Monotone center weights and the averaged odd-quintic theta gate

Scott Schubr

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Abstract

We prove positivity of the full normalized Riemann theta product moment matrix through degrees five, six and seven under the corresponding fifth-, sixth- and seventh-moment hypotheses on every nonzero nonnegative nondecreasing center weight. All finite physical parameters are included. The new degree-seven theorem follows from positive upper-tail odd-four matrices at every real cutoff, using a complete-theta unbounded tail and an exact compact matrix certificate. A second certificate proves positive means of the full two-column local Schur residual. We also prove strict regression variance for any fixed finite higher odd output list against the positive lower prefix, even for nonmonotone density weights. Polynomial weights yield positive degree-seven derivative jets for every associated theta-kernel index. Physical kernels have local positivity on fixed configurations of at most eight points. Earlier separated quintic and sextic estimates and a smooth log-concave counterexample are retained. The generic tools are classical; the quantitative signs are theta-specific.

1 Definitions and the weighted theorem

The complete theta density is

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

Jacobi symmetry makes it smooth, positive and even on $\mathbb{R}$; for $s \geq 0$ all summands are positive. Put $X = \pi e^{4s}$ and

$$p_s(t) = \frac{\Phi(s+t/2)\Phi(s-t/2)}{\Phi(s)^2}, \qquad h_{2j}(s) = (-1)^j p_s^{(2j)}(0), \qquad h_{2j+1} = 0.$$

The polynomial variable z is distinct from X. In the raw odd basis set

$$G_s = \begin{pmatrix} h_2 & h_4 \\ h_4 & h_6 \end{pmatrix}, \quad r_s = \begin{pmatrix} h_6 \\ h_8 \end{pmatrix}, \quad S_s = h_{10}, \quad \beta_s = G_s^{-1} r_s, \quad \delta_s = S_s - r_s^T G_s^{-1} r_s. \tag{2}$$

Global invertibility is proved below. Write $dm_0 = s^2 \Phi(s)^2\, ds$. An admissible weight is a finite-valued measurable nonnegative nondecreasing function W on $[0,\infty)$, not zero almost everywhere, such that

$$\int_0^\infty W(s)(1 + X(s)^5)\, dm_0(s) < \infty. \tag{3}$$

Let $Z_W = \int W\, dm_0$, $d\mu_W = W\, dm_0/Z_W$, and use $\mathbb{E}_W$ for expectation. Define $\mathcal{L}_W(z^{2j}) = \mathbb{E}_W h_{2j}$ through $j = 5$, with odd moments zero.

Theorem 1. *For every admissible W, the matrix of $\mathcal{L}_W$ on $\{z, z^3, z^5\}$ is positive definite. More precisely, with*

$$\bar{G} = \mathbb{E}_W G_s, \quad \bar{r} = \mathbb{E}_W r_s, \quad \bar{\beta} = \bar{G}^{-1}\bar{r},$$

one has

$$\begin{aligned} \Delta_W &:= \mathbb{E}_W S_s - \bar{r}^T \bar{G}^{-1} \bar{r} = \mathbb{E}_W \delta_s + V_W, \\ V_W &:= \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})] > 0, \qquad \mathbb{E}_W \delta_s > 0. \end{aligned} \tag{4}$$

Let $a = \frac{1}{4}\log(a_1/\pi)$, where

$$3.64888758188562 < a_1 < 3.64888758188563$$

is the first fixed-center quintic transition of [1]. If $W(a) > 0$, then

$$\mathbb{E}_W \delta_s > \frac{2\,000\,000\, W(a)}{Z_W}. \tag{5}$$

Corollary 2. *For $H_y(s) = \sinh(2ys)/(2ys)$, continuously defined as 1 when $ys = 0$, Theorem 1 holds for every finite $y \geq 0$. Moreover*

$$\mathbb{E}_0 \delta_s > 17\,000\,000\,000, \qquad V_y > 1\,000\,000\,000 \quad (0 \leq y \leq 1).$$

Here $Z_y = \int H_y \, dm_0 > 0$ defines normalization also at $y = 0$.

For $y > 0$ the customary unnormalized physical kernel has central value $4yZ_y$; its value vanishes at $y = 0$ and is never used as a denominator there. The normalized law has the displayed continuous endpoint.

The earlier bounded-parameter odd-quintic certificate [2] gave a stronger numerical estimate for Δ_y on $[0, 1]$. The advance here is the full parameter half-line, the monotone-weight cone, and the separation of two strictly positive mechanisms.

The moment hypotheses will remain distinct. The fifth-moment assumption (3) suffices for the full degree-five conclusion in Corollary 8. The degree-six theorem in Section 7 assumes $\int W(1 + X^6)\, dm_0 < \infty$ and adds the even polynomial z^6. Its new local sign, complete matrix tail and positive baseline are proved separately; positivity of one finite moment level is not extrapolated to the next. Section 8 adds degree seven under its own seventh-moment condition, with a different upper-tail matrix argument.

2 Classical algebra and the new theta inputs

The Schur identity (4) is completion of squares. Indeed, integrate

$$S_s - 2b^T r_s + b^T G_s b = \delta_s + (b - \beta_s)^T G_s (b - \beta_s)$$

and minimize over b. This is classical matrix-fractional geometry [4]; its positive variance alone does not determine the sign of the mean local innovation. For example, two equally weighted scalar prefixes $G_1 = G_2 = 1$, columns $r_1 = 2, r_2 = -2$, and $S_1 = S_2 = 3$ have local innovations -1 but averaged innovation 3.

The following elementary single-crossing statement is also classical; stronger integral versions occur in [7].

Lemma 3. *Suppose $f \le 0$ on $[0,a]$, $f \ge 0$ on $[a,\infty)$ and $\int f\,dm > 0$. For a nonnegative nondecreasing W with the necessary absolute integrability,*

$$\int W f\,dm \ge W(a)\int f\,dm.$$

If $W(a) = 0$, W is eventually bounded below by a positive constant, and f is strictly positive on a final half-line of positive measure, then $\int W f\,dm > 0$ as well.

Proof. The function $(W(s)-W(a))f(s)$ is nonnegative on both sides of a. If $W(a) = 0$, monotonicity makes W vanish before a, and its eventual positivity makes a strictly positive final contribution. □

We use one authenticated published result rather than reprove the entire compact classification.

Proposition 4 (Compact fixed-center premise from [1])**.** *For the exact density* (1) *and the exact signed derivatives used here, the full fixed-center moment matrix through degree four is positive definite on $3 \le X \le 12$. The minimum negative-square degree is five before a_1, at least six after a_1 through* 12*, and the degree-five matrix is positive semidefinite at a_1.*

This is the relevant portion of the published classification theorem, including its endpoint statement. Because parity separates odd and even blocks and $G_s > 0$, it implies

$$\delta_s < 0 \quad (\pi \le X < a_1), \quad \delta_s = 0 \quad (X = a_1), \quad \delta_s \ge 0 \quad (a_1 < X \le 12). \tag{6}$$

The new deductions that complete the argument are a global lower prefix, a positive full-theta tail beyond 12, and the independently integrated positive baseline. We provide their proofs and finite-arithmetic specifications below. Associated theta kernels and generalized Laguerre inequalities have a broader classical literature [3].

3 The complete-theta positive tail

Proposition 5. *For $X \ge 12$, the full odd matrix $M_o = (h_{2i+2j+2})_{i,j=0}^{2}$ satisfies*

$$M_o \succeq \frac{10}{63001}\,\mathrm{diag}(16X, 16384X^3, 67108864X^5). \tag{7}$$

Consequently $\delta_s \ge (671088640/63001)X^5 > 0$ there.

Proof. The common e^{5s} factor cancels in the paired derivative. Put

$$R_0(z) = 2z - 3, \quad R_{j+1}(z) = 2z(R_j'(z) - R_j(z)), \quad Q_k(X) = \sum_{j=0}^{2k}(-1)^{k+j}\binom{2k}{j}R_j(X)R_{2k-j}(X).$$

For the first atom, $h_{2k} = Q_k/(2X-3)^2$, and exact cancellation gives $\deg Q_k \le k+2$ for $k \le 5$. Its odd matrix determinant is

$$\frac{6039797760X^6P_9(X)}{(2X-3)^6},$$

where

$$\begin{aligned}P_9(X) = {} & 1024X^9 - 12544X^8 + 81216X^7 - 331200X^6 + 902736X^5 \\ & - 1668960X^4 + 2026620X^3 - 1519020X^2 + 637875X - 127575.\end{aligned}$$

The first two leading-minor numerator polynomials, and $P_9 - 256X^9$, have strictly positive coefficients after $X = 12 + v$. Thus the head matrix is positive definite with determinant at least $24159191040X^9$. For $d = (4, 128, 8192)$ and $D = \mathrm{diag}(d_i X^{i+1/2})_{i=0}^{2}$, its congruence by D^{-1} has determinant at least $45/32768$. Its trace is at most

$$\sum_{i=0}^{2} \sum_{j} \frac{|[X^j]Q_{2i+1}|12^{j-(2i+1)-2}}{(2-3/12)^2 d_i^2} = \frac{256401048845}{99883155456} < \frac{13}{5}.$$

Hence its least eigenvalue exceeds $1/5000$ by $\lambda_{\min} \geq \det / \operatorname{tr}^2$.

