Minimum-Degree Obstructions in the Riemann Theta Product

Classification, transition bounds and robust Gram certificates

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Luminary Oddities

October 18, 2026

Abstract

We classify the least degree of a polynomial whose square has negative signed-moment value for the normalized fixed-center Riemann theta product. On $3 \le x \le 12$ the degree has fourteen transitions, takes values between five and fifteen, and is nonmonotone. We isolate every transition and determine the degree at equality. Jacobi symmetry extends the complete classification to $\pi^2/12 \le x \le 12$. We quantify how the two- and three-summand theta models shift all fourteen crossings: the respective root errors are below 0.00049 and 2.588×10^{-10}. A constructive finite-difference bound converts polynomial obstructions into finite translation Gram witnesses. Twenty explicit rational certificates include a seven-point example stable under absolute entrywise errors of 1.94×10^{-7}. We also prove fixed-degree Gaussian limits and single-summand degree-five positivity. Infinite tails, directed interval covers and independent rational checks are included in a self-contained reproduction supplement.

All assertions concern the fixed-center product (2). **Keywords:** Riemann theta density; moment matrices; polynomial-square obstructions; Gram certificates; minimum degree; computer-assisted proof.

1 The fixed-center moment problem

Let

$$\Phi(s) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

For $x > 0$, write $s(x) = \frac{1}{4}\log(x/\pi)$ and set

$$p_x(t) = \frac{\Phi(s(x) + t/2)\Phi(s(x) - t/2)}{\Phi(s(x))^2}. \tag{2}$$

The density is even and strictly positive on $\mathbb{R}$ (Lemma 5), so the normalization is defined everywhere. Normal convergence of every fixed derivative on compact real intervals justifies differentiation. Define

$$m_{2j}(x) = (-1)^j p_x^{(2j)}(0), \quad m_{2j+1}(x) = 0, \qquad M_d(x) = (m_{j+k}(x))_{j,k=0}^{d}. \tag{3}$$

Here $m_0 = 1$. Let $\mathcal{L}_x$ be the real polynomial functional with $\mathcal{L}_x(X^j) = m_j(x)$. A polynomial P is an obstruction when $\mathcal{L}_x(P^2) < 0$, and

$$\delta(x) = \min\{\deg P : \mathcal{L}_x(P^2) < 0\}, \tag{4}$$

with value ∞ when the set is empty. For $P = \sum_{j=0}^{d} c_j X^j$, $\mathcal{L}_x(P^2) = c^{\mathsf{T}} M_d c$.

The initial explicit quintic witness appears in Schubr [8]; the October 17 independent rational reconstruction [9] gives a different verification of the same obstruction theorem. The local quintic transition

a_1 on $[3.64, 3.66]$ was established in [11] (October 17; corrected October 18, 2026). This manuscript expands the published $[3, 8]$ classification [10], whose edition 1 remains preserved. That edition supplied eleven transitions and maximum degree fourteen. Here we extend the classification to $[3, 12]$, adding three transitions and determining maximum degree fifteen. Relative to the local quintic theorem, thirteen of the fourteen transitions are additional. We quantify finite-summand root errors, construct robust finite Gram witnesses, and retain the fixed-degree constraints outside the classified domain.

Moment optimization and Schur complements are established tools [2, 3]; polynomial-square negativity tests occur in Kot et al. [5]. Csordas [1] studies associated theta kernels obtained by integrating over the center. Certified theta-derivative inequalities and derivative-sensitive tail estimates appear in Sra [12] and Planat–Solé [13]. Michałowski's July 20, 2026 version 2 [14] retains certified finite failures of Pólya-frequency order five for shifted Toeplitz minors of the density. It withdraws the earlier certified global-sign and unique-threshold claims because of an unsound derivative-tail enclosure. Its determinant functional differs from (3). Schur perturbation estimates have a classical matrix-analysis literature [6]; finite sampled positivity tests with error constraints also occur in Mari et al. [7].

