Monotone center weights and the averaged odd-quintic theta gate

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Abstract

We prove positivity of the full moment matrix through degree five for the normalized Riemann theta product averaged against every nonzero nonnegative nondecreasing center weight with a finite fifth exponential moment, and through degree six under the corresponding sixth-moment hypothesis. This includes the physical weights $\sinh(2ys)/(2ys)$ for every finite $y \geq 0$. The proof combines the previously published compact fixed-center classification with complete-theta positive matrix tails, separately certified positive means of the odd-quintic and even-sextic local Schur innovations, and a classical single-crossing argument. Both matrix-weighted regression variances are strictly positive; the quintic variance also has an independent two-region lower bound on $0 \leq y \leq 1$. For physical kernels the full degree-six result implies local positivity on fixed configurations of at most seven points. An exact initial even-sextic obstruction pays a sign interval not settled by the minimum-degree-five classification. A smooth, positive, even, uniformly log-concave density with a negative averaged quintic square shows why generic log-concavity cannot replace the theta-specific argument. All new finite arithmetic is reproducible by outward rational enclosures.

1 Definitions and the weighted theorem

The complete theta density is

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

Jacobi symmetry makes it smooth, positive and even on $\mathbb{R}$; for $s \geq 0$ all summands are positive. Put $X = \pi e^{4s}$ and

$$p_s(t) = \frac{\Phi(s + t/2)\Phi(s - t/2)}{\Phi(s)^2}, \qquad h_{2j}(s) = (-1)^j p_s^{(2j)}(0), \qquad h_{2j+1} = 0.$$

The polynomial variable z is distinct from X. In the raw odd basis set

$$G_s = \begin{pmatrix} h_2 & h_4 \\ h_4 & h_6 \end{pmatrix}, \quad r_s = \begin{pmatrix} h_6 \\ h_8 \end{pmatrix}, \quad S_s = h_{10}, \quad \beta_s = G_s^{-1} r_s, \quad \delta_s = S_s - r_s^T G_s^{-1} r_s. \tag{2}$$

Global invertibility is proved below. Write $dm_0 = s^2 \Phi(s)^2 \, ds$. An admissible weight is a finite-valued measurable nonnegative nondecreasing function W on $[0, \infty)$, not zero almost everywhere, such that

$$\int_0^\infty W(s)(1 + X(s)^5) \, dm_0(s) < \infty. \tag{3}$$

Let $Z_W = \int W \, dm_0$, $d\mu_W = W \, dm_0 / Z_W$, and use $\mathbb{E}_W$ for expectation. Define $\mathcal{L}_W(z^{2j}) = \mathbb{E}_W h_{2j}$ through $j = 5$, with odd moments zero.

Theorem 1. *For every admissible W, the matrix of $\mathcal{L}_W$ on $\{z, z^3, z^5\}$ is positive definite. More precisely, with*

$$\bar{G} = \mathbb{E}_W G_s, \quad \bar{r} = \mathbb{E}_W r_s, \quad \bar{\beta} = \bar{G}^{-1}\bar{r},$$

one has

$$\begin{aligned} \Delta_W &:= \mathbb{E}_W S_s - \bar{r}^T \bar{G}^{-1} \bar{r} = \mathbb{E}_W \delta_s + V_W, \\ V_W &:= \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})] > 0, \qquad \mathbb{E}_W \delta_s > 0. \end{aligned} \tag{4}$$

Let $a = \frac{1}{4}\log(a_1/\pi)$, where

$$3.64888758188562 < a_1 < 3.64888758188563$$

is the first fixed-center quintic transition of [1]. If $W(a) > 0$, then

$$\mathbb{E}_W \delta_s > \frac{2\,000\,000\, W(a)}{Z_W}. \tag{5}$$

Corollary 2. *For $H_y(s) = \sinh(2ys)/(2ys)$, continuously defined as 1 when $ys = 0$, Theorem 1 holds for every finite $y \geq 0$. Moreover*

$$\mathbb{E}_0 \delta_s > 17\,000\,000\,000, \qquad V_y > 1\,000\,000\,000 \quad (0 \leq y \leq 1).$$

Here $Z_y = \int H_y \, dm_0 > 0$ defines normalization also at $y = 0$.

For $y > 0$ the customary unnormalized physical kernel has central value $4yZ_y$; its value vanishes at $y = 0$ and is never used as a denominator there. The normalized law has the displayed continuous endpoint.

The earlier bounded-parameter odd-quintic certificate [2] gave a stronger numerical estimate for Δ_y on $[0, 1]$. The advance here is the full parameter half-line, the monotone-weight cone, and the separation of two strictly positive mechanisms.

The two moment hypotheses will remain distinct. The fifth-moment assumption (3) suffices for the full degree-five conclusion in Corollary 8. The degree-six theorem in Section 7 assumes $\int W(1 + X^6)\, dm_0 < \infty$ and adds the even polynomial z^6. Its new local sign, complete matrix tail and positive baseline are proved separately; positivity of one finite moment level is not extrapolated to the next.

2 Classical algebra and the new theta inputs

The Schur identity (4) is completion of squares. Indeed, integrate

$$S_s - 2b^T r_s + b^T G_s b = \delta_s + (b - \beta_s)^T G_s (b - \beta_s)$$

and minimize over b. This is classical matrix-fractional geometry [4]; its positive variance alone does not determine the sign of the mean local innovation. For example, two equally weighted scalar prefixes $G_1 = G_2 = 1$, columns $r_1 = 2, r_2 = -2$, and $S_1 = S_2 = 3$ have local innovations -1 but averaged innovation 3.

The following elementary single-crossing statement is also classical; stronger integral versions occur in [7].

Lemma 3. *Suppose $f \le 0$ on $[0,a]$, $f \ge 0$ on $[a,\infty)$ and $\int f\,dm > 0$. For a nonnegative nondecreasing W with the necessary absolute integrability,*

$$\int W f\,dm \ge W(a)\int f\,dm.$$

If $W(a) = 0$, W is eventually bounded below by a positive constant, and f is strictly positive on a final half-line of positive measure, then $\int W f\,dm > 0$ as well.

Proof. The function $(W(s)-W(a))f(s)$ is nonnegative on both sides of a. If $W(a) = 0$, monotonicity makes W vanish before a, and its eventual positivity makes a strictly positive final contribution. □

We use one authenticated published result rather than reprove the entire compact classification.

Proposition 4 (Compact fixed-center premise from [1])**.** *For the exact density* (1) *and the exact signed derivatives used here, the full fixed-center moment matrix through degree four is positive definite on $3 \le X \le 12$. The minimum negative-square degree is five before a_1, at least six after a_1 through* 12*, and the degree-five matrix is positive semidefinite at a_1.*

This is the relevant portion of the published classification theorem, including its endpoint statement. Because parity separates odd and even blocks and $G_s > 0$, it implies

$$\delta_s < 0 \quad (\pi \le X < a_1), \quad \delta_s = 0 \quad (X = a_1), \quad \delta_s \ge 0 \quad (a_1 < X \le 12). \tag{6}$$

The new deductions that complete the argument are a global lower prefix, a positive full-theta tail beyond 12, and the independently integrated positive baseline. We provide their proofs and finite-arithmetic specifications below. Associated theta kernels and generalized Laguerre inequalities have a broader classical literature [3].