We now bound every omitted atom. If $r_{jl} = [z^l]R_j$, put

$$C_j = \sum_{l} |r_{jl}|12^{l-j-1}, \qquad B_j = 2^{2j+5}.$$

For $X \geq 12$, $n^2|R_j(n^2X)| \leq C_j X^{j+1} n^{2j+4}$. For $n \geq 2$ the successive majorant ratio is at most $(3/2)^{24}e^{-60} < 1/2$ for $j \leq 10$. Therefore the derivative tail after removing the common head exponential is at most $B_j C_j X^{j+1} e^{-3X}$. Both cross terms and the tail-tail term in the paired product are retained. Set $\varepsilon = 20^{-12} > e^{-36}$. The error in the kth moment after division by X^k, before the full theta denominator correction, is bounded by

$$E_k = \frac{12^k \varepsilon}{(2-3/12)^2} \sum_{j=0}^{2k} \binom{2k}{j} C_j C_{2k-j}(2B_{2k-j} + B_j B_{2k-j}\varepsilon).$$

Here $X^k e^{-3X}$ and $(2-3/X)^{-2}$ decrease for $X \geq 12$, $k \leq 5$. Symmetry in j accounts for both cross terms. The symmetric matrix perturbation norm is at most

$$3 \max_{i,j} \frac{E_{i+j+1}}{d_i d_j} = \frac{5312409501805455073001503693}{14863564800000000000000000000000000} < \frac{1}{25000}.$$

Writing $\Phi = \Phi_1(1+R)$, the denominator correction is exactly $(1+R)^2$ and $0 < R < 1/250$ (Appendix A). Thus

$$(1+R)^2 D^{-1} M_o D^{-1} \succeq (1/5000 - 1/25000)I = I/6250.$$

Since $(1+R)^2 < (251/250)^2$, this proves (7). Minimizing the quadratic form with its monic fifth-degree coefficient fixed to one proves the Schur bound. □

Combining (6) with Proposition 5 gives one global negative-to-nonnegative crossing at $s = a$. Later higher-degree transitions in the compact classification do not change this fixed quintic conclusion.

4 Positive baseline and independent variance reserve

Proposition 6. *For the full theta density,*

$$\int_0^1 \delta_s \, dm_0 > 2\,000\,000, \qquad I_0 := \int_0^\infty \delta_s \, dm_0 > 2\,000\,000, \qquad I_0/Z_0 > 17\,000\,000\,000.$$

Proof. The first assertion is a finite interval certificate with the full inverse penalty, described in Appendix B. It gives the stronger raw lower bound 2300747. The second follows by dropping the strictly positive tail $s \geq 1$, since $\pi e^4 > 12$. The third uses the independently enclosed full normalizer. This argument does not infer a positive local mean from a known positive averaged Schur complement. □

For two disjoint regions A, B, let $Q_A = \mathbb{E}_W[1_A G_s]$ and $c_A = Q_A^{-1}\mathbb{E}_W[1_A r_s]$, and similarly for B. Completion of squares within the regions and minimization over the global coefficient give the classical parallel-sum lower bound [5, 6]

$$V_W \geq (c_A - c_B)^T (Q_A^{-1} + Q_B^{-1})^{-1}(c_A - c_B) \geq \frac{[u^T(c_A - c_B)]^2}{u^T(Q_A^{-1} + Q_B^{-1})u}. \tag{8}$$

The final inequality is matrix Cauchy–Schwarz, valid without commutativity. For the numerical estimate, use the fixed shifted basis of Appendix B, which preserves V_y; in this paragraph c_A, c_B are the corresponding coefficient vectors. For $A = [0, 3/16]$, $B = [3/16, 1]$, $u = (0, 1)^T$, the certificate encloses the matrices and columns directly over eight closed y bins and proves

$$V_y > 1916442774 > 10^9 \qquad (0 \leq y \leq 1). \tag{9}$$

The complement has nonnegative variance and is discarded. This is a physical-region computation, not the difference of two separately computed total innovations.

5 Proof of the global weighted theorem

The prefix proof in Appendix A gives $G_s > 0$, $h_2 \geq 21$, and $\det G_s \geq 4 \cdot 10^6$ for every s. The paired polynomial formulas in Section 3 give

$$\frac{h_{2j}(s)}{X^j} \longrightarrow (2j-1)!!\, 8^j \quad (1 \leq j \leq 5). \tag{10}$$

Indeed the head numerator has degree $j + 2$, and the full higher-atom correction after division by X^j is $O(X^j e^{-3X})$. Put $v = 8X$. Then $\det G_s / v^4 \to 6$ and the explicit inverse gives

$$\frac{\beta_1}{v^2} \to \frac{225 - 315}{6} = -15, \qquad \frac{\beta_2}{v} \to \frac{105 - 45}{6} = 10.$$

Thus $\beta_1/X^2 \to -960$ and $\beta_2/X \to 80$. In particular $r_s^T G_s^{-1} r_s = O(X^5)$. Smoothness on compact intervals and (3) make all the expectations in (4) absolutely convergent. Positive averaging makes $\bar{G} > 0$.

The global single crossing and Proposition 6 allow Lemma 3 with $f = \delta_s$, proving $\mathbb{E}_W \delta_s > 0$ and (5). If $V_W = 0$, positive definiteness forces $\beta_s = \bar{\beta}$ almost everywhere for μ_W. Every admissible nonzero nondecreasing weight is bounded below by a positive constant on a half-line. Continuity then forces β_s constant on that entire tail, contradicting $\beta_2/X \to 80$. Hence $V_W > 0$. Identity (4) and the Schur criterion prove Theorem 1, also for complex polynomial coefficient vectors.

Finally H_y is nondecreasing because $z\cosh z - \sinh z$ vanishes at zero and has derivative $z \sinh z \geq 0$. Its growth is at most e^{2ys}; the theta density has superexponential decay. Thus (3) holds for every fixed finite y. Propositions 6 and (9) finish Corollary 2.

6 The even block and full degree-five consequence

Proposition 7. *The complete even matrix $M_e = (h_{2i+2j})_{i,j=0}^{2}$ is positive definite for every center. On $X \geq 12$,*

$$M_e \succeq \frac{1}{4000}\operatorname{diag}(1, 1024X^2, 1048576X^4).$$

Proof. Use the recurrence of Section 3, adding $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is $393216X^3\widehat{P}_9(X)$, where

$$\begin{aligned}\widehat{P}_9 = {} & 2048X^9 - 22016X^8 + 120960X^7 - 407232X^6 + 875808X^5 \\ & - 1201680X^4 + 986904X^3 - 424980X^2 + 80190X - 8505.\end{aligned}$$

The first two leading-minor numerators have positive coefficients after $X = 12 + v$, as does

$$393216X^3\widehat{P}_9 - 3145728X^6(2X-3)^6.$$

With $D_e = \operatorname{diag}(1, 32X, 1024X^2)$, the head congruence has determinant at least $3/1024$ and trace at most $1420942925/520224768$. Its least eigenvalue is therefore at least

$$\frac{792872488009728}{2019078796107555625} > 1/3000.$$

The complete head-tail calculation of Section 3, now for $k = 0, \ldots, 4$ with $d = (1, 32, 1024)$, gives perturbation norm at most

$$\frac{5415087591445761969320387}{619315200000000000000000000000000} < 1/1000000.$$

The zeroth entry is included: before dividing by $(1+R)^2$, it changes from 1 to $(1+R)^2$. Thus the tail bound follows from $(1/3000 - 1/1000000)(250/251)^2 > 1/4000$. For $\pi \leq X \leq 12$, the even block is a principal submatrix of the positive M_4 in Proposition 4. Even reflection covers negative centers. □

Corollary 8. *For every admissible W and every nonzero complex polynomial P of degree at most five, $\mathcal{L}_W(|P|^2) > 0$. In particular this holds for every physical weight H_y, $0 \leq y < \infty$.*

Proof. Positive averaging preserves the pointwise strictly positive even block. Odd moments vanish, so a permutation of the full degree-five matrix is the direct sum of that even block and the odd block in Theorem 1. □

The even block on $\{1, z^2, z^4\}$ alone permits arbitrary nonnegative center weights with finite required moments; this does not extend the odd or even-sextic conclusions to arbitrary weights.

7 The even sextic and full degree-six theorem

Suppose, in addition to the stated measurability, sign, monotonicity and nonzero conditions on W, that

$$\int_0^\infty W(s)(1 + X(s)^6)\, dm_0(s) < \infty. \tag{11}$$

Extend $\mathcal{L}_W$ by $\mathcal{L}_W(z^{12}) = \mathbb{E}_W h_{12}$, retaining zero odd moments. Use a distinct notation for the even prefix and its next column:

$$A_s = (h_{2i+2j})_{i,j=0}^{2}, \qquad c_s = (h_6, h_8, h_{10})^T, \qquad b_s = A_s^{-1}c_s, \qquad \eta_s = h_{12} - c_s^T A_s^{-1} c_s.$$

Proposition 7 proves $A_s > 0$ at every center.