2 Complete minimum-degree classification

Theorem 1. *There are exactly fourteen transition parameters $3 < a_1 < \cdots < a_{14} < 12$ for δ on $[3, 12]$. Their strict brackets and crossing degrees d_i are given in Table 1. Put $a_0 = 3$, $a_{15} = 12$, and*

$$(v_0, \dots, v_{14}) = (5, 6, 7, 9, 10, 9, 10, 11, 13, 14, 13, 12, 13, 14, 15). \tag{5}$$

For $0 \le i \le 14$ and $a_i < x < a_{i+1}$, $\delta(x) = v_i$. Also $\delta(3) = 5$, $\delta(12) = 15$, and

$$\delta(a_i) = \max(v_{i-1}, v_i) \qquad (1 \le i \le 14). \tag{6}$$

At a_i, M_{d_i} is positive semidefinite with a one-dimensional nullspace. On a neighborhood of a_i its monic Schur contraction has a simple zero. In particular, $\max_{3 \le x \le 12} \delta(x) = 15$.

i	d_i	Lower endpoint for a_i	Upper endpoint for a_i	$\delta(a_i)$
1	5	3.64888758188562	3.64888758188563	6
2	6	4.11304745500471102656	4.11304745500471102658	7
3	7	4.14157658489452707551	4.14157658489452707553	9
4	9	4.36591915914560481792	4.36591915914560481794	10
5	9	5.38557526472222776839	5.38557526472222776841	10
6	9	6.29313187387534279042	6.29313187387534279044	10
7	10	6.90841574564289048292	6.90841574564289048294	11
8	11	7.10426848885908199356	7.10426848885908199358	13
9	13	7.30607150627930169343	7.30607150627930169345	14
10	13	7.77596605119156831048	7.77596605119156831050	14
11	12	7.79865201846424367518	7.79865201846424367520	13
12	12	8.77114292644441822659270797	8.77114292644441822659279267 5	13
13	13	9.85021335406866974640418406	9.85021335406866974640426876	14
14	14	10.92652479112676554693239122	10.92652479112676554693247592	15

Table 1: Strict transition brackets. The crossing degree is $d_i = \min(v_{i-1}, v_i)$; a null direction at equality is not a negative obstruction. Printed decimals are exact rational endpoints.

Corollary 2. *For every $x > 0$, $M_d(x) = M_d(\pi^2/x)$ and $\delta(x) = \delta(\pi^2/x)$. Thus Theorem 1 determines δ on $[\pi^2/12, 12]$; reflected transitions are π^2/a_i with reversed orientation. Throughout this domain $M_4 > 0$ and an obstruction of degree at most fifteen exists.*

2.1 Schur contractions and transition windows

Parity splits M_d into even and odd monomial blocks. When $M_{d-1} = B > 0$, write

$$M_d = \begin{pmatrix} B & w \\ w^\mathsf{T} & m_{2d} \end{pmatrix}, \qquad S_d = m_{2d} - w^\mathsf{T} B^{-1} w,$$
$$P_d = X^d - (B^{-1}w)^\mathsf{T}(1, X, \dots, X^{d-1}).$$

For every monic $P = X^d + c^\mathsf{T}(1, \dots, X^{d-1})$,

$$\mathcal{L}_x(P^2) = S_d + (c + B^{-1}w)^\mathsf{T} B (c + B^{-1}w). \tag{7}$$

Thus $S_d < 0$ gives $\delta = d$, whereas $S_d = 0$ gives a unique monic null polynomial. The latter does not determine δ until higher degrees are checked. With the positive B_0 from (11), put

$$\mathcal{N}_x = B_0^2 \mathcal{L}_x, \qquad \widetilde{S}_d = B_0^2 S_d. \tag{8}$$

This multiplication preserves signs and nullspaces.

Lemma 3. *Let $J = [l, r]$ satisfy $M_{d-1} > 0$. Suppose $\widetilde{S}_d$ has opposite endpoint signs and a derivative of constant strict sign on J. Let $h \in \{d+1, d+2\}$. Suppose the opposite-parity monomial block through degree $h-1$ is positive definite on J, and a fixed degree-h polynomial Q satisfies $\mathcal{N}_x(Q^2) < 0$ on J. Then S_d has a unique zero $a \in J$. On its negative side $\delta = d$; on its positive side $\delta = h$; and $\delta(a) = h$.*

Proof. Strict monotonicity and endpoint signs give the unique zero. Equation (7) determines the crossing parity block through degree d. That parity contributes no additional degree below h, since $h \le d + 2$. With the opposite-parity hypothesis this gives $M_{h-1} > 0$ on the positive side and $M_{h-1} \ge 0$ at a. The fixed negative degree-h witness proves the asserted minimum degrees. On the negative side, $M_{d-1} > 0$ and $S_d < 0$ give $\delta = d$. □

This accounts for the skipped minimum degrees eight and twelve at a_3 and a_8: the opposite-parity block remains positive while the next direction of the crossing parity is negative. The decreases $10 \to 9$ at a_5 and $14 \to 13$ at a_{10} show that the least degree is not monotone.