3 The complete-theta positive tail

Proposition 5. *For $X \ge 12$, the full odd matrix $M_o = (h_{2i+2j+2})_{i,j=0}^{2}$ satisfies*

$$M_o \succeq \frac{10}{63001}\,\mathrm{diag}(16X, 16384X^3, 67108864X^5). \tag{7}$$

Consequently $\delta_s \ge (671088640/63001)X^5 > 0$ there.

Proof. The common e^{5s} factor cancels in the paired derivative. Put

$$R_0(z) = 2z - 3, \quad R_{j+1}(z) = 2z(R_j'(z) - R_j(z)), \quad Q_k(X) = \sum_{j=0}^{2k}(-1)^{k+j}\binom{2k}{j}R_j(X)R_{2k-j}(X).$$

For the first atom, $h_{2k} = Q_k/(2X-3)^2$, and exact cancellation gives $\deg Q_k \le k+2$ for $k \le 5$. Its odd matrix determinant is

$$\frac{6039797760X^6P_9(X)}{(2X-3)^6},$$

where

$$\begin{aligned}P_9(X) = {} & 1024X^9 - 12544X^8 + 81216X^7 - 331200X^6 + 902736X^5 \\ & - 1668960X^4 + 2026620X^3 - 1519020X^2 + 637875X - 127575.\end{aligned}$$

The first two leading-minor numerator polynomials, and $P_9 - 256X^9$, have strictly positive coefficients after $X = 12 + v$. Thus the head matrix is positive definite with determinant at least $24159191040X^9$. For $d = (4, 128, 8192)$ and $D = \mathrm{diag}(d_i X^{i+1/2})_{i=0}^{2}$, its congruence by D^{-1} has determinant at least $45/32768$. Its trace is at most

$$\sum_{i=0}^{2} \sum_{j} \frac{|[X^j]Q_{2i+1}|12^{j-(2i+1)-2}}{(2-3/12)^2 d_i^2} = \frac{256401048845}{99883155456} < \frac{13}{5}.$$

Hence its least eigenvalue exceeds $1/5000$ by $\lambda_{\min} \geq \det / \operatorname{tr}^2$.

We now bound every omitted atom. If $r_{jl} = [z^l]R_j$, put

$$C_j = \sum_{l} |r_{jl}|12^{l-j-1}, \qquad B_j = 2^{2j+5}.$$

For $X \geq 12$, $n^2|R_j(n^2X)| \leq C_j X^{j+1} n^{2j+4}$. For $n \geq 2$ the successive majorant ratio is at most $(3/2)^{24}e^{-60} < 1/2$ for $j \leq 10$. Therefore the derivative tail after removing the common head exponential is at most $B_j C_j X^{j+1} e^{-3X}$. Both cross terms and the tail-tail term in the paired product are retained. Set $\varepsilon = 20^{-12} > e^{-36}$. The error in the kth moment after division by X^k, before the full theta denominator correction, is bounded by

$$E_k = \frac{12^k \varepsilon}{(2-3/12)^2} \sum_{j=0}^{2k} \binom{2k}{j} C_j C_{2k-j}(2B_{2k-j} + B_j B_{2k-j}\varepsilon).$$

Here $X^k e^{-3X}$ and $(2-3/X)^{-2}$ decrease for $X \geq 12$, $k \leq 5$. Symmetry in j accounts for both cross terms. The symmetric matrix perturbation norm is at most

$$3 \max_{i,j} \frac{E_{i+j+1}}{d_i d_j} = \frac{5312409501805455073001503693}{14863564800000000000000000000000000} < \frac{1}{25000}.$$

Writing $\Phi = \Phi_1(1+R)$, the denominator correction is exactly $(1+R)^2$ and $0 < R < 1/250$ (Appendix A). Thus

$$(1+R)^2 D^{-1} M_o D^{-1} \succeq (1/5000 - 1/25000)I = I/6250.$$

Since $(1+R)^2 < (251/250)^2$, this proves (7). Minimizing the quadratic form with its monic fifth-degree coefficient fixed to one proves the Schur bound. □

Combining (6) with Proposition 5 gives one global negative-to-nonnegative crossing at $s = a$. Later higher-degree transitions in the compact classification do not change this fixed quintic conclusion.

4 Positive baseline and independent variance reserve

Proposition 6. *For the full theta density,*

$$\int_0^1 \delta_s \, dm_0 > 2\,000\,000, \qquad I_0 := \int_0^\infty \delta_s \, dm_0 > 2\,000\,000, \qquad I_0/Z_0 > 17\,000\,000\,000.$$

Proof. The first assertion is a finite interval certificate with the full inverse penalty, described in Appendix B. It gives the stronger raw lower bound 2300747. The second follows by dropping the strictly positive tail $s \geq 1$, since $\pi e^4 > 12$. The third uses the independently enclosed full normalizer. This argument does not infer a positive local mean from a known positive averaged Schur complement. □

For two disjoint regions A, B, let $Q_A = \mathbb{E}_W[1_A G_s]$ and $c_A = Q_A^{-1}\mathbb{E}_W[1_A r_s]$, and similarly for B. Completion of squares within the regions and minimization over the global coefficient give the classical parallel-sum lower bound [5, 6]

$$V_W \geq (c_A - c_B)^T (Q_A^{-1} + Q_B^{-1})^{-1}(c_A - c_B) \geq \frac{[u^T(c_A - c_B)]^2}{u^T(Q_A^{-1} + Q_B^{-1})u}. \tag{8}$$

The final inequality is matrix Cauchy–Schwarz, valid without commutativity. For the numerical estimate, use the fixed shifted basis of Appendix B, which preserves V_y; in this paragraph c_A, c_B are the corresponding coefficient vectors. For $A = [0, 3/16]$, $B = [3/16, 1]$, $u = (0, 1)^T$, the certificate encloses the matrices and columns directly over eight closed y bins and proves

$$V_y > 1916442774 > 10^9 \qquad (0 \leq y \leq 1). \tag{9}$$

The complement has nonnegative variance and is discarded. This is a physical-region computation, not the difference of two separately computed total innovations.

