Moment certificates for homogeneous stable count laws

Scott Schubr
Luminary Oddities

Abstract

Let P be a probability law on nonnegative integer count vectors of total size n, with a homogeneous real-stable probability generating polynomial and positive mean vector t. We compare P with $M = \mathrm{Mult}(n, t/n)$ using the weighted covariance deficit

$$W = \frac{1}{2}\left(m - 1 - \sum_{i=1}^{m} \frac{\mathrm{Var}(A_i)}{t_i}\right).$$

We prove $\mathrm{TV}(P, M) \le \min\{1, (11/5)W\}$, independently of degree, category count and the mean vector. The argument transfers the centered elementary-symmetric-polynomial inequality of Han and Niles-Weed through Banach's theorem and an orthogonal multinomial density expansion. It retains explicit low orders, handles exceptional discrete root profiles, and combines exact polynomial certificates with positivity of the likelihood ratio. A classical hyperbolic-slice reduction also gives the stronger bound $\mathrm{TV}(P, M) \le \min\{1, 2W\}$ for every $1 \le n \le 50$. Standalone rational-arithmetic code verifies all finite and continuous-domain certificates. Every all-degree linear coefficient must be at least two. The results include nonfactorizable stable generating polynomials.

1 Setting and relation to prior work

A nonzero polynomial is *real stable* if it has real coefficients and no zero when every variable has positive imaginary part. Let

$$p(x) = \sum_{|a|=n} P(a)x^a, \qquad P(a) \ge 0, \qquad p(\mathbf{1}) = 1$$

be homogeneous and real stable. Here $|a| = \sum_i a_i$ and A has law P. Put $t_i = \mathbb{E}A_i > 0$, $q = t/n$, $L(x) = q \cdot x$, and $M = \mathrm{Mult}(n, q)$. Total variation is $\mathrm{TV}(P, M) = \frac{1}{2}\sum_a |P(a) - M(a)|$. Define

$$B = \frac{1}{2}\left(\mathrm{diag}(t) - \frac{tt^\top}{n} - \mathrm{Cov}(A)\right), \qquad W = \mathrm{Tr}(\mathrm{diag}(t)^{-1}B). \tag{1}$$

We will prove $B \succeq 0$. The identity $\sum_i t_i = n$ gives the formula for W in the abstract.

Multinomial approximation for dependent random variables is classical; Loh's Stein-method treatment [7] is a central predecessor. Ehm's binomial approximation [3] already treats the bivariate stable subclass, whose generating polynomials factor into linear forms. Roos [9] develops approximation of convolution products, including independent categorical sums.

The closest methodological predecessor is Han and Niles-Weed [5]. Their analysis of permutation mixtures combines centered elementary symmetric polynomials, Banach transfer and orthogonal chi-square expansions. Their Theorem 4.3(2), equation (27), supplies the coefficient inequality used

below. Related Banach and coefficient methods occur in Roos [10]; later centered-vector and matrix inequalities appear in [6]. Our argument adapts these tools to the centered hyperbolic expansion of an arbitrary homogeneous stable count law. For categorical observations, a single permutation mixture of independent coordinates has the same count generating polynomial as the underlying product of categorical linear forms. A general exchangeable mixture representation instead yields a mixture of independent reference laws; it does not directly give our one mean-matched multinomial or the deficit in (1).

The contribution here is this full-stable-class transfer, the refined universal covariance certificate with coefficient $11/5$, and a reproducible finite-degree verification of coefficient two. Banach's theorem, the centered ESP inequality, hyperbolic-slice extremality, and the orthogonal-expansion architecture are credited inputs rather than new general mechanisms. Fixed degree and category count are separate parameters throughout.

2 Centered expansion and the exact density identity

Degree one is exactly multinomial. In the lemmas below we therefore assume $n \geq 2$.

Lemma 1. *There are homogeneous polynomials R_k of degree k, $2 \leq k \leq n$, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L(x)^n + \sum_{k=2}^{n} R_k(x) L(x)^{n-k}. \tag{2}$$

Moreover $B \succeq 0$, $B\mathbf{1} = 0$, and $R_2(x) = -x^\top Bx$.