Proposition 4. *Where $M_{d-1} > 0$, $\mathcal{L}_x(P_d Q) = 0$ for every $\deg Q < d$, and*

$$S_d'(x) = \mathcal{L}_x'(P_d^2). \tag{9}$$

At a simple zero a, if q is the coefficient vector of P_d, the eigenvalue of M_d vanishing there has derivative $S_d'(a)/\|q\|_2^2$.

Proof. The orthogonality equations are $Bc + w = 0$. Differentiation gives (9), since $2\mathcal{L}_x(P_d \partial_x P_d) = 0$ by $\deg \partial_x P_d < d$. At the zero the nullspace is spanned by q and the other eigenvalues are positive. Differentiating the symmetric eigenproblem gives $q^\mathsf{T} M_d' q / \|q\|_2^2 = S_d'(a)/\|q\|_2^2$. □

3 Certified evaluation and proof of the classification

3.1 Derivative recurrence and infinite tails

The factorization

$$A_x(t) = \sum_{n \ge 1} n^2 (2n^2 x e^{2t} - 3) e^{-n^2 x (e^{2t} - 1)} e^{-(n^2 - 1)x}, \tag{10}$$
$$\Phi(s + t/2) = \pi e^{5s - x} e^{5t/2} A_x(t) \quad (x = \pi e^{4s})$$

gives $p_x(t) = A_x(t)A_x(-t)/A_x(0)^2$. Define

$$R_0(z) = 2z - 3, \qquad R_{k+1}(z) = 2z(R_k'(z) - R_k(z)),$$
$$B_k(x) = \sum_{n\ge 1} n^2 R_k(n^2x)e^{-(n^2-1)x}. \tag{11}$$

Then $A_x^{(k)}(0) = B_k$, $\deg R_k = k+1$, and $B_0 > 0$ for $x \ge 3$. The unnormalized moments are

$$N_{2j} = B_0^2 m_{2j} = (-1)^j \sum_{u=0}^{2j} (-1)^u \binom{2j}{u} B_u B_{2j-u}, \qquad N_{2j+1} = 0. \tag{12}$$

In particular $N_0 = B_0^2$.

The primary computation retains $1 \le n \le 7$ and encloses every omitted term before forming products. For integers $k, r \ge 0$,

$$B_k^{(r)} = \sum_{n\ge 1} n^2 e^{-(n^2-1)x} \sum_{\ell=0}^{\min(r,k+1)} \binom{r}{\ell} n^{2\ell} (-(n^2-1))^{r-\ell} R_k^{(\ell)}(n^2x). \tag{13}$$

Write $R_k(z) = \sum_{i=0}^{k+1} r_{ki} z^i$ and put

$$H_{k,r}(x) = \sum_{\ell=0}^{\min(r,k+1)} \binom{r}{\ell} \sum_{i=\ell}^{k+1} |r_{ki}| \frac{i!}{(i-\ell)!} x^{i-\ell} 8^{2i+2r-2\ell+2}, \tag{14}$$
$$\rho_{k,r}(x) = (9/8)^{2k+2r+4} e^{-17x}.$$

For $\rho_{k,r} < 1$, the absolute omitted tail is bounded by

$$\frac{e^{-63x}}{1-\rho_{k,r}(x)} H_{k,r}(x). \tag{15}$$

Indeed, replacing $(n^2-1)^{r-\ell}$ by $n^{2(r-\ell)}$ leaves exponents in n of at most $2k+2r+4$. For $n \ge 8$, each successive ratio is at most $\rho_{k,r}$. The $[3,8]$ cover uses $k \le 28$; the $[8,12]$ engine supports degrees through 22, hence $k \le 44$. With $r \le 9$ and $x \ge 3$, all these ratios are below $1/2$. For point moments and their first center derivatives, the common exponent $4 \cdot 22 + 6 = 94$ also bounds the omitted-term ratios. Taylor derivatives use the individual exponents $2k + 2r + 4$. Symmetric tail intervals are added before every product in (12), including all mixed tails.

The original cover uses 90-digit directed decimal intervals; the extension uses 140 digits. For exponential bounds, reduce t to $u = t/2^q < 1$ and sum e^u through order 80. The positive remainder is bounded by $u^{81}/(81!(1-u/82))$. Positive reciprocal bounds followed by q outward squarings enclose e^{-t}. No floating-point sign estimate enters an acceptance decision.

