $1, 2, 3$, combine ordered pairs and the fixed $q_i(z)$ basis products exactly before evaluating intervals, and enclose every $n \geq 4$ derivative monomial using ratio

$$(5/4)^{2l+2} e^{-9X} < 1$$

on its entire cell. The exponent reaches 32 at derivative fourteen. Head-tail and tail-tail errors are retained for every moment. Removing the common factor πe^{5s-X} from each derivative leaves the raw integration weight $s^2\pi^2 e^{10s-2X}$; the normalized Φ^2 cancels, so no small normalizer is inverted.

Write $F(s) = s^2 \Phi(s)^2 C^T M_7(s) C$. On each of the 1024 closed cells $[j/1024, (j+1)/1024]$, whole-cell interval arithmetic encloses every entry f_{ij} by $[f_-, f_+]$. If that cell overlaps $[0, 7/100]$ in width w, its contribution to any removed initial segment $\int_0^a f_{ij}$, $0 \leq a \leq 7/100$, is contained in

$$[w \min(0, f_-),\ w \max(0, f_+)].$$

Indeed the selected portion has any length from zero to w; the signed enclosure covers all choices. Summing gives simultaneous loss intervals for every real cutoff. Subtract them from the independently enclosed entries of $\int_0^1 F$. Discarding correlation weakens the bound safely. For the resulting symmetric interval matrix, each Gershgorin row reserve exceeds $1/50000$; the smallest exceeds $33/10^6$. This proves (17). The checker `verify_septic_compact.py 1024` also verifies the exact bridge past a_3.

For the second certificate use the same basis and raw fields, but split the cell containing $7/100$ and form the local odd-two residual. If its prefix is $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, cross columns are $(u, w)^T, (z, t)^T$, and

output entries are d_{11}, d_{12}, d_{22}, the three residual entries are

$$d_{11} - \frac{cu^2 - 2buw + aw^2}{ac - b^2},$$
$$d_{12} - \frac{cuz - b(ut + wz) + awt}{ac - b^2},$$
$$d_{22} - \frac{cz^2 - 2bzt + at^2}{ac - b^2}.$$

Both a and $ac - b^2$ are checked positive on every entire cell, with subdivision if needed. Apply the same signed-loss construction to the three weighted residual entries. Exact lower bounds for its first diagonal and determinant exceed $3/100000$ and $2399/10^{12}$ respectively. This proves the finite input of Proposition 14; `verify_septic_residual.py 1024` reconstructs it. These two representations share the outward rational interval engine, not independent arithmetic implementations. The complete tail, compact-classification dependency, layer-cake step and variance arguments supply the analytic lifting beyond the finite checks.

F Complete even-octic certificates

F.1 A sharper exponential envelope for the unbounded tail

Use the paired polynomials R_j, Q_k from Appendix E, now through derivative sixteen and $k \le 8$. Put

$$D = \operatorname{diag}(1, 32X, 1024X^2, 65536X^3, 8388608X^4).$$

All four preceding leading-minor numerators are strictly positive after shifting $X = 12 + v$. The exact reserve polynomial

$$\det(Q_{i+j})_{i,j=0}^{4} - 2^{89}X^{20}(2X - 3)^{10}$$

has nonnegative shifted coefficients, with a positive constant. Thus the head is PD and has normalized determinant at least $2^{89}X^{20}$. Its D-congruent trace is bounded by $5.292776831925055\ldots$ (an exact rational computed by the checker). AM–GM on the other four eigenvalues gives

$$\lambda_{\min} \ge \frac{256 \det(D^{-1}E_{8,1}D^{-1})}{\operatorname{tr}(D^{-1}E_{8,1}D^{-1})^4} > 1/2000000.$$

Here $E_{8,1}$ is the first-atom block; its determinant is at least $2^{89}/(1 \cdot 32 \cdot 1024 \cdot 65536 \cdot 8388608)^2$ after congruence.

For $P(L + v) = \sum_j c_j v^j$ and $c > 0$, define

$$\operatorname{Env}(P, L, c) = \sum_j |c_j|(3j/(8c))^j,$$

with the zeroth factor 1. It bounds $|P(L+v)|e^{-cv}$ for every $v \ge 0$, since $\sup v^j e^{-cv} = (j/(ce))^j$ and $e > 8/3$. Shifting before taking absolute values retains useful cancellation.