Proof. For real x, set $y = x - L(x)\mathbf{1}$. The polynomial $p(z\mathbf{1} + y)$ is monic of degree n. Every root is real: an upper-half-plane root contradicts stability, and a lower-half-plane root has an upper conjugate. Its coefficient of z^{n-1} is $t \cdot y = 0$. Write its roots as $\lambda_1, \ldots, \lambda_n$, so $\sum_j \lambda_j = 0$, and let $R_k(x)$ be its coefficient of z^{n-k}. This defines homogeneous polynomials, invariant under addition of $a\mathbf{1}$ because y is unchanged. Substitution $z = L(x)$ gives (2).

Write $H = \nabla^2 p(\mathbf{1})$. The coefficient of z^{n-2} is $\frac{1}{2} y^\top H y$. Euler's identity gives $H\mathbf{1} = (n-1)t$ and $\mathbf{1}^\top H \mathbf{1} = n(n-1)$, so

$$R_2(x) = \tfrac{1}{2} x^\top \left(H - \frac{n-1}{n} t t^\top \right) x.$$

Since $H = \mathrm{Cov}(A) + tt^\top - \mathrm{diag}(t)$, this is $-x^\top Bx$. On the other hand the root identity gives

$$R_2(x) = -\tfrac{1}{2} \sum_j \lambda_j^2, \qquad \sum_j \lambda_j^2 = 2x^\top Bx. \tag{3}$$

Thus $B \succeq 0$; the displayed formula or translation invariance gives $B\mathbf{1} = 0$. □

If $W = 0$, positive t and $B \succeq 0$ imply $B = 0$. Equation (3) then makes every centered root zero for every real x, so all R_k vanish and $P = M$. This proves rigidity without dividing by W. The cases $n = 1$ or $m = 1$ are also exact multinomial laws. Zero-mean categories can be deleted because their nonnegative counts vanish almost surely.

For a homogeneous polynomial R_k, let T_k denote its symmetric k-linear form, with $R_k(x) = T_k(x, \ldots, x)$. Our ordered-index convention means that a monomial coefficient at α is $(k!/\alpha!)T_{k,\alpha}$.

Lemma 2. *Let ν_k be the signed coefficient measure of $R_k L^{n-k}$. There exist mutually orthogonal mean-zero densities $U_k = d\nu_k/dM$ satisfying*

$$\frac{P}{M} - 1 = \sum_{k=2}^{n} U_k, \qquad \chi^2(P\|M) = \sum_{k=2}^{n} \mathbb{E}_M U_k^2, \tag{4}$$

and

$$\mathbb{E}_M U_k^2 = \binom{n}{k}^{-1} \sum_{i_1,\dots,i_k} \frac{T_k(e_{i_1},\dots,e_{i_k})^2}{q_{i_1}\cdots q_{i_k}}. \tag{5}$$

Proof. Translation invariance of R_k gives $T_k(\mathbf{1}, v_2, \dots, v_k) = 0$: first differentiate the diagonal polynomial and then polarize its remaining arguments. Let $X_1, \dots, X_n$ be independent categorical variables of law q and set

$$H_k(i_1,\dots,i_k) = \frac{T_k(e_{i_1},\dots,e_{i_k})}{q_{i_1}\cdots q_{i_k}}, \qquad U_k = \binom{n}{k}^{-1} \sum_{|S|=k} H_k(X_S).$$

Symmetry makes U_k a function of the count vector. Summing any one argument of H_k against q gives zero by the contraction identity. Hence the kernel is completely conditionally degenerate. For each fixed subset S,

$$\mathbb{E}[H_k(X_S) x^{\mathrm{counts}(X)}] = R_k(x) L(x)^{n-k}.$$

Coefficient comparison after symmetrization proves $\nu_k(a) = M(a)U_k(a)$. All $M(a)$ on $|a| = n$ are positive. In a product of two kernels associated with different subsets, an index in the symmetric difference occurs in only one kernel; integrate that independent categorical observation to obtain zero. This applies both within an order and between distinct orders. Thus only identical subsets contribute to the square norm, giving (5), while (2) gives (4). All operations are finite sums. □

3 Hilbert tensor transfer and a universal certificate

We use Banach's classical real Hilbert-space theorem [1, 2]: the norm of a symmetric multilinear form on arbitrary unit-vector tuples equals its norm on diagonal tuples. It is also the input in [5, Lemma 4.2].