Theorem 9. *For every W satisfying* (11)*, the full moment matrix $H_6 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^{6}$ is positive definite. With $\bar{b} = (\mathbb{E}_W A_s)^{-1}\mathbb{E}_W c_s$, its even Schur complement satisfies*

$$\Delta_{6,W} = \mathbb{E}_W \eta_s + \mathbb{E}_W[(b_s - \bar{b})^T A_s (b_s - \bar{b})], \tag{12}$$

and both terms on the right are strictly positive. In particular the theorem holds for every fixed finite physical parameter $y \geq 0$, including the normalized endpoint $y = 0$. Moreover

$$\int_0^\infty \eta_s \, dm_0 > 250000000, \qquad \mathbb{E}_0 \eta_s > 2000000000000. \tag{13}$$

Let $a = \frac{1}{4}\log(a_2/\pi)$, where

$$4.11304745500471102656 < a_2 < 4.11304745500471102658.$$

If $W(a) > 0$, then

$$\mathbb{E}_W \eta_s > \frac{250000000\, W(a)}{Z_W}. \tag{14}$$

7.1 The local sign and the initial obstruction

The compact classification in [1], with the normalization of Proposition 4, has minimum negative-square degree six on (a_1, a_2), at least seven on $(a_2, 12]$, and a positive-semidefinite degree-six matrix at a_2. The degree-six crossing has zero even Schur complement. Before a_1, minimum negative degree five alone does not settle the sign of the even sextic. We supply that missing interval directly.

Proposition 10. *For the actual fixed-center functional, the monic polynomial*

$$P(z) = z^6 - 167z^4 - 7243z^2 + 303062$$

satisfies $\mathcal{L}_X(P^2) < 0$ for every $157/50 \leq X \leq 73/20$.

Proof. The exact whole-cell construction in Appendix D encloses this contraction on 256 closed cells, retaining all theta atoms, and gives a strictly negative upper endpoint on each cell. No point-sample or first-atom approximation is used. □

Since $\pi > 157/50$ and $a_1 < 73/20$, Schur minimization with $A_s > 0$ gives $\eta_s \leq \mathcal{L}_X(P^2) < 0$ on $[\pi, a_1]$. On (a_1, a_2), every square of degree at most five is nonnegative by the compact classification. Thus its negative degree-six square must have a negative even component: the odd component has degree at most five, and cross parity vanishes. The positive even prefix then implies $\eta_s < 0$. At a_2 one has $\eta_s = 0$, and after a_2 through 12 the degree-six matrix is positive semidefinite, so $\eta_s \geq 0$.

Proposition 11. *For every $X \geq 12$, the complete even matrix $E_6 = (h_{2i+2j})_{i,j=0}^{3}$ satisfies*

$$E_6 \succeq \frac{1}{25000}\,\mathrm{diag}(1, 1024X^2, 1048576X^4, 1073741824X^6). \tag{15}$$

In particular $\eta_s \geq (1073741824/25000)X^6 > 0$ there.

Proof. Appendix C proves the full half-line bound by a first-atom determinant reserve, exact second-atom paired cancellation and a geometric enclosure of every remaining atom. Minimizing (15) with the coefficient of z^6 equal to one gives the innovation bound. □

Consequently η_s has the negative-to-nonnegative crossing required by Lemma 3, at $s = a$, and is strictly positive on a final half-line. The fixed-center obstruction of Proposition 10 is not a counterexample to the averaged theorem.

7.2 Integrability, baseline and proof of Theorem 9

For each fixed $k \le 6$, paired differentiation gives $h_{2k} = O(X^k)$ as $X \to \infty$. One direct justification is to factor the leading atom's paired product as

$$e^{-2X(\cosh(2t)-1)} \frac{(2Xe^{2t}-3)(2Xe^{-2t}-3)}{(2X-3)^2}.$$

Its $2k$th coefficient is a polynomial of degree at most k in X, times bounded amplitude derivatives. The complete remaining paired atoms are $O(X^{2k}e^{-3X})$ after head-square normalization; their differentiated series is uniformly summable for $X \ge \pi$. Thus Proposition 7 gives

$$c_s^T A_s^{-1} c_s \le 4000 \left(h_6^2 + \frac{h_8^2}{1024X^2} + \frac{h_{10}^2}{1048576X^4} \right) = O(X^6).$$

Compact continuity and positive definiteness give a bounded inverse on every bounded center interval. Therefore all terms of (12) are absolutely integrable under (11).

The direct local-innovation integration in Appendix D gives a raw lower bound greater than 262225959 on $0 \le s \le 1$. The remaining innovation is positive by Proposition 11; the complete normalizer upper bound then gives (13). This computation retains the whole inverse penalty; it does not obtain the local mean by subtracting a variance from an averaged innovation.

Apply Lemma 3 to η_s. If $W(a) > 0$, it gives (14). If $W(a) = 0$, monotonicity makes W vanish before a and bounds it below by a positive constant on a final half-line, which supplies a strictly positive integral. Completion of squares proves (12), and its positive mean proves the averaged even block positive definite. The sixth-moment hypothesis implies (3); Theorem 1 supplies the odd block. Parity therefore proves $H_6 > 0$, also for complex coefficient vectors.

For strict variance, the leading moments satisfy $h_{2k}/X^k \to (2k-1)!!8^k$. In scaled coordinates, the regression problem converges to the positive Gaussian moment problem of variance 8. Its monic sixth Hermite polynomial gives

$$b_0/X^3 \to 7680, \qquad b_1/X^2 \to -2880, \qquad b_2/X \to 120.$$

If the variance vanished, positive definiteness of A_s would give $b_s = \bar{b}$ almost everywhere on the final half-line where W is bounded below. Continuity would make b_s constant there, contradicting the last limit. Finally, $H_y \le e^{2ys}$ and the superexponential theta decay imply (11) for every fixed finite y. This completes the theorem.

8 The odd septic and full degree-seven theorem

Assume now

$$\int_0^\infty W(s)(1 + X(s)^7)\, dm_0(s) < \infty, \tag{16}$$

with the same sign, measurability, monotonicity and nonzero conditions as before. Extend $\mathcal{L}_W$ through h_{14}. Put

$$M_7(s) = (h_{2i+2j+2}(s))_{i,j=0}^3, \qquad T_7(a) = \int_a^\infty M_7(s)\, dm_0(s).$$

Let C have columns equal to the coefficient vectors of

$$q_0(z) = z/8, \qquad q_1(z) = (z^3 - 139z)/1024,$$
$$q_2(z) = (z^5 - 461z^3 + 32200z)/262144,$$
$$q_3(z) = (z^7 - 1015z^5 + 242645z^3 - 11576832z)/67108864$$

in the monomial basis (z, z^3, z^5, z^7). Thus $q(z) = C^T(z, z^3, z^5, z^7)^T$ and C is invertible.

Theorem 12. *Every finite $a \geq 0$ satisfies $T_7(a) > 0$. More quantitatively,*

$$C^T \left(\int_a^1 M_7(s)\, dm_0(s) \right) C \succeq \frac{1}{50000} I_4 \qquad (0 \leq a \leq 7/100). \tag{17}$$

For every W satisfying (16), the full moment matrix $H_7 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^7$ is positive definite. This includes every fixed finite physical $y \geq 0$, with the normalized endpoint $y = 0$.

The new ingredient is the odd block on four monomials. The corresponding pointwise three-monomial odd prefix is indefinite near the initial center, so its inverse cannot serve as a positive local metric. The proof uses integrated upper tails instead.

Proposition 13. *For $X \geq 12$ the complete theta matrix satisfies*

$$M_7(s) \succeq \frac{1}{150000} \operatorname{diag}(16X, 16384X^3, 67108864X^5, 1099511627776X^7). \tag{18}$$

Appendix E proves this whole-halfline estimate and (17), with all atoms and derivatives through order fourteen. The compact fixed-center classification [1] has minimum negative-square degree at least nine on $[a_3, 12]$, including its third transition, where

$$a_3 < 414157658489452707553/10^{20}.$$

This implies $M_7 \succeq 0$ there. The exact bridge

$$\pi e^{28/100} > 414157658489452707553/10^{20}$$

is checked by the compact script. Consequently $M_7(s) \succeq 0$ for every $s \geq 7/100$, and it is strictly positive definite on a terminal halfline.

Proof of Theorem 12. For $a \leq 7/100$, add the nonnegative integral over $s \geq 1$ to (17). For $a \geq 7/100$, the entire tail is nonnegative, and (18) makes its quadratic form strictly positive on a terminal interval of positive dm_0 measure for every nonzero vector. Thus $T_7(a) > 0$ for every finite cutoff.

For every fixed k, the paired first-atom argument used in Section 7 gives $h_{2k} = O(X^k)$: its amplitude derivatives remain bounded after division by $(2X-3)^2$, and its opposite-shift exponential has coefficient of degree at most k. Every remaining paired atom has polynomial growth of fixed order and a relative factor at most e^{-3X}; its differentiated all-atom series is uniformly summable for $X \geq \pi$. Hence all entries through h_{14} are absolutely integrable under (16).

