Monotone center weights and the averaged odd-quintic theta gate

Scott Schubr

October 24, 2026

Abstract

We prove positivity of the odd moment block on $\{z, z^3, z^5\}$ for the normalized Riemann theta product averaged against every nonzero nonnegative nondecreasing center weight with a finite fifth exponential moment. This includes the physical weights $\sinh(2ys)/(2ys)$ for every finite $y \geq 0$. The proof combines the previously published compact fixed-center classification with a complete-theta positive matrix tail, a separately certified positive mean of the local Schur innovation, and a classical single-crossing argument. The matrix-weighted regression variance is strictly positive and has an independent two-region lower bound on $0 \leq y \leq 1$. A separate even-block estimate gives full degree-five moment positivity and local six-point positivity for the physical kernels. A smooth, positive, even, uniformly log-concave density with a negative averaged quintic square shows why generic log-concavity cannot replace the theta-specific argument. All new finite arithmetic is reproducible by outward rational enclosures.

1 Definitions and the weighted theorem

The complete theta density is

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

Jacobi symmetry makes it smooth, positive and even on $\mathbb{R}$; for $s \geq 0$ all summands are positive. Put $X = \pi e^{4s}$ and

$$p_s(t) = \frac{\Phi(s + t/2)\Phi(s - t/2)}{\Phi(s)^2}, \qquad h_{2j}(s) = (-1)^j p_s^{(2j)}(0), \qquad h_{2j+1} = 0.$$

The polynomial variable z is distinct from X. In the raw odd basis set

$$G_s = \begin{pmatrix} h_2 & h_4 \\ h_4 & h_6 \end{pmatrix}, \quad r_s = \begin{pmatrix} h_6 \\ h_8 \end{pmatrix}, \quad S_s = h_{10}, \quad \beta_s = G_s^{-1} r_s, \quad \delta_s = S_s - r_s^T G_s^{-1} r_s. \tag{2}$$

Global invertibility is proved below. Write $dm_0 = s^2 \Phi(s)^2\, ds$. An admissible weight is a finite-valued measurable nonnegative nondecreasing function W on $[0, \infty)$, not zero almost everywhere, such that

$$\int_0^\infty W(s)(1 + X(s)^5)\, dm_0(s) < \infty. \tag{3}$$

Let $Z_W = \int W\, dm_0$, $d\mu_W = W\, dm_0/Z_W$, and use $\mathbb{E}_W$ for expectation. Define $\mathcal{L}_W(z^{2j}) = \mathbb{E}_W h_{2j}$ through $j = 5$, with odd moments zero.

Theorem 1. *For every admissible W, the matrix of $\mathcal{L}_W$ on $\{z, z^3, z^5\}$ is positive definite. More precisely, with*

$$\bar{G} = \mathbb{E}_W G_s, \quad \bar{r} = \mathbb{E}_W r_s, \quad \bar{\beta} = \bar{G}^{-1}\bar{r},$$

one has

$$\begin{aligned} \Delta_W &:= \mathbb{E}_W S_s - \bar{r}^T \bar{G}^{-1} \bar{r} = \mathbb{E}_W \delta_s + V_W, \\ V_W &:= \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})] > 0, \qquad \mathbb{E}_W \delta_s > 0. \end{aligned} \tag{4}$$

Let $a = \frac{1}{4}\log(a_1/\pi)$, where

$$3.64888758188562 < a_1 < 3.64888758188563$$

is the first fixed-center quintic transition of [1]. If $W(a) > 0$, then

$$\mathbb{E}_W \delta_s > \frac{2\,000\,000\, W(a)}{Z_W}. \tag{5}$$

Corollary 2. *For $H_y(s) = \sinh(2ys)/(2ys)$, continuously defined as 1 when $ys = 0$, Theorem 1 holds for every finite $y \geq 0$. Moreover*

$$\mathbb{E}_0 \delta_s > 17\,000\,000\,000, \qquad V_y > 1\,000\,000\,000 \quad (0 \leq y \leq 1).$$

Here $Z_y = \int H_y \, dm_0 > 0$ defines normalization also at $y = 0$.

For $y > 0$ the customary unnormalized physical kernel has central value $4yZ_y$; its value vanishes at $y = 0$ and is never used as a denominator there. The normalized law has the displayed continuous endpoint.

The earlier bounded-parameter odd-quintic certificate [2] gave a stronger numerical estimate for Δ_y on $[0, 1]$. The advance here is the full parameter half-line, the monotone-weight cone, and the separation of two strictly positive mechanisms.

2 Classical algebra and the new theta inputs

The Schur identity (4) is completion of squares. Indeed, integrate

$$S_s - 2b^T r_s + b^T G_s b = \delta_s + (b - \beta_s)^T G_s (b - \beta_s)$$

and minimize over b. This is classical matrix-fractional geometry [4]; its positive variance alone does not determine the sign of the mean local innovation. For example, two equally weighted scalar prefixes $G_1 = G_2 = 1$, columns $r_1 = 2, r_2 = -2$, and $S_1 = S_2 = 3$ have local innovations -1 but averaged innovation 3.