Lemma 3. *If $|R_k(x)| \le c(x^\top B x)^{k/2}$ for all real x, then*

$$\mathbb{E}_M U_k^2 \le \frac{c^2 (nW)^k}{\binom{n}{k}}. \tag{6}$$

Proof. For $v \in \ker B$, the polynomial $R_k(x + sv)$ is bounded independently of real s, hence is constant. Differentiation and polarization give $T_k(v, v_2, \dots, v_k) = 0$. Thus T_k descends to the real Hilbert quotient by $\ker B$, with squared norm $x^\top B x$. Banach's theorem there implies

$$|T_k(v_1,\dots,v_k)| \le c \prod_j \sqrt{v_j^\top B v_j}.$$

Apply this to coordinate vectors in (5) and sum the products to obtain

$$\mathbb{E}_M U_k^2 \le \binom{n}{k}^{-1} c^2 \left(\sum_i \frac{B_{ii}}{q_i}\right)^k = \binom{n}{k}^{-1} c^2 (nW)^k.$$

No inverse of B is used. □

Han and Niles-Weed [5, Theorem 4.3(2)] prove that a real centered vector z with $\sum_j z_j^2 = n$ satisfies $|e_k(z)|^2 \le 10\binom{n}{k}$. Scaling the centered roots in (3) therefore yields, with $u = x^\top Bx$,

$$|R_k(x)|^2 \le 10\binom{n}{k}(2u/n)^k.$$

The zero case $u = 0$ follows directly from the roots. Lemmas 2 and 3 now give

$$\mathbb{E}_M U_k^2 \le 10(2W)^k, \qquad \chi^2(P\|M) \le \frac{40W^2}{1-2W} \quad (2W < 1). \tag{7}$$

The next elementary bound retains positivity of the actual density. It is the upper-distance branch of the classical sharp relation between chi-square divergence and total variation; see Guntuboyina, Saha and Schiebinger [4, Corollary 5.3], which credits Gilardoni's Proposition 2.1. We give the short proof needed here.

Lemma 4. *For probability measures $P \ll M$, write $v = \chi^2(P\|M)$ and $\delta = \mathrm{TV}(P, M)$. If $\delta > 1/2$, then $v \ge \delta/(1-\delta)$.*

Proof. Let $Y = dP/dM - 1$ and $a = M(Y < 0)$. Then $\mathbb{E}Y_+ = \mathbb{E}Y_- = \delta$ and $Y_- \le 1$, so $a \ge \delta$. Cauchy–Schwarz on the two parts of the partition gives

$$v \ge \frac{\delta^2}{a} + \frac{\delta^2}{1-a} = \frac{\delta^2}{a(1-a)}.$$

For $a \ge \delta > 1/2$, $a(1-a) \le \delta(1-\delta)$. The degenerate partition cases force $\delta = 0$ and do not occur here. □

The uniform coefficient estimate (7) is sufficient for an absolute linear certificate. To obtain the sharper coefficient 11/5, we retain individual low-order terms before using its geometric tail. We first establish the finite-degree result and then give the full large-degree argument.

4 Exact finite-degree certificates

The coefficient extremum reduction used next is classical: see Riener and Schabert [8, Theorem 2.8 and Corollary 2.10], as well as the real-vector reduction in [5]. We include the needed special proof.