For the exact second-atom increment $C_k e^{-3X} + B_k e^{-6X}$ of Appendix E, with $L = 12$ and $\varepsilon = (3/8)^{36}$, use

$$\frac{\varepsilon \operatorname{Env}(C_k, 12, 3) + \varepsilon^2 \operatorname{Env}(B_k, 12, 6)}{12^{k+2}}.$$

This bounds the increment divided by X^{k+2} on the whole halfline without losing the near-endpoint paired cancellation. Continue the same complete $n \geq 3$ derivative-tail construction through $j = 16$. Its maximal atom power is 36, and

$$(4/3)^{36}(3/8)^{84} < 1/2.$$

Both cross terms and the tail-tail product remain. The exact symmetric five-by-five preconditioned error norm is

$$5 \max_{i,j} \frac{e_{i+j}}{d_i d_j} = 0.000000008356726546345431\ldots < 1/100000000.$$

The positive central denominator has $R < 1/250$, with ratio check $(3/2)^{36}(3/8)^{60} < 1/2$. Therefore

$$D^{-1}E_8D^{-1} \succeq (1/2000000 - 1/100000000)(250/251)^2 I \succeq I/2500000.$$

This proves Proposition 18. The exact polynomial, ratio and normalization inequalities are reconstructed by `verify_octic_tail.py`.

F.2 The compact full matrix and the separate local residual

Extend the whole-cell recurrence of Appendix E through order sixteen. There are now fifteen symmetric fields from the five even basis polynomials. Moment index zero is retained: there is no odd-basis index shift. The $n \geq 4$ monomial ratio has maximal power 36, and every complete paired error remains. On 1024 closed cells, the signed-loss construction gives simultaneous enclosures of every $\int_a^1 E_8\, dm_0$, $0 \leq a \leq 7/100$. Each congruent Gershgorin row reserve exceeds $1/100000$; the smallest exceeds $9/100000$. The checker `verify_octic_compact.py 1024` also verifies the bridge past a_3. The determinant $\det C_8 = 2^{-70}$ and the displayed Frobenius bound follow by direct rational calculation from the stated basis.

For the residual, let the raw even-three prefix be

$$\widetilde{A} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

Use all six adjugate entries $df - e^2, ce - bf, be - cd, af - c^2, bc - ae, ad - b^2$. For cross-output columns u, w and top entries d_{11}, d_{12}, d_{22}, subtract

$$\frac{u^T \operatorname{adj}(\widetilde{A})u}{\det \widetilde{A}}, \quad \frac{u^T \operatorname{adj}(\widetilde{A})w}{\det \widetilde{A}}, \quad \frac{w^T \operatorname{adj}(\widetilde{A})w}{\det \widetilde{A}}$$

from those three entries. Whole-cell Sylvester gates are $a > 0$, $ad - b^2 > 0$ and $\det \widetilde{A} > 0$. Homogeneous Schur scaling restores the same raw weight $s^2\pi^2 e^{10s-2X}$; it loses no Φ^2 factor. Split the cell containing $7/100$, retain full interval dependency errors and use the signed initial-loss intervals. The resulting two-by-two matrix has first diagonal $> 1/10000$ and determinant $> 1/100000000$. The checker `verify_octic_residual.py 1024` reconstructs the certificate and enforces its PD signs. These two compact representations share an interval engine, not independent arithmetic implementations. Their analytic terminal PSD and strictness premises were proved separately above.

G A terminating effective finite-order tail algorithm

For input $N \geq 0$ and parity $\epsilon \in \{0,1\}$, omit empty blocks and set

$$r_\epsilon = \lfloor (N-\epsilon)/2 \rfloor + 1, \quad \mathcal{B}_\epsilon = (h_{2i+2j+2\epsilon})_{i,j=0}^{r_\epsilon-1}, \quad K = r(r-1)+\epsilon r.$$

The checker `verify_finite_order_tail.py N` tests successive integers $L \geq \max(3, N+1)$ and returns rational bounds proving both parity blocks PD for every $X \geq L$. We give a computable termination bound, rather than assume eventual numerical success.

For $g_k = (2k-1)!!8^k$, $g_0 = 1$, paired cancellation gives $\deg Q_k \leq k+2$ and leading coefficient $4g_k$. Gaussian parity Hermite norms give

$$G_{\det} = 8^K \prod_{i=0}^{r-1} (2i+\epsilon)!, \qquad \kappa = G_{\det}/4.$$

The head determinant numerator has degree $K+2r$ and leading coefficient $4^r G_{\det} > 0$. Thus the reserve polynomial

$$P_{\mathrm{res}} = \det(Q_{i+j+\epsilon}) - \kappa X^K (2X-3)^{2r}$$

and every preceding leading minor have positive leading coefficient. If $P(x) = \sum_{l=0}^d a_l x^l$ with $a_d > 0$, then every coefficient of $P(L+v)$ is strictly positive whenever

$$L \geq B(P) := \left\lceil 1 + \frac{\sum_{l<d} |a_l|}{a_d} \right\rceil.$$

Indeed $[v^j]P(L+v) \geq \binom{d}{j} L^{d-j-1}(a_d L - \sum_{l<d}|a_l|) > 0$ for $j < d$; the leading coefficient is unchanged. Let B_{head} be the maximum over these finitely many polynomials for both parities.