The following elementary single-crossing statement is also classical; stronger integral versions occur in [7].

Lemma 3. *Suppose $f \leq 0$ on $[0, a]$, $f \geq 0$ on $[a, \infty)$ and $\int f \, dm > 0$. For a nonnegative nondecreasing W with the necessary absolute integrability,*

$$\int W f \, dm \geq W(a) \int f \, dm.$$

If $W(a) = 0$, W is eventually bounded below by a positive constant, and f is strictly positive on a final half-line of positive measure, then $\int W f \, dm > 0$ as well.

Proof. The function $(W(s)-W(a))f(s)$ is nonnegative on both sides of a. If $W(a)=0$, monotonicity makes W vanish before a, and its eventual positivity makes a strictly positive final contribution. □

We use one authenticated published result rather than reprove the entire compact classification.

Proposition 4 (Compact fixed-center premise from [1])**.** *For the exact density* (1) *and the exact signed derivatives used here, the full fixed-center moment matrix through degree four is positive definite on* $3 \le X \le 12$*. The minimum negative-square degree is five before* a_1*, at least six after* a_1 *through* 12*, and the degree-five matrix is positive semidefinite at* a_1*.*

This is the relevant portion of the published classification theorem, including its endpoint statement. Because parity separates odd and even blocks and $G_s > 0$, it implies

$$\delta_s < 0 \quad (\pi \le X < a_1), \quad \delta_s = 0 \quad (X = a_1), \quad \delta_s \ge 0 \quad (a_1 < X \le 12). \tag{6}$$

The new deductions that complete the argument are a global lower prefix, a positive full-theta tail beyond 12, and the independently integrated positive baseline. We provide their proofs and finite-arithmetic specifications below. Associated theta kernels and generalized Laguerre inequalities have a broader classical literature [3].

3 The complete-theta positive tail

Proposition 5. *For* $X \ge 12$*, the full odd matrix* $M_o = (h_{2i+2j+2})_{i,j=0}^{2}$ *satisfies*

$$M_o \succeq \frac{10}{63001}\,\mathrm{diag}(16X, 16384X^3, 67108864X^5). \tag{7}$$

Consequently $\delta_s \ge (671088640/63001)X^5 > 0$ *there.*

Proof. The common e^{5s} factor cancels in the paired derivative. Put

$$R_0(z) = 2z - 3, \quad R_{j+1}(z) = 2z(R_j'(z) - R_j(z)), \quad Q_k(X) = \sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_j(X) R_{2k-j}(X).$$

For the first atom, $h_{2k} = Q_k/(2X-3)^2$, and exact cancellation gives $\deg Q_k \le k+2$ for $k \le 5$. Its odd matrix determinant is

$$\frac{6039797760 X^6 P_9(X)}{(2X-3)^6},$$

where

$$\begin{aligned} P_9(X) = {} & 1024X^9 - 12544X^8 + 81216X^7 - 331200X^6 + 902736X^5 \\ & - 1668960X^4 + 2026620X^3 - 1519020X^2 + 637875X - 127575. \end{aligned}$$

The first two leading-minor numerator polynomials, and $P_9 - 256X^9$, have strictly positive coefficients after $X = 12 + v$. Thus the head matrix is positive definite with determinant at least $24159191040X^9$. For $d = (4, 128, 8192)$ and $D = \mathrm{diag}(d_i X^{i+1/2})_{i=0}^{2}$, its congruence by D^{-1} has determinant at least $45/32768$. Its trace is at most

$$\sum_{i=0}^{2} \sum_{j} \frac{|[X^j]Q_{2i+1}| 12^{j-(2i+1)-2}}{(2-3/12)^2 d_i^2} = \frac{256401048845}{99883155456} < \frac{13}{5}.$$

Hence its least eigenvalue exceeds $1/5000$ by $\lambda_{\min} \geq \det / \operatorname{tr}^2$.

We now bound every omitted atom. If $r_{jl} = [z^l]R_j$, put

$$C_j = \sum_l |r_{jl}| 12^{l-j-1}, \qquad B_j = 2^{2j+5}.$$

For $X \geq 12$, $n^2|R_j(n^2X)| \leq C_j X^{j+1} n^{2j+4}$. For $n \geq 2$ the successive majorant ratio is at most $(3/2)^{24}e^{-60} < 1/2$ for $j \leq 10$. Therefore the derivative tail after removing the common head exponential is at most $B_j C_j X^{j+1} e^{-3X}$. Both cross terms and the tail-tail term in the paired product are retained. Set $\varepsilon = 20^{-12} > e^{-36}$. The error in the kth moment after division by X^k, before the full theta denominator correction, is bounded by

$$E_k = \frac{12^k \varepsilon}{(2 - 3/12)^2} \sum_{j=0}^{2k} \binom{2k}{j} C_j C_{2k-j} (2B_{2k-j} + B_j B_{2k-j} \varepsilon).$$