3.2 Centered Taylor cover

For $J = [l, r]$, choose a rational $c \in J$ and put $h = x - c$. Taylor's theorem gives

$$B_k(c+h) \in \sum_{u=0}^{L} \frac{B_k^{(u)}(c)}{u!} h^u + \frac{B_k^{(L+1)}(J)}{(L+1)!} h^{L+1}, \tag{16}$$

with every coefficient enclosed using (15). Products are retained without truncation. The derivatives N_{2j}' are assembled separately from Taylor enclosures for B_k' and B_k, rather than by differentiating an interval remainder.

On an ordinary cell, choose fixed rational monic polynomials $q_0, \dots, q_d$. Their coefficient matrix is triangular with diagonal one; exact orthogonality is unnecessary. For fixed positive rational scales σ_i, the inequalities

$$\frac{\mathcal{N}_x(q_i^2)}{\sigma_i^2} - \sum_{\substack{j<d \\ j\neq i}} \frac{|\mathcal{N}_x(q_i q_j)|}{\sigma_i \sigma_j} > 0 \qquad (0 \le i < d) \tag{17}$$

prove positive definiteness through degree $d-1$ by symmetric strict diagonal dominance and congruence. A negative upper bound for $\mathcal{N}_x(q_d^2)$ proves $\delta = d$ on the whole cell. Taylor order $L = 8$ is used for these cells.

On each transition window, fixed-basis interval Schur elimination encloses entries and their derivatives with $L = 4$. It verifies all positive prefix pivots, a strict sign for $\widetilde{S}'_d$, and opposite endpoint signs. Separate checks establish the opposite-parity block and higher-degree negative witness in Lemma 3. These numerical derivative bounds concern $\widetilde{S}'_d$, not S'_d on the whole window.

Proof of Theorem 1. The supplied covers have 277 ordinary cells and eleven transition windows on $[3, 8]$, and 108 ordinary cells and three transition windows on $[8, 12]$. They meet only at endpoints and cover exactly $[3, 12]$. On each ordinary cell, the strict interval inequalities (17) and negative witness establish the recorded minimum degree. Each transition window satisfies Lemma 3; independent exact-rational interval checks isolate its zero in the strict bracket of Table 1. The prefix and opposite-parity checks give positive semidefiniteness below the persistent witness at equality, proving (6).

The checker verifies exact adjacency, endpoint degrees, agreement of neighboring regimes, and all root and witness bindings. Thus no intervening region or transition of δ is omitted, and the sequence is (5). Schur factorization gives one null direction in M_{d_i} at each root; the derivative bound gives simplicity. The degree-fifteen region and upper bound everywhere give the maximum. The supplement regenerates the entire cover and checks Table 1 against its explicit rational inputs. □

The independent rational implementation rounds outward to the grid 10^{-120}, retains ten theta summands, and uses an exponential series through order 120 after binary range reduction. It checks every listed root, selected degree-minimal neighborhoods, and the finite-summand and Gram comparisons below. This separately verifies those premises; it is not a second-arithmetic replay of all 385 ordinary cells. A separate 150-digit reproduction covers all 108 extension cells and three windows using the reviewed interval method.

4 Symmetry and the fixed-degree Gaussian limit

Lemma 5. *The density Φ is even and strictly positive on $\mathbb{R}$, and $p_x(t) = p_{\pi^2/x}(t)$ for $x > 0$ and real t.*

Proof. Let $\vartheta(u) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 u}$. Jacobi's transformation $\vartheta(u) = u^{-1/2}\vartheta(1/u)$ makes $F(s) = e^s\vartheta(e^{4s})$ even [4]. Direct differentiation gives $F'' - F = 16\Phi$: the zero lattice term cancels, and the two equal nonzero terms give the factor 16. Hence Φ is even. For $s \ge 0$ each summand of (1) is positive because $2\pi n^2 e^{4s} - 3 > 0$; evenness covers $s < 0$. Replacing x by π^2/x sends s to $-s$ and interchanges the numerator factors of (2). □

Proof of Corollary 2. Differentiate the identity at $t = 0$. The reflection of $[3, 12]$ is $[\pi^2/12, \pi^2/3]$; these intervals overlap. Every ordinary cell and transition window establishes $M_4 > 0$, including equality cases. The witnesses and symmetry give $\delta \le 15$ on the stated union. □

Theorem 6. *For every fixed integer $d \ge 0$, put $T_d(x) = \operatorname{diag}((8x)^{-j/2})_{j=0}^{d}$. Then*

$$T_d(x) M_d(x) T_d(x) \longrightarrow G_d \qquad (x \to \infty), \tag{18}$$

where G_d is the standard real Gaussian moment matrix. For each d there exists X_d such that $M_d(x) > 0$ for $x \ge X_d$ and for $0 < x \le \pi^2/X_d$. Thus $\delta(x)$ escapes every fixed bound at both extremes, allowing the value ∞.