Choose d_i to be the least power of two with $d_i^2 \geq g_{2i+\epsilon}$ and put $D_\epsilon = \mathrm{diag}(d_i X^{i+\epsilon/2})$. For $L \geq 3$, a trace bound is

$$T_\epsilon(L) = \sum_i \frac{\sum_l |[X^l]Q_{2i+\epsilon}| L^{l-2i-\epsilon-2}}{(2-3/L)^2 d_i^2} \leq T_\epsilon(3).$$

Once the shifted leading minors and determinant reserve pass, the head lower bound is

$$\gamma_\epsilon(L) = \frac{(r-1)^{r-1}\kappa}{(\prod_i d_i)^2 T_\epsilon(L)^{r-1}} \geq \gamma_0 := \min_\epsilon \gamma_\epsilon(3) > 0.$$

For $r = 1$, the omitted eigenvalue product and zeroth powers are defined as 1.

Use the exact second-atom envelope of Appendix F. For $j \leq 2N$ set

$$c_j(L) = \sum_l |[X^l]R_j| L^{l-j-1},$$
$$H_j(L) = \sum_l |[X^l]R_j|(1+2^{2l+2})L^{l-j-1}, \quad b_j = 2\,3^{2j+4}.$$

The full $n \geq 3$ derivative tail and all paired errors follow the same formula as Appendix E, with 12 replaced by L and $k \leq N$. Both needed geometric ratios are $< 1/2$ for $L \geq N+1$:

$$(3/2)^{4N+4}(3/8)^{5L} < 1/2, \qquad (4/3)^{4N+4}(3/8)^{7L} < 1/2.$$

The same inequality makes the remaining polynomial-exponential tails decrease from L. Let $e_k(L)$ be the complete normalized paired error, including the second-atom envelope and division by $(2-3/L)^2$. The block error is bounded by

$$\varepsilon_\epsilon(L) = r \max_{i,j} \frac{e_{i+j+\epsilon}(L)}{d_i d_j}.$$

The central ratio is bounded by $\rho(L) = [2L/(2L-3)]32(3/8)^{3L}$. If $\varepsilon_\epsilon(L) < \gamma_\epsilon(L)/2$, return

$$\mathcal{B}_\epsilon(X) \succeq D_\epsilon(X)^2 \frac{\gamma_\epsilon(L) - \varepsilon_\epsilon(L)}{(1+\rho(L))^2} > 0 \quad (X \geq L).$$

Every acceptance comparison uses exact rational arithmetic.

To bound when this must happen, write $c_j^* = \sum_l |[X^l]R_j|$ and $H_j^* = \sum_l |[X^l]R_j|(1+2^{2l+2})$, and define

$$\mathfrak{A}_c(P) = \sum_l |[X^l]P| \sum_{j=0}^{l} \binom{l}{j} (3j/(8c))^j.$$

The zeroth factor is 1. The shifted envelope divided by L^{k+2} is at most $L^k \mathfrak{A}_c(P)$ for the degrees occurring here. Put

$$\begin{aligned}
\mathfrak{K}_k &= \mathfrak{A}_3(C_k) + \mathfrak{A}_6(B_k) \\
&\quad + \sum_{j=0}^{2k} \binom{2k}{j} [H_j^* b_{2k-j} c_{2k-j}^* + b_j c_j^* H_{2k-j}^* + b_j b_{2k-j} c_j^* c_{2k-j}^*], \\
\mathfrak{K}_N^* &= \max_\epsilon r \max_{i,j} \mathfrak{K}_{i+j+\epsilon}/(d_i d_j).
\end{aligned}$$

Then $\varepsilon_\epsilon(L) \leq \mathfrak{K}_N^* L^N (3/8)^{3L}$. For $L \geq \max(1, N^2)$, the elementary $\log L \leq \sqrt{L}$ gives $N \log L \leq L$, and hence

$$L^N (3/8)^{3L} \leq e^L (3/8)^{3L} < (27/128)^L < 4^{-L}.$$

Choose an integer $q \geq 1$ with $4^q > 2\mathfrak{K}_N^*/\gamma_0$. This is computable without logarithms: if the latter rational has positive numerator u in lowest terms, take $q = \max(1, \lceil \text{bitlength}(u)/2 \rceil)$. Therefore

$$L_{\text{safe}} = \max(3, N+1, N^2, B_{\text{head}}, q)$$

pays every head, ratio and perturbation test. The successive-integer search must halt no later than this finite bound.

For illustration, exact normal/optimized checks returned

N	0	1	4	7	8	10
L	3	4	7	9	11	17

with corresponding conservative safe bounds $12, 18, 2608, 2007039, 125672971, 298569113120$. The closed-form $1000N^2$ theorem is the cleaner symbolic estimate; this exact search can certify much smaller low-order instances. Neither the finite cases nor the terminating order-dependent search settles all orders at any fixed center.