For a nonzero real vector u, use the classical layer-cake identity

$$W(s) = \int_0^\infty \mathbf{1}_{\{W(s) > t\}}\, dt.$$

A nonempty superlevel set of a nondecreasing function is a terminal interval, up to an irrelevant endpoint. Its quadratic-form integral equals $u^T T_7(a) u > 0$ at a finite cutoff. Absolute integrability justifies signed Fubini. Since W is not zero almost everywhere, a positive-measure set of levels has such a nonempty tail. Therefore $\int W u^T M_7 u\, dm_0 > 0$. This is the usual directional upper-tail criterion for the positive-semidefinite matrix cone. Theorem 9 supplies the even block under the stronger seventh-moment assumption. Parity proves $H_7 > 0$, also for complex coefficient vectors. Finally $H_y \leq e^{2ys}$ and the theta decay pay (16) for every fixed finite y. □

8.1 A stronger two-column local-residual theorem

Retain the genuinely positive pointwise prefix G_s on $U = (z, z^3)$, and add $V = (z^5, z^7)$. Write

$$M_7 = \begin{pmatrix} G_s & B_s \\ B_s^T & D_s \end{pmatrix}, \quad B_s = \begin{pmatrix} h_6 & h_8 \\ h_8 & h_{10} \end{pmatrix}, \quad D_s = \begin{pmatrix} h_{10} & h_{12} \\ h_{12} & h_{14} \end{pmatrix}, \quad R_s = D_s - B_s^T G_s^{-1} B_s.$$

Proposition 14. *Every finite cutoff satisfies $\int_a^\infty R_s \, dm_0 > 0$. Consequently $\mathbb{E}_W R_s > 0$ for every weight in Theorem 12.*

Proof. Modulo the prefix U, the last two q columns have output transformation

$$S = \begin{pmatrix} 1/262144 & -1015/67108864 \\ 0 & 1/67108864 \end{pmatrix}.$$

The Schur residual in the q basis is exactly $S^T R_s S$: invertible prefix changes and addition of prefix columns cancel in a residual. The direct whole-cell checker in Appendix E encloses, uniformly for $0 \le a \le 7/100$, the raw matrix

$$S^T \left(\int_a^1 R_s \, dm_0 \right) S.$$

Its first diagonal exceeds $3/100000$ and its determinant exceeds $2399/10^{12}$, so it is positive definite. For $s \ge 7/100$, the pointwise PSD classification and (18), together with $G_s > 0$, imply $R_s \succeq 0$ and strict positivity on a terminal halfline. This joins the compact certificate to every finite tail. The growth $B_s^T G_s^{-1} B_s = O(X^7)$ entrywise follows from the scaled Gaussian limit below; it also pays absolute integrability of the residual. The same layer-cake argument completes the claim. □

The integrated odd Schur complement therefore has two positive mechanisms. With $\beta_s = G_s^{-1} B_s$ and $\bar{\beta} = (\mathbb{E}_W G_s)^{-1} \mathbb{E}_W B_s$, completion of squares gives

$$\mathbb{E}_W D_s - (\mathbb{E}_W B_s)^T (\mathbb{E}_W G_s)^{-1} \mathbb{E}_W B_s = \mathbb{E}_W R_s + \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})]. \tag{19}$$

This variable-metric least-squares identity is classical; see [4] and the explicit averaging formulation in [9]. Positivity of the old scalar diagonal alone would not control the new coupling. Proposition 14 pays the entire residual matrix independently.

8.2 Strict variance for any finite higher odd output list

The next statement concerns variance only and does not require a monotone density weight.

Proposition 15. *Fix distinct integers $2 \le k_1 < \cdots < k_m$, take outputs z^{2k_j+1} against the prefix $U = (z, z^3)$, and form B_s and $\beta_s = G_s^{-1} B_s$ from their actual theta moments. Let W be any nonnegative measurable density weight, positive on a set of positive dm_0 measure, with*

$$\int W(s)(1 + X(s)^{2k_m+1}) \, dm_0(s) < \infty.$$

Then the m by m regression variance

$$\mathcal{V}_W = \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})]$$

is strictly positive definite.

Proof. For $\tau = 8X$, the complete paired first-atom expression gives $h_{2k}/\tau^k \to (2k-1)!!$ for every fixed k. Equivalently, after $t = u/\sqrt{8X}$ the normalized paired expression tends locally uniformly on complex compact u sets to $e^{-u^2/2}$. The remaining atoms and their fixed derivatives are exponentially small; Cauchy's formula gives the moment limit. This is a fixed-order asymptotic.

The leading Gaussian prefix is

$$G_0 = \begin{pmatrix} \tau & 3\tau^2 \\ 3\tau^2 & 15\tau^3 \end{pmatrix}.$$

For $c_k = (2k+1)!!$, the cross column is $(c_k\tau^{k+1}, (2k+3)c_k\tau^{k+2})^T$. Inversion gives the actual leading limits

$$\frac{\beta_{1,k}}{\tau^k} \longrightarrow (1-k)c_k, \qquad \frac{\beta_{2,k}}{\tau^{k-1}} \longrightarrow kc_k/3. \tag{20}$$

Indeed congruence by $\operatorname{diag}(\tau^{1/2}, \tau^{3/2})$ makes the prefix converge to $\left(\begin{smallmatrix} 1 & 3 \\ 3 & 15 \end{smallmatrix}\right) > 0$, so inversion is justified. The inverse-energy entry for outputs i, j is $O(X^{k_i+k_j+1})$; the highest displayed moment pays every entry and the square identity. Compact continuity handles bounded centers.

The complete theta sum is real analytic locally around each positive real center, by normal convergence on a sufficiently small complex neighborhood. Positivity of Φ and $\det G_s$ makes β_s real analytic on $(0, \infty)$. If $x^T \mathcal{V}_W x = 0$, the positive metric forces $\beta_s x = \bar{\beta} x$ almost everywhere where $W > 0$. Some finite interior interval contains a positive-measure subset of this support. The real-analytic identity theorem extends this equality to every $s > 0$. Divide its second row by $\tau^{k_m - 1}$ and pass to infinity in (20); the nonzero leading coefficient gives $x_m = 0$. Descending through the distinct powers gives every $x_j = 0$. Thus $\mathcal{V}_W > 0$. □

For a single atomic center the variance is zero; a density-support hypothesis cannot be omitted. The proposition supplies no uniform-in-order eigenvalue bound and does not dominate a negative residual mean at higher degree. For the two outputs z^5, z^7, the Gaussian leading regression is

$$\beta_0(\tau) = \begin{pmatrix} -15\tau^2 & -210\tau^3 \\ 10\tau & 105\tau^2 \end{pmatrix}, \quad \det(\beta_0(\tau_2) - \beta_0(\tau_1)) = 525(\tau_2 - \tau_1)^4.$$

Thus actual full-rank contrasts occur at sufficiently large separated centers. A useful quantitative regional bound still requires paid regional masses and errors. In particular a nonmonotone compactly supported density need not charge such distant regions; its strictness above follows from analyticity instead. Applied to (19), Proposition 15 makes the second term strictly positive as well as the first.

8.3 All associated indices at a fixed finite moment degree

For an integer $n \geq 1$ define the associated kernel in the normalization of [3] by

$$K_n(t) = \int_{\mathbb{R}} \Phi(s+t)\Phi(s-t)s^{2n}\, ds, \qquad J_n(t) = \int_0^\infty s^{2n}\Phi(s+t/2)\Phi(s-t/2)\, ds.$$

Corollary 16. *Every $n \geq 1$ has a positive-definite normalized derivative-moment matrix of degree seven at the origin for K_n. If $O_n(J_n)$ denotes the raw odd-four moment block of J_n, then*

$$C^T O_n(J_n) C \succeq \frac{(7/100)^{2n-2}}{50000} I_4. \tag{21}$$

Proof. Take $W(s) = s^{2n-2}$, with $W \equiv 1$ for $n = 1$. Every such weight is admissible under (16). Evenness gives exactly $J_n(t) = K_n(t/2)/2$. Normalized moments differ by the positive diagonal

congruence $\operatorname{diag}(2^{-j})_{j=0}^{7}$, so Theorem 12 proves the jet claim. For $n = 1$, (21) is the cutoff-zero reserve. For $p = 2n - 2 > 0$, signed Fubini and the positive Stieltjes representation of s^p give

$$O_n(J_n) = \int_0^\infty p a^{p-1} T_7(a)\, da.$$

Insert the congruent reserve for $0 \le a \le 7/100$ and drop the PSD remaining tails. Their early scalar mass is $(7/100)^p$. □

Here n ranges over associated indices, not derivative orders. The theorem uses associated indices $n \ge 1$. The case $n = 0$ would require $W = s^{-2}$; its classical autocorrelation positivity has a different proof. Csordas's Theorem 4.5 gives admissibility of the associated theta kernels, while Theorem 4.6 and Open Problem 4.7 address global positive definiteness and Fourier/Laguerre inequalities. The positive-K_1, negative-K_2 example there uses a different Gaussian-polynomial density.

9 Local positivity for at most eight physical kernel points

Define, only for the physical weights and each fixed finite $y \ge 0$,

$$k_y(t) = \frac{1}{Z_y} \int_0^\infty s^2 H_y(s) \Phi(s + t/2) \Phi(s - t/2)\, ds.$$

For $y > 0$ this is the customary physical kernel divided by its nonzero central value. At $y = 0$ it is $Q_1(t)/Q_1(0)$, where $Q_1(t) = 2 \int s^2 \Phi(s + t/2) \Phi(s - t/2)\, ds$.