Proof. The single-summand normalized product is

$$p_x^{[1]}(t) = \left(1 - \frac{12x(\cosh(2t) - 1)}{(2x-3)^2}\right) e^{-2x(\cosh(2t)-1)} \qquad (x \geq 3). \tag{19}$$

Under $t = z/\sqrt{8x}$ its quadratic exponent is $-z^2/2$, each exponent coefficient of order $2k > 2$ is $O(x^{1-k})$, and every fixed nonconstant coefficient of the prefactor tends to zero. Coefficientwise at each fixed order this gives $p_x^{[1]}(z/\sqrt{8x}) \to e^{-z^2/2}$, hence $m_{2j}^{[1]}(x)/(8x)^j \to (2j-1)!!$.

For fixed k and $x \geq 3$, polynomial growth gives

$$B_k(x) = R_k(x) + O_k(x^{k+1}e^{-3x}). \tag{20}$$

Factor e^{-3x} from $n \geq 2$ and bound the remaining polynomial-weighted series by its convergent value at $x = 3$. Since $B_0 \sim 2x$, substitution into (12), including the denominator change, gives

$$m_{2j}(x) - m_{2j}^{[1]}(x) = O_j(x^{2j}e^{-3x}).$$

After division by $(8x)^j$ this proves (18); odd entries vanish identically. For every nonzero real polynomial P of degree at most d, the Gaussian integral of P^2 is positive, so $G_d > 0$. Entrywise convergence in fixed finite dimension preserves strict positive definiteness for sufficiently large x. Invertible congruence and Lemma 5 finish the proof. □

The threshold depends on d. The theorem asserts neither positivity of all degrees at one center nor existence of a finite obstruction at every center outside the classified domain.

5 The two-summand quintic mechanism

Write $p_x^{[N]}$ for the normalized product retaining $n \leq N$ in (1), and use $[N]$ for its moments and contractions.

Theorem 7. *For all $x \geq 3$, $M_5^{[1]}(x) > 0$. At $x = 91/25$, the degree-four prefix is positive definite in the one-, two-, and three-summand models and in the full model, while*

$$\begin{aligned}
2418877279 &< S_5^{[1]} < 2418877280,\\
-43193403 &< S_5^{[2]} < -43193402,\\
-45634396 &< S_5^{[3]} < -45634395,\\
-45634396 &< S_5 < -45634395,\\
-2440994 &< S_5 - S_5^{[2]} < -2440993.
\end{aligned}$$

Thus two summands already produce the quintic obstruction at this center, and the remaining infinite sum preserves it.

Proof. Put $D = (2x-3)^2$ and

$$Q_j = (-1)^j \sum_{u=0}^{2j} (-1)^u \binom{2j}{u} R_u R_{2j-u}, \qquad m_{2j}^{[1]} = Q_j/D.$$

For $e \in \{0,1\}$ and $k \leq 3$, the leading parity minor is

$$D^{-k} \det(Q_{i+j+e})_{i,j=0}^{k-1} = D^{-k} c_{e,k} (2x-3)^{h_{e,k}} F_{e,k}(x-3).$$

Exact polynomial arithmetic gives Table 2. Every coefficient of each $F_{e,k}$ is strictly positive. Thus all six leading parity minors are positive for $x \geq 3$, and Sylvester's criterion gives $M_5^{[1]} > 0$. The supplement

records all coefficients, reconstructs the determinants, and independently checks the moment numerators from (19).