H The all-order monotone cone fails

The positive conclusions above have a genuine quantifier boundary. For every fixed degree there is a positive terminal center region. There is nevertheless no theorem making every upper tail positive at every degree. The result below concerns the *complete* theta density, not a finite-atom surrogate.

For $A \geq \pi$ write $a(A) = \frac{1}{4}\log(A/\pi)$ and define the unnormalized tail kernel and its raw moments by

$$J_A(t) = \int_{a(A)}^{\infty} s^2 \Phi(s + t/2)\Phi(s - t/2)\, ds, \qquad m_{2k}(A) = (-1)^k J_A^{(2k)}(0), \quad m_{2k+1} = 0.$$

Our Fourier convention is $\widehat{J}(x) = \int_{\mathbb{R}} J(t)e^{ixt}\, dt$. The raw polynomial functional is $\mathcal{Q}_A(P) = \sum_{i,j} p_i p_j m_{i+j}(A)$ for $P(z) = \sum_i p_i z^i$; no centerwise optimization is interchanged with its integral.

Theorem 26 (Uniform eventual cutoff obstruction)**.** *Put $d = \pi/\log 2$ and*

$$\tau(A) = \left(2\left\lfloor \frac{5A}{2d} \right\rfloor + 1\right) d.$$

There exists a finite A_0 such that, for every real $A \geq A_0$,

$$\widehat{J}_A(2\tau(A)) < 0.$$

For each such cutoff there is a finite nonzero polynomial with rational coefficients for which $\mathcal{Q}_A(P) < 0$. At least one parity admits such a polynomial. Any such polynomial degree N satisfies $N > \sqrt{A/1000}$.

The threshold A_0 and these polynomial coefficients are not numerically specified by this asymptotic theorem. Thus the weight $W = \mathbf{1}_{[a(A),\infty)}$ disproves the stronger all-order monotone-weight assertion, while satisfying every finite moment-integrability condition.

H.1 Exact grouping of all atom products

Write $X = \pi e^{4s}$. The nth theta atom is

$$\Phi_n(s) = \pi n^2 e^{5s}(2n^2 X - 3)e^{-n^2 X}.$$

For $q = nm$, $Y = qX$ and $u = 2t + \log(n/m)$, the paired exponential is $e^{-2Y\cosh u}$ and its polynomial factor is $4Y^2 - 12Y\cosh u + 9$. The standard integral representation of K [13] gives the exact Fourier factor

$$e^{-i\tau\log(n/m)}\mathcal{H}_\tau(Y), \qquad \mathcal{H}_\tau(Y) = (4Y^2 + 9)K_{i\tau}(2Y) + 6Y\partial_Y K_{i\tau}(2Y), \quad x = 2\tau.$$

Here $6Y\partial_Y K_{i\tau}(2Y) = 12YK'_{i\tau}(2Y)$; the prime differentiates the Bessel argument. Also

$$s^2\pi^2 e^{10s}\, ds = \frac{X^{3/2}\log^2(X/\pi)}{64\sqrt{\pi}}\, dX.$$

Define the real divisor phase sum and tail integral

$$D_q(\tau) = \sum_{nm=q} e^{-i\tau\log(n/m)}, \qquad |D_q| \leq d(q) \leq q,$$

$$I_q(\tau, b) = \int_b^{\infty} Y^{3/2}\log^2\left(\frac{Y}{q\pi}\right)\mathcal{H}_\tau(Y)\, dY.$$

Then

$$64\sqrt{\pi}\,\widehat{J}_A(2\tau) = \sum_{q\ge 1} q^{-1/2} D_q(\tau) I_q(\tau, qA). \tag{29}$$

All sum/integral interchanges are absolute: before oscillatory cancellation the pair transform is bounded by $(4Y^2+9)K_0(2Y) + 12YK_1(2Y)$. Its lower endpoint is qA, its other factors have polynomial/logarithmic growth, and its exponential decay sums over all q. This argument does not discard atoms whose shifted arguments are negative.