5 Proof of the global weighted theorem

The prefix proof in Appendix A gives $G_s > 0$, $h_2 \geq 21$, and $\det G_s \geq 4 \cdot 10^6$ for every s. The paired polynomial formulas in Section 3 give

$$\frac{h_{2j}(s)}{X^j} \longrightarrow (2j-1)!!\, 8^j \quad (1 \leq j \leq 5). \tag{10}$$

Indeed the head numerator has degree $j + 2$, and the full higher-atom correction after division by X^j is $O(X^j e^{-3X})$. Put $v = 8X$. Then $\det G_s / v^4 \to 6$ and the explicit inverse gives

$$\frac{\beta_1}{v^2} \to \frac{225 - 315}{6} = -15, \qquad \frac{\beta_2}{v} \to \frac{105 - 45}{6} = 10.$$

Thus $\beta_1/X^2 \to -960$ and $\beta_2/X \to 80$. In particular $r_s^T G_s^{-1} r_s = O(X^5)$. Smoothness on compact intervals and (3) make all the expectations in (4) absolutely convergent. Positive averaging makes $\bar{G} > 0$.

The global single crossing and Proposition 6 allow Lemma 3 with $f = \delta_s$, proving $\mathbb{E}_W \delta_s > 0$ and (5). If $V_W = 0$, positive definiteness forces $\beta_s = \bar{\beta}$ almost everywhere for μ_W. Every admissible nonzero nondecreasing weight is bounded below by a positive constant on a half-line. Continuity then forces β_s constant on that entire tail, contradicting $\beta_2/X \to 80$. Hence $V_W > 0$. Identity (4) and the Schur criterion prove Theorem 1, also for complex polynomial coefficient vectors.

Finally H_y is nondecreasing because $z\cosh z - \sinh z$ vanishes at zero and has derivative $z \sinh z \geq 0$. Its growth is at most e^{2ys}; the theta density has superexponential decay. Thus (3) holds for every fixed finite y. Propositions 6 and (9) finish Corollary 2.

6 The even block and full degree-five consequence

Proposition 7. *The complete even matrix $M_e = (h_{2i+2j})_{i,j=0}^{2}$ is positive definite for every center. On $X \geq 12$,*

$$M_e \succeq \frac{1}{4000}\operatorname{diag}(1, 1024X^2, 1048576X^4).$$

Proof. Use the recurrence of Section 3, adding $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is $393216X^3\widehat{P}_9(X)$, where

$$\begin{aligned}\widehat{P}_9 = {} & 2048X^9 - 22016X^8 + 120960X^7 - 407232X^6 + 875808X^5 \\ & - 1201680X^4 + 986904X^3 - 424980X^2 + 80190X - 8505.\end{aligned}$$

The first two leading-minor numerators have positive coefficients after $X = 12 + v$, as does

$$393216X^3\widehat{P}_9 - 3145728X^6(2X-3)^6.$$

With $D_e = \operatorname{diag}(1, 32X, 1024X^2)$, the head congruence has determinant at least $3/1024$ and trace at most $1420942925/520224768$. Its least eigenvalue is therefore at least

$$\frac{792872488009728}{2019078796107555625} > 1/3000.$$

The complete head-tail calculation of Section 3, now for $k = 0, \ldots, 4$ with $d = (1, 32, 1024)$, gives perturbation norm at most

$$\frac{5415087591445761969320387}{619315200000000000000000000000000} < 1/1000000.$$

The zeroth entry is included: before dividing by $(1+R)^2$, it changes from 1 to $(1+R)^2$. Thus the tail bound follows from $(1/3000 - 1/1000000)(250/251)^2 > 1/4000$. For $\pi \leq X \leq 12$, the even block is a principal submatrix of the positive M_4 in Proposition 4. Even reflection covers negative centers. □

Corollary 8. *For every admissible W and every nonzero complex polynomial P of degree at most five, $\mathcal{L}_W(|P|^2) > 0$. In particular this holds for every physical weight H_y, $0 \leq y < \infty$.*

Proof. Positive averaging preserves the pointwise strictly positive even block. Odd moments vanish, so a permutation of the full degree-five matrix is the direct sum of that even block and the odd block in Theorem 1. □

The even block on $\{1, z^2, z^4\}$ alone permits arbitrary nonnegative center weights with finite required moments; this does not extend the odd or even-sextic conclusions to arbitrary weights.

7 The even sextic and full degree-six theorem

Suppose, in addition to the stated measurability, sign, monotonicity and nonzero conditions on W, that

$$\int_0^\infty W(s)(1 + X(s)^6)\, dm_0(s) < \infty. \tag{11}$$

Extend $\mathcal{L}_W$ by $\mathcal{L}_W(z^{12}) = \mathbb{E}_W h_{12}$, retaining zero odd moments. Use a distinct notation for the even prefix and its next column:

$$A_s = (h_{2i+2j})_{i,j=0}^{2}, \qquad c_s = (h_6, h_8, h_{10})^T, \qquad b_s = A_s^{-1}c_s, \qquad \eta_s = h_{12} - c_s^T A_s^{-1} c_s.$$

Proposition 7 proves $A_s > 0$ at every center.

Theorem 9. *For every W satisfying* (11)*, the full moment matrix $H_6 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^{6}$ is positive definite. With $\bar{b} = (\mathbb{E}_W A_s)^{-1}\mathbb{E}_W c_s$, its even Schur complement satisfies*

$$\Delta_{6,W} = \mathbb{E}_W \eta_s + \mathbb{E}_W[(b_s - \bar{b})^T A_s (b_s - \bar{b})], \tag{12}$$

and both terms on the right are strictly positive. In particular the theorem holds for every fixed finite physical parameter $y \geq 0$, including the normalized endpoint $y = 0$. Moreover

$$\int_0^\infty \eta_s \, dm_0 > 250000000, \qquad \mathbb{E}_0 \eta_s > 2000000000000. \tag{13}$$

Let $a = \frac{1}{4}\log(a_2/\pi)$, where

$$4.11304745500471102656 < a_2 < 4.11304745500471102658.$$

If $W(a) > 0$, then

$$\mathbb{E}_W \eta_s > \frac{250000000\, W(a)}{Z_W}. \tag{14}$$

7.1 The local sign and the initial obstruction

The compact classification in [1], with the normalization of Proposition 4, has minimum negative-square degree six on (a_1, a_2), at least seven on $(a_2, 12]$, and a positive-semidefinite degree-six matrix at a_2. The degree-six crossing has zero even Schur complement. Before a_1, minimum negative degree five alone does not settle the sign of the even sextic. We supply that missing interval directly.

Proposition 10. *For the actual fixed-center functional, the monic polynomial*

$$P(z) = z^6 - 167z^4 - 7243z^2 + 303062$$

satisfies $\mathcal{L}_X(P^2) < 0$ for every $157/50 \leq X \leq 73/20$.

Proof. The exact whole-cell construction in Appendix D encloses this contraction on 256 closed cells, retaining all theta atoms, and gives a strictly negative upper endpoint on each cell. No point-sample or first-atom approximation is used. □

Since $\pi > 157/50$ and $a_1 < 73/20$, Schur minimization with $A_s > 0$ gives $\eta_s \leq \mathcal{L}_X(P^2) < 0$ on $[\pi, a_1]$. On (a_1, a_2), every square of degree at most five is nonnegative by the compact classification. Thus its negative degree-six square must have a negative even component: the odd component has degree at most five, and cross parity vanishes. The positive even prefix then implies $\eta_s < 0$. At a_2 one has $\eta_s = 0$, and after a_2 through 12 the degree-six matrix is positive semidefinite, so $\eta_s \geq 0$.