Lemma 5. *For fixed $n \ge 3$ and $3 \le k \le n$, the maximum of $|e_k(\lambda)|$ on $\sum_i \lambda_i = 0$, $\sum_i \lambda_i^2 = 2$ is attained at a vector with two distinct values.*

Proof. The slice is nonempty and compact. Maximize either sign of e_k, and among maximizers select one with the largest number d of distinct roots of $f(z) = \prod_i (z - \lambda_i)$. If $3 \le d < n$, write $f = hg$, where g contains each distinct linear factor once and h retains multiplicity minus one. For either sufficiently small sign of ε, $g + \varepsilon$ has d simple real roots near the old roots. Thus $f + \varepsilon h = h(g + \varepsilon)$ is real-rooted and preserves the first two coefficients, since $\deg h = n - d \le n - 3$. If the target coefficient of h is nonzero, one sign improves the objective. Otherwise the objective is constant. Every repeated old root remains through h and also supplies a new nearby root through $g + \varepsilon$; no new root equals an old one. Taking disjoint neighborhoods of distinct old roots shows the distinct-root count strictly increases, a contradiction. If $d = n$, perturbing by εz^{n-k} preserves the first two coefficients and real-rootedness for both small signs, but changes the objective. Hence $d \le 2$. The positive square sum rules out $d = 1$. Apply this argument to both signs to obtain the absolute-value assertion. □

For $r+s=n$, a normalized two-valued vector has values a repeated r times and $b=-ra/s$ repeated s times, with $a^2=2s/(nr)$. Thus define the exact rational constants

$$c_{n,k}^2=\max_{1\le r<n}\left(\frac{2s}{nr}\right)^k\left[\sum_{j=\max(0,k-s)}^{\min(k,r)}\binom{r}{j}\binom{s}{k-j}(-r/s)^{k-j}\right]^2, \tag{8}$$

$$A_{n,k}=\frac{n^k c_{n,k}^2}{\binom{n}{k}},\qquad V_n(w)=\sum_{k=2}^{n}A_{n,k}w^k.$$

For $k=2$, $e_2=-1$ on the normalized slice, so the same formula applies. Homogeneity, Lemma 5 and the tensor transfer prove

$$\chi^2(P\|M)\le V_n(W). \tag{9}$$

These constants come from proved extrema and finite arithmetic.

Proposition 6. *If*

$$S_n=\sum_{k=2}^{n}A_{n,k}4^{-(k-2)}\le 16,\qquad D_n=\sum_{k=2}^{n}A_{n,k}\frac{(k-1)^{k-1}}{2^{k-1}k^k}\le 2,$$

then every law of degree n in our setting satisfies $\mathrm{TV}(P,M)\le\min\{1,2W\}$.

Proof. For $0<W\le 1/4$, nonnegative coefficients give $V_n(W)\le S_nW^2\le 16W^2$, and $\delta\le\sqrt{v}/2$ proves $\delta\le 2W$. For $1/4<W<1/2$, differentiation gives

$$\max_{0\le w\le 1/2}w^{k-1}(1-2w)=\frac{(k-1)^{k-1}}{2^{k-1}k^k}.$$

Consequently $(1-2W)V_n(W)/W\le D_n\le 2$, hence $v\le 2W/(1-2W)$. If $\delta\le 1/2$ the conclusion is immediate. Otherwise Lemma 4 gives $\delta\le v/(1+v)\le 2W$. For $W\ge 1/2$ use $\delta\le 1$, and for $W=0$ use rigidity. □

Theorem 7. *For every degree $1\le n\le 50$, every category count, and every positive mean vector in the stable setting,*

$$\mathrm{TV}(P,\mathrm{Mult}(n,t/n))\le\min\{1,2W\}.$$

Proof. Degree one is exact. For $2\le n\le 50$, the accompanying standalone program `verify_coefficients.py` evaluates every candidate in (8) with integer and rational arithmetic, forms S_n, D_n, and checks their exact inequalities against 16 and 2. All 49 degrees pass. The largest S_n is $981/200$ at $n=6$, and all $D_n<141/100<2$. Proposition 6 proves the result.