At $x = 91/25$, evaluate (11)–(12) with the indicated finite sums and full tail bounds. With $u = m_2$, $v = m_4$, $w = m_6$, $y = m_8$, $z = m_{10}$,

$$S_5 = z - \frac{w^3 - 2vwy + uy^2}{uw - v^2}. \tag{21}$$

The interval checks give $u > 0$, $uw - v^2 > 0$, $v - u^2 > 0$, and $\det \left(\begin{smallmatrix} 1 & u & v \\ u & v & w \\ v & w & y \end{smallmatrix} \right) > 0$, which establish the degree-four prefix. Formula (21) gives the printed strict bounds. A separate exact-rational implementation reproduces these contraction bounds. □

e	k	$c_{e,k}$	$h_{e,k}$	$\deg F_{e,k}$	Minimum coefficient of $F_{e,k}$
0	1	1	2	0	1
0	2	32	0	6	64
0	3	393216	0	12	2048
1	1	8	0	3	4
1	2	6144	0	8	64
1	3	6039797760	0	15	1024

Table 2: Exact single-summand certificate after $x = 3 + y$. The minimum ranges over all integer coefficients, including the constant term.

The comparison concerns the entire normalized two-summand product. It does not isolate cross terms from the second-summand square. Small scalar corrections need not be small in the high derivatives and Schur cancellations determining the witness degree.

6 Quantitative finite-summand transition bounds

Let $\widetilde{M}_d = B_0^2 M_d$ and let $\widetilde{M}_d^{[N]}$ use the first N terms of (11). Their last monic Schur contractions are $\widetilde{S}_d$ and $\widetilde{S}_d^{[N]}$. The normalizing scalars are positive, so the roots and definiteness agree with the corresponding normalized matrices. Derivative bounds in this section are unnormalized.

Lemma 8. *Let two real symmetric block matrices be*

$$K_N = \begin{pmatrix} H_N & b_N \\ b_N^{\mathsf{T}} & w_N \end{pmatrix}, \qquad K = K_N + E,$$

in the same fixed basis, with last Schur contractions T_N, T. Suppose

$$H_N \geq \lambda I, \quad \|E_H\|_2 \leq \epsilon < \lambda, \quad \|b_N\|_2 \leq \beta, \quad \|E_b\|_2 \leq \gamma, \quad |E_w| \leq \zeta.$$

Then $H > 0$ and

$$|T - T_N| \leq B := \zeta + \frac{2\beta\gamma + \gamma^2}{\lambda - \epsilon} + \frac{\beta^2 \epsilon}{\lambda(\lambda - \epsilon)}. \tag{22}$$

If $T_N(\alpha) = 0$, $T(a) = 0$, both roots lie in an interval J where $H > 0$, and T' has constant sign with $|T'| \geq \rho > 0$, then a bound (22) at α gives

$$|a - \alpha| \leq B/\rho. \tag{23}$$

Existence of the roots is an additional hypothesis.

Proof. The prefix eigenvalue bound gives $\|H^{-1}\|_2 \leq (\lambda - \epsilon)^{-1}$. Expand the perturbed quadratic $b^{\mathsf{T}} H^{-1} b$ and use $H^{-1} - H_N^{-1} = -H^{-1} E_H H_N^{-1}$. The column terms are bounded by $(2\beta\gamma + \gamma^2)/(\lambda - \epsilon)$, and the inverse correction by the last term of (22). This argument requires positive prefixes, not positivity of the terminal matrices; compare the classical perturbation setting in [6]. At α, $|T(\alpha)| \leq B$. Integrating the signed derivative between a and α gives (23). □

An exact residual formula also records how the correction enters. Put $u_N = -H_N^{-1} b_N$, $q_N = (u_N, 1)$, $r = E_H u_N + E_b$ and $z = H^{-1} r$. Completing the square gives

$$T - T_N = q_N^{\mathsf{T}} E q_N - r^{\mathsf{T}} H^{-1} r. \tag{24}$$

For C^1 families the envelope identity yields

$$|T' - T_N'| \le |q_N^{\mathsf{T}} E' q_N| + \|K'\|_2 \big(2\|q_N\|_2 \|z\|_2 + \|z\|_2^2\big). \tag{25}$$

Indeed, the full minimizing vector is $q_N - (z, 0)$; substitution in $T' = q^{\mathsf{T}} K' q$ proves the bound. No derivative of the optimizing coefficients is needed. The numerical certificates below use (22) and directly enclosed Schur derivatives, rather than assuming the correction is small from the size of the undifferentiated theta remainder.