Proposition 11. *For every $X \geq 12$, the complete even matrix $E_6 = (h_{2i+2j})_{i,j=0}^{3}$ satisfies*

$$E_6 \succeq \frac{1}{25000}\,\mathrm{diag}(1, 1024X^2, 1048576X^4, 1073741824X^6). \tag{15}$$

In particular $\eta_s \geq (1073741824/25000)X^6 > 0$ there.

Proof. Appendix C proves the full half-line bound by a first-atom determinant reserve, exact second-atom paired cancellation and a geometric enclosure of every remaining atom. Minimizing (15) with the coefficient of z^6 equal to one gives the innovation bound. □

Consequently η_s has the negative-to-nonnegative crossing required by Lemma 3, at $s = a$, and is strictly positive on a final half-line. The fixed-center obstruction of Proposition 10 is not a counterexample to the averaged theorem.

7.2 Integrability, baseline and proof of Theorem 9

For each fixed $k \le 6$, paired differentiation gives $h_{2k} = O(X^k)$ as $X \to \infty$. One direct justification is to factor the leading atom's paired product as

$$e^{-2X(\cosh(2t)-1)} \frac{(2Xe^{2t}-3)(2Xe^{-2t}-3)}{(2X-3)^2}.$$

Its $2k$th coefficient is a polynomial of degree at most k in X, times bounded amplitude derivatives. The complete remaining paired atoms are $O(X^{2k}e^{-3X})$ after head-square normalization; their differentiated series is uniformly summable for $X \ge \pi$. Thus Proposition 7 gives

$$c_s^T A_s^{-1} c_s \le 4000 \left(h_6^2 + \frac{h_8^2}{1024X^2} + \frac{h_{10}^2}{1048576X^4} \right) = O(X^6).$$

Compact continuity and positive definiteness give a bounded inverse on every bounded center interval. Therefore all terms of (12) are absolutely integrable under (11).

The direct local-innovation integration in Appendix D gives a raw lower bound greater than 262225959 on $0 \le s \le 1$. The remaining innovation is positive by Proposition 11; the complete normalizer upper bound then gives (13). This computation retains the whole inverse penalty; it does not obtain the local mean by subtracting a variance from an averaged innovation.

Apply Lemma 3 to η_s. If $W(a) > 0$, it gives (14). If $W(a) = 0$, monotonicity makes W vanish before a and bounds it below by a positive constant on a final half-line, which supplies a strictly positive integral. Completion of squares proves (12), and its positive mean proves the averaged even block positive definite. The sixth-moment hypothesis implies (3); Theorem 1 supplies the odd block. Parity therefore proves $H_6 > 0$, also for complex coefficient vectors.

For strict variance, the leading moments satisfy $h_{2k}/X^k \to (2k-1)!!8^k$. In scaled coordinates, the regression problem converges to the positive Gaussian moment problem of variance 8. Its monic sixth Hermite polynomial gives

$$b_0/X^3 \to 7680, \qquad b_1/X^2 \to -2880, \qquad b_2/X \to 120.$$

If the variance vanished, positive definiteness of A_s would give $b_s = \bar{b}$ almost everywhere on the final half-line where W is bounded below. Continuity would make b_s constant there, contradicting the last limit. Finally, $H_y \le e^{2ys}$ and the superexponential theta decay imply (11) for every fixed finite y. This completes the theorem.

8 Local positivity for at most seven physical kernel points

Define, only for the physical weights and each fixed finite $y \ge 0$,

$$k_y(t) = \frac{1}{Z_y} \int_0^\infty s^2 H_y(s) \Phi(s + t/2) \Phi(s - t/2)\, ds.$$

For $y > 0$ this is $K_y(t)/K_y(0)$ in the customary physical normalization. At $y = 0$ it is $Q_1(t)/Q_1(0)$, where $Q_1(t) = 2\int s^2 \Phi(s + t/2)\Phi(s - t/2)\, ds$.

Corollary 12. *Fix $1 \le n \le 7$ and distinct real numbers $r_0, \dots, r_{n-1}$. For every fixed finite $y \ge 0$ there exists $\epsilon_0 > 0$ such that*

$$[k_y(\epsilon(r_i - r_j))]_{i,j=0}^{n-1} > 0 \qquad (0 < |\epsilon| < \epsilon_0).$$

With $\mathcal{V}(r)=\prod_{i<j}(r_j-r_i)$ *and* $H_{n-1}=(\mathcal{L}_y(z^{i+j}))_{i,j=0}^{n-1}$,

$$\det[k_y(\epsilon(r_i-r_j))]=\frac{\epsilon^{n(n-1)}\mathcal{V}(r)^2\det H_{n-1}}{\prod_{j=0}^{n-1}(j!)^2}+o(\epsilon^{n(n-1)}).$$

The denominator is 1194393600 *for* $n=6$ *and* 619173642240000 *for* $n=7$.

Proof. For every fixed derivative order j and $u\geq 1$, the complete derivative-polynomial series gives $|\Phi^{(j)}(u)|\leq C_j e^{(4j+9)u}e^{-\pi e^{4u}}$. For $|t|\leq T$ and $s\geq T/2+1$, the differentiated paired product is therefore bounded by $C_{j,T}e^{C_j s}e^{-c_T e^{4s}}$, with $c_T=2\pi e^{-2T}>0$. The factor $s^2H_y(s)\leq s^2e^{2ys}$ preserves integrability. On the remaining compact interval smooth theta convergence supplies domination. Consequently k_y is smooth, real and even, $k_y(0)=1$, and $k_y^{(2j)}(0)=(-1)^j\mathcal{L}_y(z^{2j})$.

Put $m=n-1\leq 6$. We give the classical Taylor-confluence argument using the leading moment matrix H_m. Let $V_{ia}=r_i^a$ for $0\leq i,a\leq m$ and $C_\epsilon=\operatorname{diag}(1,\epsilon^{-1},\dots,\epsilon^{-m})V^{-1}$. Taylor expansion through order $2m$ has remainder $o(\epsilon^{2m})$ and bivariate terms

$$\epsilon^{p+q}r_i^p r_j^q\frac{(-1)^q k_y^{(p+q)}(0)}{p!q!},\qquad p+q\leq 2m.$$

Multiplication by V^{-1} maps the evaluation vector of degree $p\leq m$ to its coordinate vector. For $p>m$, its coordinate $a\leq m$ acquires a factor ϵ^{p-a} tending to zero; p,q cannot both exceed m. The scaled remainder is $o(1)$ since $a+b\leq 2m$. Hence the real congruence of the kernel matrix tends entrywise to

$$\left(\frac{(-1)^b k_y^{(a+b)}(0)}{a!b!}\right)_{a,b=0}^{m}=JH_mJ^*,\qquad J=\operatorname{diag}(i^a/a!)_{a=0}^m.$$

This is positive definite by Theorem 9; the complex phases express a real matrix because cross-parity moments vanish. Openness of positive definiteness and invertibility of C_ϵ prove the claim. Finally $\det C_\epsilon=\epsilon^{-m(m+1)/2}/\mathcal{V}(r)$, proving the determinant asymptotic. □

This classical consequence concerns the stated local point configuration. It is asserted only for the physical family, whose locally uniform shifted-derivative domination through order twelve was just proved.