H.2 Complete Mellin mass and a uniform oscillatory subtraction

For $\mu > 0$ the classical Mellin formula [13, 10.43.19], with the $2Y$ scaling retained, is

$$M_\mu(\tau) = \int_0^\infty Y^{\mu-1} K_{i\tau}(2Y)\, dY = \tfrac{1}{4}\Gamma((\mu+i\tau)/2)\Gamma((\mu-i\tau)/2).$$

Integration by parts and the gamma recurrence give

$$F_\mu(\tau) := \int_0^\infty Y^{\mu-1}\mathcal{H}_\tau(Y)\, dY = 4M_{\mu+2} + (9-6\mu)M_\mu = [(\mu-3)^2+\tau^2]M_\mu.$$

Logarithmic differentiation under the convergent integrals yields

$$I_q(\tau,0) = (\partial_\mu - \log(q\pi))^2 F_\mu(\tau)\big|_{\mu=5/2}.$$

Set

$$B(\tau) = 2\pi(\tau/2)^{7/2} e^{-\pi\tau/2}\log^2\tau > 0.$$

Differentiated Stirling asymptotics, locally uniform near $\mu = 5/2$, imply $I_q(\tau,0)/B(\tau) = 1 + O(1/\log\tau)$ for each fixed q. In particular $F_{5/2} = (\tau^2/4 + 1/16)|\Gamma(5/4 + i\tau/2)|^2$, fixing the leading sign and constant.

We require a uniform estimate for the *lower* incomplete integral. For each fixed $r < 1$, $\mu > 0$, and integer $0 \le l \le 2$,

$$\int_0^{r\tau/2} Y^{\mu-1}\log^l(Y/(q\pi))K_{i\tau}(2Y)\, dY = O\Big(e^{-\pi\tau/2}\tau^{\mu-3/2}(1+\log\tau)^l\Big). \tag{30}$$

The constants are uniform when the upper ratio varies in a fixed compact subset of $(0,1)$. Here and below q is fixed when this estimate is used.

For completeness we specify the classical large-order interface needed for this endpoint estimate. On $0 < z \le r < 1$, the oscillatory branches in Dunster's Theorem 1, equations (17)–(19), (25)–(27) of [12], together with the exact I–K connection [13, 10.27.4], give

$$e^{\pi\tau/2}K_{i\tau}(\tau z) = \sqrt{2\pi/\tau}(1-z^2)^{-1/4}\cos(\tau\psi(z) - \pi/4) + O(\tau^{-3/2}), \tag{31}$$

uniformly down to $z = 0$, where $\psi(z) = \operatorname{arcosh}(1/z) - \sqrt{1-z^2}$ and $\psi'(z) = -\sqrt{1-z^2}/z$. This is the oscillatory I-branch continuation, not an application of the recessive K formula across a turning point. In the source notation the differential error functions F_s are $O(z^2)$ at zero and the error-path differential has size $O(dz/z)$; hence their products are integrable there. The integrated coefficients E_s need not vanish at zero. The source remainder integrals and connection normalizers therefore have uniform $O(1/\tau)$ relative error on $0 < z \le r$. This yields the absolute remainder in (31) without differentiating it.

After $Y = \tau z/2$, integrate the leading cosine once by parts using ψ'. The lower endpoint vanishes because $z^\mu(1+|\log z|)^l \to 0$, and the derivative of amplitude divided by ψ' is integrable on $(0, r]$. The remainder in (31) is integrated absolutely. Both terms give (30).

For $\mathcal{H}_\tau$, integrate its derivative-K term by parts on the lower finite interval first. The resulting integrals have $\mu = 9/2, 5/2$ and logarithmic powers at most two. The retained endpoint is

$$6b^{5/2}\log^2(b/(q\pi))K_{i\tau}(2b),$$

which is $O(e^{-\pi\tau/2}\tau^2\log^2\tau)$ when $2b/\tau$ stays below turning. Consequently, uniformly for $19/100 \le A/\tau \le 21/100$,

$$I_q(\tau, qA)/B(\tau) = 1 + O(1/\log\tau) + O(\tau^{-1/2}), \qquad q = 1, 2. \tag{32}$$

Their lower ratios lie in $[.38, .42]$ and $[.76, .84]$. Only the subtracted incomplete integrals use the below-turning expansion; the exact complete Mellin expression already includes the turning region.

H.3 A uniform bound for the entire omitted product sum

Choose one fixed contour angle

$$v = \arcsin(7/8), \qquad c = \cos v = \sqrt{15}/8.$$

Shifting the defining K integral by iv gives

$$|K_{i\tau}(2Y)| \le e^{-\tau v}K_0(2cY), \qquad |\partial_Y K_{i\tau}(2Y)| \le 2e^{-\tau v}K_1(2cY).$$

The vertical sides vanish exponentially; $|\cosh(u+iv)| \le \cosh u$ pays the differentiated amplitude. For $q \ge 3$ and $A/\tau \ge 19/100$, the exponent at $Y \ge qA$ is at least $\tau(v + 19qc/50)$. At the first omitted product its gap above $\pi\tau/2$ is strictly greater than $\tau/960$. Indeed, for $\alpha = \arccos(7/8)$,

$$\cos(8/15) \le 1 - \tfrac{1}{2}(8/15)^2 + \tfrac{1}{24}(8/15)^4 = \frac{130787}{151875} < \frac{7}{8},$$

so $\alpha < 8/15$, whereas $57\sqrt{15}/400 > 171/320$ and $171/320 - 8/15 = 1/960$. Each further q gains an exponent exceeding $57\tau/320$.