Theorem 9. *For each crossing a_i in Table 1 and $N \in \{2, 3\}$, there is a certified locally matched simple root $a_i^{[N]}$ of $\widetilde{S}_{d_i}^{[N]}$. The full prefix is positive definite and its Schur derivative has constant strict sign on the interval joining the two isolated roots. The corresponding entry of Table 3 is a strict upper bound for $|a_i - a_i^{[N]}|$. In particular, for all fourteen crossings,*

$$|a_i - a_i^{[2]}| < 0.00049, \qquad |a_i - a_i^{[3]}| < 2.588 \times 10^{-10}.$$

Proof. Finite sums use the exact derivative formula (13) restricted to $n \le N$; the full matrices use (15). Centered fourth-order Taylor enclosures give both matrices in a common fixed monic polynomial basis. Positive rational prefix scales preserve the monic Schur contraction. Gershgorin row margins give λ; maximum absolute row sums bound ϵ by symmetry, and column one-norms bound β, γ. Directed evaluation gives ζ and (22) on the finite-root bracket.

For each of the 28 comparisons, opposite endpoint signs and positive prefix pivots certify a root in each bracket; a strict derivative certifies simplicity. A separate full derivative enclosure on the entire connecting interval supplies ρ. Lemma 8 then proves the displacement. The displayed bounds are rounded upward; for the three new crossings the reporting bound also allows both rational bracket widths. Independent rational calculations reconstruct the root brackets and check the norm and quotient inequalities. These statements give local root correspondence, not a complete classification of the truncated models. □

i	d_i	Two-summand root error	Three-summand root error
1	5	0.0004562803	2.812445×10^{-11}
2	6	0.00008398458	1.808747×10^{-12}
3	7	0.0001093035	1.796750×10^{-11}
4	9	0.0004899474	2.587364×10^{-10}
5	9	0.00002086967	1.318493×10^{-14}
6	9	1.643740×10^{-7}	3.232979×10^{-19}
7	10	1.871803×10^{-9}	4.412471×10^{-21}
8	11	1.658166×10^{-7}	1.826244×10^{-20}
9	13	1.421748×10^{-7}	1.037696×10^{-19}
10	13	4.826886×10^{-9}	9.954380×10^{-22}
11	12	2.597710×10^{-8}	1.166489×10^{-22}
12	12	5.339×10^{-11}	8.103×10^{-28}
13	13	1.613×10^{-12}	2.430×10^{-32}
14	14	4.064×10^{-14}	9.711×10^{-37}

Table 3: Strict upper bounds for locally matched root displacements. These compare roots of the full and finite-summand monic Schur contractions, not uniform errors between the product curves.

7 Explicit robust finite Gram certificates

For an even real C^{2d+2} function p near zero, define the signed derivative matrix as in (3).

Proposition 10. *Suppose $q_j \in \mathbb{Q}$ vanish unless $j \equiv e \pmod 2$, with $e \in \{0,1\}$, and $q^{\mathsf{T}} M_d q \le -\eta < 0$. Let $R > 0$ and $C_n \ge \sup_{|t|\le R} |p^{(n)}(t)|$ for the even derivative orders required in*

$$B_G = \frac{1}{2} \sum_{j,k=0}^{d} |q_j q_k| \max(j,k)^2 C_{j+k+2}. \tag{26}$$

For rational $h > 0$ with $dh \le R$ and $h^2 B_G \le \eta/2$, put

$$c_r = \sum_{\substack{r \le j \le d \\ j \equiv e \ (2)}} q_j (-1)^{(j-e)/2} h^{-j} (-1)^{j-r} \binom{j}{r} \quad (0 \le r \le d). \tag{27}$$

Then $G_{rs} = p((r-s)h)$ satisfies $c^{\mathsf{T}} G c \le -\eta/2$.

Proof. Let $(F_h)_{j,r} = h^{-j}(-1)^{j-r}\binom{j}{r}$ for $r \le j$, and zero otherwise. Repeated finite differences give

$$(F_h G F_h^{\mathsf{T}})_{jk} = (-1)^k \int_{[0,1]^{j+k}} p^{(j+k)} \left(h \left(\sum_{a=1}^{j} u_a - \sum_{b=1}^{k} v_b \right) \right) du\, dv.$$

For $j = k = 0$ use evaluation at zero. Every argument has absolute value at most $\max(j,k)h$. For the nonzero terms in q, $j + k$ is even, so $p^{(j+k+1)}(0) = 0$. Taylor's theorem bounds the entry error from $(-1)^k p^{(j+k)}(0)$ by $h^2 \max(j,k)^2 C_{j+k+2}/2$. For $U = \operatorname{diag}(1, i, \dots, i^d)$, unitary congruence changes the limiting matrix to M_d. The row $q^{\mathsf{T}} U F_h$ has common phase i^e and real component (27). Thus $|c^{\mathsf{T}} G c - q^{\mathsf{T}} M_d q| \le h^2 B_G$. □

Every negative polynomial witness for an even functional has a negative parity component. Rational approximation preserves strict negativity, and halving a rational h eventually meets the conditions. Consequently, every $x \in [\pi^2/12, 12]$ has a negative finite translation Gram matrix, with at most sixteen points. This is an upper bound on witness size.