9 Why generic log-concavity is insufficient

Arbitrary nonnegative weights are excluded: the known negative fixed-center square near $X=13/4$ remains negative under a smooth bump supported in a sufficiently small neighborhood. More fundamentally, even smooth positive even log-concavity of a density does not imply the averaged odd-quintic gate.

Proposition 13. *The density* $\varphi(s)=e^{-s^2}(1+\cos(4s)/10)$ *is positive, even, Schwartz and uniformly strictly log-concave. At* $y=0$ *its actual normalized averaged paired-moment functional has a positive odd two-by-two prefix but gives a negative square to*

$$P(z)=z^5-45z^3+59z.$$

Proof. For $c=\cos(4s)$ and $\epsilon=1/10$,

$$(\log\varphi)''=-2-16\epsilon\frac{c+\epsilon}{(1+\epsilon c)^2}\leq -2/9.$$

If $c+\epsilon \geq 0$ this is immediate; otherwise the positive correction is at most $16\epsilon/(1-\epsilon) = 16/9$. Let $J(t) = \int_{\mathbb{R}} s^2\varphi(s+t/2)\varphi(s-t/2)\,ds$ and $F(\xi) = \int \varphi(s)e^{i\xi s}\,ds$. The substitution $a = s+t/2, b = s-t/2$ has Jacobian one and gives

$$\widehat{J}(\xi) = \tfrac{1}{2}(F'(\xi)^2 - F(\xi)F''(\xi)), \qquad F(\xi) = \sqrt{\pi}\left(e^{-\xi^2/4} + \frac{e^{-(\xi-4)^2/4} + e^{-(\xi+4)^2/4}}{20}\right).$$

All operations are justified for Schwartz functions. Up to a common positive factor the signed Fourier density is

$$\begin{aligned}\nu(\xi) = (\tfrac{1}{2} - \tfrac{3}{80}e^{-8})e^{-\xi^2/2} + \tfrac{1}{800}\{e^{-(\xi-4)^2/2} + e^{-(\xi+4)^2/2}\} \\ - \tfrac{3}{20}e^{-2}\{e^{-(\xi-2)^2/2} + e^{-(\xi+2)^2/2}\}.\end{aligned}$$

For $J/J(0)$ the polynomial moment functional equals $\int P^2\nu / \int \nu$. Evenness makes this the same normalized half-line $y = 0$ functional. Variance-one Gaussians with means $0, 2, 4$ give respectively

$$\mathbb{E}P^2 = 11191, \quad 334171, \quad 2544391.$$

Thus the numerator in Gaussian units, with $q = e^{-2}$, is

$$N(q) = \frac{4782591}{400} - \frac{1002513}{10}q - \frac{33573}{80}q^4 < -1600.$$

The positive Taylor series for e^2 through degree 20, with omitted-term ratio at most $2/22$, proves $1353/10000 < q < 677/5000$. Substitution of these rational endpoints proves the strict displayed inequality. The same exact Gaussian moment calculation gives mass $> 2/5$, unnormalized second moment $> 1/3$, and odd prefix determinant $> 5/2$. Division by the positive mass preserves its definiteness and the negative square. These explicit checks are included in the companion arithmetic script. □

This example does not refute the actual theta result or stronger curvature criteria for three-point positivity. The associated-kernel Fourier identities are classical [3]; the example serves to distinguish their hypotheses from generic log-concavity.

A Global lower prefix: finite and unbounded enclosures

We give the analytic bounds behind the lower-prefix certificate. All derivatives in this appendix are physical s derivatives unless polynomial arguments are displayed. Let $b = 3/2$, $\tau_n = n^2 - 1$, $\Phi = \Phi_1(1+R)$, and $\ell = \log\Phi$. Algebra gives the positive Laplace representation

$$R = \int_3^\infty e^{-vX}\,d\sigma(v), \quad d\sigma = \sum_{n\geq 2}\left(n^4\delta_{\tau_n} + bn^2\tau_n e^{b(v-\tau_n)}1_{v\geq\tau_n}\,dv\right).$$

Define $P_0 = 1$, $P_{j+1}(z) = z(P_j - P_j')$. The physical derivative of e^{-vX} is $(-4)^j P_j(vX)e^{-vX}$. For $j \leq 6$, both $P_j(9+w)$ and $[(z^j - P_j(z))]_{z=9+w}$ have nonnegative coefficients; so do the shifted ordinary derivatives of P_4, P_6. Hence the derivatives R_j alternate in sign and are bounded by the highest Laplace moments.

For every $n \geq 2$ their majorants are

$$M_{n,j}(X) = 4^j X^j e^{-\tau_n X}\left[n^4\tau_n^j + bn^2\tau_n\sum_{l=0}^{j}\binom{j}{l}\frac{\tau_n^{j-l}l!}{(X-b)^{l+1}}\right].$$

They decrease on $X \ge 3$ for $j \le 6$. Their complete $n \ge 3$ tails are bounded by

$$\mathcal{T}_j = \frac{C_j 3^{4+2j}}{20^8(1-q_j)}, \quad C_j = 12^j \left(1 + \sum_{l=0}^{j} \binom{j}{l} l!(2/27)^l\right), \quad q_j = (4/3)^{4+2j}/20^7 < 1.$$

The ratio bound retains every atom. In particular

$$\begin{aligned} |R_1| &\le 1104/8000 + (100/99)24 \cdot 3^6/2^{24} < 1/4, \\ 0 < R_2 &\le 44736/8000 + (100/99)288 \cdot 3^8/2^{24} < 6, \\ |R_3| &\le M_{2,3}(3)e^9/8000 + \mathcal{T}_3 = 34736311628179317/147622497952000 < 256, \\ R &\le 28/8000 + 2 \cdot 3^4/[20^8(1-(4/3)^4/20^7)] < 1/250. \end{aligned}$$

Here $e^3 > 20$ follows from its Taylor polynomial through degree eight. The first two weaker tails use $e^{-24} < 2^{-24}$ and ratio $< 1/100$.