The elementary Gaussian bounds for K_0, K_1 at large positive argument, together with $d(q) \le q$, now bound the *complete* $q \ge 3$ sum by a polynomial in $\tau, \log\tau$ times $e^{-(\pi/2+1/960)\tau}$. Constants are uniform for $A/\tau \in [.19, .21]$: after $Y = qA + w$, the logarithm is $\log(A/\pi) + \log(1 + w/(qA))$, and the remaining powers of q are dominated by the geometric exponent just proved. In particular the diagonal product $q = 4$ is included in this negligible remainder.

Proof of Theorem 26. The nearest odd grid point obeys $|\tau(A) - 5A| \le d$ and $\cos(\tau(A)\log 2) = -1$, ties included. Since $d < 6$, for $A \ge 26$ one has $A/\tau(A) \in [.19, .21]$. This numerical bound establishes uniformity only, not negativity already at 26.

In (29), $D_1 = 1$ and $D_2 = 2\cos(\tau\log 2)$. Equation (32) and the infinite remainder estimate imply, uniformly along this piecewise constant frequency choice,

$$\frac{64\sqrt{\pi}\,\widehat{J}_A(2\tau(A))}{B(\tau(A))} = 1 - \sqrt{2} + O(1/\log\tau(A)) + O(\tau(A)^{-1/2}) < 0$$

for every sufficiently large A. Section H.4 proves the finite-polynomial consequence. Finally, if $A \ge 1000N^2$, the proved local degree-N theorem makes its raw integral over the entire tail strictly positive. Any negative degree therefore satisfies $A < 1000N^2$. □

H.4 Why a Fourier sign forces a finite polynomial failure

For each fixed $A \geq \pi$, J_A is real even, integrable on the real axis, and analytic in $|\operatorname{Im} t| < \pi/4$. On compact subsets of this strip, $\cos(2 \operatorname{Im} t) > 0$ gives normally convergent shifted theta sums and superexponential domination in the center integral. Evenness and theta decay also pay the real-t integral. Moreover $J_A(0) > 0$.

Suppose every finite matrix of the normalized moments $m_k(A)/J_A(0)$ were PSD. The Hamburger moment theorem supplies a positive probability measure with those moments. Cauchy's estimate for the actual analytic germ gives $m_{2k}/J_A(0) \leq C(2k)! r^{-2k}$ for some $r > 0$. Monotone convergence applied to the series for cosh gives an exponential moment at each smaller rate. The representing characteristic function is therefore analytic in a strip and has exactly the same derivatives at zero as $J_A/J_A(0)$. Local equality and analytic continuation give equality everywhere on the real axis. Thus J_A would be positive definite; its continuous Fourier transform would be nonnegative, a contradiction.

Some finite real quadratic form is therefore strictly negative. Rational coefficient vectors are dense, so rational approximation preserves strict negativity. Odd moments vanish, so splitting a polynomial into its even and odd parts leaves at least one negative parity square. The moment/positive-definite bridge is classical; the complete-theta Fourier sign is the new input. Related associated-kernel questions are formulated in [3, Sections 3–4].

H.5 An explicit certified complete-theta Fourier obstruction

Proposition 27. *For the exact cutoff $A = 23/2$,*

$$-8 \cdot 10^{-49} < \widehat{J}_A(154) < -3 \cdot 10^{-49}.$$

For every x in the closed interval $[769999/5000, 770001/5000]$,

$$-10^{-48} < \widehat{J}_A(x) < -10^{-50}.$$

The same interval conclusion holds after reflection about zero.

Here is the full certification mechanism, implemented in the two portable supplements. Scale (29) by $e^{\pi\tau/2}$ and retain $q = 1, \ldots, 8$ through $Y = 256$. After one integration by parts, the retained approximation to each integral is

$$\int_{qA}^{256} Y^{3/2}\{(4Y^2 - 6)L^2 - 12L\}K_{i\tau}(2Y)\, dY - 6(qA)^{5/2} \log^2(A/\pi) K_{i\tau}(2qA), \quad L = \log(Y/(q\pi)).$$

Outward Arb analytic quadrature at 256 bits evaluates this expression at $\tau = 77$. Its analytic callback requires $\operatorname{Re} Y > 0$, paying the logarithm, square-root and Bessel branches. A finite enclosure is used regardless of whether requested integration tolerances are met; only the strict enclosure gate accepts a sign.