Here $X^k e^{-3X}$ and $(2 - 3/X)^{-2}$ decrease for $X \geq 12$, $k \leq 5$. Symmetry in j accounts for both cross terms. The symmetric matrix perturbation norm is at most

$$3 \max_{i,j} \frac{E_{i+j+1}}{d_i d_j} = \frac{5312409501805455073001503693}{14863564800000000000000000000000000} < \frac{1}{25000}.$$

Writing $\Phi = \Phi_1(1 + R)$, the denominator correction is exactly $(1 + R)^2$ and $0 < R < 1/250$ (Appendix A). Thus

$$(1 + R)^2 D^{-1} M_o D^{-1} \succeq (1/5000 - 1/25000) I = I/6250.$$

Since $(1 + R)^2 < (251/250)^2$, this proves (7). Minimizing the quadratic form with its monic fifth-degree coefficient fixed to one proves the Schur bound. □

Combining (6) with Proposition 5 gives one global negative-to-nonnegative crossing at $s = a$. Later higher-degree transitions in the compact classification do not change this fixed quintic conclusion.

4 Positive baseline and independent variance reserve

Proposition 6. *For the full theta density,*

$$\int_0^1 \delta_s \, dm_0 > 2\,000\,000, \qquad I_0 := \int_0^\infty \delta_s \, dm_0 > 2\,000\,000, \qquad I_0/Z_0 > 17\,000\,000\,000.$$

Proof. The first assertion is a finite interval certificate with the full inverse penalty, described in Appendix B. It gives the stronger raw lower bound 2300747. The second follows by dropping the strictly positive tail $s \geq 1$, since $\pi e^4 > 12$. The third uses the independently enclosed full normalizer. This argument does not infer a positive local mean from a known positive averaged Schur complement. □

For two disjoint regions A, B, let $Q_A = \mathbb{E}_W[1_A G_s]$ and $c_A = Q_A^{-1} \mathbb{E}_W[1_A r_s]$, and similarly for B. Completion of squares within the regions and minimization over the global coefficient give the classical parallel-sum lower bound [5, 6]

$$V_W \geq (c_A - c_B)^T (Q_A^{-1} + Q_B^{-1})^{-1} (c_A - c_B) \geq \frac{[u^T(c_A - c_B)]^2}{u^T(Q_A^{-1} + Q_B^{-1})u}. \tag{8}$$

The final inequality is matrix Cauchy–Schwarz, valid without commutativity. For the numerical estimate, use the fixed shifted basis of Appendix B, which preserves V_y; in this paragraph c_A, c_B are the corresponding coefficient vectors. For $A = [0, 3/16]$, $B = [3/16, 1]$, $u = (0, 1)^T$, the certificate encloses the matrices and columns directly over eight closed y bins and proves

$$V_y > 1916442774 > 10^9 \qquad (0 \le y \le 1). \tag{9}$$

The complement has nonnegative variance and is discarded. This is a physical-region computation, not the difference of two separately computed total innovations.

5 Proof of the global weighted theorem

The prefix proof in Appendix A gives $G_s > 0$, $h_2 \ge 21$, and $\det G_s \ge 4 \cdot 10^6$ for every s. The paired polynomial formulas in Section 3 give

$$\frac{h_{2j}(s)}{X^j} \longrightarrow (2j-1)!!\, 8^j \quad (1 \le j \le 5). \tag{10}$$

Indeed the head numerator has degree $j + 2$, and the full higher-atom correction after division by X^j is $O(X^j e^{-3X})$. Put $v = 8X$. Then $\det G_s / v^4 \to 6$ and the explicit inverse gives

$$\frac{\beta_1}{v^2} \to \frac{225 - 315}{6} = -15, \qquad \frac{\beta_2}{v} \to \frac{105 - 45}{6} = 10.$$

Thus $\beta_1/X^2 \to -960$ and $\beta_2/X \to 80$. In particular $r_s^T G_s^{-1} r_s = O(X^5)$. Smoothness on compact intervals and (3) make all the expectations in (4) absolutely convergent. Positive averaging makes $\bar{G} > 0$.

The global single crossing and Proposition 6 allow Lemma 3 with $f = \delta_s$, proving $\mathbb{E}_W \delta_s > 0$ and (5). If $V_W = 0$, positive definiteness forces $\beta_s = \bar{\beta}$ almost everywhere for μ_W. Every admissible nonzero nondecreasing weight is bounded below by a positive constant on a half-line. Continuity then forces β_s constant on that entire tail, contradicting $\beta_2/X \to 80$. Hence $V_W > 0$. Identity (4) and the Schur criterion prove Theorem 1, also for complex polynomial coefficient vectors.

Finally H_y is nondecreasing because $z \cosh z - \sinh z$ vanishes at zero and has derivative $z \sinh z \ge 0$. Its growth is at most e^{2ys}; the theta density has superexponential decay. Thus (3) holds for every fixed finite y. Propositions 6 and (9) finish Corollary 2.