Corollary 17. *Fix $1 \le n \le 8$ and distinct real numbers $r_0, \dots, r_{n-1}$. For every fixed finite $y \ge 0$ there exists $\epsilon_0 > 0$ such that*

$$[k_y(\epsilon(r_i - r_j))]_{i,j=0}^{n-1} > 0 \qquad (0 < |\epsilon| < \epsilon_0).$$

With $\mathcal{V}(r) = \prod_{i<j}(r_j - r_i)$ and $H_{n-1} = (\mathcal{L}_y(z^{i+j}))_{i,j=0}^{n-1}$,

$$\det[k_y(\epsilon(r_i - r_j))] = \frac{\epsilon^{n(n-1)} \mathcal{V}(r)^2 \det H_{n-1}}{\prod_{j=0}^{n-1} (j!)^2} + o(\epsilon^{n(n-1)}).$$

The denominator is 1194393600 *for $n = 6$ and* 619173642240000 *for $n = 7$.*

Proof. For every fixed derivative order j and $u \ge 1$, the complete derivative-polynomial series gives $|\Phi^{(j)}(u)| \le C_j e^{(4j+9)u} e^{-\pi e^{4u}}$. For $|t| \le T$ and $s \ge T/2 + 1$, the differentiated paired product is therefore bounded by $C_{j,T} e^{C_j s} e^{-c_T e^{4s}}$, with $c_T = 2\pi e^{-2T} > 0$. The factor $s^2 H_y(s) \le s^2 e^{2ys}$ preserves integrability. On the remaining compact interval smooth theta convergence supplies domination. Consequently k_y is smooth, real and even, $k_y(0) = 1$, and $k_y^{(2j)}(0) = (-1)^j \mathcal{L}_y(z^{2j})$.

Put $m = n - 1 \le 7$. We give the classical Taylor-confluence argument using the leading moment matrix H_m. Let $V_{ia} = r_i^a$ for $0 \le i, a \le m$ and $C_\epsilon = \operatorname{diag}(1, \epsilon^{-1}, \dots, \epsilon^{-m}) V^{-1}$. Taylor expansion through order $2m$ has remainder $o(\epsilon^{2m})$ and bivariate terms

$$\epsilon^{p+q} r_i^p r_j^q \frac{(-1)^q k_y^{(p+q)}(0)}{p! q!}, \qquad p + q \le 2m.$$

Multiplication by V^{-1} maps the evaluation vector of degree $p \le m$ to its coordinate vector. For $p > m$, its coordinate $a \le m$ acquires a factor ϵ^{p-a} tending to zero; p, q cannot both exceed m. The

scaled remainder is $o(1)$ since $a+b \le 2m$. Hence the real congruence of the kernel matrix tends entrywise to

$$\left(\frac{(-1)^b k_y^{(a+b)}(0)}{a!b!} \right)_{a,b=0}^{m} = J H_m J^*, \qquad J = \operatorname{diag}(i^a/a!)_{a=0}^{m}.$$

This is positive definite by Theorem 12; the complex phases express a real matrix because cross-parity moments vanish. Openness of positive definiteness and invertibility of C_ϵ prove the claim. Finally $\det C_\epsilon = \epsilon^{-m(m+1)/2}/\mathcal{V}(r)$, proving the determinant asymptotic. □

This classical consequence concerns the stated local point configuration. It is asserted only for the physical family, whose locally uniform shifted-derivative domination through order fourteen was just proved.

10 Why generic log-concavity is insufficient

Arbitrary nonnegative weights are excluded: the known negative fixed-center square near $X = 13/4$ remains negative under a smooth bump supported in a sufficiently small neighborhood. More fundamentally, even smooth positive even log-concavity of a density does not imply the averaged odd-quintic gate.

Proposition 18. *The density $\varphi(s) = e^{-s^2}(1+\cos(4s)/10)$ is positive, even, Schwartz and uniformly strictly log-concave. At $y = 0$ its actual normalized averaged paired-moment functional has a positive odd two-by-two prefix but gives a negative square to*

$$P(z) = z^5 - 45z^3 + 59z.$$

Proof. For $c = \cos(4s)$ and $\epsilon = 1/10$,

$$(\log \varphi)'' = -2 - 16\epsilon \frac{c+\epsilon}{(1+\epsilon c)^2} \le -2/9.$$

If $c + \epsilon \ge 0$ this is immediate; otherwise the positive correction is at most $16\epsilon/(1-\epsilon) = 16/9$. Let $J(t) = \int_{\mathbb{R}} s^2 \varphi(s+t/2)\varphi(s-t/2)\, ds$ and $F(\xi) = \int \varphi(s) e^{i\xi s}\, ds$. The substitution $a = s+t/2, b = s-t/2$ has Jacobian one and gives

$$\widehat{J}(\xi) = \tfrac{1}{2}(F'(\xi)^2 - F(\xi)F''(\xi)), \qquad F(\xi) = \sqrt{\pi}\left(e^{-\xi^2/4} + \frac{e^{-(\xi-4)^2/4} + e^{-(\xi+4)^2/4}}{20} \right).$$

All operations are justified for Schwartz functions. Up to a common positive factor the signed Fourier density is

$$\begin{aligned} \nu(\xi) = (\tfrac{1}{2} - \tfrac{3}{80}e^{-8})e^{-\xi^2/2} + \tfrac{1}{800}\{e^{-(\xi-4)^2/2} + e^{-(\xi+4)^2/2}\} \\ - \tfrac{3}{20}e^{-2}\{e^{-(\xi-2)^2/2} + e^{-(\xi+2)^2/2}\}. \end{aligned}$$

For $J/J(0)$ the polynomial moment functional equals $\int P^2\nu / \int \nu$. Evenness makes this the same normalized half-line $y = 0$ functional. Variance-one Gaussians with means $0, 2, 4$ give respectively

$$\mathbb{E}P^2 = 11191, \quad 334171, \quad 2544391.$$

Thus the numerator in Gaussian units, with $q = e^{-2}$, is

$$N(q) = \frac{4782591}{400} - \frac{1002513}{10}q - \frac{33573}{80}q^4 < -1600.$$

The positive Taylor series for e^2 through degree 20, with omitted-term ratio at most $2/22$, proves $1353/10000 < q < 677/5000$. Substitution of these rational endpoints proves the strict displayed inequality. The same exact Gaussian moment calculation gives mass $> 2/5$, unnormalized second moment $> 1/3$, and odd prefix determinant $> 5/2$. Division by the positive mass preserves its definiteness and the negative square. These explicit checks are included in the companion arithmetic script. □

This example does not refute the actual theta result or stronger curvature criteria for three-point positivity. The associated-kernel Fourier identities are classical [3]; the example serves to distinguish their hypotheses from generic log-concavity.

A Global lower prefix: finite and unbounded enclosures

We give the analytic bounds behind the lower-prefix certificate. All derivatives in this appendix are physical s derivatives unless polynomial arguments are displayed. Let $b = 3/2$, $\tau_n = n^2 - 1$, $\Phi = \Phi_1(1 + R)$, and $\ell = \log \Phi$. Algebra gives the positive Laplace representation

$$R = \int_3^\infty e^{-vX}\, d\sigma(v), \quad d\sigma = \sum_{n\ge 2} \left(n^4 \delta_{\tau_n} + bn^2 \tau_n e^{b(v-\tau_n)} 1_{v\ge\tau_n}\, dv \right).$$

Define $P_0 = 1$, $P_{j+1}(z) = z(P_j - P_j')$. The physical derivative of e^{-vX} is $(-4)^j P_j(vX)e^{-vX}$. For $j \le 6$, both $P_j(9 + w)$ and $[(z^j - P_j(z))]_{z=9+w}$ have nonnegative coefficients; so do the shifted ordinary derivatives of P_4, P_6. Hence the derivatives R_j alternate in sign and are bounded by the highest Laplace moments.

For every $n \ge 2$ their majorants are

$$M_{n,j}(X) = 4^j X^j e^{-\tau_n X} \left[n^4 \tau_n^j + bn^2 \tau_n \sum_{l=0}^{j} \binom{j}{l} \frac{\tau_n^{j-l} l!}{(X-b)^{l+1}} \right].$$

They decrease on $X \ge 3$ for $j \le 6$. Their complete $n \ge 3$ tails are bounded by

$$\mathcal{T}_j = \frac{C_j 3^{4+2j}}{20^8(1-q_j)}, \quad C_j = 12^j \left(1 + \sum_{l=0}^{j} \binom{j}{l} l!(2/27)^l \right), \quad q_j = (4/3)^{4+2j}/20^7 < 1.$$

The ratio bound retains every atom. In particular

$$\begin{aligned} |R_1| &\le 1104/8000 + (100/99)24 \cdot 3^6/2^{24} < 1/4, \\ 0 < R_2 &\le 44736/8000 + (100/99)288 \cdot 3^8/2^{24} < 6, \\ |R_3| &\le M_{2,3}(3)e^9/8000 + \mathcal{T}_3 = 34736311628179317/147622497952000 < 256, \\ R &\le 28/8000 + 2 \cdot 3^4/[20^8(1 - (4/3)^4/20^7)] < 1/250. \end{aligned}$$

Here $e^3 > 20$ follows from its Taylor polynomial through degree eight. The first two weaker tails use $e^{-24} < 2^{-24}$ and ratio $< 1/100$.