For both discarded tails, shift by $i\pi/3$. For $Y \geq 1$,

$$e^{\pi\tau/2}|K_{i\tau}(2Y)| \leq e^{\pi\tau/6}\sqrt{\pi/(2Y)}e^{-Y},$$

and for $Y \geq 2$ its Y derivative is at most four times this envelope. The full $\mathcal{H}$ amplitude is then at most $37Y^2$ times the envelope. Using $L^2 \leq (Y/(q\pi))^2$, $d(q) \leq q$, $\int_b^\infty Y^5 e^{-Y} dY \leq 2b^5 e^{-b}$ for $b \geq 11$,

and the geometric ratio $(10/9)^4 e^{-A} < 1/2$, the total scaled error is at most

$$E_q = \frac{148\sqrt{\pi/2}}{\pi^2} A^5\, 9^4 e^{\pi\tau/6 - 9A},$$
$$E_Y = \frac{222\sqrt{\pi/2}}{\pi^2} 256^5 e^{\pi\tau/6 - 256}.$$

The second bound also pays the integrated-by-parts integrand, whose smaller amplitude constant is 22; $\sum_{q=1}^{8} q^{-3/2} < 3$. At 77, $E_q < 10^{-17}$ and $E_Y < 10^{-80}$. The resulting unscaled endpoints are approximately $-7.351 \cdot 10^{-49}$ and $-3.166 \cdot 10^{-49}$, proving the point bounds. A distinct implementation uses the original $\mathcal{H}$ integrand and $\partial_Y K_{i\tau}(2Y) = -K_{i\tau-1}(2Y) - K_{i\tau+1}(2Y)$, without the endpoint integration by parts, and independently certifies strict negativity.

To pay the interval statement rather than assume a bandwidth from plots, put $S(\tau) = 64\sqrt{\pi} e^{\pi\tau/2} \widehat{J}_A(2\tau)$ and let T be its retained finite-Y, finite-q part. Fix $v = 3/2$, $c = \cos v$, $\delta = \pi/2 - v$. Shifting the full-line K integral and differentiating the scaled transform give, for $76 \le \tau \le 78$,

$$|e^{\pi\tau/2} K_{i\tau}(2Y)| \le e^{78\delta - 2cY} \sqrt{\pi/(4cY)},$$
$$|\partial_\tau (e^{\pi\tau/2} K_{i\tau}(2Y))| \le e^{78\delta - 2cY} \{\delta \sqrt{\pi/(4cY)} + (2cY)^{-1}\}.$$

These follow from the Gaussian zeroth and first absolute moments on the shifted contour, not from differentiating a Bessel asymptotic. Also $|D_q'| \le q \log q$. Let $b = qA$, $d_0 = \log(A/\pi) - 1 > 0$,

$$C_q = (\log q + \delta)\sqrt{\pi/(4c)} + (2c)^{-1}, \qquad G_n(b) = e^{-2cb} \sum_{j=0}^{n} \frac{n!}{j!} \frac{b^j}{(2c)^{n+1-j}}.$$

Concavity gives $L \le Y/(qA) + d_0$. The absolute derivative of each retained integral is bounded by

$$22\sqrt{q} e^{78\delta} C_q \{G_5(b)/(qA)^2 + 2d_0 G_4(b)/(qA) + d_0^2 G_3(b)\}.$$

Its endpoint derivative is at most

$$6 b^{5/2} \log^2(A/\pi) \sqrt{q} e^{78\delta - 2cb} \{(\log q + \delta)\sqrt{\pi/(4cb)} + (2cb)^{-1}\}.$$

Summing these positive bounds gives $|T'| < 10045000000$. The program checks this strict comparison with outward arithmetic. For $h = 1/10000$, add $h \sup |T'| + 2(E_q(78) + E_Y(78))$ to the point enclosure of $S(77)$, then multiply by the positive factor $e^{-\pi\tau/2}/(64\sqrt{\pi})$ over $\tau \in [77-h, 77+h]$. The resulting endpoints lie strictly between -10^{-48} and -10^{-50}. This proves the stated closed interval with every infinite remainder paid.

Proposition 27 proves finite rational polynomial existence at a specific cutoff via Section H.4. It does not identify coefficients or a first failing degree. A rigorous finite-atom prefix check or a numerical no-hit at a large degree must not be substituted for complete-theta positivity.

H.6 Comparison and the remaining physical question

The classical associated-kernel program [3] and correlation identities [15] concern different global kernels or Wronskians. Planat's phase-aligned decomposition [14] proves negative individual blocks and cancellation between sectors of the physical growth derivative. Conversely our step-weight obstruction does not determine the sign for physical H_y. The imaginary-order Bessel asymptotics, Mellin transforms, contour shifts and moment bridge are classical ingredients. The contribution is

their complete-theta product grouping, uniform cutoff synchronization and actual total-tail obstruction, including the certified explicit point.