6 The even block and full degree-five consequence

Proposition 7. *The complete even matrix $M_e = (h_{2i+2j})_{i,j=0}^2$ is positive definite for every center. On $X \ge 12$,*

$$M_e \succeq \frac{1}{4000} \operatorname{diag}(1, 1024X^2, 1048576X^4).$$

Proof. Use the recurrence of Section 3, adding $Q_0 = (2X - 3)^2$. The first-atom even determinant numerator is $393216X^3 \widehat{P}_9(X)$, where

$$\begin{aligned} \widehat{P}_9 = {} & 2048X^9 - 22016X^8 + 120960X^7 - 407232X^6 + 875808X^5 \\ & - 1201680X^4 + 986904X^3 - 424980X^2 + 80190X - 8505. \end{aligned}$$

The first two leading-minor numerators have positive coefficients after $X = 12 + v$, as does

$$393216X^3 \widehat{P}_9 - 3145728X^6(2X - 3)^6.$$

With $D_e = \mathrm{diag}(1, 32X, 1024X^2)$, the head congruence has determinant at least $3/1024$ and trace at most $1420942925/520224768$. Its least eigenvalue is therefore at least

$$\frac{792872488009728}{2019078796107555625} > 1/3000.$$

The complete head-tail calculation of Section 3, now for $k = 0, \ldots, 4$ with $d = (1, 32, 1024)$, gives perturbation norm at most

$$\frac{54150875914457619693203 87}{619315200000000000000000000000000} < 1/1000000.$$

The zeroth entry is included: before dividing by $(1 + R)^2$, it changes from 1 to $(1 + R)^2$. Thus the tail bound follows from $(1/3000 - 1/1000000)(250/251)^2 > 1/4000$. For $\pi \le X \le 12$, the even block is a principal submatrix of the positive M_4 in Proposition 4. Even reflection covers negative centers. □

Corollary 8. *For every admissible W and every nonzero complex polynomial P of degree at most five, $\mathcal{L}_W(|P|^2) > 0$. In particular this holds for every physical weight H_y, $0 \le y < \infty$.*

Proof. Positive averaging preserves the pointwise strictly positive even block. Odd moments vanish, so a permutation of the full degree-five matrix is the direct sum of that even block and the odd block in Theorem 1. □

The even block alone permits arbitrary nonnegative center weights with finite required moments; this does not extend the odd conclusion to arbitrary weights.

7 A local six-point consequence for the physical kernels

Define, only for the physical weights and each fixed finite $y \ge 0$,

$$k_y(t) = \frac{1}{Z_y} \int_0^\infty s^2 H_y(s) \Phi(s + t/2) \Phi(s - t/2)\, ds.$$

For $y > 0$ this is $K_y(t)/K_y(0)$ in the customary physical normalization. At $y = 0$ it is $Q_1(t)/Q_1(0)$, where $Q_1(t) = 2 \int s^2 \Phi(s + t/2) \Phi(s - t/2)\, ds$.

Corollary 9. *Fix $1 \le n \le 6$ and distinct real numbers $r_0, \ldots, r_{n-1}$. For every fixed finite $y \ge 0$ there exists $\epsilon_0 > 0$ such that*

$$[k_y(\epsilon(r_i - r_j))]_{i,j=0}^{n-1} > 0 \qquad (0 < |\epsilon| < \epsilon_0).$$

For $n = 6$, with $\mathcal{V}(r) = \prod_{i<j}(r_j - r_i)$ and $H_5 = (\mathcal{L}_y(z^{i+j}))_{i,j=0}^5$,

$$\det[k_y(\epsilon(r_i - r_j))] = \frac{\epsilon^{30} \mathcal{V}(r)^2 \det H_5}{1194393600} + o(\epsilon^{30}).$$

Proof. For every fixed derivative order j and $u \ge 1$, the complete derivative-polynomial series gives $|\Phi^{(j)}(u)| \le C_j e^{(4j+9)u} e^{-\pi e^{4u}}$. For $|t| \le T$ and $s \ge T/2 + 1$, the differentiated paired product is therefore bounded by $C_{j,T} e^{C_j s} e^{-c_T e^{4s}}$, with $c_T = 2\pi e^{-2T} > 0$. The factor $s^2 H_y(s) \le s^2 e^{2ys}$ preserves integrability. On the remaining compact interval smooth theta convergence supplies domination. Consequently k_y is smooth, real and even, $k_y(0) = 1$, and $k_y^{(2j)}(0) = (-1)^j \mathcal{L}_y(z^{2j})$.

We give the classical Taylor-confluence argument for $n = 6$; the smaller cases use the corresponding leading principal matrix. Let $V_{ia} = r_i^a$ for $0 \le i, a \le 5$ and $C_\epsilon = \mathrm{diag}(1, \epsilon^{-1}, \dots, \epsilon^{-5})V^{-1}$. Taylor expansion through order ten has remainder $o(\epsilon^{10})$ and bivariate terms

$$\epsilon^{p+q} r_i^p r_j^q \frac{(-1)^q k_y^{(p+q)}(0)}{p!q!}, \qquad p+q \le 10.$$