For reproducibility, the program uses integer roots s repeated r times and $-r$ repeated s times. Their squared sum is nrs; their degree-k elementary coefficient is

$$b_{r,s,k}=\sum_j\binom{r}{j}\binom{s}{k-j}s^j(-r)^{k-j}.$$

The normalized squared candidate is $b_{r,s,k}^2/(nrs/2)^k$. This is algebraically identical to (8), avoids square roots entirely, and is the full finite computation required for the theorem. The program enumerates all $r=1,\dots,n-1$ and all $k=2,\dots,n$, using Python's standard-library arbitrary-precision integers and `Fraction`. It checks the stated maximum of S_n and the stronger common rational bound $D_n<141/100$. There is no floating-point decision in the certificate. The analytic reduction and finite code are both parts of this proof; no inference to untested degrees is made. □

For illustration, at $n = 8$ the coefficients $A_{8,2}, \dots, A_{8,8}$ are

$$\frac{16}{7},\ \frac{256}{49},\ \frac{288}{35},\ \frac{4096}{189},\ \frac{256}{7},\ \frac{12288}{125},\ 256.$$

Here $S_8 = 12558191/2646000$ and $D_8 = 15114877380793361149/15738151816396800000$. Although $V_8(1/2)/(1/2)^2 > 16$, the positive-density criterion succeeds. Thus discarding positivity at the last TV step would leave an avoidable obstruction.

5 Sharper low orders in large degree

For the remaining argument suppose $n \geq 51$. We normalize centered real root vectors by $\sum_i z_i^2 = n$. If

$$\frac{|e_k(z)|^2}{\binom{n}{k}} \leq b_k, \tag{10}$$

then scaling (3) and applying Lemma 3 gives $\mathbb{E}_M U_k^2 \leq 2^k b_k W^k$. All root extrema below use Lemma 5 one coefficient at a time; their maximizing profiles need not coincide.

5.1 Discrete exceptional profiles

Lemma 8. *If a centered, variance-normalized two-valued vector has either root value with multiplicity one or two, then* (10) *holds with* $b_k = 1$ *at every order.*

Proof. Negation and permutation allow us to treat the smaller specified block. For multiplicity one the values are $\sqrt{n-1}$ once and $-1/\sqrt{n-1}$ repeated $n-1$ times. Expansion gives

$$e_k = (-1)^k (1-k)\binom{n}{k}(n-1)^{-k/2}, \qquad \frac{e_k^2}{\binom{n}{k}} = \frac{(k-1)^2\binom{n}{k}}{(n-1)^k}.$$

The ratios at $k = 0, 1$ are one and zero. At $k = 2$ the ratio is $n/[2(n-1)] \leq 1$. For fixed $k \geq 3$, regard the last expression as a function of real $n \geq k$. Its logarithmic derivative is

$$-\frac{1}{n(n-1)} + \sum_{j=2}^{k-1} \frac{j-1}{(n-j)(n-1)} > 0,$$

because the $j = 2$ term exceeds the negative term. It therefore increases to $(k-1)^2/k! \leq 2/3$. This last sequence starts at $2/3$ for $k = 3$ and decreases thereafter, since its next-term ratio is $k^2/[(k+1)(k-1)^2] < 1$.

For multiplicity two, assume $n \geq 4$; the case $n = 3$ is already multiplicity one after reflection. The values are $\sqrt{(n-2)/2}$ twice and $-\sqrt{2/(n-2)}$ repeated $n-2$ times. Direct expansion gives

$$\frac{e_k^2}{\binom{n}{k}} = \binom{n}{k}\left(\frac{2}{n-2}\right)^k F(n,k)^2, \qquad F(n,k) = \frac{(k-1)((k-4)n+4)}{4(n-1)}. \tag{11}$$

The orders zero and one are immediate. At order two the general centered identity gives ratio $n/[2(n-1)] \leq 1$. For order three, two-valued extrema have $e_3 = nc/3$, with $c = (s-r)/\sqrt{rs}$, and $c^2 \leq (n-2)^2/(n-1)$. Hence the ratio is at most $\frac{2}{3}n(n-2)/(n-1)^2 < 1$. For order four, the centered fourth-moment bound proved below gives ratio at most $3n(n-2)/[8(n-1)(n-3)] \leq 1$ for $n \geq 4$; the cleared difference $5n^2 - 26n + 24$ is zero at four and increases thereafter.