Proposition 11. *If $c^{\mathsf{T}} G c \le -\eta_G < 0$ and E is real symmetric with $\max_{r,s} |E_{rs}| \le \varepsilon$, then*

$$c^{\mathsf{T}} (G + E) c \le -\eta_G + \varepsilon \|c\|_1^2. \tag{28}$$

In particular, $\varepsilon < \eta_G / \|c\|_1^2$ preserves a negative direction.

Proof. Expand $c^{\mathsf{T}} E c$ and apply the triangle inequality. □

Theorem 12. *At $x = 73/20$, let $h = 7/100$ and*

$$c = (0.092261117166, -0.398268873146, 0.812961093973, -1,\\ 0.779991144295, -0.365581365903, 0.080511577478).$$

All displayed coefficients are exact rationals. For $G_{rs} = p_{73/20}((r-s)h)$, $0 \le r, s \le 6$,

$$c^{\mathsf{T}} G c < -4.8371210608 \times 10^{-6}, \qquad \|c\|_1 = 3.529575171961.$$

The matrix remains indefinite under every real symmetric perturbation whose absolute entrywise error is at most 1.94×10^{-7}. The supplement gives twenty rational certificates, including six with twelve-decimal coefficients and a sixteen-point witness at $x = 12$.

Proof. With $A(y) = B_0(y)$, the exact sampled product is

$$p_x(t) = \frac{A(xe^{2t})A(xe^{-2t})}{A(x)^2} e^{-x(e^{2t}+e^{-2t}-2)}. \tag{29}$$

Retain $1 \le n \le 7$ in $A(y)$. For $y > 3/128$ the omitted terms are positive and bounded by $2yn^4e^{-(n^2-1)y}$. If $r_y = (9/8)^4e^{-17y} < 1$, their sum lies in

$$\left[0, \frac{2y\,8^4e^{-63y}}{1-r_y}\right]. \tag{30}$$

The retained terms use their actual signs. Each sampled argument is checked against these hypotheses, and the positive normalizing denominator is enclosed before division. Directed evaluation of the literal vector gives

$$-4.837121060813 \times 10^{-6} < c^{\mathsf{T}}Gc < -4.837121060812 \times 10^{-6}.$$

The stated upper bound follows. Inequality (28) and the exact norm give the inclusive error tolerance with strict remaining margin. The diagonal entries remain positive, so the perturbed matrix is indefinite. A separate Fraction implementation retains ten summands and recomputes the quadratic value and tolerance for every literal certificate. The twenty certificates include alternative vectors at repeated centers; they do not represent twenty distinct centers. Error tolerances are absolute entrywise bounds and vary substantially near transitions. These are theta-specific instances of finite sampled positivity tests; related error-aware tests occur in [7]. □

Reproducibility and scope

The standard-library supplement has one source-only reproduction command and a manuscript-binding checker. Table 4 gives the main theorem-to-program correspondence; paths are relative to `supplement/`. The fresh run regenerates 385 ordinary cells, fourteen transition windows, 28 finite/full root comparisons and all twenty Gram certificates. Independent rational root and Gram calculations use separate arithmetic. Source identities, execution records, rejected malformed inputs and separate proof, code and literature reviews are retained with the package.

Result	Principal generator or verifier
Classification and equality	`baseline/reproduce_all.py`; `extension/classification/reproduce.py`
Single-summand positivity	`baseline/one_summand.py`
Finite-summand root bounds	`extension/transition/summand_transition_bounds.py`; `extension/transition/extension_transition_bounds.py`
Literal Gram certificates	`extension/gram/replay_gram.py`; `extension/review/rational_gram.py`
Independent root checks	`extension/review/rational_transitions.py`

Table 4: Proof dependencies and executable certificate locations.

The classification is complete on the stated compact domain.

AI assistance. AI assisted derivation, verification code, and preparation; the author takes responsibility for the manuscript.

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