Multiplication by V^{-1} maps the evaluation vector of degree $p \le 5$ to its coordinate vector. For $p \ge 6$, its coordinate $a \le 5$ acquires a factor ϵ^{p-a} tending to zero; p, q cannot both exceed five. The scaled remainder is $o(1)$ since $a + b \le 10$. Hence the real congruence of the kernel matrix tends entrywise to

$$\left(\frac{(-1)^b k_y^{(a+b)}(0)}{a!b!} \right)_{a,b=0}^{5} = J H_5 J^*, \qquad J = \mathrm{diag}(i^a/a!)_{a=0}^5.$$

This is positive definite by Corollary 8; the complex phases express a real matrix because cross-parity moments vanish. Openness of positive definiteness and invertibility of C_ϵ prove the claim. Finally $\det C_\epsilon = \epsilon^{-15}/\mathcal{V}(r)$ and $\prod_{a=0}^5 (a!)^2 = 1194393600$, proving the determinant asymptotic. □

This classical consequence concerns the stated local point configuration. It is asserted only for the physical family, whose locally uniform shifted-derivative domination was just proved.

8 Why generic log-concavity is insufficient

Arbitrary nonnegative weights are excluded: the known negative fixed-center square near $X = 13/4$ remains negative under a smooth bump supported in a sufficiently small neighborhood. More fundamentally, even smooth positive even log-concavity of a density does not imply the averaged odd-quintic gate.

Proposition 10. *The density $\varphi(s) = e^{-s^2}(1+\cos(4s)/10)$ is positive, even, Schwartz and uniformly strictly log-concave. At $y = 0$ its actual normalized averaged paired-moment functional has a positive odd two-by-two prefix but gives a negative square to*

$$P(z) = z^5 - 45z^3 + 59z.$$

Proof. For $c = \cos(4s)$ and $\epsilon = 1/10$,

$$(\log \varphi)'' = -2 - 16\epsilon \frac{c+\epsilon}{(1+\epsilon c)^2} \le -2/9.$$

If $c + \epsilon \ge 0$ this is immediate; otherwise the positive correction is at most $16\epsilon/(1-\epsilon) = 16/9$. Let $J(t) = \int_{\mathbb{R}} s^2 \varphi(s+t/2)\varphi(s-t/2)\, ds$ and $F(\xi) = \int \varphi(s) e^{i\xi s}\, ds$. The substitution $a = s+t/2, b = s-t/2$ has Jacobian one and gives

$$\widehat{J}(\xi) = \tfrac{1}{2}(F'(\xi)^2 - F(\xi)F''(\xi)), \qquad F(\xi) = \sqrt{\pi}\left(e^{-\xi^2/4} + \frac{e^{-(\xi-4)^2/4} + e^{-(\xi+4)^2/4}}{20} \right).$$

All operations are justified for Schwartz functions. Up to a common positive factor the signed Fourier density is

$$\begin{aligned} \nu(\xi) = {} & (\tfrac{1}{2} - \tfrac{3}{80}e^{-8})e^{-\xi^2/2} + \tfrac{1}{800}\{e^{-(\xi-4)^2/2} + e^{-(\xi+4)^2/2}\} \\ & - \tfrac{3}{20}e^{-2}\{e^{-(\xi-2)^2/2} + e^{-(\xi+2)^2/2}\}. \end{aligned}$$

For $J/J(0)$ the polynomial moment functional equals $\int P^2\nu/\int\nu$. Evenness makes this the same normalized half-line $y=0$ functional. Variance-one Gaussians with means $0,2,4$ give respectively

$$\mathbb{E}P^2 = 11191, \quad 334171, \quad 2544391.$$

Thus the numerator in Gaussian units, with $q=e^{-2}$, is

$$N(q) = \frac{4782591}{400} - \frac{1002513}{10}q - \frac{33573}{80}q^4 < -1600.$$

The positive Taylor series for e^2 through degree 20, with omitted-term ratio at most $2/22$, proves $1353/10000 < q < 677/5000$. Substitution of these rational endpoints proves the strict displayed inequality. The same exact Gaussian moment calculation gives mass $> 2/5$, unnormalized second moment $> 1/3$, and odd prefix determinant $> 5/2$. Division by the positive mass preserves its definiteness and the negative square. These explicit checks are included in the companion arithmetic script. □

This example does not refute the actual theta result or stronger curvature criteria for three-point positivity. The associated-kernel Fourier identities are classical [3]; the example serves to distinguish their hypotheses from generic log-concavity.

A Global lower prefix: finite and unbounded enclosures

We give the analytic bounds behind the lower-prefix certificate. All derivatives in this appendix are physical s derivatives unless polynomial arguments are displayed. Let $b=3/2$, $\tau_n = n^2-1$, $\Phi = \Phi_1(1+R)$, and $\ell = \log\Phi$. Algebra gives the positive Laplace representation

$$R = \int_3^\infty e^{-vX}\,d\sigma(v), \quad d\sigma = \sum_{n\ge 2}\left(n^4\delta_{\tau_n} + bn^2\tau_n e^{b(v-\tau_n)}1_{v\ge\tau_n}\,dv\right).$$

Define $P_0=1$, $P_{j+1}(z) = z(P_j - P_j')$. The physical derivative of e^{-vX} is $(-4)^j P_j(vX)e^{-vX}$. For $j\le 6$, both $P_j(9+w)$ and $[(z^j - P_j(z))]_{z=9+w}$ have nonnegative coefficients; so do the shifted ordinary derivatives of P_4, P_6. Hence the derivatives R_j alternate in sign and are bounded by the highest Laplace moments.