For $5 \le k \le 11$, clearing positive denominators in (11) reduces the assertion to

$$16k!(n-1)^2(n-2)^k - 2^k(n)_k(k-1)^2((k-4)n+4)^2 \ge 0. \tag{12}$$

Put $n = k + x$. The companion exact program expands these seven integer polynomials. Every coefficient is strictly positive; their minimum coefficients, in order $k = 5, \dots, 11$, are

$$1408,\ 5120,\ 39168,\ 444416,\ 4986880,\ 55074816,\ 628633600.$$

The program constructs both products in (12) by integer linear-factor multiplication and checks every coefficient, so this proves the inequality for all real $x \ge 0$, not just a finite list of n.

For $n \ge k \ge 12$, the factor F is nonnegative and is at most $nk(k-1)/[4(n-1)] \le (3/11)k^2 < k^2/3$. Also $\binom{n}{k} \le n^k/k!$ and

$$\left(\frac{n}{n-2}\right)^k \le \left(\frac{k}{k-2}\right)^k < 9.$$

Indeed $t\log(t/(t-2))$ decreases for $t > 2$, and at twelve $6^{12} < 9 \cdot 5^{12}$. Thus the ratio in (11) is at most $2^k k^4/k! < 1$. This holds at twelve because $2^{12}12^4 < 12!$ and decreases thereafter, since its next-term ratio is $2(k+1)^3/k^4 < 1$. This completes the all-degree exceptional-profile proof. □

5.2 Orders two through six

Lemma 9. *For $n \ge 51$, the orthogonal density components satisfy $\mathbb{E}_M U_k^2 \le a_k W^k$ at orders two through six, where*

$$a_2 = \frac{51}{25}, \quad a_3 = \frac{16}{3}, \quad a_4 = \frac{2499}{400}, \quad a_5 = 15, \quad a_6 = 50.$$

Proof. Write $p_j = \sum_i z_i^j$, so $p_1 = 0, p_2 = n$. At order two $e_2 = -n/2$, giving $a_2 = 2n/(n-1) \le 51/25$. At order three the computation in Lemma 8 bounds the scalar ratio by $\frac{2}{3}n(n-2)/(n-1)^2 \le 2/3$, giving $a_3 \le 16/3$.

At order four Newton's identity gives $e_4 = (n^2 - 2p_4)/8$. Centering implies $z_i^2 \le n-1$, by Cauchy–Schwarz on the other coordinates. Therefore $n \le p_4 \le n(n-1)$ and $|e_4| \le n(n-2)/8$. After multiplication by 2^4 and division by $\binom{n}{4}$, the bound is $6n(n-2)/[(n-1)(n-3)]$. This decreases for $n \ge 51$ and its value at 51 is $2499/400$.

For higher orders use the two-valued reduction with $a = \sqrt{s/r}$, $b = -\sqrt{r/s}$, $r + s = n$, and $c = a + b$. Its generating function $F(T) = (1+aT)^r(1+bT)^s$ obeys

$$(1 + cT - T^2)F'(T) = -nTF(T), \qquad (k+1)e_{k+1} = -cke_k - (n-k+1)e_{k-1}. \tag{13}$$

Put $x = c^2$ and $a_0 = 5n - 12$. Recurrence (13) gives $e_5 = nc(6x - a_0)/30$, with $0 \le x \le (n-2)^2/(n-1)$. The maxima of $x(6x-a_0)^2$ on this interval occur at its right endpoint or at $a_0/18$, if that point belongs to the interval: its derivative is $(6x-a_0)(18x-a_0)$, while zero and $a_0/6$ give zero values. At the right endpoint the scalar ratio is

$$\frac{2}{15}\frac{n(n-2)(n-3)(n-4)}{(n-1)^4} < \frac{2}{15}.$$

At $a_0/18$ it is

$$\frac{4}{1215}\frac{n(5n-12)^3}{(n-1)(n-2)(n-3)(n-4)} \le \frac{100}{243}\frac{n(n-2)^2}{(n-1)(n-3)(n-4)}.$$

The right side decreases with n, since it is a constant times the product $n/(n-1)$, $(n-2)/(n-3)$ and $(n-2)/(n-4)$. Multiplying by 2^5, its value at 51 is $163268/11421 < 15$. The other endpoint contributes less than $64/15 < 15$. This proves $a_5 = 15$.