Set $U = 1 + R$. For the leading atom

$$\ell_1'' = -16[X + T_2(b/X)], \quad \ell_1'''' = -256[X + T_4(b/X)], \quad \ell_1^{(6)} = -4096[X + T_6(b/X)],$$

where

$$T_2(r) = \frac{r}{(1-r)^2}, \quad T_4(r) = \frac{r(1+4r+r^2)}{(1-r)^4}, \quad T_6(r) = \frac{r(1+26r+66r^2+26r^3+r^4)}{(1-r)^6}.$$

The fourth logarithmic correction is

$$(\log U)'''' = R_4/U - (4R_1R_3 + 3R_2^2)/U^2 + 12R_1^2R_2/U^3 - 6R_1^4/U^4.$$

It is at most $R_4 + 9/2$. The qualitative sign $\ell'''' < 0$ is also proved in Umihara [8]; we include the quantitative independent bounds needed here. The positive decreasing $n = 2$ fourth contribution is $C_4(X)e^{-3X}$, where

$$C_j(X) = 4^j \left[16P_j(3X) + \frac{18}{X-b} \sum_{l=0}^{j} P_j^{(l)}(3X) \left(\frac{X}{X-b}\right)^l\right] \quad (j = 4, 6).$$

Indeed $(P_4 - P_4')(9+w) = 1162 + 921w + 241w^2 + 26w^3 + w^4 > 0$, so its integrated fourth contribution decreases. Its $n \ge 3$ fourth tail is less than

$$\frac{100}{99}\frac{62208 \cdot 3^{12}}{20^8} = \frac{14348907}{11000000} < 2.$$

The exact endpoint coefficients are $C_4(3) = 43140096$, $C_4(16/5) = 42636828818522112/887410625$, $C_4(7/2) = 59421432$ and $C_4(4) = 278436163584/3125$. Direct substitution yields

X interval	upper for $C_4(X)e^{-3X}$	lower for $256[X + T_4(b/X)]$
$[3, 16/5]$	5393	5430
$[16/5, 7/2]$	3337	3801
$[7/2, 4]$	1748	2557
$[4, \infty)$	557	1024

The exponential lower bounds are respectively $8000, 14400, 34000, 160000$. They follow from $e^3 > 20$, $e^{3/5} > 9/5$, $e^{3/2} > 17/4$. Each final column exceeds the middle column plus $2 + 9/2$.

For the determinant, put $A = -\ell''$, $d = -\ell'''' > 0$, $e = \ell^{(6)}$. Direct paired differentiation gives

$$h_2 = -\ell''/2, \quad h_4 = (\ell'''' + 6\ell''^2)/8, \quad h_6 = -(\ell^{(6)} + 30\ell''\ell'''' + 60\ell''^3)/32,$$

$$64 \det G = 24A^4 - 18A^2 d - d^2 - Ae. \tag{16}$$

The fourth logarithmic error budget is $46595/128$. Expanding $\log(1 + R)$ through order six and dropping its negative groups gives

$$(\log U)^{(6)} \le R_6 + (15/8)R_4 + 841095/16.$$

The constants follow respectively from $4(1/4)256 + 3 \cdot 6^2 + 6(1/4)^4$ and $120(1/4)6 \cdot 256 + 30 \cdot 6^3 + 360(1/4)^4 6$. For a cell $[L, H]$, bound R_4 below by $C_4(H)e^{-3H}$ and above by $C_4(L)e^{-3L} + \mathcal{T}_4$. Bound R_6 above by $C_6^+(L, H)e^{-3L} + \mathcal{T}_6$, where C_6^+ replaces polynomial arguments by $3H$, denominators by $L - b$, and $X/(X - b)$ by $L/(L - b)$. Positive shifted derivative coefficients justify the enclosure. With these bounds $r_{4,-}, r_{4,+}, r_{6,+}$ define

$$A_- = 16[L + T_2(b/H)] - 6, \qquad A_+ = 16[H + T_2(b/L)] + 1/16,$$
$$d_+ = 256[H + T_4(b/L)] - r_{4,-}/(251/250) + 46595/128,$$
$$e_+ = -4096[L + T_6(b/H)] + r_{6,+} + (15/8)r_{4,+} + 841095/16.$$

On the 96 closed cells $L = 3 + j/32$, $H = L + 1/32$, rational arithmetic proves $A_- > 0$, $d_+ > 0$, $A_-^2 \ge 3d_+/8$ and

$$24A_-^4 - 18A_-^2 d_+ - d_+^2 - (A_+ \text{ if } e_+ \ge 0 \text{ else } A_-)e_+ > 256000000.$$

The minimum lower floor is 264806350, and the minimum guard floor is 3928. The guard justifies the monotonic substitution in A; the other signs justify every remaining substitution. Exponential enclosures here use $\sum_{j=0}^{80} q^j/j!$ and its remainder $(q^{81}/81!)/(1 - q/82)$ for $0 \le q \le 18$. For $X \ge 6$, the sixth envelope at 6 gives $e < 150000$; also $16X - 6 \le A \le 16X + 33$ and $d \le 256X + 782$. The resulting lower polynomial in $v = X - 6$ has coefficients

$$(1211952476, 958667264, 274580480, 34209792, 1572864),$$

and its monotonicity guard has coefficients $(28923/4, 2784, 256)$. These are positive. Equation (16) proves globally $\det G \ge 4 \cdot 10^6$ and $h_2 \ge 21$. Thus both the finite interval and the remaining unbounded domain are covered.

B Whole-cell baseline and regional certificate

The separate baseline uses 1024 closed cells $[j/1024, (j + 1)/1024]$, not point evaluations. Work in the unit-triangular basis

$$U = z, \quad V = z^3 - 139z, \quad R = z^5 - 461z^3 + 32200z.$$

The scalar monic innovation and matrix-weighted residual variance are invariant under these fixed lower-degree basis shifts. Let $b(s) = \pi e^{5s-X}$ and use tildes for moment numerators multiplied by Φ^2/b^2. Then

$$s^2\Phi^2\delta_s = s^2b^2(\widetilde{S} - \widetilde{r}^T\widetilde{G}^{-1}\widetilde{r}). \tag{17}$$

This exact cancellation avoids division by a small density.