For every $n\ge 2$ their majorants are

$$M_{n,j}(X) = 4^j X^j e^{-\tau_n X}\left[n^4\tau_n^j + bn^2\tau_n\sum_{l=0}^{j}\binom{j}{l}\frac{\tau_n^{j-l}l!}{(X-b)^{l+1}}\right].$$

They decrease on $X\ge 3$ for $j\le 6$. Their complete $n\ge 3$ tails are bounded by

$$\mathcal{T}_j = \frac{C_j 3^{4+2j}}{20^8(1-q_j)}, \quad C_j = 12^j\left(1+\sum_{l=0}^{j}\binom{j}{l}l!(2/27)^l\right), \quad q_j = (4/3)^{4+2j}/20^7 < 1.$$

The ratio bound retains every atom. In particular

$$\begin{aligned}
|R_1| &\le 1104/8000 + (100/99)24\cdot 3^6/2^{24} < 1/4,\\
0 < R_2 &\le 44736/8000 + (100/99)288\cdot 3^8/2^{24} < 6,\\
|R_3| &\le M_{2,3}(3)e^9/8000 + \mathcal{T}_3 = 34736311628179317/147622497952000 < 256,\\
R &\le 28/8000 + 2\cdot 3^4/[20^8(1-(4/3)^4/20^7)] < 1/250.
\end{aligned}$$

Here $e^3 > 20$ follows from its Taylor polynomial through degree eight. The first two weaker tails use $e^{-24} < 2^{-24}$ and ratio $< 1/100$.

Set $U = 1 + R$. For the leading atom

$$\ell_1'' = -16[X + T_2(b/X)], \quad \ell_1'''' = -256[X + T_4(b/X)], \quad \ell_1^{(6)} = -4096[X + T_6(b/X)],$$

where

$$T_2(r) = \frac{r}{(1-r)^2}, \quad T_4(r) = \frac{r(1+4r+r^2)}{(1-r)^4}, \quad T_6(r) = \frac{r(1+26r+66r^2+26r^3+r^4)}{(1-r)^6}.$$

The fourth logarithmic correction is

$$(\log U)'''' = R_4/U - (4R_1R_3 + 3R_2^2)/U^2 + 12R_1^2R_2/U^3 - 6R_1^4/U^4.$$

It is at most $R_4 + 9/2$. The qualitative sign $\ell'''' < 0$ is also proved in Umihara [8]; we include the quantitative independent bounds needed here. The positive decreasing $n = 2$ fourth contribution is $C_4(X)e^{-3X}$, where

$$C_j(X) = 4^j \left[16P_j(3X) + \frac{18}{X-b} \sum_{l=0}^{j} P_j^{(l)}(3X) \left(\frac{X}{X-b} \right)^l \right] \quad (j = 4, 6).$$

Indeed $(P_4 - P_4')(9+w) = 1162 + 921w + 241w^2 + 26w^3 + w^4 > 0$, so its integrated fourth contribution decreases. Its $n \geq 3$ fourth tail is less than

$$\frac{100}{99} \frac{62208 \cdot 3^{12}}{20^8} = \frac{14348907}{11000000} < 2.$$

The exact endpoint coefficients are $C_4(3) = 43140096$, $C_4(16/5) = 42636828818522112/887410625$, $C_4(7/2) = 59421432$ and $C_4(4) = 278436163584/3125$. Direct substitution yields

X interval	upper for $C_4(X)e^{-3X}$	lower for $256[X + T_4(b/X)]$
$[3, 16/5]$	5393	5430
$[16/5, 7/2]$	3337	3801
$[7/2, 4]$	1748	2557
$[4, \infty)$	557	1024

The exponential lower bounds are respectively $8000, 14400, 34000, 160000$. They follow from $e^3 > 20$, $e^{3/5} > 9/5$, $e^{3/2} > 17/4$. Each final column exceeds the middle column plus $2 + 9/2$.

For the determinant, put $A = -\ell''$, $d = -\ell'''' > 0$, $e = \ell^{(6)}$. Direct paired differentiation gives

$$h_2 = -\ell''/2, \quad h_4 = (\ell'''' + 6\ell''^2)/8, \quad h_6 = -(\ell^{(6)} + 30\ell''\ell'''' + 60\ell''^3)/32,$$

$$64 \det G = 24A^4 - 18A^2d - d^2 - Ae. \tag{11}$$

The fourth logarithmic error budget is $46595/128$. Expanding $\log(1 + R)$ through order six and dropping its negative groups gives

$$(\log U)^{(6)} \leq R_6 + (15/8)R_4 + 841095/16.$$

The constants follow respectively from $4(1/4)256 + 3 \cdot 6^2 + 6(1/4)^4$ and $120(1/4)6 \cdot 256 + 30 \cdot 6^3 + 360(1/4)^4 6$. For a cell $[L, H]$, bound R_4 below by $C_4(H)e^{-3H}$ and above by $C_4(L)e^{-3L} + \mathcal{T}_4$. Bound