At order six set $y = c^2/n$, $z_0 = 1/n$. The recurrence gives

$$\frac{e_6}{n^3} = f_6(y, z_0) = -\frac{y^2}{6} + \frac{13 - 36z_0}{72}y - \frac{(1-4z_0)(1-2z_0)}{48}.$$

Here $0 \le y \le 1$, $0 \le z_0 \le 1/51$. This is concave in y. Its unrestricted vertex value is $(97 - 504z_0 + 720z_0^2)/3456 \le 97/3456$. The endpoint values are $-(1-4z_0)(1-2z_0)/48$ and $-(1+54z_0+24z_0^2)/144$, both at least $-1/48$ on the specified interval. Thus $|f_6| \le 97/3456$. Since $\prod_{j=0}^{5}(1 - jz_0)$ decreases with z_0 there,

$$a_6 \le \frac{2^6\, 6!(97/3456)^2}{\prod_{j=0}^{5}(1 - j/51)} = \frac{13359434513}{271201280} < 50.$$

The tensor transfer now proves each assertion. □

5.3 Orders seven through ten and exact rectangle certificates

For two-valued profiles with $r, s \ge 3$, assume $r \le s$ by negation. Then

$$0 \le y = \frac{c^2}{n} = \frac{(s-r)^2}{nrs} \le \frac{1}{r} \le \frac{1}{3}, \qquad 0 < z_0 = \frac{1}{n} \le \frac{1}{51}.$$

Define polynomials P_k by

$$\frac{e_k}{n^{k/2}} = y^{(k \bmod 2)/2} P_k(y, z_0), \quad P_0 = 1,\ P_1 = 0,$$
$$(k+1)P_{k+1} = -ky^{k \bmod 2} P_k - (1 - (k-1)z_0)P_{k-1}. \tag{14}$$

This follows from (13), with $y = 0$ covered by polynomial continuation. With $D_k(z_0) = \prod_{j=0}^{k-1}(1 - jz_0) > 0$, the scalar ratio is exactly

$$\frac{e_k^2}{\binom{n}{k}} = \frac{k! y^{k \bmod 2} P_k(y, z_0)^2}{D_k(z_0)}.$$

The code verifies that, for $k = 7, 8, 9, 10$,

$$G_k(y, z_0) = D_k(z_0) - k! y^{k \bmod 2} P_k(y, z_0)^2 \ge 0 \quad \text{on } [0, 1/3] \times [0, 1/51]. \tag{15}$$

This is an exact polynomial inequality on a continuous rectangle, not a sample of root profiles. Combined with Lemma 8, it covers every discrete multiplicity and proves

$$\mathbb{E}_M U_k^2 \le 2^k W^k, \qquad 7 \le k \le 10, \tag{16}$$

for every $n \ge 51$.

We specify the finite certificate mechanism. On $[a, b]$ substitute $x = a + (b-a)t$. If a polynomial of degree d becomes $\sum_j v_j t^j$, its Bernstein coefficients are

$$\beta_i = \sum_{j=0}^{i} v_j \frac{\binom{i}{j}}{\binom{d}{j}}, \qquad p(a + (b-a)t) = \sum_{i=0}^{d} \beta_i \binom{d}{i} t^i (1-t)^{d-i}. \tag{17}$$

All basis functions are nonnegative on $[0,1]$. The bivariate transform applies this formula in each coordinate. The standalone code constructs (14) and (15) with rational arithmetic, transforms the gaps on the specified rectangles, and accepts a box only if every Bernstein coefficient is nonnegative. Otherwise it divides the box into two exact halves and certifies both children; thus the accepted leaves cover the original rectangle. Orders seven and eight require one box each, and orders nine and ten require two each. The optional JSON output records all power coefficients, boxes and Bernstein arrays. This finite computation proves (15); no floating-point or midpoint test can accept a box.