Set $U = 1 + R$. For the leading atom

$$\ell_1'' = -16[X + T_2(b/X)], \quad \ell_1'''' = -256[X + T_4(b/X)], \quad \ell_1^{(6)} = -4096[X + T_6(b/X)],$$

where

$$T_2(r)=\frac{r}{(1-r)^2},\quad T_4(r)=\frac{r(1+4r+r^2)}{(1-r)^4},\quad T_6(r)=\frac{r(1+26r+66r^2+26r^3+r^4)}{(1-r)^6}.$$

The fourth logarithmic correction is

$$(\log U)''''=R_4/U-(4R_1R_3+3R_2^2)/U^2+12R_1^2R_2/U^3-6R_1^4/U^4.$$

It is at most $R_4+9/2$. The qualitative sign $\ell''''<0$ is also proved in Umihara [8]; we include the quantitative independent bounds needed here. The positive decreasing $n=2$ fourth contribution is $C_4(X)e^{-3X}$, where

$$C_j(X)=4^j\left[16P_j(3X)+\frac{18}{X-b}\sum_{l=0}^{j}P_j^{(l)}(3X)\left(\frac{X}{X-b}\right)^l\right]\quad(j=4,6).$$

Indeed $(P_4-P_4')(9+w)=1162+921w+241w^2+26w^3+w^4>0$, so its integrated fourth contribution decreases. Its $n\ge3$ fourth tail is less than

$$\frac{100}{99}\frac{62208\cdot3^{12}}{20^8}=\frac{14348907}{11000000}<2.$$

The exact endpoint coefficients are $C_4(3)=43140096$, $C_4(16/5)=42636828818522112/887410625$, $C_4(7/2)=59421432$ and $C_4(4)=278436163584/3125$. Direct substitution yields

X interval	upper for $C_4(X)e^{-3X}$	lower for $256[X+T_4(b/X)]$
$[3,16/5]$	5393	5430
$[16/5,7/2]$	3337	3801
$[7/2,4]$	1748	2557
$[4,\infty)$	557	1024

The exponential lower bounds are respectively $8000,14400,34000,160000$. They follow from $e^3>20$, $e^{3/5}>9/5$, $e^{3/2}>17/4$. Each final column exceeds the middle column plus $2+9/2$.

For the determinant, put $A=-\ell''$, $d=-\ell''''>0$, $e=\ell^{(6)}$. Direct paired differentiation gives

$$h_2=-\ell''/2,\quad h_4=(\ell''''+6\ell''^2)/8,\quad h_6=-(\ell^{(6)}+30\ell''\ell''''+60\ell''^3)/32,$$

$$64\det G=24A^4-18A^2d-d^2-Ae.\tag{22}$$

The fourth logarithmic error budget is $46595/128$. Expanding $\log(1+R)$ through order six and dropping its negative groups gives

$$(\log U)^{(6)}\le R_6+(15/8)R_4+841095/16.$$

The constants follow respectively from $4(1/4)256+3\cdot6^2+6(1/4)^4$ and $120(1/4)6\cdot256+30\cdot6^3+360(1/4)^46$. For a cell $[L,H]$, bound R_4 below by $C_4(H)e^{-3H}$ and above by $C_4(L)e^{-3L}+\mathcal{T}_4$. Bound R_6 above by $C_6^+(L,H)e^{-3L}+\mathcal{T}_6$, where C_6^+ replaces polynomial arguments by $3H$, denominators by $L-b$, and $X/(X-b)$ by $L/(L-b)$. Positive shifted derivative coefficients justify the enclosure. With these bounds $r_{4,-},r_{4,+},r_{6,+}$ define

$$A_-=16[L+T_2(b/H)]-6,\qquad A_+=16[H+T_2(b/L)]+1/16,$$
$$d_+=256[H+T_4(b/L)]-r_{4,-}/(251/250)+46595/128,$$
$$e_+=-4096[L+T_6(b/H)]+r_{6,+}+(15/8)r_{4,+}+841095/16.$$

On the 96 closed cells $L = 3 + j/32$, $H = L + 1/32$, rational arithmetic proves $A_- > 0$, $d_+ > 0$, $A_-^2 \geq 3d_+/8$ and

$$24A_-^4 - 18A_-^2 d_+ - d_+^2 - (A_+ \text{ if } e_+ \geq 0 \text{ else } A_-)e_+ > 256000000.$$

The minimum lower floor is 264806350, and the minimum guard floor is 3928. The guard justifies the monotonic substitution in A; the other signs justify every remaining substitution. Exponential enclosures here use $\sum_{j=0}^{80} q^j/j!$ and its remainder $(q^{81}/81!)/(1-q/82)$ for $0 \leq q \leq 18$. For $X \geq 6$, the sixth envelope at 6 gives $e < 150000$; also $16X - 6 \leq A \leq 16X + 33$ and $d \leq 256X + 782$. The resulting lower polynomial in $v = X - 6$ has coefficients

$$(1211952476, 958667264, 274580480, 34209792, 1572864),$$

and its monotonicity guard has coefficients $(28923/4, 2784, 256)$. These are positive. Equation (22) proves globally $\det G \geq 4 \cdot 10^6$ and $h_2 \geq 21$. Thus both the finite interval and the remaining unbounded domain are covered.

B Whole-cell baseline and regional certificate

The separate baseline uses 1024 closed cells $[j/1024, (j+1)/1024]$, not point evaluations. Work in the unit-triangular basis

$$U = z, \quad V = z^3 - 139z, \quad R = z^5 - 461z^3 + 32200z.$$

The scalar monic innovation and matrix-weighted residual variance are invariant under these fixed lower-degree basis shifts. Let $b(s) = \pi e^{5s-X}$ and use tildes for moment numerators multiplied by Φ^2/b^2. Then

$$s^2\Phi^2\delta_s = s^2b^2(\widetilde{S} - \widetilde{r}^T\widetilde{G}^{-1}\widetilde{r}). \tag{23}$$

This exact cancellation avoids division by a small density.

To specify the complete derivative enclosures, use $q_0(z) = 2z - 3$, $q_{j+1}(z) = (5 + 4z\partial_z - 4z)q_j(z)$ through $j = 10$. The head derivative is

$$d_j = \sum_{n=1}^{3} n^2 q_j(n^2X)e^{-(n^2-1)X}.$$

For a whole cell with $X \in [X_-, X_+]$, the omitted derivative tail is bounded by the sum over monomials $q_{jl}z^l$ of

$$|q_{jl}|X_+^l 4^{2l+2}e^{-15X_-} \Big/ \left[1 - (5/4)^{2l+2}e^{-9X_-}\right].$$

Every denominator is positive. The paired head moment polynomial is formed before interval evaluation:

$$\widetilde{m}_{2k}^{(n,m)} = \frac{(-1)^k}{4^k}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}[n^2q_j(n^2X)][m^2q_{2k-j}(m^2X)]e^{-(n^2+m^2-2)X},$$

summed for $1 \leq n \leq m \leq 3$ with multiplicity two off the diagonal. With head and tail derivative bounds, the absolute moment error is

$$4^{-k}\sum_{j=0}^{2k}\binom{2k}{j}(|d_j|t_{2k-j} + t_j|d_{2k-j}| + t_jt_{2k-j}).$$

All cross terms are retained. The six fields are the linear combinations

$$\begin{aligned} a &= m_2, \\ b &= m_4 - 139m_2, \\ c &= m_6 - 278m_4 + 19321m_2, \\ r_1 &= m_6 - 461m_4 + 32200m_2, \\ r_2 &= m_8 - 600m_6 + 96279m_4 - 4475800m_2, \\ S &= m_{10} - 922m_8 + 276921m_6 - 29688400m_4 + 1036840000m_2. \end{aligned}$$

All are formed at numerator level before enclosure. Absolute errors are combined using absolute coefficients. On every cell the lower bounds for a and $ac - b^2$ are positive. The inverse penalty is at most the upper bound of $(cr_1^2 - 2br_1r_2 + ar_2^2)/(ac - b^2)$, and also at most $2(cr_1^2 + ar_2^2)/(ac - b^2)$ with absolute upper bounds. Use the smaller of these valid upper bounds, and subtract it from the lower endpoint for S in (23). Integrating the resulting interval times s^2b^2 proves the baseline. The recorded rational lower exceeds 2300747; no adaptive subdivision is needed.

For normalization, $B_0 = 81/40$ and the positive theta series give, for $s \geq 1$,

$$\Phi(s) \leq 2B_0\pi X e^{5s-X}, \qquad H_y(s) \leq e^{2s} \quad (0 \leq y \leq 1).$$

Indeed $|2n^2X - 3| \leq B_0n^2X$ for $X > 120$, and $\sum n^4e^{-(n^2-1)X} < 2$ by the successive ratio $16e^{-3X} < 1/2$. Therefore

$$s^2H_y\Phi^2 \leq 4B_0^2\,4^4s^2e^{20s-6e^{4s}}.$$

Since $e^4 > 40$ and $e^3 > 20$, $20s - 6e^{4s} \leq -220 - 940(s-1)$ and $e^{-220} < 20^{-73}$. The full normalizer tail is at most

$$\frac{4B_0^2\,4^4}{20^{73}}\left(\frac{1}{940} + \frac{2}{940^2} + \frac{2}{940^3}\right). \tag{24}$$

The compact denominator interval for the whole bin $0 \leq y \leq 1/8$ contains Z_0 after adding this tail. Interval division of the baseline by this full denominator proves the 17 billion bound.

For the variance, integrate the five fields of G, r separately on $A = [0, 3/16]$ and $B = [3/16, 1]$, with the common factor $s^2b^2H_y$. Use the eight closed bins $j/8 \leq y \leq (j+1)/8$, $j = 0, \ldots, 7$. Regional interval inverses have positive determinant. Enclose the second-coordinate contrast in (8), square its minimum absolute value, and divide by upper bounds for the inverse sum and full Z_y. The binwise gain floors exceed, respectively,

$$\begin{aligned} &1966012227,\ 1955643969,\ 1946391148,\ 1938242732, \\ &1931186825,\ 1925210688,\ 1920300777,\ 1916442774. \end{aligned}$$

This includes the $1/Z_y$ scaling of raw regional matrices. It certifies (9) independently of a total averaged innovation.