R_6 above by $C_6^+(L,H)e^{-3L}+\mathcal{T}_6$, where C_6^+ replaces polynomial arguments by $3H$, denominators by $L-b$, and $X/(X-b)$ by $L/(L-b)$. Positive shifted derivative coefficients justify the enclosure. With these bounds $r_{4,-}, r_{4,+}, r_{6,+}$ define

$$A_- = 16[L+T_2(b/H)]-6, \qquad A_+ = 16[H+T_2(b/L)]+1/16,$$
$$d_+ = 256[H+T_4(b/L)]-r_{4,-}/(251/250)+46595/128,$$
$$e_+ = -4096[L+T_6(b/H)]+r_{6,+}+(15/8)r_{4,+}+841095/16.$$

On the 96 closed cells $L=3+j/32$, $H=L+1/32$, rational arithmetic proves $A_->0$, $d_+>0$, $A_-^2\ge 3d_+/8$ and

$$24A_-^4-18A_-^2d_+-d_+^2-(A_+ \text{ if } e_+\ge 0 \text{ else } A_-)e_+ > 256000000.$$

The minimum lower floor is 264806350, and the minimum guard floor is 3928. The guard justifies the monotonic substitution in A; the other signs justify every remaining substitution. Exponential enclosures here use $\sum_{j=0}^{80} q^j/j!$ and its remainder $(q^{81}/81!)/(1-q/82)$ for $0\le q\le 18$. For $X\ge 6$, the sixth envelope at 6 gives $e<150000$; also $16X-6\le A\le 16X+33$ and $d\le 256X+782$. The resulting lower polynomial in $v=X-6$ has coefficients

$$(1211952476, 958667264, 274580480, 34209792, 1572864),$$

and its monotonicity guard has coefficients $(28923/4, 2784, 256)$. These are positive. Equation (11) proves globally $\det G\ge 4\cdot 10^6$ and $h_2\ge 21$. Thus both the finite interval and the remaining unbounded domain are covered.

B Whole-cell baseline and regional certificate

The separate baseline uses 1024 closed cells $[j/1024,(j+1)/1024]$, not point evaluations. Work in the unit-triangular basis

$$U=z,\quad V=z^3-139z,\quad R=z^5-461z^3+32200z.$$

The scalar monic innovation and matrix-weighted residual variance are invariant under these fixed lower-degree basis shifts. Let $b(s)=\pi e^{5s-X}$ and use tildes for moment numerators multiplied by Φ^2/b^2. Then

$$s^2\Phi^2\delta_s = s^2b^2(\widetilde{S}-\widetilde{r}^T\widetilde{G}^{-1}\widetilde{r}). \tag{12}$$

This exact cancellation avoids division by a small density.

To specify the complete derivative enclosures, use $q_0(z)=2z-3$, $q_{j+1}(z)=(5+4z\partial_z-4z)q_j(z)$ through $j=10$. The head derivative is

$$d_j=\sum_{n=1}^{3} n^2 q_j(n^2X)e^{-(n^2-1)X}.$$

For a whole cell with $X\in[X_-,X_+]$, the omitted derivative tail is bounded by the sum over monomials $q_{jl}z^l$ of

$$|q_{jl}|X_+^l 4^{2l+2}e^{-15X_-}\Big/\left[1-(5/4)^{2l+2}e^{-9X_-}\right].$$

Every denominator is positive. The paired head moment polynomial is formed before interval evaluation:

$$\widetilde{m}_{2k}^{(n,m)} = \frac{(-1)^k}{4^k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} [n^2 q_j(n^2 X)][m^2 q_{2k-j}(m^2 X)] e^{-(n^2+m^2-2)X},$$

summed for $1 \le n \le m \le 3$ with multiplicity two off the diagonal. With head and tail derivative bounds, the absolute moment error is

$$4^{-k} \sum_{j=0}^{2k} \binom{2k}{j} (|d_j| t_{2k-j} + t_j |d_{2k-j}| + t_j t_{2k-j}).$$

All cross terms are retained. The six fields are the linear combinations

$$\begin{aligned}
a &= m_2, \\
b &= m_4 - 139 m_2, \\
c &= m_6 - 278 m_4 + 19321 m_2, \\
r_1 &= m_6 - 461 m_4 + 32200 m_2, \\
r_2 &= m_8 - 600 m_6 + 96279 m_4 - 4475800 m_2, \\
S &= m_{10} - 922 m_8 + 276921 m_6 - 29688400 m_4 + 1036840000 m_2.
\end{aligned}$$

All are formed at numerator level before enclosure. Absolute errors are combined using absolute coefficients. On every cell the lower bounds for a and $ac - b^2$ are positive. The inverse penalty is at most the upper bound of $(cr_1^2 - 2br_1r_2 + ar_2^2)/(ac - b^2)$, and also at most $2(cr_1^2 + ar_2^2)/(ac - b^2)$ with absolute upper bounds. Use the smaller of these valid upper bounds, and subtract it from the lower endpoint for S in (12). Integrating the resulting interval times $s^2 b^2$ proves the baseline. The recorded rational lower exceeds 2300747; no adaptive subdivision is needed.