6 The universal coefficient eleven fifths

Theorem 10. *For every homogeneous stable count law in our setting,*

$$\mathrm{TV}(P, \mathrm{Mult}(n, t/n)) \le \min\{1, (11/5)W\}.$$

Proof. If $n \le 50$, apply Theorem 7. Suppose $n \ge 51$, and let $a_2, \dots, a_6$ be as in Lemma 9, with $a_k = 2^k$ for $7 \le k \le 10$. Define

$$V(w) = \sum_{k=2}^{10} a_k w^k + \frac{20480 w^{11}}{1-2w}, \qquad N(w) = (1-2w)V(w).$$

Orthogonality, Lemma 9, (16), and the credited Han–Niles-Weed estimate $\mathbb{E}U_k^2 \le 10(2W)^k$ for $k \ge 11$ give $v = \chi^2(P\|M) \le V(W)$ for $0 \le W < 1/2$. The finite tail is bounded by the infinite geometric tail, since all its terms are nonnegative.

Put $C = 11/5$. On $0 < W \le 1/(2C) = 5/22$, $V(W)/W^2$ increases, and its exact value at $5/22$ is less than $4C^2 = 484/25$. The accompanying code checks this rational comparison directly. Hence $\delta \le \sqrt{v}/2 \le CW$ throughout this interval.

On $5/22 < W < 5/11$, suppose $\delta > CW$. Lemma 4 implies $v > CW/(1-CW)$. The opposite weak inequality follows from $v \le V(W)$ and

$$Q(W) = C(1-2W) - (1-CW)\frac{N(W)}{W} \ge 0. \tag{18}$$

Here $N(W)$ is divisible by W^2, so Q is a polynomial; both original rational denominators are positive. The code constructs Q directly from the displayed constants and certifies (18) on $[5/22, 5/11]$ by the exact Bernstein transformation (17). Three covering subintervals suffice. It checks their endpoints and exact adjacency, and every coefficient on every accepted interval is nonnegative. Therefore (18) is a proved continuous-domain inequality, contradicting the assumed strict lower bound on v. Finally, $W \ge 5/11$ is covered by $\delta \le 1 \le CW$, and $W = 0$ by rigidity. □

The program `verify_refined.py` checks every finite computation used here. These are the degrees two through fifty, the seven polynomial identities for multiplicity two, the four rectangle gaps, and the final global gap and endpoint comparison. The program is self-contained and uses only the Python standard library. The option `--output`, followed by a filename, saves every exact rational certificate. Explicit failure checks remain active under Python optimization. The analytic reductions, the factorial tail for multiplicity two, and the finite certificate calculations are all dependencies of the theorem. A finite root sample or extrapolation cannot replace them.

7 Limitations and the coefficient-two question

Every universal coefficient is at least two. For even n, take the stable monomial $p(x_1, x_2) = x_1^{n/2} x_2^{n/2}$. Then $W = 1/2$ and

$$\mathrm{TV}(P, \mathrm{Mult}(n, 1/2)) = 1 - 2^{-n} \binom{n}{n/2} \longrightarrow 1.$$

This elementary binomial example gives the lower barrier. Consequently the optimal all-degree coefficient satisfies

$$2 \leq C_{\mathrm{opt}} \leq \frac{11}{5}.$$

The number fifty is a verified computational cutoff.

The mean vector and covariance are treated as known inputs. We prove no sample-complexity result for estimating W, no practical testing guarantee, no sampler construction and no computational speedup. Stability is a substantive hypothesis; matching a few moments without it does not certify the full law. The finite result and universal theorem allow singular covariance deficit, repeated roots and zero coefficients. Signed centered-expansion terms are not themselves probability laws, so the positivity lemma is applied only to the full density P/M.

Computational and AI assistance

The certificates use only the accompanying standard-library Python sources, which terminate with an explicit failure if a required inequality fails. The finite-degree calculation is in `verify_coefficients.py`; the complete refined verification, including that calculation, is independently self-contained in `verify_refined.py`. The latter optionally writes all exact rational certificate data to a reader-selected output file. No random sampling or external numerical package is used. Generative AI systems assisted with mathematical exploration, literature discovery, proof drafting, exact-code development and independent adversarial proof checks. These checks are internal AI assistance. Mathematical claims are presented with their proofs and reproducible computation so that they can be assessed independently; the author retains responsibility for the manuscript. Classical inputs and the close Han–Niles-Weed methodology are explicitly attributed.

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