To specify the complete derivative enclosures, use $q_0(z) = 2z - 3$, $q_{j+1}(z) = (5 + 4z\partial_z - 4z)q_j(z)$ through $j = 10$. The head derivative is

$$d_j = \sum_{n=1}^{3} n^2 q_j(n^2 X) e^{-(n^2-1)X}.$$

For a whole cell with $X \in [X_-, X_+]$, the omitted derivative tail is bounded by the sum over monomials $q_{jl} z^l$ of

$$|q_{jl}| X_+^l 4^{2l+2} e^{-15X_-} \Big/ \left[1 - (5/4)^{2l+2} e^{-9X_-}\right].$$

Every denominator is positive. The paired head moment polynomial is formed before interval evaluation:

$$\widetilde{m}_{2k}^{(n,m)} = \frac{(-1)^k}{4^k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} [n^2 q_j(n^2 X)][m^2 q_{2k-j}(m^2 X)] e^{-(n^2+m^2-2)X},$$

summed for $1 \le n \le m \le 3$ with multiplicity two off the diagonal. With head and tail derivative bounds, the absolute moment error is

$$4^{-k} \sum_{j=0}^{2k} \binom{2k}{j} (|d_j| t_{2k-j} + t_j |d_{2k-j}| + t_j t_{2k-j}).$$

All cross terms are retained. The six fields are the linear combinations

$$\begin{aligned}
a &= m_2, \\
b &= m_4 - 139 m_2, \\
c &= m_6 - 278 m_4 + 19321 m_2, \\
r_1 &= m_6 - 461 m_4 + 32200 m_2, \\
r_2 &= m_8 - 600 m_6 + 96279 m_4 - 4475800 m_2, \\
S &= m_{10} - 922 m_8 + 276921 m_6 - 29688400 m_4 + 1036840000 m_2.
\end{aligned}$$

All are formed at numerator level before enclosure. Absolute errors are combined using absolute coefficients. On every cell the lower bounds for a and $ac - b^2$ are positive. The inverse penalty is at most the upper bound of $(cr_1^2 - 2br_1r_2 + ar_2^2)/(ac - b^2)$, and also at most $2(cr_1^2 + ar_2^2)/(ac - b^2)$ with absolute upper bounds. Use the smaller of these valid upper bounds, and subtract it from the lower endpoint for S in (17). Integrating the resulting interval times $s^2 b^2$ proves the baseline. The recorded rational lower exceeds 2300747; no adaptive subdivision is needed.

For normalization, $B_0 = 81/40$ and the positive theta series give, for $s \ge 1$,

$$\Phi(s) \le 2B_0 \pi X e^{5s - X}, \qquad H_y(s) \le e^{2s} \quad (0 \le y \le 1).$$

Indeed $|2n^2 X - 3| \le B_0 n^2 X$ for $X > 120$, and $\sum n^4 e^{-(n^2-1)X} < 2$ by the successive ratio $16e^{-3X} < 1/2$. Therefore

$$s^2 H_y \Phi^2 \le 4 B_0^2 \, 4^4 s^2 e^{20s - 6e^{4s}}.$$

Since $e^4 > 40$ and $e^3 > 20$, $20s - 6e^{4s} \le -220 - 940(s-1)$ and $e^{-220} < 20^{-73}$. The full normalizer tail is at most

$$\frac{4 B_0^2 \, 4^4}{20^{73}} \left(\frac{1}{940} + \frac{2}{940^2} + \frac{2}{940^3} \right). \tag{18}$$

The compact denominator interval for the whole bin $0 \le y \le 1/8$ contains Z_0 after adding this tail. Interval division of the baseline by this full denominator proves the 17 billion bound.

For the variance, integrate the five fields of G, r separately on $A = [0, 3/16]$ and $B = [3/16, 1]$, with the common factor $s^2 b^2 H_y$. Use the eight closed bins $j/8 \le y \le (j+1)/8$, $j = 0, \dots, 7$. Regional interval inverses have positive determinant. Enclose the second-coordinate contrast in (8), square its minimum absolute value, and divide by upper bounds for the inverse sum and full Z_y. The binwise gain floors exceed, respectively,

$$1966012227,\ 1955643969,\ 1946391148,\ 1938242732,$$
$$1931186825,\ 1925210688,\ 1920300777,\ 1916442774.$$

This includes the $1/Z_y$ scaling of raw regional matrices. It certifies (9) independently of a total averaged innovation.

The interval script uses exact rational operations with outward rounding at 80 decimal places; the other finite checks use exact rational polynomial arithmetic. Pi is enclosed by the Machin alternating series; exponentials use a positive Taylor series after range reduction, with an explicit geometric remainder; the hyperbolic quotient uses its positive series. They reconstruct polynomials and tails instead of reading stored numerical enclosures. Explicit exception gates, active also under optimized Python, require raw baseline $> 2 \cdot 10^6$, normalized baseline $> 17 \cdot 10^9$, and every variance bound $> 10^9$. The public scripts are `verify_prefix.py` (lower-order controls and the 96-cell prefix), `verify_tail.py` (odd complete tail), `verify_even.py` (even complete tail), `verify_separated.py` (baseline and regional variance), and `verify_counterexample.py` (signed Gaussian control). The accompanying README specifies exact commands. Arithmetic checking supports the displayed analytic proof; the compact classification theorem is a cited input. These finite checks support the analytic argument.

C Complete even-sextic tail certificate

Use R_j, Q_k from Section 3, now through $j = 12$ and $k = 6$, and $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is

$$\det(Q_{i+j})_{i,j=0}^{3} = 4638564679680 X^6 P_{14}(X),$$

where

$$\begin{aligned} P_{14} = {} & 131072X^{14} - 2293760X^{13} + 20570112X^{12} - 120004608X^{11} \\ & + 493764096X^{10} - 1486503936X^9 + 3312458496X^8 - 5440255488X^7 \\ & + 6483101760X^6 - 5448021120X^5 + 3089651040X^4 - 1112123520X^3 \\ & + 247531950X^2 - 39803400X + 3444525. \end{aligned}$$

After $X = 12 + v$, each of its first three leading-minor numerators has nonnegative coefficients with a positive constant coefficient. The reserve polynomial

$$\det(Q_{i+j})_{i,j=0}^{3} - 2^{50} X^{12} (2X-3)^8$$

also has nonnegative shifted coefficients. These exact coefficient checks prove the head matrix positive definite and its determinant at least $2^{50} X^{12}$ on the whole half-line. They do not infer definiteness from a determinant alone.

Let $D = \mathrm{diag}(1, 32X, 1024X^2, 32768X^3)$, with constant factors $d = (1, 32, 1024, 32768)$. For the preconditioned head, the determinant is at least $1/1024$ and the trace is at most

$$T = \sum_{i=0}^{3} \sum_{l} \frac{|[X^l]Q_{2i}|12^{l-2i-2}}{(2-3/12)^2 d_i^2} < 8.093.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$. Hence its least eigenvalue is at least

$$\frac{27}{1024T^3} = \frac{29057903779751680918020367814555074 56}{583946075616356139859073544032736515742 29} > \frac{1}{21000}.$$

Here and below T denotes the displayed exact rational sum, before its decimal upper bound.