For normalization, $B_0 = 81/40$ and the positive theta series give, for $s \ge 1$,

$$\Phi(s) \le 2B_0 \pi X e^{5s - X}, \qquad H_y(s) \le e^{2s} \quad (0 \le y \le 1).$$

Indeed $|2n^2 X - 3| \le B_0 n^2 X$ for $X > 120$, and $\sum n^4 e^{-(n^2-1)X} < 2$ by the successive ratio $16e^{-3X} < 1/2$. Therefore

$$s^2 H_y \Phi^2 \le 4 B_0^2 \, 4^4 s^2 e^{20s - 6e^{4s}}.$$

Since $e^4 > 40$ and $e^3 > 20$, $20s - 6e^{4s} \le -220 - 940(s-1)$ and $e^{-220} < 20^{-73}$. The full normalizer tail is at most

$$\frac{4 B_0^2 \, 4^4}{20^{73}} \left(\frac{1}{940} + \frac{2}{940^2} + \frac{2}{940^3} \right). \tag{13}$$

The compact denominator interval for the whole bin $0 \le y \le 1/8$ contains Z_0 after adding this tail. Interval division of the baseline by this full denominator proves the 17 billion bound.

For the variance, integrate the five fields of G, r separately on $A = [0, 3/16]$ and $B = [3/16, 1]$, with the common factor $s^2 b^2 H_y$. Use the eight closed bins $j/8 \le y \le (j+1)/8$, $j = 0, \dots, 7$. Regional interval inverses have positive determinant. Enclose the second-coordinate contrast in (8), square its minimum absolute value, and divide by upper bounds for the inverse sum and full Z_y. The binwise gain floors exceed, respectively,

$$\begin{gathered}
1966012227,\ 1955643969,\ 1946391148,\ 1938242732, \\
1931186825,\ 1925210688,\ 1920300777,\ 1916442774.
\end{gathered}$$

This includes the $1/Z_y$ scaling of raw regional matrices. It certifies (9) independently of a total averaged innovation.

The interval script uses exact rational operations with outward rounding at 80 decimal places; the other finite checks use exact rational polynomial arithmetic. Pi is enclosed by the Machin alternating series; exponentials use a positive Taylor series after range reduction, with an explicit geometric remainder; the hyperbolic quotient uses its positive series. They reconstruct polynomials and tails instead of reading stored numerical enclosures. Explicit exception gates, active also under optimized Python, require raw baseline $> 2 \cdot 10^6$, normalized baseline $> 17 \cdot 10^9$, and every variance bound $> 10^9$. The public scripts are `verify_prefix.py` (lower-order controls and the 96-cell prefix), `verify_tail.py` (odd complete tail), `verify_even.py` (even complete tail), `verify_separated.py` (baseline and regional variance), and `verify_counterexample.py` (signed Gaussian control). The accompanying README specifies exact commands. Arithmetic checking supports the displayed analytic proof; the compact classification theorem is a cited input. These finite checks support the analytic argument.

C The monotone-weight condition

The monotone-weight condition is sufficient. The theorem concerns the odd block on $\{z, z^3, z^5\}$. The physical local six-point corollary has the fixed-shape scope stated above. The full degree-five consequence uses the separate even-block argument above. The single-crossing lemma, Schur identity and parallel-sum bound are classical tools; the contribution is their theta-specific hypotheses, the new unbounded tail and separated baseline, and the resulting weighted parameter extension.

References

[1] S. Schubr, *Minimum-Degree Obstructions in the Riemann Theta Product: Classification, transition bounds and robust Gram certificates*, edition 2, October 18, 2026, `https://luminaryoddities.com/researcharchive/minimum-degree-obstructions-riemann-theta-product/v/2`.

[2] S. Schubr, *Averaged Quintic Gate by Paired Kernel and Endpoint Conversion*, unpublished manuscript, 2026.

[3] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ-function*, Comput. Methods Funct. Theory **15** (2015), 373–391, arXiv:1309.0055.

[4] J. Gallier, *The Schur complement and symmetric positive semidefinite (and definite) matrices*, author notes, `https://www.cis.upenn.edu/~jean/schur-comp.pdf`.

[5] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594, doi:10.1016/0022-247X(69)90200-5.

[6] Z. Tarcsay, *On the parallel sum of positive operators, forms, and functionals*, arXiv:1501.01922, 2015.

[7] L. Smith, *Monotone comparative statics under uncertainty: interval dominance order*, author lecture notes, Integral Single Crossing Corollary, `https://www.lonessmith.com/wp-content/uploads/2025/03/1-Monotone-comparative-statics.pdf`.

[8] H. Umihara, *A Fourth-Derivative Inequality for the Riemann Xi Kernel*, preprint, August 2026, doi:10.5281/zenodo.21902312.