For physical weights, if $F(z) = \int_{\mathbb{R}} \Phi(s) e^{izs} ds = \xi((1+iz)/2)/4$, then

$$\widehat{J}_y(x) = \frac{\partial_y |F(x+iy)|^2}{8y}, \qquad y > 0.$$

The known zero strip and paired canonical product give strict positivity for every $y \geq 1$, a classical zero-free-half-plane consequence [16]. Hence the remaining physical all-order parameter window is $0 \leq y < 1$, with the continuous normalized endpoint. The negative step weights of this section do not belong to that family. We assert neither global physical positivity nor its negation.

I Scope

The monotone-weight condition is sufficient. Degrees five through eight retain their respective integrability hypotheses. Every finite physical parameter is included; physical local fixed-shape positivity extends to nine points with a shape- and parameter-dependent scale. The positive-prefix rigidity theorem assumes a genuinely PD pointwise prefix, including the zero center, and proves variance strictness for density weights; it does not pay a negative mean residual. Associated-kernel corollaries range over all indices at fixed finite derivative degree.

The general Hermite theorem supplies a degree-dependent far-center region $X \geq 1000N^2$, not all orders at one fixed center. Its moving-basis reserve must not be read as a common monomial eigenvalue floor. The exact finite-order algorithm gives constructive tail certificates, not an all-order averaged theorem. The stronger all-order monotone-weight target is disproved by Theorem 26; the degree-dependent positive terminal region is fully consistent with that obstruction. Classical Schur, layer-cake, Hermite/Cauchy, Gaussian projection, coefficient-shift, analytic-rigidity and confluence mechanisms are credited.

References

[1] S. Schubr, *Minimum-Degree Obstructions in the Riemann Theta Product: Classification, transition bounds and robust Gram certificates*, edition 2, October 18, 2026, `https://luminaryoddities.com/researcharchive/minimum-degree-obstructions-riemann-theta-product/v/2`.

[2] S. Schubr, *Averaged Quintic Gate by Paired Kernel and Endpoint Conversion*, unpublished manuscript, 2026.

[3] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ-function*, Comput. Methods Funct. Theory **15** (2015), 373–391, arXiv:1309.0055.

[4] J. Gallier, *The Schur complement and symmetric positive semidefinite (and definite) matrices*, author notes, `https://www.cis.upenn.edu/~jean/schur-comp.pdf`.

[5] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594, doi:10.1016/0022-247X(69)90200-5.

[6] Z. Tarcsay, *On the parallel sum of positive operators, forms, and functionals*, arXiv:1501.01922, 2015.

[7] L. Smith, *Monotone comparative statics under uncertainty: interval dominance order*, author lecture notes, Integral Single Crossing Corollary, `https://www.lonessmith.com/wp-content/uploads/2025/03/1-Monotone-comparative-statics.pdf`.

[8] H. Umihara, *A Fourth-Derivative Inequality for the Riemann Xi Kernel*, preprint, August 2026, doi:10.5281/zenodo.21902312.

[9] GettysburgResearch, *Reviewer B—independent first-pass scientific review*, September 5, 2026, Section 6, repository report (variable-metric Schur averaging identity), `https://github.com/GettysburgResearch/riemann/blob/main/reviews/B/REPORT.md`.

[10] M. Griffin, K. Ono, L. Rolen and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Natl. Acad. Sci. USA **116** (2019), 11103–11110, arXiv:1902.07321.

[11] C. O'Sullivan, *Zeros of Jensen polynomials and asymptotics for the Riemann xi function*, manuscript dated November 29, 2020, `https://fsw01.bcc.cuny.edu/cormac.osullivan/Research/xi-revised.pdf`.

[12] T. M. Dunster, *Uniform asymptotic expansions for Bessel functions of imaginary order and their zeros*, arXiv:2412.12595, version 1, December 17, 2024, `https://arxiv.org/html/2412.12595v1`.

[13] F. W. J. Olver et al., eds., *NIST Digital Library of Mathematical Functions*, Sections 10.27, 10.32 and 10.43, `https://dlmf.nist.gov/`.

[14] M. Planat, *Nonlocal Cancellation in a Theta-Kernel Decomposition of the Riemann Ξ-Growth Derivative: An Obstruction to Phase-Aligned Blockwise Positivity*, preprint, August 25, 2026, doi:10.20944/preprints202608.1692.v1. See also *A Theta-Kernel Reformulation of Riemann-Ξ Growth and the Obstruction to Blockwise Positivity*, Symmetry **18** (2026), 1283, doi:10.3390/sym18081283.

[15] D. K. Dimitrov and Y. Xu, *Wronskians of Fourier and Laplace transforms*, arXiv:1606.05011, 2016, https://arxiv.org/abs/1606.05011.

[16] J. Sondow and C. Dumitrescu, *A monotonicity property of Riemann's xi function and a reformulation of the Riemann hypothesis*, Period. Math. Hungar. **60** (2010), 37–40, https://arxiv.org/abs/1005.1104.