The interval script uses exact rational operations with outward rounding at 80 decimal places; the other finite checks use exact rational polynomial arithmetic. Pi is enclosed by the Machin alternating series; exponentials use a positive Taylor series after range reduction, with an explicit geometric remainder; the hyperbolic quotient uses its positive series. They reconstruct polynomials and tails instead of reading stored numerical enclosures. Explicit exception gates, active also under optimized Python, require raw baseline $> 2 \cdot 10^6$, normalized baseline $> 17 \cdot 10^9$, and every variance bound $> 10^9$. The public scripts are `verify_prefix.py` (lower-order controls and the 96-cell prefix), `verify_tail.py` (odd complete tail), `verify_even.py` (even complete tail), `verify_separated.py`

(baseline and regional variance), and `verify_counterexample.py` (signed Gaussian control). The accompanying README specifies exact commands. Arithmetic checking supports the displayed analytic proof; the compact classification theorem is a cited input. These finite checks support the analytic argument.

C Complete even-sextic tail certificate

Use R_j, Q_k from Section 3, now through $j = 12$ and $k = 6$, and $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is

$$\det(Q_{i+j})_{i,j=0}^{3} = 4638564679680X^6 P_{14}(X),$$

where

$$\begin{aligned} P_{14} = {} & 131072X^{14} - 2293760X^{13} + 20570112X^{12} - 120004608X^{11} \\ & + 493764096X^{10} - 1486503936X^9 + 3312458496X^8 - 5440255488X^7 \\ & + 6483101760X^6 - 5448021120X^5 + 3089651040X^4 - 1112123520X^3 \\ & + 247531950X^2 - 39803400X + 3444525. \end{aligned}$$

After $X = 12 + v$, each of its first three leading-minor numerators has nonnegative coefficients with a positive constant coefficient. The reserve polynomial

$$\det(Q_{i+j})_{i,j=0}^{3} - 2^{50}X^{12}(2X-3)^8$$

also has nonnegative shifted coefficients. These exact coefficient checks prove the head matrix positive definite and its determinant at least $2^{50}X^{12}$ on the whole half-line. They do not infer definiteness from a determinant alone.

Let $D = \operatorname{diag}(1, 32X, 1024X^2, 32768X^3)$, with constant factors $d = (1, 32, 1024, 32768)$. For the preconditioned head, the determinant is at least $1/1024$ and the trace is at most

$$T = \sum_{i=0}^{3} \sum_{l} \frac{|[X^l]Q_{2i}|\, 12^{l-2i-2}}{(2-3/12)^2 d_i^2} < 8.093.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$. Hence its least eigenvalue is at least

$$\frac{27}{1024T^3} = \frac{29057903779751680918020367814555074 56}{58394607561635613985907354403273651574229} > \frac{1}{21000}.$$

Here and below T denotes the displayed exact rational sum, before its decimal upper bound.

To retain enough cancellation at $X = 12$, treat the second atom by its paired polynomials rather than by absolute derivative products. Put

$$\begin{aligned} R_{j,n}(X) &= n^2 R_j(n^2 X), \\ C_k(X) &= 2\sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_{j,1}(X) R_{2k-j,2}(X), \\ B_k(X) &= 16 Q_k(4X). \end{aligned}$$

The additional retained numerator is exactly $C_k(X)e^{-3X} + B_k(X)e^{-6X}$: both ordered cross pairs and the diagonal second-atom pair are included once. Write

$$\varepsilon = (3/8)^{36}, \quad \varepsilon_8 = (3/8)^{96}, \quad c_j = \sum_l |[X^l]R_j| 12^{l-j-1},$$

$$H_j = \sum_l |[X^l]R_j|(1 + 2^{2l+2})12^{l-j-1}, \qquad b_j = 2 \cdot 3^{2j+4}.$$

The elementary inequality $e > 8/3$ gives $e^{-3X} \le \varepsilon$ and $e^{-8X} \le \varepsilon_8$ for $X \ge 12$. The retained two-atom derivative head is bounded by $H_j X^{j+1}$. For the remaining atoms $n \ge 3$, the ratio of successive $n^{2j+4}e^{-(n^2-1)X}$ terms is at most

$$(4/3)^{28}e^{-7X} \le (4/3)^{28}(3/8)^{84} < 1/2 \qquad (j \le 12).$$

Thus their complete derivative tail is at most $b_j c_j X^{j+1} e^{-8X}$.

For $k = 0, \dots, 6$, define the rational numerator-error bound

$$e_k = \frac{1}{(2 - 3/12)^2} \Bigg\{ \varepsilon \sum_l |[X^l]C_k| 12^{l-k-2} + \varepsilon^2 \sum_l |[X^l]B_k| 12^{l-k-2} + 12^k \varepsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \big[H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} + b_j b_{2k-j} c_j c_{2k-j} \varepsilon_8 \big] \Bigg\}.$$

This bounds the omitted moment numerator after division by $X^k(2X-3)^2$. The first line retains the exact second-atom cancellation; the remaining lines retain both head-tail terms and the tail-tail term. Every normalized monomial $X^a e^{-3X}$ arising in the first line decreases from 12, since $a \le k \le 6$; the terms with e^{-6X} also decrease. The other terms use the decreasing functions $X^k e^{-8X}$ and $X^k e^{-16X}$. These facts justify the uniform substitution $X = 12$, rather than merely an endpoint test.

The symmetric preconditioned perturbation consequently has norm at most

$$N = 4 \max_{0 \le i,j \le 3} \frac{e_{i+j}}{d_i d_j} < \frac{1}{180000}.$$

The exact value is approximately 0.00000529965257745. The zeroth moment is included in this bound. As in the earlier tails, before the full denominator is restored its zeroth entry is $(1+R)^2$, not one. For the denominator itself,

$$0 < R \le \frac{2X}{2X-3} 32e^{-3X} < 1/250.$$

The geometric ratio here is bounded by $(3/2)^{28}e^{-5X} < 1/2$; the exponent 28 safely covers the zeroth derivative as well. Both rational endpoint ratio checks use $e > 8/3$. Therefore

$$D^{-1}E_6D^{-1} \succeq \left(\frac{1}{21000} - \frac{1}{180000}\right)\left(\frac{250}{251}\right)^2 I \succ \frac{1}{25000} I,$$

which proves Proposition 11. The standalone checker `verify_sextic_tail.py` reconstructs the polynomials, leading minors, determinant reserve and every rational bound in this appendix.

D Whole-cell sextic sign and mean certificates

The arithmetic uses the interval method and paired-polynomial construction of Appendix B, extended through physical derivative twelve and moments $m_0, \dots, m_{12}$. In particular $q_{j+1} = (5 + 4z\partial_z - 4z)q_j$ is now continued through $j = 12$; the same geometric formula encloses every $n \geq 4$ derivative. Every head-tail and tail-tail product remains in the moment error. The zeroth moment is treated by the same calculation.

For Proposition 10, divide $[157/50, 73/20]$ into 256 equal closed X intervals. The coefficients of the square of the displayed monic polynomial, in ascending even powers, are

$$91846575844,\ -4390156132,\ -48761659,\ 3025286,$$
$$13403,\ -334,\ 1.$$

Combine these with the paired moment polynomials before enclosure. On every whole cell, the sum of its head upper bound and its full absolute tail error is strictly negative. Dividing by the positive complete density square proves the stated fixed-center sign. The script `verify_sextic_initial.py` supplies the exact upper bound and rejects empty partitions.

For the mean, use the fixed scaled basis

$$\begin{aligned} V_0 &= 1, \\ V_1 &= z^2/64 - 1, \\ V_2 &= z^4/4096 - 6z^2/64 + 3, \\ V_3 &= z^6/262144 - 15z^4/4096 + 45z^2/64 - 15. \end{aligned}$$

The residual innovation for V_3 is $\eta_s/262144^2$. Lower-degree shifts do not affect this residual, and the scaling is restored at the end. Work in raw paired numerators as in (23). If their prefix is

$$\widetilde{A} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \qquad \widetilde{c} = (u, v, w)^T,$$

its six adjugate entries are

$$df - e^2, \quad ce - bf, \quad be - cd, \quad af - c^2, \quad bc - ae, \quad ad - b^2.$$

The whole-cell lower endpoints of a, $ad - b^2$ and $\det \widetilde{A}$ are positive. Contract the interval adjugate with (u, v, w) on both sides to obtain a numerator upper bound U. If $D_- > 0$ is the determinant lower bound, then

$$\widetilde{c}^T \widetilde{A}^{-1} \widetilde{c} \leq \max(0, U)/D_-.$$

Subtract this entire inverse penalty from the lower top-square endpoint, multiply by the interval for $s^2\pi^2 e^{10s-2X}$ and the cell width, and sum. Exact outward arithmetic on the 1024 closed intervals $[j/1024, (j+1)/1024]$ gives, after multiplication by 262144^2,

$$\int_0^1 \eta_s \, dm_0 > 262225959 > 250000000.$$

The positive remainder for $s \geq 1$ follows from Proposition 11. An independent compact normalizer enclosure plus the full tail (24) gives

$$\frac{\int_0^1 \eta_s \, dm_0}{Z_0} > 2049676440449 > 2000000000000.$$

The checker `verify_sextic_mean.py`, run with argument `1024`, reconstructs these enclosures with the same outward 80-decimal rational arithmetic, Machin bound for pi and Taylor/geometric bounds for exponentials. Explicit exception gates retain all prefix signs and require the conservative raw and normalized thresholds in (13). All three new scripts may be run normally and under optimized Python. The paired recurrences, infinite tails, sign transfer and inverse-energy arguments above are the analytic justification for their finite checks.