To retain enough cancellation at $X = 12$, treat the second atom by its paired polynomials rather than by absolute derivative products. Put

$$\begin{aligned} R_{j,n}(X) &= n^2 R_j(n^2 X), \\ C_k(X) &= 2\sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_{j,1}(X) R_{2k-j,2}(X), \\ B_k(X) &= 16 Q_k(4X). \end{aligned}$$

The additional retained numerator is exactly $C_k(X)e^{-3X} + B_k(X)e^{-6X}$: both ordered cross pairs and the diagonal second-atom pair are included once. Write

$$\varepsilon = (3/8)^{36}, \quad \varepsilon_8 = (3/8)^{96}, \quad c_j = \sum_{l} |[X^l]R_j| 12^{l-j-1},$$

$$H_j = \sum_{l} |[X^l]R_j|(1 + 2^{2l+2})12^{l-j-1}, \qquad b_j = 2 \cdot 3^{2j+4}.$$

The elementary inequality $e > 8/3$ gives $e^{-3X} \le \varepsilon$ and $e^{-8X} \le \varepsilon_8$ for $X \ge 12$. The retained two-atom derivative head is bounded by $H_j X^{j+1}$. For the remaining atoms $n \ge 3$, the ratio of successive $n^{2j+4}e^{-(n^2-1)X}$ terms is at most

$$(4/3)^{28} e^{-7X} \le (4/3)^{28}(3/8)^{84} < 1/2 \qquad (j \le 12).$$

Thus their complete derivative tail is at most $b_j c_j X^{j+1} e^{-8X}$.

For $k = 0, \dots, 6$, define the rational numerator-error bound

$$\begin{aligned} e_k = \frac{1}{(2-3/12)^2} \Bigg\{ & \varepsilon \sum_{l} |[X^l]C_k| 12^{l-k-2} + \varepsilon^2 \sum_{l} |[X^l]B_k| 12^{l-k-2} \\ & + 12^k \varepsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \big[H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} \\ & + b_j b_{2k-j} c_j c_{2k-j} \varepsilon_8 \big] \Bigg\}. \end{aligned}$$

This bounds the omitted moment numerator after division by $X^k(2X-3)^2$. The first line retains the exact second-atom cancellation; the remaining lines retain both head-tail terms and the tail-tail term. Every normalized monomial $X^a e^{-3X}$ arising in the first line decreases from 12, since

$a \le k \le 6$; the terms with e^{-6X} also decrease. The other terms use the decreasing functions $X^k e^{-8X}$ and $X^k e^{-16X}$. These facts justify the uniform substitution $X = 12$, rather than merely an endpoint test.

The symmetric preconditioned perturbation consequently has norm at most

$$N = 4 \max_{0 \le i,j \le 3} \frac{e_{i+j}}{d_i d_j} < \frac{1}{180000}.$$

The exact value is approximately 0.00000529965257745. The zeroth moment is included in this bound. As in the earlier tails, before the full denominator is restored its zeroth entry is $(1 + R)^2$, not one. For the denominator itself,

$$0 < R \le \frac{2X}{2X - 3}\, 32e^{-3X} < 1/250.$$

The geometric ratio here is bounded by $(3/2)^{28} e^{-5X} < 1/2$; the exponent 28 safely covers the zeroth derivative as well. Both rational endpoint ratio checks use $e > 8/3$. Therefore

$$D^{-1} E_6 D^{-1} \succeq \left(\frac{1}{21000} - \frac{1}{180000}\right)\left(\frac{250}{251}\right)^2 I \succ \frac{1}{25000} I,$$

which proves Proposition 11. The standalone checker `verify_sextic_tail.py` reconstructs the polynomials, leading minors, determinant reserve and every rational bound in this appendix.

D Whole-cell sextic sign and mean certificates

The arithmetic uses the interval method and paired-polynomial construction of Appendix B, extended through physical derivative twelve and moments $m_0, \dots, m_{12}$. In particular $q_{j+1} = (5 + 4z\partial_z - 4z)q_j$ is now continued through $j = 12$; the same geometric formula encloses every $n \ge 4$ derivative. Every head-tail and tail-tail product remains in the moment error. The zeroth moment is treated by the same calculation.

For Proposition 10, divide $[157/50, 73/20]$ into 256 equal closed X intervals. The coefficients of the square of the displayed monic polynomial, in ascending even powers, are

$$91846575844,\ -4390156132,\ -48761659,\ 3025286,$$
$$13403,\ -334,\ 1.$$

Combine these with the paired moment polynomials before enclosure. On every whole cell, the sum of its head upper bound and its full absolute tail error is strictly negative. Dividing by the positive complete density square proves the stated fixed-center sign. The script `verify_sextic_initial.py` supplies the exact upper bound and rejects empty partitions.

For the mean, use the fixed scaled basis

$$\begin{aligned} V_0 &= 1, \\ V_1 &= z^2/64 - 1, \\ V_2 &= z^4/4096 - 6z^2/64 + 3, \\ V_3 &= z^6/262144 - 15z^4/4096 + 45z^2/64 - 15. \end{aligned}$$

The residual innovation for V_3 is $\eta_s/262144^2$. Lower-degree shifts do not affect this residual, and the scaling is restored at the end. Work in raw paired numerators as in (17). If their prefix is

$$\widetilde{A} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \qquad \widetilde{c} = (u, v, w)^T,$$

its six adjugate entries are

$$df - e^2, \quad ce - bf, \quad be - cd, \quad af - c^2, \quad bc - ae, \quad ad - b^2.$$

The whole-cell lower endpoints of a, $ad - b^2$ and $\det \widetilde{A}$ are positive. Contract the interval adjugate with (u, v, w) on both sides to obtain a numerator upper bound U. If $D_- > 0$ is the determinant lower bound, then

$$\widetilde{c}^T \widetilde{A}^{-1} \widetilde{c} \le \max(0, U)/D_-.$$

Subtract this entire inverse penalty from the lower top-square endpoint, multiply by the interval for $s^2\pi^2 e^{10s-2X}$ and the cell width, and sum. Exact outward arithmetic on the 1024 closed intervals $[j/1024, (j+1)/1024]$ gives, after multiplication by 262144^2,

$$\int_0^1 \eta_s \, dm_0 > 262225959 > 250000000.$$

The positive remainder for $s \ge 1$ follows from Proposition 11. An independent compact normalizer enclosure plus the full tail (18) gives

$$\frac{\int_0^1 \eta_s \, dm_0}{Z_0} > 2049676440449 > 2000000000000.$$

The checker `verify_sextic_mean.py`, run with argument `1024`, reconstructs these enclosures with the same outward 80-decimal rational arithmetic, Machin bound for pi and Taylor/geometric bounds for exponentials. Explicit exception gates retain all prefix signs and require the conservative raw and normalized thresholds in (13). All three new scripts may be run normally and under optimized Python. The paired recurrences, infinite tails, sign transfer and inverse-energy arguments above are the analytic justification for their finite checks.

E The monotone-weight condition

The monotone-weight condition is sufficient. The full degree-five conclusion requires the fifth-moment hypothesis (3); the full degree-six conclusion requires the stronger sixth-moment hypothesis (11). Both apply to every finite physical parameter. The physical local seven-point corollary has the fixed-shape scope stated above and includes the earlier six-point consequence. The single-crossing lemma, Schur identity, Hermite regression, Taylor confluence and parallel-sum bound are classical tools; the contribution is the theta-specific signs, complete matrix tails and separated baselines that justify the stated weighted extensions.

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