E Complete odd-septic and all-cutoff certificates

E.1 The entire local tail beyond twelve

Cancel the e^{5s} factors in the opposite paired shifts. Set

$$R_0(X) = 2X - 3, \quad R_{j+1} = 2X(R_j' - R_j), \quad Q_k = \sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_j R_{2k-j} \quad (0 \le k \le 7).$$

The first-atom matrix has entries $Q_{i+j+1}/(2X-3)^2$, with $\deg Q_k \le k+2$. Exact arithmetic reconstructs

$$\det(Q_{i+j+1}) = 1994953897436774400 X^{10} P_{14}(X),$$

where the ascending coefficients of P_{14} are

$$442047375, -3150592200, 11194876350, -25569432000,$$
$$39943380960, -44288622720, 35956699200, -21886364160,$$
$$10134653184, -3589152768, 967872000, -195207168,$$
$$28237824, -2686976, 131072.$$

After $X = 12+v$, the first three leading-minor numerators have nonnegative coefficients and positive constants. The polynomial

$$\det(Q_{i+j+1}) - 2^{68} X^{16} (2X-3)^8$$

also has nonnegative shifted coefficients. Sylvester's criterion and this reserve give head PD and normalized determinant at least $2^{68}X^{16}$.

Let $d = (4, 128, 8192, 1048576)$ and $D = \operatorname{diag}(d_i X^{i+1/2})_{i=0}^3$. The preconditioned head determinant is at least $1/65536$, and its trace is bounded by

$$T = \sum_{i=0}^{3} \frac{\sum_l |[X^l] Q_{2i+1}| 12^{l-2i-3}}{(2-3/12)^2 d_i^2} = \frac{46528533188040845}{14728370570919936}.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$, so the least eigenvalue exceeds $27/(65536T^3) > 1/80000$.

Retain the second atom in paired form. Put $R_{j,n} = n^2 R_j(n^2 X)$ and

$$C_k = 2\sum_j (-1)^{k+j} \binom{2k}{j} R_{j,1} R_{2k-j,2}, \qquad B_k = 16 Q_k(4X).$$

The additional numerator is $C_k e^{-3X} + B_k e^{-6X}$. Its normalized degrees are at most k, so every monomial times the relevant exponential decreases for $X \ge 12$. With $\epsilon = (3/8)^{36}$ and $\epsilon_8 = (3/8)^{96}$, the exact inequality $e > 8/3$ supplies the endpoint majorant

$$A_k = \epsilon \sum_l |[X^l] C_k| 12^{l-k-2} + \epsilon^2 \sum_l |[X^l] B_k| 12^{l-k-2}.$$

For $0 \le j \le 14$, define

$$c_j = \sum_l |[X^l]R_j| 12^{l-j-1},$$
$$H_j = \sum_l |[X^l]R_j|(1+2^{2l+2})12^{l-j-1}, \qquad b_j = 2\,3^{2j+4}.$$

The retained derivative head is bounded by $H_j X^{j+1}$. The entire $n \ge 3$ derivative tail is bounded by $b_j c_j X^{j+1} e^{-8X}$: its successive-atom ratio is at most $(4/3)^{32}e^{-7X} < (4/3)^{32}(3/8)^{84} < 1/2$. This pays every atom through derivative fourteen. Both cross terms and the tail-tail product remain in

$$F_k = 12^k \epsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \left(H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} + b_j b_{2k-j} c_j c_{2k-j} \epsilon_8\right).$$

Set $e_k = (A_k + F_k)/(2-3/12)^2$. A symmetric four-by-four entry bound gives the operator error

$$N = 4 \max_{0 \le i,j \le 3} \frac{e_{i+j+1}}{d_i d_j} < \frac{1}{200000}.$$

All comparisons are exact rational gates, not rounded decimal tests. Finally, writing $\Phi = \Phi_1(1+R)$, the positive zeroth tail has

$$R \le \frac{2X}{2X-3}\, 32e^{-3X} < 1/250.$$

The geometric test $(3/2)^{32}(3/8)^{60} < 1/2$ pays that zeroth sum. Restoring $(1+R)^{-2}$ after the perturbation yields

$$\frac{1/80000 - 1/200000}{(1+1/250)^2} > 1/150000,$$

which proves (18). The standalone checker `verify_septic_tail.py` reconstructs these polynomials, shifted minors and rational budgets.

E.2 Every real short cutoff, rather than sampled cutoffs

For the compact integrators use the physical derivative recurrence $q_{j+1} = (5 + 4X\partial_X - 4X)q_j$ through $j = 14$, beginning at $q_0 = 2X - 3$, as in Appendix B. Retain atoms $1, 2, 3$, combine ordered pairs and the fixed $q_i(z)$ basis products exactly before evaluating intervals, and enclose every $n \ge 4$ derivative monomial using ratio

$$(5/4)^{2l+2}e^{-9X} < 1$$

on its entire cell. The exponent reaches 32 at derivative fourteen. Head-tail and tail-tail errors are retained for every moment. Removing the common factor πe^{5s-X} from each derivative leaves the raw integration weight $s^2\pi^2 e^{10s-2X}$; the normalized Φ^2 cancels, so no small normalizer is inverted.

Write $F(s) = s^2\Phi(s)^2 C^T M_7(s) C$. On each of the 1024 closed cells $[j/1024, (j+1)/1024]$, whole-cell interval arithmetic encloses every entry f_{ij} by $[f_-, f_+]$. If that cell overlaps $[0, 7/100]$ in width w, its contribution to any removed initial segment $\int_0^a f_{ij}$, $0 \le a \le 7/100$, is contained in

$$[w\min(0, f_-),\ w\max(0, f_+)].$$

Indeed the selected portion has any length from zero to w; the signed enclosure covers all choices. Summing gives simultaneous loss intervals for every real cutoff. Subtract them from the independently enclosed entries of $\int_0^1 F$. Discarding correlation weakens the bound safely. For the resulting

symmetric interval matrix, each Gershgorin row reserve exceeds $1/50000$; the smallest exceeds $33/10^6$. This proves (17). The checker `verify_septic_compact.py 1024` also verifies the exact bridge past a_3.

For the second certificate use the same basis and raw fields, but split the cell containing $7/100$ and form the local odd-two residual. If its prefix is $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, cross columns are $(u,w)^T, (z,t)^T$, and output entries are d_{11}, d_{12}, d_{22}, the three residual entries are

$$d_{11} - \frac{cu^2 - 2buw + aw^2}{ac - b^2},$$
$$d_{12} - \frac{cuz - b(ut + wz) + awt}{ac - b^2},$$
$$d_{22} - \frac{cz^2 - 2bzt + at^2}{ac - b^2}.$$

Both a and $ac - b^2$ are checked positive on every entire cell, with subdivision if needed. Apply the same signed-loss construction to the three weighted residual entries. Exact lower bounds for its first diagonal and determinant exceed $3/100000$ and $2399/10^{12}$ respectively. This proves the finite input of Proposition 14; `verify_septic_residual.py 1024` reconstructs it. These two representations share the outward rational interval engine, not independent arithmetic implementations. The complete tail, compact-classification dependency, layer-cake step and variance arguments supply the analytic lifting beyond the finite checks.

F Scope

The monotone-weight condition is sufficient. Degrees five, six and seven retain their respective fifth-, sixth- and seventh-moment hypotheses. Every finite physical parameter is included, and local fixed-shape positivity extends to eight physical kernel points. The finite-output variance proposition permits more general density weights but does not extend the mean-residual theorem to arbitrary nonmonotone weights. The associated-kernel corollary ranges over every index at one fixed finite derivative degree; it proves no global kernel or Fourier positivity. The layer-cake criterion, Schur identity, Hermite regression, analytic rigidity, Taylor confluence and parallel sum are classical. The contribution consists of the complete-theta matrix estimates, all-cutoff certificates and resulting weighted extensions.

References

[1] S. Schubr, *Minimum-Degree Obstructions in the Riemann Theta Product: Classification, transition bounds and robust Gram certificates*, edition 2, October 18, 2026, `https://luminaryoddities.com/researcharchive/minimum-degree-obstructions-riemann-theta-product/v/2`.

[2] S. Schubr, *Averaged Quintic Gate by Paired Kernel and Endpoint Conversion*, unpublished manuscript, 2026.

[3] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ-function*, Comput. Methods Funct. Theory **15** (2015), 373–391, arXiv:1309.0055.

[4] J. Gallier, *The Schur complement and symmetric positive semidefinite (and definite) matrices*, author notes, `https://www.cis.upenn.edu/~jean/schur-comp.pdf`.

[5] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594, doi:10.1016/0022-247X(69)90200-5.

[6] Z. Tarcsay, *On the parallel sum of positive operators, forms, and functionals*, arXiv:1501.01922, 2015.

[7] L. Smith, *Monotone comparative statics under uncertainty: interval dominance order*, author lecture notes, Integral Single Crossing Corollary, `https://www.lonessmith.com/wp-content/uploads/2025/03/1-Monotone-comparative-statics.pdf`.

[8] H. Umihara, *A Fourth-Derivative Inequality for the Riemann Xi Kernel*, preprint, August 2026, doi:10.5281/zenodo.21902312.

[9] GettysburgResearch, *Reviewer B—independent first-pass scientific review*, September 5, 2026, Section 6, repository report (variable-metric Schur averaging identity), `https://github.com/GettysburgResearch/riemann/blob/main/reviews/B/REPORT.md`.